AIMS Mathematics, 10(7): 16957–16993.
DOI:10.3934/math.2025762
Received: 26 March 2025
Revised: 13 June 2025
Accepted: 23 June 2025
Published: 29 July 2025

https://www.aimspress.com/journal/Math

Research article

Stability, chaos and bifurcations by explicit criterions of a discrete tuberculosis epidemic model

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Abstract: In this paper, we investigated the local stability, chaotic dynamics, and bifurcations of a discrete tuberculosis (TB) epidemic model, which provides crucial insights into the complex behavior of infectious disease spread in discrete-time settings. Specifically, we established the existence of a disease-free fixed point for all parametric values and demonstrated that an endemic fixed point emerges under specific parametric condition. The local stability at these fixed points was examined using the linear stability theory, revealing important thresholds for disease persistence or eradication. A comprehensive bifurcation analysis was conducted, showing that while no flip bifurcation occurred at the disease-free fixed point, both Neimark-Sacker and flip bifurcations took place at the endemic fixed point, and we studied said bifurcations by the explicit criterions. To further understand the model's nonlinear dynamics, we explored chaotic behavior by a feedback control strategy to mitigate chaotic oscillations. Numerical simulations are presented to validate our theoretical findings, demonstrating the model's capacity to capture real-world epidemiological patterns. Finally, theoretical results are also interpreted biologically/medically.

Keywords: tuberculosis model; explicit criterions; numerical simulation; bifurcation
Mathematics Subject Classification: 40A05, 70K50, 92D25

1. Introduction

Tuberculosis is an infectious disease caused by bacteria. Mycobacterium tuberculosis often infects the lungs but can also infect the other parts of the body. Most commonly is when a person breathes in air containing the TB germs, which can enter the lungs and settle there, and then start to multiply. From the lungs, it can transfer and infect other parts of the body through the blood, such as the kidney, spine, and

brain. TB germs can live in the body without causing a disease, called inactive TB. People with inactive TB germs do not have the symptoms of TB disease and cannot spread the germs to others, but without treatment, these people can develop active TB disease at any time and become sick. When the immune system of a person cannot stop the multiplication and growth of TB germs, it becomes active and starts to increase its number and spread to other persons. The most common symptoms of TB are chronic cough with sputum containing blood, fever, weight loss, and night sweat. Globally, an estimated 10.8 million people are affected by TB, with 1.5 million deaths. Different factors involved in the spread of TB include poverty, ineffective diagnostic practices, drug-resistant TB, and endemicity of instrumental agents [1]. It is largely unknown how many identified cases in high-burden situations are reported, though, as less than one-third of the projected number of detected cases are reported annually [2]. To better understand how infectious diseases spread, and to provide a convenient summary of epidemiological statistics, the use of mathematical models and simulations to study the dynamics of infectious diseases has gained significant attention over the years [3]. In order to control the underlying disease, it is helpful to forecast upcoming epidemics as the infection worsens and to make up to date conclusions. We refer to models to investigate the spread of infectious diseases as dynamic epidemiological models because they help to detect the disease [4]. Disease modelers attempt to take into account the potential for some degree of infection heterogeneity in every population when creating mathematical models, which leads to a group of infectious disease models identified as compartmental models. Compartmental models can be formulated with the help of stochastic DEs or a deterministic approach using systems of ODEs and a stochastic approach using continuous-time Markov chains. The background for creating comparable stochastic models is provided by the ODE epidemic model, which is also a primary cause of analogy with stochastic epidemic models. Due to the stochastic nature of epidemic processes, stochastic models aid in understanding that disparities in disease blowout may be attributed to chances of fluctuations alone than varying virulence or infectiousness [5]. Moreover, stochastic models are useful for studying outbreaks in small communities and for determining the dynamics of infection in the early stages [6]. This shows that deterministic epidemic models are resilient to random fluctuations in larger populations. A universal group of stochastic models, with or without consistent socializing, has an infinite population limit, which is appropriately represented by the deterministic model [7, 8]. According to a deterministic model, a diverse population is further subdivided into a limited amount of consistent parts. Then, using ODEs, the widespread dynamics are commonly modeled with a undertaking among sub-groups. It has long been practiced to model contiguous diseases using the fatalistic approach. In a variety of settings, these disease models have been widely utilized to study other infectious diseases like measles, chickenpox, influenza, and many more [9]. For example, contact number, thresholds, the basic reproductive number, and replacement number are theoretical outcomes of deterministic models. When it comes to infectious diseases, these disease models assist states, zones, and communities in creating suitable preventative measures to decline the possibility of contamination. Using various classes of epidemic compartmental models, various studies have investigated tuberculosis infection from a modeling perspective. However, the group of susceptible-exposed-infected-recovered (SEIR) compartmental models is typically used in the procedure of examining the impact of unprotected entities on the inclusive TB contamination dynamics of epidemic models, in contrast to the susceptible-infected-recovered (SIR) models, which are typically accepted when the covert phase of the disease is overlooked. Most TB researchers, using the SEIR model, examined global analysis at the growth, contiguous, and improved phases using

either infectious drive or non-sequential incidence rate. Thus, in recent years, many mathematicians examined the dynamical behavior, particular bifurcation phenomena, of certain tuberculosis epidemic models. For instance, Wangari and Stone [10] examined backward bifurcation analysis of the following SEIR epidemic model:

$$\begin{cases} \dot{S} = \bigwedge -(\mu + \lambda)S, \\ \dot{E} = (1-q)\lambda S - (p\lambda + \mu + k)E + (1-\sigma)\lambda\theta R, \\ \dot{I} = q\lambda S + (k + p\lambda)E + \sigma\lambda\theta R - (\mu_d + r + \mu)I, \\ \dot{R} = -(\mu + \lambda\theta)R + rI, \end{cases} \tag{1.1}$$

where $\bigwedge$ is recruitment by migration or birth, λ denotes incident rate dependent on frequency, θ is recurrent TB by reinfection, q is basic evolution rate, k denotes internal reactivation rate, p is the erotogenic re-infection, σ is chance of fast progression after recontamination, μ_d is death rate due to disease, μ is natural death rate, and r denotes per-head recovery rate. Porco and Blower [11] examined quantitative analysis of the following TB disease model without considering the treatment:

$$\begin{cases} \dot{X} = \prod -\mu X - \beta X T_i, \\ \dot{L} = (1-p)\beta X T_i - (\mu + \nu)L, \\ \dot{T}_i = \beta f p X T_i + \omega R + q\nu L - (\mu_T + c + \mu)T_i, \\ \dot{T}_n = (1-f)\beta p X T_i + \nu(1-q)L - (\mu_T + c + \mu)T_n + \omega R, \\ \dot{R} = c(T_i + T_n) - (\mu + 2\omega)R, \end{cases} \tag{1.2}$$

where X, L, T_i, T_n, and R denote susceptible, latently infected, number of individuals with lung infectious TB, and number of sick members who are not spreading and immunized, respectively. Moreover, c, 2ω, β, ν, q, $\prod$, $\frac{1}{\mu}$, and f denote per member recovery rate, active TB relapse, transmission factor, possibility of acquiring and transmitting TB, chance of getting infectious TB in a situation an individual has delayed TB, the rate recruitment, average life anticipation and likelihood of developing contagious TB, and chance of getting infectious TB in a situation an individual has viral TB, respectively. Ullah et al. [12] investigated dynamics of the following SLITR fractional epidemic model:

$$\begin{cases} {}^{CF}D_t^{\alpha} S = \Lambda - \frac{\beta}{N} S I - \mu S, \\ {}^{CF}D_t^{\alpha} L = \frac{\beta}{N} S I - (\epsilon + \mu)L + \delta(1-\eta)T, \\ {}^{CF}D_t^{\alpha} I = \epsilon L + \delta\eta T - (\tau_1 + \gamma + \mu)I, \\ {}^{CF}D_t^{\alpha} T = \gamma I - (\tau_2 + \alpha + \delta + \mu)T, \\ {}^{CF}D_t^{\alpha} R = \alpha T - \mu R, \end{cases} \tag{1.3}$$

where S, L, I, T, and R denote the susceptible, latent, infected, under treatment, and recovered members, respectively. Furthermore, the parameters η, γ, ϵ, τ_1, δ, α, τ_2, β, μ, N, and Λ show therapy failure rate, transfer from I group to T, transmission rate from L group to I, mortality rate of infected members caused by disease, percentage of individuals to leave T, continuation from T group to R, mortality rate caused by disease in class T, the infection incidence rate, natural mortality rate, total

inhabitants, and birth rate, respectively. Bowong [13] examined dynamics and control of the following TB model:

