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Research article

Integral inequalities involving a new class of generalized strongly modified (p, h)-convex functions

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Abstract: A novel class of generalized strongly modified (GSM) (p, h)-convex functions (CFs) was presented in paper and its fundamental properties were established. Schur, Hermite-Hadamard (H-H), and Fejér inequalities were proved for this new notion of convexity. Several illustrations have been incorporated by selecting several GSM (p, h)-CFs to substantiate the existence and feasibility of Schur, H-H, and Fejér-type inequalities. These inequalities are valuable resources for analyzing the characteristics of newly defined GSM (p, h)-CFs. A comparison was given to show that the results of this study represent a significant improvement over those of earlier publications.

Keywords: convex function; Schur-type inequalities; H-H inequalities; Fejér-type inequalities; optimization
Mathematics Subject Classification: 26D07, 26D15, 26E70

1. Introduction

An essential notion in arithmetics is convexity, having critical importance in many fields, together with optimization [1], analysis [2, 3], and geometry [4]. The concept pertains to the characteristic of a function or a set, wherein the line segment linking any two points on the function or within the set remains completely above the function or within the set. This unique behavior leads to numerous crucial inequalities which have full-size programs in unique regions [5–8].

Schur-type inequalities refer to a category of mathematical inequalities, which are named after the mathematician Issai Schur [9]. These inequalities establish relationships between the values of variables in a given system. The unique shape of the inequalities is based on the context and the variables concerned [10]. Schur-type inequalities provide bounds or regulations on the values of variables based on their relationships with different variables or parameters inside the system.

These inequalities are substantially implemented in diverse disciplines of mathematics, along with analysis [11, 12], optimization [13, 14], and mathematical physics. Schur-type inequalities have applications in numerous areas, such as matrix principles [15] and mathematical modeling. Many researchers have frequently used these inequalities to establish optimization criteria, or to derive bounds on certain quantities of interest. Regularly, Schur-type inequalities play a considerable role in mathematical research and provide valuable insights into the relationships among variables in a given system.

H-H inequalities were first proposed by Charles Hermite and Jacques Hadamard in 1885 [16]. For a convex function $\xi : B_1 \subset \mathbb{R} \to \mathbb{R}$, these inequalities are presented as [17–20]:

$$\xi\left(\frac{\phi_1+\phi_2}{2}\right) \le \frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2} \xi(\alpha)d\alpha \le \frac{\xi(\phi_1)+\xi(\phi_2)}{2}, \qquad for\, any \quad \phi_1, \phi_2 \in B_1.$$

Moreover, Fejér-type inequalities are classified as a set of inequalities that extend and broaden the notion of convexity [21, 22]. This inequality is a mathematical result for the convex function that establishes a maximum limit on the difference between the function's value at the average of its arguments and the function's average of its values [20]. In simpler terms, it measures the proximity of the average of a convex function to the function's evaluation at the average of its inputs.

For a convex function, this inequality is expressed as [20]:

$$\xi\left(\frac{\phi_1+\phi_2}{2}\right)\int_{\phi_1}^{\phi_2} \Phi(\alpha)d\alpha \le \int_{\phi_1}^{\phi_2} \xi(\alpha)\Phi(\alpha)d\alpha \le \frac{\xi(\phi_1)+\xi(\phi_2)}{2}\int_{\phi_1}^{\phi_2} \Phi(\alpha)d\alpha, \quad for\, any \quad \phi_1, \phi_2 \in B_1,$$

where, $\Phi : B_1 \subset \mathbb{R} \to \mathbb{R}$ is a non-negative integrable function, which is symmetric about $\frac{\phi_1+\phi_2}{2}$. The applications of these inequalities extends to fourier analysis [23] and harmonic analysis [24]. Comprehending the characteristics of functions and their approximations necessitates a recognition of these inequalities.

Furthermore, concerning the different classes of convexity, S. Varošanec proposed the perception of h-CFs [25]. G. Toader first proposed the theory of modified h-CFs [26]. Zhang et al. proposed p-CFs [27]. Feng, B. et al. (2020) established the notion of modified (p, h)-CFs [28]. The notion of generalized p-CFs was pioneered by Jung et al. in 2020. Saleem et. al. (2020) introduced generalized strongly (GS) p-CFs. GS modified h-CFs were explored by Zhao et al. in 2020. Motivated by these established classes of convex functions, we have pioneered a generalized class of convexity that generalizes these existing classes.

The objective of the paper is to investigate GSM (p, h)-CFs, as well as to explore their fundamental properties. Furthermore, we aim to establish the Schur, H-H, and Fejér-type inequalities for this new notion of convexity.

The paper is categorized as follows: Section 2 describes different classes of CFs. In Section 3, we define and prove the properties of GSM (p, h)-CFs. Section 4 presents the proof of the Schur inequality for the pristine class of convexity. Section 5 provides the proof of H-H inequalities for the established notion of CFs. In Section 6, we demonstrate the Fejér-type inequalities. Finally, Section 8 gives an overview of the the entire work.

2. Preliminaries

The following details are employed throughout the paper and results.

(a) Assume that $h : [0, 1] \to \mathbb{R}^+$ is a non-negative function;
(b) Let $\eta : J \times K \to \mathbb{R}$ be a bi-function for appropriate $J, K \subset \mathbb{R}$;
(c) Let $\Phi : B_1 \subset \mathbb{R} \to \mathbb{R}$ be a real-valued function;
(d) Assume $p \geq 1$ and $r \in [0, 1]$.

Now, some definitions of introduced classes of convex functions are recalled.
Convex function [29]:
A function Φ is a convex function if

$$\Phi(ru_1 + (1 - r)u_2) \leq r\Phi(u_1) + (1 - r)\Phi(u_2) \tag{2.1}$$

holds, $\forall\, u_1, u_2 \in B_1$.
Modified h-convex function [26]:
A function Φ is a modified h-CF if

$$\Phi(ru_1 + (1 - r)u_2) \leq h(r)\Phi(u_1) + (1 - h(r))\Phi(u_2) \tag{2.2}$$

holds, $\forall\, u_1, u_2 \in B_1$.
p-convex function [30]:
The function Φ is a p-CF if

$$\Phi\left((ru_1^p + (1 - r)u_2^p)^{1/p}\right) \leq r\Phi(u_1) + (1 - r)\Phi(u_2) \tag{2.3}$$

holds, $\forall\, u_1, u_2 \in B_1$.
Modified (p, h)-convex function [28]:
The function Φ is a modified (p, h)-CF if

$$\Phi\left((ru_1^p + (1 - r)u_2^p)^{1/p}\right) \leq h(r)\Phi(u_1) + (1 - h(r))\Phi(u_2) \tag{2.4}$$

holds, $\forall\, u_1, u_2 \in B_1$.
Generalized modified h-convex function [31]:
A function Φ is a generalized modified h-CF if

$$\Phi(ru_1 + (1 - r)u_2) \leq \Phi(u_2) + h(r)\eta(\Phi(u_1), \Phi(u_2)) \tag{2.5}$$

holds, $\forall\, u_1, u_2 \in B_1$.
GSM h-convex function [31]:
A function Φ is a GSM h-CF with modulus l_1 if

$$\Phi(ru_1 + (1 - r)u_2) \leq \Phi(u_2) + h(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1 - r)(u_1 - u_2)^2 \tag{2.6}$$

holds, $\forall\, u_1, u_2 \in B_1$ and $l_1 > 0$.
Generalized p-convex function [32]:
A function Φ is a generalized p-CF if

$$\Phi\left((ru_1^p + (1 - r)u_2^p)^{1/p}\right) \leq \Phi(u_2) + r\eta(\Phi(u_1), \Phi(u_2)) \tag{2.7}$$

holds, $\forall\, u_1, u_2 \in B_1$.
Generalized strongly p-convex function [33]:
A function Φ is a GS p-CF with modulus l_1 if

$$\Phi\left((r{u_1}^p + (1-r){u_2}^p)^{1/p}\right) \leq \Phi(u_2) + r\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1-r)({u_1}^p - {u_2}^p)^2 \tag{2.8}$$

holds, $\forall\, u_1, u_2 \in B_1$ and $l_1 > 0$.
Super multiplicative function [15]:
A function Φ is called super multiplicative if

$$\Phi(u_1 u_2) \geq \Phi(u_1)\Phi(u_2),\ \forall\, u_1, u_2 \in B_1. \tag{2.9}$$

3. Main results

Now, we begin with the thought of GSM (p, h)-CFs in the following definition.
A function Φ is a GSM (p, h)-CF with modulus l_1 if

$$\Phi\left((ru_1^p + (1-r)u_2^p)^{1/p}\right) \leq \Phi(u_2) + h(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1-r)(u_1^p - u_2^p)^2 \tag{3.1}$$

holds for all $u_1, u_2 \in B_1$.

Remark 3.1. *(a) By assuming $p = 1$ in* (3.1)*, one gets the GSM h-CF.*
(b) By substituting $p = 1$ and $l_1 = 0$ in (3.1)*, one gets the generalized modified h-CF.*
(c) By assuming $h(r) = r$ in (3.1)*, one has the generalized strongly p-CF.*
(d) By putting $h(r) = r$ and $l_1 = 0$ in (3.1)*, one has the generalized p-CF.*
(e) By assuming $h(r) = r$, $p = 1$, $l_1 = 0$, and $\eta(a, b) = a - b$ in (3.1)*, one gets the convex function.*

Note:

(a) The x-axis presents the values of u_1.
(b) The values of u_2 are present along the y-axis.
(c) The values of expression involving on the left-hand and right-hand sides of inequalities (3.2)–(3.4) are given along the z-axis.

Example 3.2. *For $u_1, u_2 \in [1, \infty)$, $r \in (0, 1]$, $l_1 > 0$, $p \geq 1$, $h(r) \geq r$, and $\eta(u_1, u_2) = 2u_1 + u_2$, the function $\Phi(x) = x^4$ is a GSM (p, h)-CF.*

If we suppose $u_1, u_2 \in [1, \infty)$, $r = \frac{1}{2}$, $p = 2$, $h(r) = \frac{1}{r^2}$, and $l_1 = \frac{1}{2}$ in (3.1)*, then*

$$\frac{u_1^4}{4} + \frac{u_2^4}{4} + \frac{u_1^2 u_2^2}{2} \leq 8u_1^4 + 5u_2^4 - \frac{u_1^4}{8} - \frac{u_2^4}{8} + \frac{u_1^2 u_2^2}{4}. \tag{3.2}$$

The validity of inequality (3.2) is depicted in Figure 1.

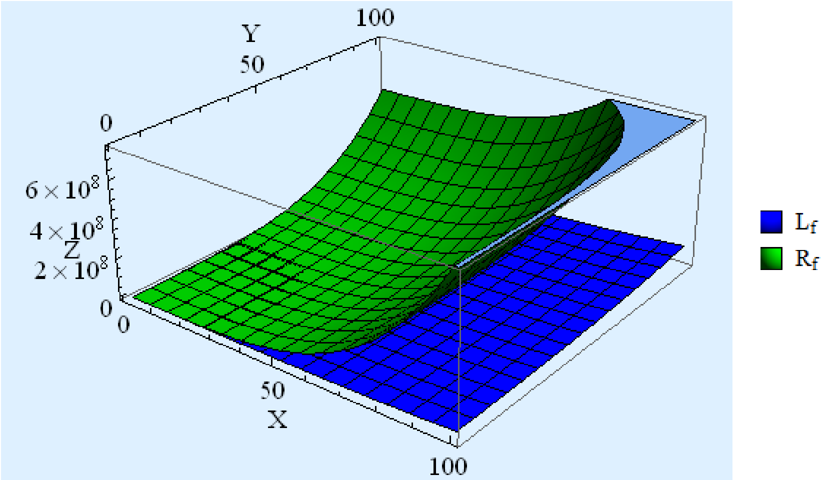

Figure 1. The graphical representations of functions involved on both sides of inequality (3.2).

Example 3.3. *Suppose* $u_1, u_2 \in [1, \infty)$, $r \in (0, 1]$, $l_1 > 0$, $p \geq 1$, $h(r) \geq r$, *and* $\eta(u_1, u_2) = \frac{3}{u_1^2} + \frac{3}{u_2^2}$. *Then the function* $\Phi(x) = x^{-4}$ *is a GSM* (p, h)*-CF.*
Choose $u_1, u_2 \in [1, \infty)$, $r = \frac{1}{2}$, $p = 2$, $h(r) = \frac{1}{r^2}$, *and* $l_1 = \frac{1}{2}$ *in* (3.1) *to get*

$$\left(\frac{2}{u_1^2 + u_2^2}\right)^2 \leq 12u_1^4 + 12u_2^4 + \frac{1}{u_2^4} - \frac{u_1^4}{8} - \frac{u_2^4}{8} + \frac{u_1^2 u_2^2}{4}. \tag{3.3}$$

The viability of inequality (3.3) is shown in Figure 2.

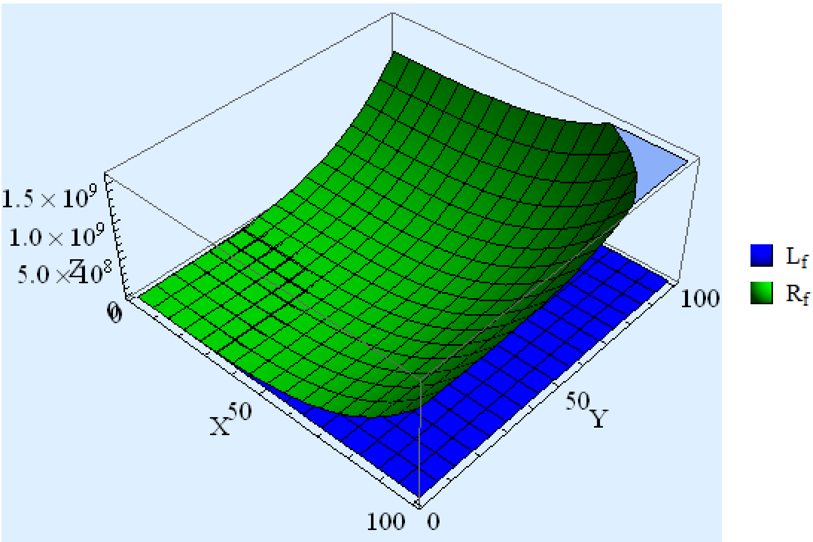

Figure 2. The graph of inequality (3.3).