$$\begin{cases} \dot{S} = \bigwedge - (\lambda + \mu)S, \\ \dot{E} = \lambda(1-p)S + r_2 I - \lambda\sigma(1-r_1)E - (k(1-r_1)+\mu)E, \\ \dot{I} = \lambda p S + (1-r_1)(\lambda\sigma + k)E - (d + r_2 + \mu)I, \end{cases} \tag{1.4}$$

where S, E, and I are susceptible, exposed, and infected individuals, respectively. Additionally, r_1, μ, k, r_2, σ, $\bigwedge$, p, and d denote chemoprophylaxis of latently infectious members, death by nature, slow way to activate TB group, healing rate of the contagious, recontamination parameter of latently infectious members, recovery rate of group S, fast way to infectious group, and mortality due TB, respectively. Hu et al. [14] examined the local dynamic, one-parameter bifurcations of a discrete epidemic SI model:

$$\begin{cases} S_{t+1} = & S_t + h\left(A - dS_t - \lambda S_t I_t\right), \\ I_{t+1} = & I_t + h\left(\lambda S_t I_t - (d+r) I_t\right), \end{cases} \tag{1.5}$$

where A, d, λ, r, and h denote inclusion rate in population, natural mortality rate, bilinear incidence rate, recovery rate of individual I, and integral step size, respectively. Parvin et al. [15] investigated the stability analysis about fixed points of a discrete SIR model:

$$\begin{cases} S_{t+1} = & S_t + h\left(A - \delta_0 S_t - \frac{\alpha S_t I_t}{1+\beta S_t + \gamma I_t}\right), \\ I_{t+1} = & I_t + h\left(\frac{\alpha S_t I_t}{1+\beta S_t + \gamma I_t} - \delta_0 I_t - \delta_1 I_t - \delta_2 I_t - \frac{a I_t}{1+b I_t}\right), \\ R_{t+1} = & R_t + h\left(\delta_2 I_t - \delta_0 R_t - \frac{a I_t}{1+b I_t}\right), \end{cases} \tag{1.6}$$

where δ_0, α, γ, β, δ_1, δ_2, and h denote natural mortality rate, transmission rate, measure of inhibition effect, inhibitory effect, mortality rate of I due to infection, recovery rate of I class due to immunity, and integral step size, respectively.

Elaydi and Lozi [16] first reformulated discrete TB models, and then global behavior was explored for these models. Monisha and Devi [17] explored the fuzzy reproduction number, local and global behavior at fixed points, and bifurcation analysis of a fuzzy mathematical model with TB transmission. Ginting et al. [18] gave the deterministic mathematical model to analyze the TB dynamics by considering reinfection and vaccination. Gemeda et al. [19] gave the fractional-order TB model using a caputo fractional approach to discuss TB transmission dynamics incorporating vaccination and treatments. Yao et al. [20] explored global bifurcation behaviors of an in-host mycobacterium TB model. Kotola et al. [21] explored the optimal control strategy and bifurcation behavior of a COVID-19 and HIV/AIDS co-infection model. Panigoro et al. [22] studied the bifurcation behavior and global dynamics of a discrete SIS model with saturated incidence and reinfection. Toufga et al. [23] investigated optimal control in a discrete TB mathematical model. Jajarmi et al. [24] introduced a new and efficient numerical method to model fractional-order TB mathematical models. Choiński et al. [25] examined the dynamical properties of the SIS model, where the discretization process is carried out by the nonstandard finite difference scheme. The aforementioned studies motivated us to explore the dynamic analysis of a tuberculosis epidemic model where, for the completion of this section, we give the mathematical reformulation of the SEIR tuberculosis epidemic model. In order to reformulate the SEIR tuberculosis epidemic model, we divide the total population N into four classes: Susceptible, S,

exposed, E, infectious, I, and recovered, R, individuals. Moreover, from the compartmental diagram, which is drawn in Figure 1, and under the demography assumption that birth rate Λ is equal to death rate μ, one has the following equations [26]:

$$\begin{cases} \dot{S} = & \Lambda N - \mu S - \frac{\alpha S I}{N}, \ \dot{E} = \frac{\alpha S I}{N} - (\epsilon + \mu)E, \\ \dot{I} = & -(\mu + \beta)I + \epsilon E, \ \dot{R} = -\mu R + \beta I, \end{cases} \tag{1.7}$$

where β, ϵ, and α denote the recovery rate of individual I, rate under which individual moves from E to I, and the infection rate, respectively. Further, it is noted that (1.7) becomes the form:

$$\begin{cases} \dot{s} = & \Lambda - \alpha s i - \mu s, \ \dot{e} = -(\epsilon + \mu)e + \alpha s i, \\ \dot{i} = & -(\beta + \mu)i + \epsilon e, \ \dot{r} = -\mu r + \beta i, \end{cases} \tag{1.8}$$

by the following rescaling parameters:

$$s = \frac{S}{N}, i = \frac{I}{N}, r = \frac{R}{N}, e = \frac{E}{N}, \tag{1.9}$$

where s, e, i, and r represent susceptible, exposed, infectious, and recovered proportion of individuals S, E, I, and R with $s + i + e + r = 1$. Now, setting $r = 1 - s - e - i$ and making r the subject instead of (1.8), it is enough to study behavior of the following sei tuberculosis disease model:

$$\dot{s} = \Lambda - \alpha s i - \mu s, \ \dot{e} = -(\mu + \epsilon)e + \alpha s i, \ \dot{i} = -(\mu + \beta)i + \epsilon e. \tag{1.10}$$

It is noted here that the original model, which is depicted in (1.10), is reformulated in a continuous-time framework, but our aim is to investigate the dynamics of its discrete counterpart for several reasons. First, many real-world processes, including the spread of infectious diseases and treatment interventions, are inherently discrete in nature, occurring at specific intervals (for example, daily, monthly, weekly). Second, discrete models are particularly useful in numerical simulations and can offer insights into complex dynamical behaviors, such as bifurcations and chaos, which may not be easily captured in the continuous setting. Last, the discretization enables us to explore the mathematical properties of the system from a different perspective, potentially revealing novel dynamics and contributing to the broader understanding of the modeled phenomenon. Due to the aforementioned reasons, by Euler's forward formula, the desired discrete version of the sei tuberculosis epidemic model (1.10) becomes

$$\begin{aligned} s_{t+1} &= h\Lambda + (1 - \mu h)s_t - \alpha h s_t i_t, \ e_{t+1} = (1 - \mu h - \epsilon h)e_t + \alpha h s_t i_t, \\ i_{t+1} &= (1 - h\mu - h\beta)i_t + h\epsilon e_t, \end{aligned} \tag{1.11}$$

where h denotes the integral step size. More specifically, in the discretization process, h represents the length of the discrete-time interval, which corresponds to the unit of time used to observe changes in the TB epidemic. Biologically, since tuberculosis progresses relatively slowly compared to many acute infectious diseases, it is appropriate to choose a step size $h = 1$ representing one time unit as one month. This monthly time scale aligns with the typical time frame over which transitions such as progression from exposed to infectious and initiation of treatment occur in TB dynamics. Thus, the chosen step size ensures a realistic and biologically meaningful discretization of the continuous-time

TB model (1.10). Furthermore, it is important here to emphasize that a specific discrete tuberculosis epidemic model, which is depicted in (1.11), considered for our study has not been previously subjected to such a detailed and rigorous dynamic analysis, particularly involving the combination of local stability, explicit bifurcation criteria, and chaos control. However, it is true that the general structure of compartmental epidemic models (such as SEIR-type models) is widely known and has been extensively studied in recent years. Moreover, our understudied discrete-time sei model (1.11) incorporates specific features tailored to TB dynamics, including the transition from exposed to infectious compartments, and the absence of a recovery equation due to the closed population assumption with $s + i + e + r = 1$. This enables the model to be effectively reduced to a discrete three-dimensional sei system, which we analyze in depth. The novelty of our work lies not in merely adopting a known model structure but in the comprehensive and explicit characterization of the model's nonlinear behavior using tools that go beyond standard linear analysis. We provide clear and mathematically rigorous conditions for the occurrence of Neimark-Sacker and flip bifurcations, and we demonstrate that these bifurcations arise under specific parametric constraints only at the endemic fixed point-not at the disease-free equilibrium. Such explicit bifurcation conditions are rarely derived in earlier studies on discrete TB models, many of which either stop at linear stability analysis or rely purely on numerical exploration without establishing analytical thresholds for dynamic transitions. Furthermore, our investigation into chaotic dynamics and its suppression using feedback control strategy adds a distinct and innovative aspect to the study. Chaos in discrete epidemiological models is an emerging area with significant implications for understanding real-world complexities in disease transmission. By incorporating the feedback control strategy, we also identify the chaotic regimes. For the numerical simulations, we bridge the theoretical and applied aspects of the work, offering epidemiologically relevant insights that reinforce the usefulness of our approach. Thus, the numerical simulation section, to verify the theoretical results, will be organized in two ways: First, the appropriate data, and second, fitting the real epidemiological data from Turkey and the Ashanti region of Ghana to the understudied discrete tuberculosis epidemic model (1.11) by choosing step size $h = 1$. In summary, although the formulation of the discrete model follows a recognizable framework, the analytical depth, explicit derivation of bifurcation conditions, and incorporation of chaos control techniques collectively mark a significant advancement over existing studies. These contributions distinguish our work and highlight its relevance in both the theoretical and applied study of infectious disease dynamics, particularly for tuberculosis in discrete-time settings. Our key aims include:

- To explore the local dynamics at fixed points based on the linear stability theory.
- Construction of basic reproduction number by the next-generation approach.
- To explore the codimension-one bifurcation by the bifurcation theory.
- Study of chaos by the feedback control strategy.
- Validation of theoretical results numerically in two ways: For appropriate data and fitting the real epidemiological data from Turkey and the Ashanti region of Ghana.
- Medical/biological explanation of theoretical results.