Example 3.4. *Assume* $u_1, u_2 \in [1, 13]$, $r \in (0, 1]$, $l_1 > 0$, $p \geq 1$, $h(r) \geq r$, *and* $\eta(u_1, u_2) = 8u_1 + 5u_2$, *and then the function* $\Phi(x) = (1 + x)^2$ *is a GSM* (p, h)*-CF.*

Suppose $u_1, u_2 \in [1, 13]$, $r = \frac{1}{2}$, $p = 2$, $h(r) = \frac{1}{r}$, *and* $l_1 = \frac{1}{2}$ *in* (3.1), *and then*

$$\left(1 + \left(\frac{u_1^2}{2} + \frac{u_2^2}{2}\right)^{1/2}\right)^2 \leq 32(1 + u_1)^2 + 21(1 + u_2)^2 - \frac{(u_1^2 - u_2^2)^2}{8}. \tag{3.4}$$

Figure 3 reflects the credibility of inequality (3.4).

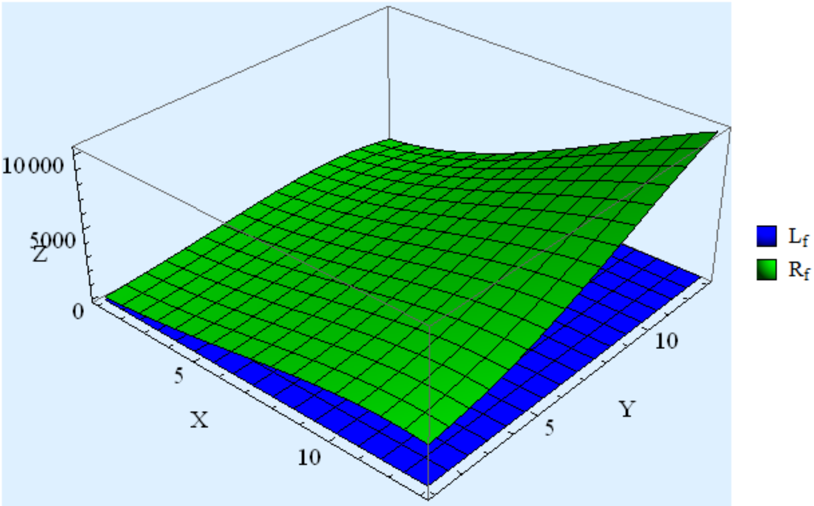

Figure 3. The figure of inequality (3.4).

A few fundamental characteristics of this new class of convexity are now demonstrated.

Lemma 3.5. *Let* Φ *and* ξ *be GSM* (p, h)*-CFs, and then their sum is also a GSM* (p, h)*-CF.*

Proof. For $u_1, u_2 \in B_1$ and $l_1 > 0$,

$$(\Phi + \xi)\Big((ru_1^p + (1 - r)u_2^p)^{1/p}\Big) = \Phi\Big((ru_1^p + (1 - r)u_2^p)^{1/p}\Big) + \xi\Big((ru_1^p + (1 - r)u_2^p)^{1/p}\Big).$$

As Φ and ξ are a GSM (p, h)-CF,

$$\begin{aligned}&(\Phi + \xi)\Big((ru_1^p + (1 - r)u_2^p)^{1/p}\Big)\\ &\leq (\Phi(u_2) + h(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1 - r)(u_1^p - u_2^p)^2)\\ &+ (\xi(u_2) + h(r)\eta(\xi(u_1), \xi(u_2)) - l_1 r(1 - r)(u_1^p - u_2^p)^2)\\ &\leq (\Phi + \xi)(u_2) + h(r)\eta((\Phi + \xi)(u_1), (\Phi + \xi)(u_2)) - (l_1 r(1 - r)(u_1^p - u_2^p)^2).\end{aligned}$$

□

Lemma 3.6. *Assume that* Φ *is a GSM* (p, h)*-CF, and then for* $n > 0$, $n\Phi$ *is also a GSM* (p, h)*-CF.*

Proof. For $u_1, u_2 \in B_1$ and $l_1 > 0$,

$$\begin{aligned}&n\Phi\left((ru_1^p + (1 - r)u_2^p)^{1/p}\right)\\ &\leq n(\Phi(u_2) + h(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1 - r)(u_1^p - u_2^p)^2)\\ &\leq n\Phi(u_2) + h(r)\eta(n \cdot \Phi(u_1), n \cdot \Phi(u_2)) - nl_1 r(1 - r)(u_1^p - u_2^p)^2.\end{aligned}$$

□

Lemma 3.7. *Assume that h_1, h_2 are two non-zero functions such that $h_2(r) \leq h_1(r)$. Let $\Phi : B_1 \subset \mathbb{R} \to \mathbb{R}$ be a GSM (p, h_2)-CF, and then Φ is also GSM a (p, h_1)-CF.*

Proof. For $u_1, u_2 \in B_1$ and $l_1 > 0$,

$$\begin{aligned}
&\Phi\left((ru_1^p + (1-r)u_2^p)^{1/p}\right) \\
&\leq \Phi(u_2) + h_2(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1-r)(u_1^p - u_2^p)^2 \\
&\leq \Phi(u_2) + h_1(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1-r)(u_1^p - u_2^p)^2.
\end{aligned}$$

□

Lemma 3.8. *Let $\Phi_l : B_1 \subset \mathbb{R} \to \mathbb{R}$ be GSM (p, h)-CFs for $l \in \{1, 2, \ldots, d\}$ and $\sum_{l=1}^{d} n_l = 1$ for $d \in \mathbb{N}$, and then their linear combination $W : \mathbb{R} \to \mathbb{R}$ defined by $W(s) = \sum_{l=1}^{d} n_l \Phi_l(s)$ is also a GSM (p, h)-CF.*

Proof. By choosing $u_1, u_2 \in B_1$, $l_1 > 0$ and $s = (ru_1^p + (1-r)u_2^p)^{1/p}$ in

$$W(s) = \sum_{l=1}^{d} n_l \Phi_l(s),$$

one gets

$$\begin{aligned}
&W\Big((ru_1^p + (1-r)u_2^p)^{1/p}\Big) \\
&= \sum_{l=1}^{d} n_l \left(\Phi_l\Big((ru_1^p + (1-r)u_2^p)^{1/p}\Big)\right) \\
&\leq \sum_{l=1}^{d} n_l(\Phi_l(u_2)) + h(r)\eta\left(\sum_{l=1}^{d} n_l(\Phi_l(u_1)), \sum_{l=1}^{d} n_l(\Phi_l(u_2))\right) - \sum_{l=1}^{d} n_l l_1 r(1-r)(u_1^p - u_2^p)^2 \\
&= W(u_2) + h(r)\eta(W(u_1), W(u_2)) - l_1 r(1-r)(u_1^p - u_2^p)^2.
\end{aligned}$$

□

Lemma 3.9. *Let $\{\Phi_s : B_1 \subset \mathbb{R} \to \mathbb{R}; s \in \mathbb{N}\}$ be a non-empty collection of GSM (p, h)-CFs and $\forall$ $w \in B_1$, $\sup_{s\in\mathbb{N}} \Phi_s(w)$ exists in $\mathbb{R}$. Subsequently, the function $\Phi : B_1 \subset \mathbb{R} \to \mathbb{R}$ defined by*

$$\Phi(w) = \sup_{s\in\mathbb{N}} \Phi_s(w) \tag{3.5}$$

for all $w \in B_1$ is also a GSM (p, h)-CF.

Proof. For $u_1, u_2 \in B_1$, $l_1 > 0$, and $w = (ru_1^p + (1-r)u_2^p)^{1/p}$ in (3.5),

$$\begin{aligned}
&\Phi\Big((ru_1^p + (1-r)u_2^p)^{1/p}\Big) \\
&= \sup_{s\in\mathbb{N}} \Phi_s\Big((ru_1^p + (1-r)u_2^p)^{1/p}\Big) \\
&\leq \sup_{s\in\mathbb{N}} \Phi_s(u_2) + h(u)\eta\left(\sup_{s\in\mathbb{N}} \Phi_s(u_1), \sup_{s\in\mathbb{N}} \Phi_s(u_2)\right) - l_1 r(1-r)(u_1^p - u_2^p)^2 \\
&= \Phi(u_2) + h(r)\eta\left(\Phi(u_1), \Phi(u_2)\right) - l_1 r(1-r)(u_1^p - u_2^p)^2.
\end{aligned}$$

□

Lemma 3.10. *Let Φ be a GSM (p,h)-CF and presume that $\eta(s,w) = -\eta(w,s)$. Then $\Phi(u_1^p + u_2^p - w^p)^{1/p} \leq \Phi(u_1) + \Phi(u_2) - \Phi(w)$ for all $w \in [u_1, u_2]$, where, $w^p = ru_1^p + (1-r)u_2^p$.*

Proof. For $w \in [u_1, u_2]$,

$$\begin{aligned}
&\Phi\left((u_1^p + u_2^p - w^p)^{1/p}\right) = \Phi\left(((1-r)u_1^p + ru_2^p)^{1/p}\right)\\
&\leq \Phi(u_1) + h(r)\eta(\Phi(u_2), \Phi(u_1)) - l_1 r(1-r)(u_2^p - u_1^p)^2\\
&\leq \Phi(u_1) + \Phi(u_2) - \Phi(u_2) - h(r)\eta(\Phi(u_1), \Phi(u_2)) + l_1 r(1-r)(u_1^p - u_2^p)^2\\
&= \Phi(u_1) + \Phi(u_2) - \left(\Phi(u_2) + h(r)\eta(\Phi(u_1), \Phi(u_2)) - l_1 r(1-r)(u_1^p - u_2^p)^2\right)\\
&= \Phi(u_1) + \Phi(u_2) - \Phi(w).
\end{aligned}$$

□

4. Schur inequality

While studying different kinds of inequalities in mathematics, one cannot ignore the Schur inequality because of its magnificent properties. The inequality involves the algebraic expression which always remains greater than or equal to zero. For the GSM (p,h)-CF, the Schur inequality is established in the next theorem.

Theorem 4.1. *Assume that Φ be a GSM (p,h)-CF, and then for $u_1, u_2, u_3 \in B_1$ such that $u_1 < u_2 < u_3$, $u_3 - u_1, u_3 - u_2, u_2 - u_1 \in B_1$, $p \geq 1$, we have*

$$\begin{aligned}
\Phi\left((u_2^p)^{1/p}\right) \times h(u_3^p - u_1^p) \quad &\leq h(u_3^p - u_1^p)\Phi(u_3) + h(u_3^p - u_2^p)\eta(\Phi(u_1), \Phi(u_3))\\
&- h(u_3^p - u_1^p) l_1 (u_3^p - u_2^p)(u_2^p - u_1^p).
\end{aligned} \tag{4.1}$$

Proof. Let $u_1, u_2, u_3 \in B_1$ be such that $\left(\dfrac{u_3 - u_2}{u_3 - u_1}\right) \in (0,1)$, $\left(\dfrac{u_2 - u_1}{u_3 - u_1}\right) \in (0,1)$, and $\left(\dfrac{u_3 - u_2}{u_3 - u_1} + \dfrac{u_2 - u_1}{u_3 - u_1}\right) = 1$, and then

$$h(u_3 - u_2) = h\left(\frac{u_3 - u_2}{u_3 - u_1} \times (u_3 - u_1)\right) \leq h\left(\frac{u_3 - u_2}{u_3 - u_1}\right) \times h(u_3 - u_1).$$

Suppose $h(u_3 - u_2) > 0$, and by the definition of Φ,

$$\Phi\left((rs^p + (1-r)w^p)^{1/p}\right) \leq \Phi(w) + h(r)\eta(\Phi(s), \Phi(w)) - l_1 r(1-r)(s^p - w^p)^2. \tag{4.2}$$

By assuming $\dfrac{(u_3^p - u_2^p)}{(u_3^p - u_1^p)} = r$, $\quad s = u_1$, and $w = u_3$ in (4.2),

$$\begin{aligned}
\Phi\left(\left(u_2^p\right)^{1/p}\right) &\leq \Phi(u_3) + h\left(\frac{u_3^p - u_2^p}{u_3^p - u_1^p}\right)\eta(\Phi(u_1), \Phi(u_3)) - l_1 \frac{(u_3^p - u_2^p)(u_2^p - u_1^p)}{(u_3^p - u_1^p)^2}(u_1^p - u_3^p)^2\\
&= \Phi(u_3) + \left(\frac{h(u_3^p - u_2^p)}{h(u_3^p - u_1^p)}\right)\eta(\Phi(u_1), \Phi(u_3)) - l_1 (u_3^p - u_2^p)(u_2^p - u_1^p).
\end{aligned}$$

$$\Phi\left((u_2^p)^{1/p}\right) \times h(u_3^p - u_1^p) \le h(u_3^p - u_1^p)\Phi(u_3) + h(u_3^p - u_2^p)\eta(\Phi(u_1), \Phi(u_3)) - h(u_3^p - u_1^p)l_1(u_3^p - u_2^p)(u_2^p - u_1^p). \tag{4.3}$$

Conversely,

$$\Phi\left((u_2^p)^{1/p}\right) \times h(u_3^p - u_1^p) \le h(u_3^p - u_1^p)\Phi(u_3) + h(u_3^p - u_2^p)\eta(\Phi(u_1), \Phi(u_3)) - h(u_3^p - u_1^p)l_1(u_3^p - u_2^p)(u_2^p - u_1^p).$$

$$\Phi\left((u_2^p)^{1/p}\right) \le \Phi(u_3) + h\left(\frac{u_3^p - u_2^p}{u_3^p - u_1^p}\right)\eta(\Phi(u_1), \Phi(u_3)) - l_1(u_3^p - u_2^p)(u_2^p - u_1^p). \tag{4.4}$$

By putting $\frac{u_3^p - u_2^p}{u_3^p - u_1^p} = r,\ s = u_1$, and $w = u_3$ in (4.4),

$$\Phi\left((rs^p + (1-r)w^p)^{1/p}\right) \le \Phi(w) + h(r)\eta(\Phi(s), \Phi(w)) - l_1 r(1-r)(s^p - w^p)^2. \tag{4.5}$$

Thus, Φ is a GSM (p, h)-CF. □

Remark 4.2. *(a) By putting $p = 1$ in (4.1) of Theorem 4.1, one has Theorem 1 of [31].*

(b) By choosing $p = 1$, $l_1 = 0$, and $\eta(u_1, u_2) = u_1 - u_2$ in (4.1) of Theorem 4.1, one has Theorem 2 of [26].

(c) By putting $p = 1$ and $h(t) = t$ in (4.1) of Theorem 4.1, one gets the Schur inequality for GS convex functions.