The paper is further organized as follows: In Section 2, we study the linearized form, basic reproduction number, and existence of fixed points. We study local dynamics in Section 3. In Section 4, we provide an in-depth analysis of bifurcation. The chaos control by the feedback control strategy is presented in Section 5. In Section 6, we present the numerical simulations, while the conclusion is provided in Section 7.

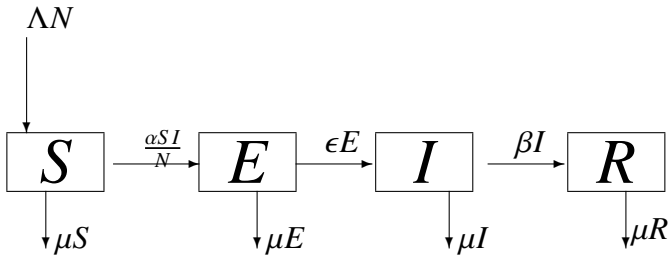


Figure 1. Flow diagram for epidemic model.

2. Fixed points, basic reproduction number and linearized system

In this section, we study the fixed points, basic reproduction number, and linearized system of the discrete tuberculosis epidemic model (1.11). More precisely, we divide this section into the following subsections:

2.1. Existence of disease free fixed point (DFEP)

Here, we present the existence of disease free fixed point for the discrete tuberculosis epidemic model (1.11) as the following Lemma:

Lemma 2.1. *$\forall\ h, \alpha, \Lambda, \mu, \epsilon, \beta$, the discrete tuberculosis epidemic model (1.11) has DFEP:* $\Omega_1 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$.

Proof. If $\Omega = (s, e, i)$ denotes fixed point of model (1.11) then

$$s = h\Lambda + (1 - \mu h)s - \alpha hsi,\ e = (1 - \mu h - \epsilon h)e + \alpha hsi,\ i = (1 - h\mu - h\beta)i + h\epsilon e. \tag{2.1}$$

Since, for $(\frac{\Lambda}{\mu}, 0, 0)$, system (2.1) is satisfied, and so the discrete tuberculosis epidemic model (1.11) has DFEP $\Omega_1 = \left(\frac{\Lambda}{\mu}, 0, 0\right)\ \forall\ h$, α, Λ, μ, ϵ, and β. □

Before examining the endemic fixed point (EFP), we first determine the basic reproduction number $\mathcal{R}_0$ by the next-generation approach based on the existing theory [27–30].

2.2. Basic reproduction number

It is noted that for the understudied model (1.11), the said approach is performed by choosing the matrices $\Gamma = \begin{pmatrix} \alpha h s_t i_t \\ 0 \end{pmatrix}$ and $\Upsilon = \begin{pmatrix} (\epsilon + \mu) h e_t \\ -\epsilon h e_t + (\beta + \mu) h i_t \end{pmatrix}$, where for DFEP, the corresponding variational matrices are $\hat{\Gamma}|_{\text{DFEP}} = \begin{pmatrix} 0 & \frac{\alpha\Lambda h}{\mu} \\ 0 & 0 \end{pmatrix}$ and $\hat{\Upsilon}|_{\text{DFEP}} = \begin{pmatrix} (\epsilon + \mu)h & 0 \\ -\epsilon h & (\beta + \mu)h \end{pmatrix}$. Now, on computing $\hat{\Gamma}|_{\text{DFEP}}\hat{\Upsilon}^{-1}|_{\text{DFEP}} = \begin{pmatrix} \frac{\Lambda\alpha\epsilon}{\mu(\beta+\mu)(\epsilon+\mu)} & \frac{\Lambda\alpha}{\mu(\mu+\beta)} \\ 0 & 0 \end{pmatrix}$, one has the dominant eigenvalue $\frac{\Lambda\alpha\epsilon}{\mu(\epsilon+\mu)(\beta+\mu)}$ and hence, the basic reproduction number for the model (1.11) can be expressed as $\mathcal{R}_0 = \frac{\Lambda\alpha\epsilon}{\mu(\epsilon+\mu)(\beta+\mu)}$.

Hereafter, we identify the EFP of the discrete tuberculosis epidemic model (1.11) in terms of $\mathcal{R}_0$ as a following Lemma:

Lemma 2.2. *If*

$$\mathcal{R}_0 := \frac{\Lambda\alpha\epsilon}{\mu(\epsilon+\mu)(\beta+\mu)} > 1, \tag{2.2}$$

then model (1.11) has EFP $\Omega_2 = \left(\frac{(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}, \frac{\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon(\mu+\epsilon)}, \frac{\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon)}{\alpha(\beta+\mu)(\mu+\epsilon)}\right)$.

Proof. In order to obtain EFP, from (2.1), one has

$$(\alpha i+\mu)s = \Lambda,\ (\mu+\epsilon)e = \alpha si,\ (\mu+\beta)i = \epsilon e. \tag{2.3}$$

From (2.3), one can also write

$$i = \frac{\epsilon e}{\beta+\mu}, \tag{2.4}$$

$$s = \frac{(\mu+\epsilon)e}{\alpha i}, \tag{2.5}$$

and

$$(\mu+\epsilon)e + \mu s = \Lambda. \tag{2.6}$$

Using (2.4) into (2.5), one gets

$$s = \frac{(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}. \tag{2.7}$$

Using (2.7) into (2.6), one gets

$$e = \frac{\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon(\mu+\epsilon)}. \tag{2.8}$$

Now, using (2.8) into (2.4), one has

$$i = \frac{\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon)}{\alpha(\beta+\mu)(\mu+\epsilon)}. \tag{2.9}$$

Finally, from (2.7)–(2.9), one can summarize that EFP of the tuberculosis epidemic model (1.11) is $\Omega_2 = \left(\frac{(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}, \frac{\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon(\mu+\epsilon)}, \frac{\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon)}{\alpha(\beta+\mu)(\mu+\epsilon)}\right)$, if $\Lambda > \frac{\mu(\mu+\epsilon)(\mu+\beta)}{\alpha\epsilon}$, that is, $\mathcal{R}_0 = \frac{\Lambda\alpha\epsilon}{\mu(\epsilon+\mu)(\beta+\mu)} > 1$. □

2.3. Linearized system

Now, the variation matrix $V|_\Omega$ evaluated at Ω of a linearized system of discrete model (1.11) under the map $(f_1, f_2, f_3) \mapsto (s_{t+1}, e_{t+1}, i_{t+1})$ is

$$\Psi_{t+1} = V|_\Omega \Psi_t, \tag{2.10}$$

where

$$V|_\Omega := \begin{pmatrix} 1-\mu h-\alpha hi & 0 & -\alpha hs \\ \alpha hi & 1-\mu h-\epsilon h & \alpha hs \\ 0 & \epsilon h & 1-\mu h-\beta h \end{pmatrix}, \tag{2.11}$$

and

$$f_1 = h\Lambda + (1-\mu h)s - \alpha hsi,\ f_2 = (1-\mu h-\epsilon h)e + \alpha hsi,\ f_3 = (1-h\mu-h\beta)i + h\epsilon e. \tag{2.12}$$

3. Local behavior

By the stability theory [31–34], we study local behavior at $\Omega_{1,2}$ of the discrete tuberculosis epidemic model (1.11). Thus, for Ω_1, (2.11) becomes

$$V|_{\Omega_1} := \begin{pmatrix} 1-h\mu & 0 & -\frac{h\alpha\Lambda}{\mu} \\ 0 & -h(\epsilon+\mu)+1 & \frac{h\alpha\Lambda}{\mu} \\ 0 & h\epsilon & -h(\beta+\mu)+1 \end{pmatrix}. \tag{3.1}$$