Example 4.3. *If we choose $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $r = \frac{1}{2}$, $l_1 = \frac{1}{2}$, $p = 2$, $h(r) = \frac{1}{r}$, $\eta(u_1, u_2) = 2u_1 + u_2$, and $\Phi(x) = x^4$ in (4.1), we get*

$$2 \le 25.7875.$$

Thus, this validates the existence of Theorem 4.1.

5. Hermite-Hadamard inequalities

The H-H inequality for the introduced class of convexity is presented in Theorem 5.1.

Theorem 5.1. *Assume that $\Phi : B_1 = [u_1, u_2] \subset \mathbb{R} \to \mathbb{R}$ is a GSM (p, h)-CF on $[u_1, u_2] \subseteq \mathbb{R}$ with $u_1 < u_2$, and then*

$$\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right) - h(1/2)E_\eta + \frac{l_1}{12}(u_1^p - u_2^p)^2 \le \frac{p}{(u_2^p - u_1^p)}\int_{u_1}^{u_2} \frac{\Phi(s)}{s^{1-p}}ds \le \Phi(u_2) + I_\eta - \frac{l_1}{6}(u_2^p - u_1^p)^2, \tag{5.1}$$

where

$$E_\eta = \int_0^1 \eta(\Phi(ru_1^p + (1-r)u_2^p)^{1/p}, \Phi((1-r)u_1^p + ru_2^p)^{1/p})dr;$$

$$I_\eta = \int_0^1 h(r)\eta(\Phi(u_1), \Phi(u_2))dr.$$

Proof. For $s, w \in B_1$, $p \geq 1$, $l_1 > 0$, and $n_1 \in [0,1]$, we have

$$\Phi\left((n_1 s^p + (1-n_1)w^p)^{1/p}\right) \leq \Phi(w) + h(n_1)\eta(\Phi(s), \Phi(w)) - l_1 n_1(1-n_1)(s^p - w^p)^2. \tag{5.2}$$

Putting, $n_1 = 1/2$ in (5.2),

$$\Phi\left(\left(\frac{s^p + w^p}{2}\right)^{1/p}\right) \leq \Phi(w) + h(1/2)\eta(\Phi(s), \Phi(w)) - \frac{l_1}{4}(s^p - w^p)^2. \tag{5.3}$$

Putting, $s^p = ru_1^p + (1-r)u_2^p$ and $w^p = (1-r)u_1^p + ru_2^p$ in (5.3),

$$\begin{aligned}\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right) &\leq \Phi\left(\left((1-r)u_1^p + ru_2^p\right)^{1/p}\right)\\ &+ h(1/2)\eta(\Phi\left((ru_1^p + (1-r)u_2^p)^{1/p}\right), \Phi((1-r)u_1^p + ru_2^p)^{1/p})\\ &- \frac{l_1}{4}((ru_1^p + (1-r)u_2^p) - ((1-r)u_1^p + ru_2^p))^2.\end{aligned} \tag{5.4}$$

By integrating (5.4) with respect to r from 0 to 1,

$$\begin{aligned}\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right) &\leq \int_0^1 \Phi\left(((1-r)u_1^p + ru_2^p)^{1/p}\right)dr\\ &+ h(1/2)\int_0^1 \eta(\Phi\left((ru_1^p + (1-r)u_2^p)^{1/p}\right), \Phi((1-r)u_1^p + ru_2^p)^{1/p})dr\\ &- \frac{l_1}{4}\int_0^1 ((ru_1^p + (1-r)u_2^p) - ((1-r)u_1^p + ru_2^p))^2 dr.\end{aligned} \tag{5.5}$$

Putting, $s^p = (1-r)u_1^p + ru_2^p$,

$$\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right) \leq \frac{p}{(u_2^p - u_1^p)}\int_{u_1}^{u_2} \frac{\Phi(s)}{s^{1-p}}ds + h(1/2)E_\eta - \frac{l_1}{12}(u_1^p - u_2^p)^2,$$

$$\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right) - h(1/2)E_\eta + \frac{l_1}{12}(u_1^p - u_2^p)^2 \leq \frac{p}{(u_2^p - u_1^p)}\int_{u_1}^{u_2} \frac{\Phi(s)}{s^{1-p}}ds. \tag{5.6}$$

Put $s^p = ru_1^p + (1-r)u_2^p$ on the right-hand side of (5.6) to get

$$\frac{p}{(u_2^p - u_1^p)} \int_{u_1}^{u_2} \frac{\Phi(s)}{s^{1-p}} ds = \int_0^1 \Phi\left((ru_1^p + (1-r)u_2^p)^{1/p}\right) dr.$$

Since Φ is a GSM (p,h)-CF, then

$$\frac{p}{(u_2^p - u_1^p)} \int_{u_1}^{u_2} \frac{\Phi(s)}{s^{1-p}} ds \leq \int_0^1 \Phi(u_2) dr + \int_0^1 h(u_1)\eta(\Phi(u_1), \Phi(u_2)) dr - \frac{l_1}{6}(u_2^p - u_1^p)^2. \tag{5.7}$$

From (5.6) and (5.7), we get

$$\begin{aligned} \Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right) - h(1/2)E_\eta + \frac{l_1}{12}(u_1^p - u_2^p)^2 \quad &\leq \frac{p}{(u_2^p - u_1^p)} \int_{u_1}^{u_2} \frac{\Phi(s)}{s^{1-p}} ds \\ &\leq \Phi(u_2) + I_\eta - \frac{l_1}{6}(u_2^p - u_1^p)^2. \end{aligned}$$

□

Remark 5.2. *(a) Put $p = 1$ in Theorem 5.1 to obtain Theorem 2 of [31].*
(b) Choose $p = 1$, $l_1 = 0$, and $\eta(u_1, u_2) = u_1 - u_2$ in Theorem 5.1 to have Theorem 3 of [26].
(c) Assume $p = 1$ and $h(t) = t$ in Theorem 5.1 to get Theorem 2.1 of [34].
(d) Suppose $p = 1$, $l_1 = 0$, $h(t) = t$, and $\eta(u_1, u_2) = u_1 - u_2$ in Theorem 5.1, one obtains the H-H inequality for the CF (see [17]).

Example 5.3. *If we choose $u_1 = 1$, $u_2 = 2$, $r = \frac{1}{2}$, $l_1 = \frac{1}{2}$, $p = 2$, $h(r) = \frac{1}{r^2}$, $\eta(u_1, u_2) = 2u_1 + u_2$, and $\Phi(x) = x^4$ in (5.1), we get*

$$\begin{aligned} &\Phi\left(\left(\frac{1+4}{2}\right)^{\frac{1}{2}}\right) - 4\int_0^1 \eta\left(\left(\Phi\left(\frac{1}{2} + \frac{4}{2}\right)^{\frac{1}{2}}\right), \left(\Phi\left(\frac{1}{2} + \frac{4}{2}\right)^{\frac{1}{2}}\right)\right) dr + \frac{1}{24}(1-4)^2 \\ &\leq \frac{2}{(4-1)} \int_1^2 \frac{s^4}{s^{-1}} ds \leq (2)^4 + 4\int_0^1 \eta(\Phi(1), \Phi(2)) dr - \frac{1}{12}(4-1)^2. \end{aligned}$$

Hence,

$$-68.375 \leq 7 \leq 87.25.$$

Thus, this shows the validity of Theorem 5.1.

6. Fejér-type inequalities

Theorem 6.1. *Suppose that $\Phi : B_1 = [u_1, u_2] \subset \mathbb{R} \to \mathbb{R}$ is a GSM (p, h)-CF and let $\xi : B_1 = [u_1, u_2] \subset \mathbb{R} \to \mathbb{R}$ be non-negative, integrable, and symmetric with respect to $\frac{(u_1 + u_2)}{2}$. Then*

$$\begin{aligned}\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right)\int_{u_1}^{u_2} \xi(s)ds + \frac{l_1}{4}\int_{u_1}^{u_2} (u_1^p + u_2^p - 2s^p)^2 \xi(s)ds - J\eta(u_1, u_2) \le \int_{u_1}^{u_2} \Phi(s)\xi(s)ds \\ \le \frac{\Phi(u_1) + \Phi(u_2)}{2}\int_{u_1}^{u_2} \xi(s)ds + K\eta(u_1, u_2) - l_1 \int_{u_1}^{u_2} ((s^p - u_2^p)(u_1^p - s^p))\xi(s)ds,\end{aligned} \tag{6.1}$$

where

$$J\eta(u_1, u_2) = h(1/2)\int_{u_1}^{u_2} \eta(\Phi(u_1 + u_2 - s), \Phi(s))\xi(s)ds;$$

$$K\eta(u_1, u_2) = \frac{(\eta(\Phi(u_1), \Phi(u_2) + \eta(\Phi(u_2), \Phi(u_1))}{2}\int_{u_1}^{u_2} h\left(\frac{s^p - u_2^p}{u_1^p - u_2^p}\right)\xi(s)ds.$$

Proof. As Φ is a GSM (p, h)-CF, therefore

$$\begin{aligned}&\Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right)\int_{u_1}^{u_2} \xi(s)ds = \int_{u_1}^{u_2} \Phi\left(\left(\frac{u_1^p + u_2^p + s^p - s^p}{2}\right)^{1/p}\right)\xi(s)ds \\ &= \int_{u_1}^{u_2} \Phi\left(\left(\frac{u_1^p + u_2^p - s^p}{2} + \frac{s^p}{2}\right)^{1/p}\right)\xi(s)ds \\ &\le \int_{u_1}^{u_2} \Phi(s)\xi(s)ds + h(1/2)\int_{u_1}^{u_2} \eta(\Phi(u_1 + u_2 - s), \Phi(s))\xi(s)ds - \frac{l_1}{4}\int_{u_1}^{u_2} (u_1^p + u_2^p - 2s^p)^2 \xi(s)ds.\end{aligned}$$

$$\Longrightarrow \Phi\left(\left(\frac{u_1^p + u_2^p}{2}\right)^{1/p}\right)\int_{u_1}^{u_2} \xi(s)ds + \frac{l_1}{4}\int_{u_1}^{u_2} (u_1^p + u_2^p - 2s^p)^2 \xi(s)ds - J\eta(u_1, u_2) \le \int_{u_1}^{u_2} \Phi(s)\xi(s)ds. \tag{6.2}$$

By choosing, $s = (ru_1^p + (1 - r)u_2^p)^{1/p}$ on the right-hand side of (6.2),

$$\int_{u_1}^{u_2} \Phi(s)\xi(s)ds = \left(\frac{u_2^p - u_1^p}{ps^{p-1}}\right)\int_0^1 (\Phi(ru_1^p + (1 - r)u_2^p)^{1/p} \times \xi\Big((ru_1^p + (1 - r)u_2^p)^{1/p}\Big))dr.$$

Since Φ is a GSM (p,h)-CF, then

$$\frac{ps^{p-1}}{u_2^p-u_1^p}\int_{u_1}^{u_2}\Phi(s)\xi(s)ds \leq \int_0^1 \Phi(u_1)\times\xi\Big((ru_1^p+(1-r)u_2^p)^{1/p}\Big)dr$$
$$+\eta(\Phi(u_2),\Phi(u_1)\int_0^1 h(r)\times\xi\Big((ru_1^p+(1-r)u_2^p)^{1/p}\Big)dr \tag{6.3}$$
$$-l_1(u_2^p-u_1^p)^2\int_0^1 (r(1-r))\times\xi\Big((ru_1^p+(1-r)u_2^p)^{1/p}\Big)dr.$$

Putting, $s=(ru_2^p+(1-r)u_1^p)^{1/p}$ on the right-hand side of (6.2),

$$\int_{u_1}^{u_2}\Phi(s)\xi(s)ds=\left(\frac{u_2^p-u_1^p}{ps^{p-1}}\right)\int_0^1(\Phi(ru_2^p+(1-r)u_1^p)^{1/p}\times\xi(ru_2^p+(1-r)u_1^p)^{1/p})dr.$$

As Φ is a GSM (p,h)-CF, therefore

$$\frac{ps^{p-1}}{u_2^p-u_1^p}\int_{u_1}^{u_2}\Phi(s)\xi(s)ds \leq \int_0^1 \Phi(u_2)\times\xi(ru_2^p+(1-r)u_1^p)^{1/p})dr$$
$$+\eta(\Phi(u_1),\Phi(u_2)\int_0^1 h(r)\times\xi(ru_2^p+(1-r)u_1^p)^{1/p})dr \tag{6.4}$$
$$-l_1(u_2^p-u_1^p)^2\int_0^1 (r(1-r))\times\xi(ru_2^p+(1-r)u_1^p)^{1/p})dr.$$

By adding (6.3) and (6.4), we get

$$\frac{2ps^{p-1}}{u_2^p-u_1^p}\int_{u_1}^{u_2}\Phi(s)\xi(s)ds \leq (\Phi(u_1)+\Phi(u_2))\int_0^1 \xi\Big((ru_1^p+(1-r)u_2^p)^{1/p}\Big)dr$$
$$+(\eta(\Phi(u_1),\Phi(u_2)+\eta(\Phi(u_2),\Phi(u_1))\int_0^1 h(r)\times\xi\Big((ru_1^p+(1-r)u_2^p)^{1/p}\Big)dr \tag{6.5}$$
$$-2l_1(u_2^p-u_1^p)^2\int_0^1 (r(1-r))\times\xi\Big((ru_1^p+(1-r)u_2^p)^{1/p}\Big)dr.$$

Put $s = \left((ru_1^p + (1-r)u_2^p)^{1/p}\right)$ on the right-hand side of (6.5) to get

$$\int_{u_1}^{u_2} \Phi(s)\xi(s)ds \le \left(\frac{\Phi(u_1)+\Phi(u_2)}{2}\right)\int_{u_1}^{u_2} \xi(s)ds$$

$$+\left(\frac{\eta(\Phi(u_1),\Phi(u_2))+\eta(\Phi(u_2),\Phi(u_1))}{2}\right)\int_{u_1}^{u_2} h\left(\frac{s^p-u_2^p}{u_1^p-u_2^p}\right)\times\xi(s)ds$$

$$-\,l_1\int_0^1 ((s^p-u_2^p)(u_1^p-s^p))\times\xi(s)ds$$

$$\Longrightarrow \int_{u_1}^{u_2} \Phi(s)\xi(s)ds \le \frac{\Phi(u_1)+\Phi(u_2)}{2}\int_{u_1}^{u_2}\xi(s)ds$$

$$+K\eta(u_1,u_2) - l_1\int_{u_1}^{u_2} ((s^p-u_2^p)(u_1^p-s^p))\xi(s)ds. \tag{6.6}$$

From (6.2) and (6.6), we get

$$\Phi\left(\left(\frac{u_1^p+u_2^p}{2}\right)^{1/p}\right)\int_{u_1}^{u_2}\xi(s)ds + \frac{l_1}{4}\int_{u_1}^{u_2}(u_1^p+u_2^p-2s^p)^2\xi(s)ds - J\eta(u_1,u_2)$$

$$\le \int_{u_1}^{u_2}\Phi(s)\xi(s)ds$$

$$\le \frac{\Phi(u_1)+\Phi(u_2)}{2}\int_{u_1}^{u_2}\xi(s)ds + K\eta(u_1,u_2) - l_1\int_{u_1}^{u_2}((s^p-u_2^p)(u_1^p-s^p))\xi(s)ds.$$

□

Remark 6.2. *(a) Assume $p = 1$ in (6.1) of Theorem 6.1, and we have Theorem 1 of [31].*
(b) If we assume $p = 1$ and $h(t) = t$ in (6.1) of Theorem 6.1, one obtains Theorem 2.3 of [34].