The characteristics equation of $V|_{\Omega_1}$ at Ω_1 is

$$\lambda^3 + \varphi_1\lambda^2 + \varphi_2\lambda + \varphi_3 = 0, \tag{3.2}$$

where

$$\begin{cases} \varphi_1 &= h(\beta+3\mu+\epsilon)-3, \\ \varphi_2 &= \frac{h^2(\mu(2\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2)-\alpha\Lambda\epsilon)-2h\mu(\beta+3\mu+\epsilon)+3\mu}{\mu}, \\ \varphi_3 &= \frac{(h\mu-1)(h^2(\mu(\beta+\mu)(\mu+\epsilon)-\alpha\Lambda\epsilon)-h\mu(\beta+2\mu+\epsilon)+\mu)}{\mu}, \end{cases} \tag{3.3}$$

and

$$\lambda_1 = \ 1-h\mu,\ \lambda_{2,3} = \ \frac{2-h(\beta+2\mu+\epsilon)\pm\sqrt{\Delta}}{2\sqrt{\mu}}, \tag{3.4}$$

with

$$\Delta = \ \frac{h^2\left(4\alpha\Lambda\epsilon+\mu(\beta-\epsilon)^2\right)}{\mu} > 0. \tag{3.5}$$

Theorem 3.1. *Ω_1 of the discrete tuberculosis model (1.11) is*

(i) a stable node if

$$0 < h < \min\left\{\frac{2}{\mu}, \frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu) \\ +\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}, \frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu) \\ -\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}\right\}; \tag{3.6}$$

(ii) an unstable node if

$$h > \max\left\{\frac{2}{\mu}, \frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu) \\ +\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}, \frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu) \\ -\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}\right\}; \tag{3.7}$$

(iii) a saddle node if

$$\max\left\{\frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu) \\ +\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}, \frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu) \\ -\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}\right\} < h < \frac{2}{\mu}, \tag{3.8}$$

or

$$\frac{2}{\mu} < h < \min\left\{\frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu)\\ +\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}, \frac{4\sqrt{\mu}}{\left\{\begin{array}{c}\sqrt{\mu}(\beta+\epsilon+2\mu)\\ -\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}\end{array}\right\}}\right\}; \tag{3.9}$$

(iv) a non-hyperbolic if

$$h = \frac{2}{\mu}, \tag{3.10}$$

or

$$h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \tag{3.11}$$

or

$$h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}. \tag{3.12}$$

Proof. If one has $|\lambda_1| = |1-h\mu| < 1$ and $|\lambda_{2,3}| = \left|\frac{2-h(\beta+2\mu+\epsilon)\pm h\sqrt{\Delta}}{2\sqrt{\mu}}\right| < 1$, that is, $0<h < \min\left\{\frac{2}{\mu}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, then Ω_1 is a stable node. Similarly, Ω_1 is unstable node if $h > \max\left\{\frac{2}{\mu}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, saddle node if $\max\left\{\frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\} < h < \frac{2}{\mu}$ or $\frac{2}{\mu} < h < \min\left\{\frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, and non-hyperbolic if $h = \frac{2}{\mu}$ or $h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}$ or $h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}$. □

Example 1. *If $\beta = 1.3$, $\epsilon = 3.0$, $\alpha = 4.6$, and $\Lambda = 1.9$, then at Ω_1, dynamics of the tuberculosis model (1.11) is given in Figure 2.*

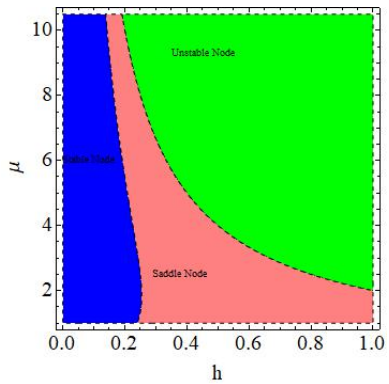


Figure 2. Dynamics of Ω_1 with $h \in (0.0, 1.0)$, and $\mu \in (1.0, 10.5)$.

Example 2. *If $\beta = 1.3$, $\epsilon = 3.0$, and $\alpha = 4.6$, then at Ω_1, the dynamics of the tuberculosis model (1.11) is given in Figure 3.*

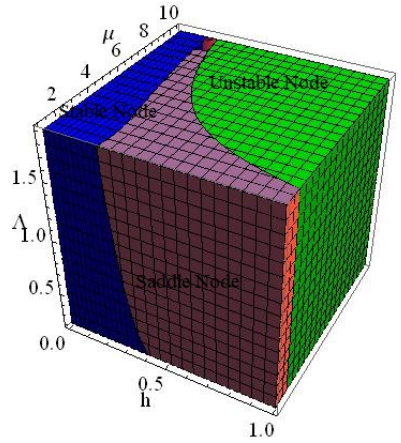


Figure 3. Dynamics of Ω_1 with $h \in (0.0, 1.0)$, $\mu \in (1.0, 10.5)$, and $\Lambda \in (0.2, 1.9)$.

Theorem 3.2. *Necessary and sufficient (N and S) conditions for Ω_1, under which roots of (3.2) inside the open unit disc are*

$$\frac{\mu(h(\mu+\beta)-2)(h(\mu+\epsilon)-2)}{\alpha h^2\epsilon} < \Lambda < \frac{\mu(\mu+\beta)(\mu+\epsilon)}{\alpha\epsilon}, \tag{3.13}$$

$$\frac{\mu\mathcal{H}_1}{\alpha h\epsilon(3h\mu-4)} < \Lambda < \frac{\mu\mathcal{H}_2}{\alpha\epsilon(3h\mu-2)}, \tag{3.14}$$

and

$$\Lambda < \min\left\{\frac{\mu(\beta(h\mu-1)+\mu(h\mu-2))(\mu(h\mu-2)+\epsilon(h\mu-1))}{\alpha\epsilon(h\mu-1)^2}, \frac{\mu(\beta(h(\mu+\epsilon)-1)+h\mu(\mu+\epsilon)-2\mu-\epsilon)}{\alpha h\epsilon}\right\}, \tag{3.15}$$

where

$$\begin{cases} \mathcal{H}_1 &= \beta(3h^2\mu(\mu+\epsilon)-4h(2\mu+\epsilon)+4)+(h\mu-2)(3\mu(h\mu-2)+\epsilon(3h\mu-2)), \\ \mathcal{H}_2 &= -2\beta\epsilon+3\mu^2(h(\beta+\epsilon)-2)+\mu(\beta(3h\epsilon-4)-4\epsilon)+3h\mu^3. \end{cases} \tag{3.16}$$

Proof. By the linear stability theory,

$$\begin{aligned} |\varphi_1+\varphi_3| &= \left|h(\beta+3\mu+\epsilon)-3+\frac{h\mu-1}{\mu}\Big(h^2(\mu(\mu+\beta)(\mu+\epsilon)-\alpha\Lambda\epsilon)-h\mu(\beta+2\mu+\epsilon)+\mu\Big)\right| < 1+\varphi_2 \\ &= 1+\frac{1}{\mu}\Big(h^2\left(\mu\left(2\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2\right)-\alpha\Lambda\epsilon\right)-2h\mu(\beta+3\mu+\epsilon)+3\mu\Big), \end{aligned}$$

that is,

$$\frac{\mu(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)}{\alpha h^2\epsilon} < \Lambda < \frac{\mu(\mu+\beta)(\mu+\epsilon)}{\alpha\epsilon},$$

$$\begin{aligned} |\varphi_1-3\varphi_3| &= \left|h(\beta+3\mu+\epsilon)-3-\frac{3(h\mu-1)}{\mu}\Big(h^2(\mu(\mu+\beta)(\mu+\epsilon)-\alpha\Lambda\epsilon)-h\mu(\beta+2\mu+\epsilon)+\mu\Big)\right| < 3-\varphi_2 \\ &= 3-\frac{1}{\mu}\Big(h^2\Big(\mu\Big(2\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2\Big)-\alpha\Lambda\epsilon\Big)-2h\mu(\beta+3\mu+\epsilon)+3\mu\Big), \end{aligned}$$

one gets

$$\frac{\mu\mathcal{H}_1}{\alpha h\epsilon(3h\mu-4)} < \Lambda < \frac{\mu\mathcal{H}_2}{\alpha\epsilon(3h\mu-2)}.$$

Finally,

$$\begin{aligned}\varphi_3^2+\varphi_2-\varphi_3\varphi_1 =&\Big(\frac{(h\mu-1)(h^2(\mu(\mu+\beta)(\mu+\epsilon)-\alpha\Lambda\epsilon)-h\mu(\beta+2\mu+\epsilon)+\mu)}{\mu}\Big)^2\\ &+\frac{h^2(\mu(2\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2)-\alpha\Lambda\epsilon)-2h\mu(\beta+3\mu+\epsilon)+3\mu}{\mu}\\ &-\Big(\frac{(h\mu-1)(h^2(\mu(\beta+\mu)(\mu+\epsilon)-\alpha\Lambda\epsilon)-h\mu(\beta+2\mu+\epsilon)+\mu)}{\mu}\Big)\Big(h(\beta+3\mu+\epsilon)-3\Big)\\ <&1,\end{aligned}$$

that is, the calculation yields that

$$\Lambda < \min\left\{\frac{\mu(\beta(h\mu-1)+\mu(h\mu-2))(\mu(h\mu-2)+\epsilon(h\mu-1))}{\alpha\epsilon(h\mu-1)^2}, \frac{\mu(\beta(h(\mu+\epsilon)-1)+h\mu(\mu+\epsilon)-2\mu-\epsilon)}{\alpha h\epsilon}\right\}$$

are the desired computable N and S parametric condition under which characteristics roots of (3.2) are inside the open unit disc. □

Theorem 3.3. *Sufficient condition under which roots of (3.2) inside the open unit disc is*

$$0 < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}. \tag{3.17}$$

Proof. Again by stability theory,

$$\begin{aligned}|\varphi_1|+|\varphi_2|+|\varphi_3| =& |h(\beta+3\mu+\epsilon)-3|+\left|\frac{h^2}{\mu}\Big(\mu\Big(2\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2\Big)-\alpha\Lambda\epsilon\Big)-2h\mu(\beta+3\mu+\epsilon)+3\mu\right|\\ &+\left|\frac{(h\mu-1)}{\mu}\Big(h^2(\mu(\mu+\beta)(\mu+\epsilon)-\alpha\Lambda\epsilon)-h\mu(\beta+2\mu+\epsilon)+\mu\Big)\right| < 1,\end{aligned}$$

that is, $0 < \Lambda < \frac{\mu(\mu+\beta)(\mu+\epsilon)}{\alpha\epsilon}$ is the desired computable condition under which roots of (3.2) inside the open unit disc. □

Example 3. *If $\beta = 1.0$, $\epsilon = 3.3$, $\alpha = 4.5$ and $\mu = 1.1$, then at Ω_1, the region of stability for the tuberculosis model (1.11) is given in Figure 4.*

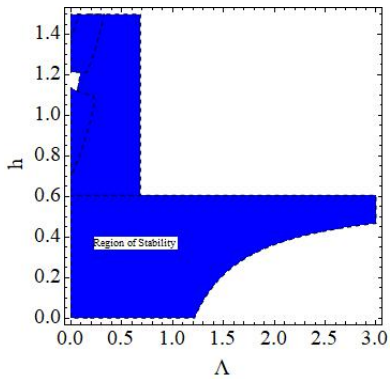


Figure 4. Stability region of Ω_1 with $\Lambda \in (0.0, 3.0)$, and $h \in (0.0, 1.5)$.

Example 4. *If $\beta = 1.0$, $\epsilon = 3.3$ and $\alpha = 4.5$, then at Ω_1, the region of stability for the tuberculosis model (1.11) is given in Figure 5.*

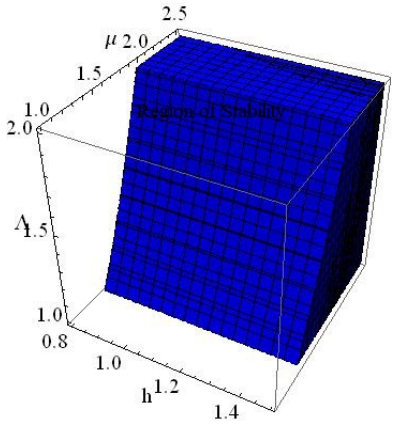


Figure 5. Stability region of Ω_1 with $h \in (0.8, 1.5)$, $\mu \in (1.0, 2.5)$, and $\Lambda \in (1.0, 2.0)$.

At Ω_2, (2.11) becomes

$$V|_{\Omega_2} := \begin{pmatrix} 1 - h\left(\mu + \frac{\alpha\epsilon\Lambda - \mu(\mu+\beta)(\epsilon+\mu)}{(\mu+\beta)(\epsilon+\mu)}\right) & 0 & -\frac{h(\mu+\beta)(\epsilon+\mu)}{\epsilon} \\ \frac{h(\alpha\epsilon\Lambda - \mu(\mu+\beta)(\epsilon+\mu))}{(\mu+\beta)(\epsilon+\mu)} & -h(\epsilon+\mu)+1 & \frac{h(\mu+\beta)(\epsilon+\mu)}{\epsilon} \\ 0 & h\epsilon & -h(\mu+\beta)+1 \end{pmatrix}. \tag{3.18}$$

The characteristics equation of $V|_{\Omega_2}$ at Ω_2 is

$$\lambda^3 + \varphi_1\lambda^2 + \varphi_2\lambda + \varphi_3 = 0, \tag{3.19}$$

where

$$\begin{cases} \varphi_1 &= h\left(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)} + \beta + 2\mu + \epsilon\right) - 3, \\ \varphi_2 &= \frac{\left\{\begin{array}{l}\beta\left(\epsilon\left(\alpha h^2\Lambda - 8h\mu + 3\right) + 3\mu(1-2h\mu) - 2h\epsilon^2\right) + \alpha h^2\Lambda\epsilon(2\mu+\epsilon) \\ \qquad - 2h(\alpha\Lambda\epsilon + \mu(\mu+\epsilon)(2\mu+\epsilon)) - 2\beta^2 h(\mu+\epsilon) + 3\mu(\mu+\epsilon)\end{array}\right\}}{(\beta+\mu)(\mu+\epsilon)}, \\ \varphi_3 &= h^3(\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon)) - \frac{\alpha h^2\Lambda\epsilon(\beta+2\mu+\epsilon)}{(\beta+\mu)(\mu+\epsilon)} + h\left(\frac{\epsilon(\alpha\Lambda+(\beta+\mu)(\mu+\epsilon))}{(\beta+\mu)(\mu+\epsilon)} + \beta + 2\mu\right) - 1. \end{cases} \tag{3.20}$$

Theorem 3.4. *Necessary and sufficient (N and S) conditions for Ω_2 under which roots of (3.19) inside the open unit disc are*

$$\frac{(\mu+\beta)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon) + 8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} < \Lambda < \frac{\mu(\mu+\beta)(\mu+\epsilon)}{\alpha\epsilon}, \tag{3.21}$$

$$\begin{aligned} &\frac{(\mu+\beta)(\mu+\epsilon)\left(\beta\left(3h^2\mu(\mu+\epsilon) - 4\right) + 3h^2\mu^2(\mu+\epsilon) - 4(2\mu+\epsilon)\right)}{\alpha\epsilon\left(3h^2(\mu+\beta)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon) + 4\right)} \\ <&\Lambda < \frac{3h\mu(\beta+\mu)^2(\mu+\epsilon)^2}{\alpha\epsilon(\beta(3h(\mu+\epsilon)-2) + 3h\mu(\mu+\epsilon) - 2(2\mu+\epsilon))}, \end{aligned} \tag{3.22}$$

and

$$\Lambda < \min\left\{\frac{\xi - \sqrt{\chi}}{\mathcal{L}_5}, \frac{\xi + \sqrt{\chi}}{\mathcal{L}_5}\right\}, \tag{3.23}$$

where

$$\begin{cases} \chi = & \alpha^2\epsilon^2(\beta+\mu)^2(\mu+\epsilon)^2\Big(\beta^2\Big(6h^2\mu^4 + \epsilon^2(h\epsilon-3)(h\epsilon-1) + 2h\mu^3(9h\epsilon-8) + \mu^2 \\ & (h\epsilon-1)(19h\epsilon-11) + 2\mu\epsilon(h\epsilon(4h\epsilon-9)+4)\Big) + \beta^4(h(\mu+\epsilon)-1)^2 + 2\beta^3(h(\mu \\ & +\epsilon)-1)\Big(\mu(2h\mu-3) + h\epsilon^2 + \epsilon(3h\mu-1)\Big) + 2\beta\Big(\mu^3(h\mu-1)(2h\mu-3) + h\epsilon^4 \\ & (h\mu-1) + \epsilon^3(h\mu(5h\mu-7)+1) + \mu\epsilon^2(3h\mu-4)(3h\mu-1) + \mu^2\epsilon(7h\mu(h\mu-2) \\ & +4)\Big) + \Big(\mu^2(h\mu-1) + \epsilon^2(h\mu-1) + \mu\epsilon(2h\mu-3)\Big)^2\Big), \\ \xi = & \alpha\epsilon(\beta+\mu)(\mu+\epsilon)\Big(\beta^2(h(\mu+\epsilon)-1)\Big(2h^2\mu(\mu+\epsilon)-1\Big) + \beta\Big(\mu(2h\mu-3)\Big(2h^2\mu^2 \\ & -1\Big) + h\epsilon^2(2h\mu(2h\mu-1)-1) + \epsilon(h\mu(8h\mu(h\mu-1)-3)+1)\Big) + \epsilon^2(h\mu-1) \\ & \Big(2h^2\mu^2-1\Big) + \mu\epsilon(2h\mu-3)\Big(2h^2\mu^2-1\Big) + \mu^2(h\mu-1)(2h\mu(h\mu-1)-3)\Big), \\ \mathcal{L}_5 = & 2\alpha^2\epsilon^2(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)(\beta(h(\mu+\epsilon)-1) + h\mu(\mu+\epsilon) - 2\mu - \epsilon). \end{cases} \tag{3.24}$$