Example 6.3. *If we choose $u_1 = 2$, $u_2 = 4$, $s = 3$, $r = \frac{1}{2}$, $l_1 = \frac{1}{2}$, $p = 2$, $h(r) = \frac{1}{r^2}$, $\eta(u_1,u_2) = 2u_1 + u_2$, $\Phi(x) = x^4$, and $\xi = x^2$ in (6.1), we get*

$$\Phi\left(\left(\frac{1+4}{2}\right)^{1/2}\right)\int_2^4 s^2ds + \frac{1}{8}\int_2^4((4+16-18)^2(s^2))ds - 4\int_2^4\eta(\Phi(3),\Phi(3))s^2ds$$

$$\le \int_2^4(s^4\times s^2)ds \le \frac{\Phi(2)+\Phi(4)}{2}\int_2^4 s^2ds + \frac{(\eta(\Phi(2),\Phi(4))+(\eta(\Phi(4),\Phi(2))}{2}\times\int_2^4 h(\frac{(9-16)}{(4-16)})\times s^2ds.$$

$$\Longrightarrow -15.685 \le 2322.2857 \le 23,715.384.$$

Hence, Theorem 6.1 holds.

7. Comparison

The following example compares the generalized class of convexity with an existing class named the GSM h-convex function.

Example 7.1. *If we choose* $\Phi(x) = x^4$, $r = \frac{1}{2}$, $p = 2.7$, $h(r) = \frac{1}{r^2}$, $l_1 = \frac{1}{2}$, *and* $\eta(u_1, u_2) = 2u_1 + u_2$ *in* (3.1), *we get*

$$\left(\frac{u_1^{2.7} + u_2^{2.7}}{2}\right)^{1.48} \leqslant 8u_1^4 + 5u_2^4 - \frac{(u_1^{2.7} - u_2^{2.7})^2}{8}. \tag{7.1}$$

Put $u_1 = 2$ *in* (7.1) *to have*

$$\left(\frac{6.498 + u_2^{2.7}}{2}\right)^{1.48} \leqslant 128 + 5u_2^4 - \frac{(6.498 - u_2^{2.7})^2}{8}. \tag{7.2}$$

By choosing $u_2 = 8$ *in* (7.2), *one has*

$$1,507.537 \leq 11,382.31. \tag{7.3}$$

Similarly, by assuming $\Phi(x) = x^4$, $r = \frac{1}{2}$, $h(r) = \frac{1}{r^2}$, $l_1 = \frac{1}{2}$, *and* $\eta(u_1, u_2) = 2u_1 + u_2$ *in* (2.6), *one has*

$$\left(\frac{u_1 + u_2}{2}\right)^4 \leqslant 8u_1^4 + 5u_2^4 - \frac{(u_1 - u_2)^2}{8}. \tag{7.4}$$

Put $u_1 = 2$ *in* (7.4) *to have*

$$\left(\frac{2 + u_2}{2}\right)^4 \leqslant 128 + 5u_2^4 - \frac{(2 - u_2)^2}{8}. \tag{7.5}$$

By choosing $u_2 = 8$ *in* (7.5), *one has*

$$625 \leq 20,603. \tag{7.6}$$

Figure 4 presents the comparison of (7.2) and (7.5).

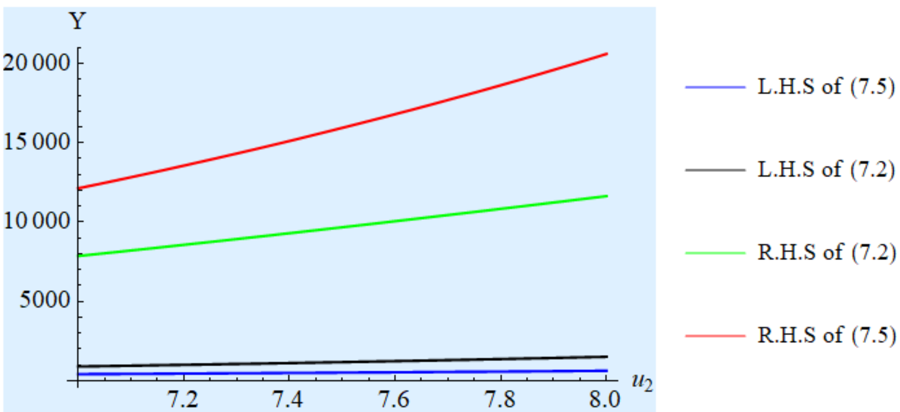

Figure 4. The graph of inequalities (7.2) and (7.5).

Remark 7.2. *The difference of bounds of* (7.3) *is* $9,874.773$*, while the difference of bounds of* (7.6) *is* $19,978$*. Therefore, the novel class reduces the gap of bounds and thus refines the result of the existing class.*

8. Conclusions

The extension of GSM h-CFs, GS h-CFs, GSM p-CFs and generalized p-CFs, is presented in this paper. The study of GSM (p,h)-CFs has yielded various important results and inequalities such as the Schur, H-H, and Fejér inequalities. The current findings expand upon the Schur inequalities discussed in [26,31], the H-H inequalities presented in [26,31,34], and the Fejér inequalities presented in [31,34]. We have established the credibility of the findings by presenting some numerical instances and graphs of GSM (p,h)-CFs. Also, by using GSM (p,h)-CFs, we can find the bounds of other companion inequalities. Moreover, we can also improve the bounds of the Schur, H-H, and Fejér inequalities by utilizing newly defined convexities along with fractional integrals.