Proof. By linear stability theory,

$$\begin{aligned} |\varphi_1 + \varphi_3| = & \left|h\Big(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)} + \beta + 2\mu + \epsilon\Big) - 3 + h^3(\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon))\right. \\ & \left. - \frac{\alpha h^2\Lambda\epsilon(\beta+2\mu+\epsilon)}{(\beta+\mu)(\mu+\epsilon)} + h\Big(\frac{\epsilon(\alpha\Lambda + (\beta+\mu)(\mu+\epsilon))}{(\beta+\mu)(\mu+\epsilon)} + \beta + 2\mu\Big) - 1\right| \\ < & 1 + \varphi_2 \\ = & 1 + \frac{1}{(\beta+\mu)(\mu+\epsilon)}\Big(\beta\Big(\epsilon\Big(\alpha h^2\Lambda - 8h\mu + 3\Big) + 3\mu(1-2h\mu) - 2h\epsilon^2\Big) \\ & + \alpha h^2\Lambda\epsilon(2\mu+\epsilon) - 2h(\alpha\Lambda\epsilon + \mu(\mu+\epsilon)(2\mu+\epsilon)) - 2\beta^2 h(\mu+\epsilon) + 3\mu(\mu+\epsilon)\Big), \end{aligned}$$

that is, one has

$$\frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon) + 8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon},$$

$$\begin{aligned} |\varphi_1 - 3\varphi_3| = & \left|h\Big(\frac{\alpha\lambda\epsilon}{(\mu+\beta)(\mu+\epsilon)} + \beta + 2\mu + \epsilon\Big) - 3 - 3(h^3(\alpha\Lambda\epsilon - \mu(\mu+\beta)(\mu+\epsilon))\right. \\ & \left. - \frac{\alpha h^2\Lambda\epsilon(\beta+2\mu+\epsilon)}{(\mu+\beta)(\mu+\epsilon)} + h\Big(\frac{\epsilon(\alpha\Lambda + (\mu+\beta)(\mu+\epsilon))}{(\mu+\beta)(\mu+\epsilon)} + \beta + 2\mu\Big) - 1)\right| \\ < & 3 - \varphi_2 = 3 - \frac{1}{(\mu+\beta)(\mu+\epsilon)}\Big(\beta\Big(\epsilon\Big(\alpha h^2\Lambda - 8h\mu + 3\Big) + 3\mu(1-2h\mu) - 2h\epsilon^2\Big) \\ & + \alpha h^2\Lambda\epsilon(2\mu+\epsilon) - 2h(\alpha\Lambda\epsilon + \mu(\mu+\epsilon)(2\mu+\epsilon)) - 2\beta^2 h(\mu+\epsilon) + 3\mu(\mu+\epsilon)\Big), \end{aligned}$$

so, one has

$$\begin{aligned} & \frac{(\beta+\mu)(\mu+\epsilon)\left(\beta\left(3h^2\mu(\mu+\epsilon) - 4\right) + 3h^2\mu^2(\mu+\epsilon) - 4(2\mu+\epsilon)\right)}{\alpha\epsilon\left(3h^2(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon) + 4\right)} \\ < & \Lambda < \frac{3h\mu(\beta+\mu)^2(\mu+\epsilon)^2}{\alpha\epsilon(\beta(3h(\mu+\epsilon)-2) + 3h\mu(\mu+\epsilon) - 2(2\mu+\epsilon))}, \end{aligned}$$

and

$$\begin{aligned}
&\varphi_3^2+\varphi_2-\varphi_3\varphi_1\\
=&\Big(h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))-\frac{\alpha h^2\Lambda\epsilon(\beta+2\mu+\epsilon)}{(\beta+\mu)(\mu+\epsilon)}+h\Big(\frac{\epsilon(\alpha\Lambda+(\beta+\mu)(\mu+\epsilon))}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu\Big)-1\Big)^2\\
&+\frac{\left\{\begin{array}{l}\beta\Big(\epsilon\Big(\alpha h^2\Lambda-8h\mu+3\Big)+3\mu(1-2h\mu)-2h\epsilon^2\Big)+\alpha h^2\Lambda\epsilon(2\mu+\epsilon)\\ \qquad -2h(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))-2\beta^2h(\mu+\epsilon)+3\mu(\mu+\epsilon)\end{array}\right\}}{(\beta+\mu)(\mu+\epsilon)}\\
&-\Big(h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))-\frac{\alpha h^2\Lambda\epsilon(\beta+2\mu+\epsilon)}{(\beta+\mu)(\mu+\epsilon)}\\
&+h\Big(\frac{\epsilon(\alpha\Lambda+(\beta+\mu)(\mu+\epsilon))}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu\Big)-1\Big)\Big(h\Big(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu+\epsilon\Big)-3\Big)<1,
\end{aligned}$$

gives $\Lambda<\min\left\{\frac{\xi-\sqrt{\chi}}{\mathcal{L}_5},\frac{\xi+\sqrt{\chi}}{\mathcal{L}_5}\right\}$ are the desired computable N and S parametric conditions under which characteristic roots of (3.19) inside the open unit disc. □

Theorem 3.5. *Sufficient condition under which roots of (3.19) inside the open unit disc is*

$$0<\Lambda<\frac{\mu(\mu+\beta)(\mu+\epsilon)}{\alpha\epsilon}. \tag{3.25}$$

Proof. Again by stability theory,

$$\begin{aligned}
&|\varphi_1|+|\varphi_2|+|\varphi_3|=\Big|h\Big(\frac{\alpha\lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu+\epsilon\Big)-3\Big|\\
&+\Big|\frac{1}{(\beta+\mu)(\mu+\epsilon)}\Big(\beta\Big(\epsilon\Big(\alpha h^2\Lambda-8h\mu+3\Big)+3\mu(1-2h\mu)-2h\epsilon^2\Big)+\alpha h^2\lambda\epsilon(2\mu+\epsilon)\\
&-2h(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))-2\beta^2h(\mu+\epsilon)+3\mu(\mu+\epsilon)\Big)\Big|\\
&+\Big|h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))-\frac{\alpha h^2\Lambda\epsilon(\beta+2\mu+\epsilon)}{(\beta+\mu)(\mu+\epsilon)}+h\Big(\frac{\epsilon(\alpha\Lambda+(\beta+\mu)(\mu+\epsilon))}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu\Big)-1\Big|<1,
\end{aligned}$$

that is, $0<\Lambda<\frac{\mu(\mu+\beta)(\mu+\epsilon)}{\alpha\epsilon}$ is the desired computable condition under which roots of (3.19) are inside the open unit disc. □

Example 5. *If $\beta=1.3$, $\epsilon=3.0$, $\alpha=4.6$, and $\mu=1.1$, then at Ω_2, the stability region for the tuberculosis model (1.11) is given in Figure 6.*

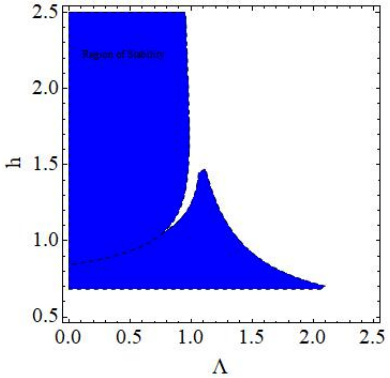


Figure 6. Stability region of Ω_2 with $\Lambda\in(0.0,2.5)$, and $h\in(0.5,2.5)$.

Example 6. *If $\beta = 1.3$, $\epsilon = 3.0$, and $\alpha = 4.6$, then at Ω_2, the stability region for the tuberculosis model (1.11) is given in Figure 7.*

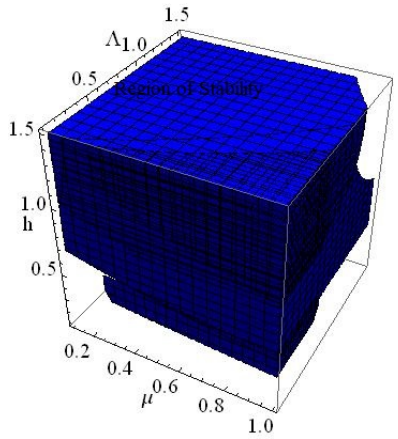


Figure 7. Stability region of Ω_2 with $\mu \in (0.1, 1.0)$, $\Lambda \in (0.1, 1.5)$, and $h \in (0.1, 1.5)$.

4. Bifurcations analysis

In this section, we study bifurcations at $\Omega_{1,2}$ of a tuberculosis model (1.11). However, straightforward analysis imply that at Ω_1, no flip bifurcation exists if $(\mu, \beta, \epsilon, \alpha, h, \Lambda) \in F|_{\Omega_1} = \left\{(\mu, \beta, \epsilon, \alpha, h, \Lambda) : h = \frac{2}{\mu}\right\}$, $(\mu, \beta, \epsilon, \alpha, h, \Lambda) \in F|_{\Omega_1} = \left\{(\mu, \beta, \epsilon, \alpha, h, \Lambda) : h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$ as well as the set $(\mu, \beta, \epsilon, \alpha, h, \Lambda) \in F|_{\Omega_1} = \left\{(\mu, \beta, \epsilon, \alpha, h, \Lambda) : h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, respectively. In the rest of the section, we choose Λ as a bifurcation parameter to investigate one-parameter bifurcations by explicit criterions. Before studying the one-parameter bifurcations, we first state the criteria for N-S and flip bifurcations separately as the following lemmas:

Lemma 4.1. *[35–37] If one has the t-dimensional system, then*

$$X_{t+1} = f_\Lambda(X_t), \tag{4.1}$$

where the bifurcation parameter is $\Lambda \in \mathbb{R}$. Furthermore, the characteristic polynomial of t-dimensional system of $V|_X$ at X is

$$P(\lambda) = \lambda^t + \varphi_1\lambda^{t-1} + \varphi_2\lambda^{t-2} + \cdots + \varphi_t. \tag{4.2}$$

Now, considering the determinants: $\Delta_0^{\pm}(\Lambda) = 1$, $\Delta_1^{\pm}(\Lambda), \cdots, \Delta_t^{\pm}(\Lambda)$, which can be expressed as

$$\Delta_j^{\pm}(\Lambda) = \left\| \begin{pmatrix} 1 & \varphi_1 & \varphi_2 & \cdots & \varphi_{j-1} \\ 0 & 1 & \varphi_1 & \cdots & \varphi_{j-2} \\ 0 & 0 & 1 & \cdots & \varphi_{j-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \pm \begin{pmatrix} \varphi_{t-j+1} & \varphi_{t-j+2} & \cdots & \varphi_{t-1} & \varphi_t \\ \varphi_{t-j+2} & \varphi_{t-j+3} & \cdots & \varphi_t & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{t-1} & \varphi_t & \cdots & 0 & 0 \\ \varphi_t & 0 & \cdots & 0 & 0 \end{pmatrix} \right\|, \tag{4.3}$$

where $j = 1, \cdots, t$. Moreover, at critical value $\Lambda = \Lambda_0$, N-S bifurcation exists if following quantities must hold:

Γ_1 : *Assignment of Eigenvalue:* $\Delta_{t-1}^{-}(\Lambda_0) = 0, \Delta_{t-1}^{+}(\Lambda_0) > 0, P_{\Lambda_0}(1) > 0, (-1)^t P_{\Lambda_0}(-1) > 0, \Delta_j^{\pm}(\Lambda_0) > 0$ *where* $j = t-3, t-5, \cdots, 1$ *(or 2), whenever t is even or odd, respectively.*

Γ_2 : *Condition of Transversality:* $\frac{d}{d\Lambda}\Delta_{t-1}^{-}(\Lambda_0) \neq 0$.

Γ_3 : *Resonance condition:* $\cos\left(\frac{2\pi}{l}\right) = 1 - 0.5P_{\Lambda}(1)\frac{\Delta_{t-3}^{-}(\Lambda_0)}{\Delta_{t-2}^{+}(\Lambda_0)}$ *or condition of Nonresonance:* $\cos(\frac{2\pi}{l}) \neq 1 - 0.5P_{\Lambda}(1)\frac{\Delta_{t-3}^{-}(\Lambda_0)}{\Delta_{t-2}^{+}(\Lambda_0)}$ *where* $l = 3, 4, \cdots$.

Lemma 4.2. *[38] One has system (4.1) with characteristic polynomial is depicted in (4.2). Furthermore, we reconsider the determinants* $\Delta_0^{\pm}(\Lambda) = 1$, $\Delta_1^{\pm}(\Lambda), \cdots, \Delta_t^{\pm}(\Lambda)$ *that are expressed in (4.3). The flip bifurcation exists at* $\Lambda = \Lambda_0$ *if the following quantities hold:*

Γ_1 : *Assignment of eigenvalue:* $P_{\Lambda_0}(-1) = 0$, $P_{\Lambda_0}(1) > 0$, $\Delta_{t-1}^{\pm}(\Lambda_0) > 0$, $\Delta_j^{\pm}(\Lambda_0) > 0$ *where* $j = t-3, t-5, \cdots, 1$ *(or 2) whenever t is even or odd, respectively.*

Γ_2 : *Condition of Transversality:* $\frac{\sum_{j=1}^{t}(-1)^{t-i}\varphi_j^{'}}{\sum_{j=1}^{t}(-1)^{t-i}(t-j+1)\varphi_{j-1}} \neq 0$ *where* $\varphi_j^{'}$ *denotes derivative w.r.t* Λ *at* $\Lambda = \Lambda_0$.

Theorem 4.1. *If*

$$\Lambda = \frac{\mathcal{L}_2 + \sqrt{\mathcal{L}_1}}{2\alpha^2\epsilon^2(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)(\beta(h(\mu+\epsilon)-1)+h\mu(\mu+\epsilon)-2\mu-\epsilon)} = \Lambda_0, \tag{4.4}$$

$$\Lambda > \max\left\{\frac{\mathcal{L}_4 + \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}, \frac{\mathcal{L}_4 - \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\}, \tag{4.5}$$

$$\Lambda > \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}, \tag{4.6}$$

$$\Lambda > \frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon)+8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)}, \tag{4.7}$$

$$\begin{aligned}&\frac{h^3}{(\beta+\mu)^2(\mu+\epsilon)^2}\Big(\alpha\epsilon(\beta+\mu)(\mu+\epsilon)(\beta^2(h(\mu+\epsilon)-1)(2h^2\mu(\mu+\epsilon)-1)+\beta(\mu(2h\mu-3)\\ &(2h^2\mu^2-1)+h\epsilon^2(2h\mu(2h\mu-1)-1)+\epsilon(h\mu(8h\mu(h\mu-1)-3)+1))+\epsilon^2(h\mu-1)\\ &(2h^2\mu^2-1)+\mu\epsilon(2h\mu-3)(2h^2\mu^2-1)+\mu^2(h\mu-1)(2h\mu(h\mu-1)-3))-\mathcal{L}_2-\sqrt{\mathcal{L}_1}\Big)\neq 0,\end{aligned} \tag{4.8}$$

and

$$\cos\tfrac{2\pi}{l} \neq \frac{h^2(\mu+\epsilon)(\mu+\beta)(\mu(\mu+\beta)(\mu+\epsilon)-\alpha\Lambda\epsilon)}{\left\{\begin{array}{c}2(\beta(\epsilon^2(h^2(\alpha\Lambda-2\mu^2)+1)-2h^2\mu^4+\epsilon(h\mu-1)(\alpha h\Lambda-4h\mu^2-4\mu)\\ +3\mu^2)+\beta^2(-(\mu+\epsilon))(h^2\mu(\mu+\epsilon)-1)-\mu(\mu+\epsilon)\\ (h^2\mu^3+h^2\mu^2\epsilon-2\mu-\epsilon)+\alpha\Lambda\epsilon(h\mu-1)(h(\mu+\epsilon)-1))\end{array}\right\}}+1, \tag{4.9}$$