Author contributions

M.H.B., A.N. and K.A.K.: Conceptualization; A.N. and K.A.K.: Methodology; S.E. and T.A.: Software; K.A.K., S.E. and T.A.: Validation; A.N. and S.E.: Formal analysis; A.N. and M.H.B.: Investigation; M.H.B.: Writing – original draft preparation; K.A.K. and T.A: Writing – review and editing; S.E. and T.A.: Visualization; A.N.: Supervision; S.E.: Project administration; T.A.: Funding acquisition, All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. D. Bertsekas, A. Nedic, A. Ozdaglar, *Convex analysis and optimization*, Athena Scientific, 2003.
2. S. I. Butt, H. Budak, M. Tariq, M. Nadeem, Integral inequalities for n-polynomial s-type preinvex functions with applications, *Math. Method. Appl. Sci.*, **44** (2021), 11006–11021. https://doi.org/10.1002/mma.7465

3. A. Nosheen, S. Ijaz, K. A. Khan, K. M. Awan, M. A. Albahar, M. Thanoon, Some q-symmetric integral inequalities involving s-convex functions, *Symmetry*, **15** (2023), 1169. https://doi.org/10.3390/sym15061169
4. I. J. Bakelman, *Convex analysis and nonlinear geometric elliptic equations*, Springer Science and Business Media, 2012.
5. G. G. Magaril-Il'yaev, V. M. Tikhomirov, *Convex analysis: Theory and applications*, American Mathematical Society, **222** (2003).
6. K. Mehrez, P. Agarwal, New Hermite–Hadamard type integral inequalities for convex functions and their applications, *J. Comput. Appl. Math.*, **350** (2019), 274–285. https://doi.org/10.1016/j.cam.2018.10.022
7. S. K. Sahoo, H. Ahmad, M. Tariq, B. Kodamasingh, H. Aydi, M. De la Sen, Hermite–Hadamard type inequalities involving k-fractional operator for (h, m)-convex functions, *Symmetry*, **13** (2021), 1686. https://doi.org/10.3390/sym13091686
8. M. Vivas-Cortez, M. A. Ali, H. Budak, H. Kalsoom, P. Agarwal, Some new Hermite–Hadamard and related inequalities for convex functions via (p, q)-integral, *Entropy*, **23** (2021), 828. https://doi.org/10.3390/e23070828
9. M. Niezgoda, Group majorization and Schur type inequalities, *Linear Algebra Appl.*, **268** (1998), 9–30. https://doi.org/10.1016/S0024-3795(97)89322-6
10. A. N. Koam, A. Nosheen, K. A. Khan, M. H. Bukhari, A. Ahmad, M. S. Alatawi, On Riemann–Liouville integral via strongly modified (h, m)-convex functions, *Fractal Fract.*, **8** (2024), 680. https://doi.org/10.3390/fractalfract8120680
11. A. Nosheen, K. A. Khan, M. H. Bukhari, M. K. Kahungu, A. F. Aljohani, On Riemann–Liouville integrals and Caputo fractional derivatives via strongly modified (p, h)-convex functions, *PloS One*, **19** (2024), e0311386. https://doi.org/10.1371/journal.pone.0311386
12. F. Zhang, *The Schur complement and its applications*, Springer Science and Business Media, **4** (2006).
13. H. T. Jongen, T. Möbert, J. Rückmann, K. Tammer, On inertia and Schur complement in optimization, *Linear Algebra Appl.*, **95** (1987), 97–109. https://doi.org/10.1016/0024-3795(87)90028-0
14. M. B. Khan, H. M. Srivastava, P. O. Mohammed, K. Nonlaopon, Y. S. Hamed, Some new Jensen, Schur and Hermite–Hadamard inequalities for log convex fuzzy interval-valued functions, *AIMS Math.*, **7** (2022), 4338–4358. https://doi.org/10.3934/math.2022226
15. Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, S. M. Kang, Hermite–Hadamard-type inequalities for functions whose derivatives are η-convex via fractional integrals, *J. Inequal. Appl.*, **2019** (2019), 1–16. https://doi.org/10.1186/s13660-019-1993-y
16. C. P. Niculescu, L. E. Persson, Old and new on the Hermite–Hadamard inequality, *Real Anal. Exch.*, **29** (2003), 663–685. https://doi.org/10.14321/realanalexch.29.2.0663
17. A. El Farissi, Simple proof and refinement of Hermite–Hadamard inequality, *J. Math. Inequal.*, **4** (2010), 365–369. https://doi.org/10.7153/jmi-04-33
18. S. Jain, K. Mehrez, D. Baleanu, P. Agarwal, Certain Hermite–Hadamard inequalities for logarithmically convex functions with applications, *Mathematics*, **7** (2019), 163. https://doi.org/10.3390/math7020163

19. M. Kadakal, İ. İşcan, P. Agarwal, M. Jleli, Exponential trigonometric convex functions and Hermite–Hadamard type inequalities. *Math. Slovaca*, **71** (2021), 43–56. https://doi.org/10.1515/ms-2017-0410
20. M. R. Delavar, M. De La Sen, Some generalizations of Hermite–Hadamard type inequalities, *SpringerPlus*, **5** (2016), 1661. https://doi.org/10.1186/s40064-016-3301-3
21. H. M. Srivastava, M. Tariq, P. O. Mohammed, H. Alrweili, E. Al-Sarairah, M. De La Sen, On modified integral inequalities for a generalized class of convexity and applications, *Axioms*, **12** (2023), 162. https://doi.org/10.3390/axioms12020162
22. K. L. Tseng, S. R. Hwang, S. S. Dragomir, Fejér-type inequalities (I), *J. Inequal. Appl.*, **2010** (2010), 1–7. https://doi.org/10.1155/2010/531976
23. U. Goginava, The weak type inequality for the maximal operator of the Marcinkiewicz–Fejér means of the two-dimensional Walsh-Fourier series, *J. Approx. Theory*, **154** (2008), 161–180. https://doi.org/10.1016/j.jat.2008.03.012
24. M. A. Latif, S. S. Dragomir, E. Momoniat, Some Fejér type inequalities for harmonically convex functions with applications to special means, *Int. J. Anal. Appl.*, **13** (2017), 1–14.
25. S. Varošanec, On h-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086
26. M. A. Noor, K. I. Noor, M. U. Awan, Hermite Hadamard inequalities for modified h-convex functions, *Transylv. J. Math. Mech.*, **6** (2014), 1–10.
27. M. A. Noor, M. U. Awan, K. I. Noor, M. Postolache, Some integral inequalities for p-convex functions, *Filomat*, **30** (2016), 2435–2444. https://doi.org/10.2298/FIL1609435N
28. B. Feng, M. Ghafoor, Y. M. Chu, M. I. Qureshi, X. Feng, C. Yao, et al., Hermite–Hadamard and Jensen's type inequalities for modified (p,h)-CFs, *AIMS Math.*, **5** (2020), 6959–6971. https://doi.org/10.3934/math.2020446
29. R. T. Rockafellar, *Convex analysis*, Princeton: Princeton University Press, 1997.
30. K. S. Zhang, J. P. Wan, p-Convex functions and their properties, *Pure Appl. Math. J.*, **23** (2007), 130–133.
31. T. Zhao, M. S. Saleem, W. Nazeer, I. Bashir, I. Hussain, On generalized strongly modified h-convex functions, *J. Inequal. Appl.*, **2020** (2020), 1–12. https://doi.org/10.1186/s13660-020-2281-6
32. C. Y. Jung, M. S. Saleem, W. Nazeer, M. S. Zahoor, A. Latif, S. M. Kang, Unification of generalized and p-convexity, *J. Funct. Space.*, **2020** (2020), 4016386. https://doi.org/10.1155/2020/4016386
33. M. S. Saleem, Y. M. Chu, N. Jahangir, H. Akhtar, C. Y. Jung, On generalized strongly p-convex functions of higher order, *J. Math.*, **2020** (2020), 1–8. https://doi.org/10.1155/2020/8381431
34. M. U. Awan, M. A. Noor, K. I. Noor, F. Safdar, On strongly generalized convex functions, *Filomat*, **31** (2017), 5783–5790. https://doi.org/10.2298/FIL1718783A