where

$$
\begin{cases}
\mathcal{L}_1 = & \alpha^2\epsilon^2(\beta+\mu)^2(\mu+\epsilon)^2(\beta^2(6h^2\mu^4+\epsilon^2(h\epsilon-3)(h\epsilon-1)+2h\mu^3(9h\epsilon-8)+\mu^2 \\
& (h\epsilon-1)(19h\epsilon-11)+2\mu\epsilon(h\epsilon(4h\epsilon-9)+4))+\beta^4(h(\mu+\epsilon)-1)^2+2\beta^3(h(\mu+\epsilon) \\
& -1)(\mu(2h\mu-3)+h\epsilon^2+\epsilon(3h\mu-1))+2\beta(\mu^3(h\mu-1)(2h\mu-3)+h\epsilon^4(h\mu-1)+ \\
& \epsilon^3(h\mu(5h\mu-7)+1)+\mu\epsilon^2(3h\mu-4)(3h\mu-1)+\mu^2\epsilon(7h\mu(h\mu-2)+4))+(\mu^2(h\mu \\
& -1)+\epsilon^2(h\mu-1)+\mu\epsilon(2h\mu-3))^2), \\
\mathcal{L}_2 = & \alpha\epsilon(\beta+\mu)(\mu+\epsilon)(\beta^2(h(\mu+\epsilon)-1)(2h^2\mu(\mu+\epsilon)-1)+\beta(\mu(2h\mu-3)(2h^2\mu^2 \\
& -1)+h\epsilon^2(2h\mu(2h\mu-1)-1)+\epsilon(h\mu(8h\mu(h\mu-1)-3)+1))+\epsilon^2(h\mu-1)(2h^2\mu^2 \\
& -1)+\mu\epsilon(2h\mu-3)(2h^2\mu^2-1)+\mu^2(h\mu-1)(2h\mu(h\mu-1)-3)), \\
\mathcal{L}_3 = & \alpha^2\epsilon^2(\beta+\mu)^2(\mu+\epsilon)^2(h^6(\beta+\mu)^2(\mu+\epsilon)^2(10\mu(\beta+\epsilon)+(\beta+\epsilon)^2+17\mu^2)- \\
& 2h^5(\beta+\mu)(\mu+\epsilon)(\beta^3+2\beta^2(8\mu+5\epsilon)+\beta(4\mu+\epsilon)(13\mu+10\epsilon)+41\mu^3+\epsilon^3+16\mu\epsilon^2 \\
& +52\mu^2\epsilon)+h^4(\beta^4+26\beta^3\epsilon+\mu^2(179\beta^2+536\beta\epsilon+179\epsilon^2)+2\mu(\beta+\epsilon)(15\beta^2+109 \\
& \beta\epsilon+15\epsilon^2)+75\beta^2\epsilon^2+26\beta\epsilon^3+318\mu^3(\beta+\epsilon)+169\mu^4+\epsilon^4)-8h^3(\beta^3+\mu(13\beta^2+ \\
& 42\beta\epsilon+13\epsilon^2)+10\beta^2\epsilon+10\beta\epsilon^2+35\mu^2(\beta+\epsilon)+24\mu^3+\epsilon^3)+8h^2(3\beta^2+17\mu(\beta+ \\
& \epsilon)+11\beta\epsilon+17\mu^2+3\epsilon^2)-32h(\beta+2\mu+\epsilon)+16), \\
\mathcal{L}_4 = & \alpha\epsilon(\beta+\mu)(\mu+\epsilon)(2h^5\mu(\beta+\mu)^2(\mu+\epsilon)^2-2h^4\mu(\beta+\mu)(\mu+\epsilon)(\beta+2\mu+\epsilon) \\
& -3h^3(\beta+\mu)(\mu+\epsilon)(\beta+\mu+\epsilon)+h^2(3\beta^2+17\mu(\beta+\epsilon)+11\beta\epsilon+17\mu^2+3\epsilon^2) \\
& -8h(\beta+2\mu+\epsilon)+4),
\end{cases}
\tag{4.10}
$$

then at $\Lambda = \Lambda_0$, the tuberculosis model (1.11) undergoes N-S bifurcation.

Proof. By using Lemma 4.1 for $t = 3$, one gets:

$$
\begin{aligned}
\Delta_2^-(\Lambda) =& 1-\varphi_2+\varphi_3(\varphi_1-\varphi_3) \\
=& h^3\Big(\frac{1}{(\beta+\mu)^2(\mu+\epsilon)^2}\Big(\alpha^2\Lambda^2\epsilon^2(\beta+2\mu+\epsilon)+\alpha\Lambda\epsilon(\beta+\mu)(\mu+\epsilon)(\beta^2+3\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2+\epsilon^2) \\
& +\mu(\mu+\beta)^3(\mu+\epsilon)^3\Big)-h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))^2 \\
& -\frac{2\alpha h^2\Lambda\epsilon}{(\mu+\beta)(\mu+\epsilon)}((\beta+2\mu+\epsilon)(\mu(\beta+\mu)(\mu+\epsilon)-\alpha\Lambda\epsilon)) \\
& -h(\frac{\alpha^2\Lambda^2\epsilon^2}{(\mu+\beta)^2(\mu+\epsilon)^2}((\beta^2+5\mu(\beta+\epsilon)+3\beta\epsilon+5\mu^2+\epsilon^2))-\alpha\Lambda\epsilon(\beta+\mu+\epsilon) \\
& +\mu(\mu+\beta)(\mu+\epsilon)(\beta+2\mu+\epsilon))\Big) \\
=& 0
\end{aligned}
$$

implies that its one root is

$$
\Lambda = \frac{\mathcal{L}_2+\sqrt{\mathcal{L}_1}}{2\alpha^2\epsilon^2(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)(\beta(h(\mu+\epsilon)-1)+h\mu(\mu+\epsilon)-2\mu-\epsilon)},
$$

$$\begin{aligned}\Delta_2^+(\Lambda) =&1+\varphi_2-\varphi_3(\varphi_1+\varphi_3)\\ =&\frac{1}{(\beta+\mu)(\mu+\epsilon)}(\beta(\epsilon(\alpha h^2\Lambda-8h\mu+3)+3\mu(1-2h\mu)-2h\epsilon^2)+\alpha h^2\Lambda\epsilon(2\mu+\epsilon)\\ &-2h(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))-2\beta^2h(\mu+\epsilon)+3\mu(\mu+\epsilon))-(h^3(\alpha\Lambda\epsilon-\mu(\mu+\beta)(\mu+\epsilon))\\ &-\frac{\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}(\beta+2\mu+\epsilon)+\frac{2h}{(\mu+\beta)(\mu+\epsilon)}(\alpha\Lambda\epsilon+\beta^2(\mu+\epsilon)+\beta(\mu+\epsilon)(3\mu+\epsilon)\\ &+\mu(\mu+\epsilon)(2\mu+\epsilon))-4)(h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))-\frac{\alpha h^2}{(\beta+\mu)(\mu+\epsilon)}\Lambda\epsilon(\beta+2\mu+\epsilon)\\ &+h(\frac{\epsilon}{(\beta+\mu)(\mu+\epsilon)}(\alpha\Lambda+(\mu+\beta)(\mu+\epsilon))+\beta+2\mu)-1)+1\\ >&0\end{aligned}$$

implies

$$\Lambda>\max\left\{\frac{\mathcal{L}_4+\sqrt{\mathcal{L}_3}}{\mathcal{L}_5},\frac{\mathcal{L}_4-\sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\},$$

$$P_\Lambda(1)=1+\varphi_1+\varphi_2+\varphi_3=h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))>0$$

implies

$$\Lambda>\frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon},$$

$$\begin{aligned}(-1)^3P_\Lambda(-1)=&1-\varphi_1+\varphi_2-\varphi_3\\ =&h^3(\mu(\beta+\mu)(\mu+\epsilon)-\alpha\Lambda\epsilon)+\frac{2\alpha h^2\Lambda\epsilon}{(\mu+\beta)(\mu+\epsilon)}(\beta+2\mu+\epsilon)\\ &-h(\frac{4\epsilon}{(\mu+\beta)(\mu+\epsilon)}(\alpha\Lambda+(\beta+\mu)(\mu+\epsilon))-4\beta-8\mu)+8>0\end{aligned}$$

implies

$$\Lambda>\frac{(\mu+\beta)(\mu+\epsilon)(h^3\mu(\mu+\beta)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+8)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)},$$

$$\begin{aligned}\frac{d}{d\Lambda}(\Delta_2^-(\Lambda))|_{\Lambda=\Lambda_0}=&\frac{d}{d\Lambda}(1-\varphi_2+\varphi_3(\varphi_1-\varphi_3))|_{\Lambda=\Lambda_0}\\ =&\frac{h^3}{(\mu+\beta)^2(\mu+\epsilon)^2}\Big(\alpha\epsilon(\mu+\beta)(\mu+\epsilon)(\beta^2(h(\mu+\epsilon)-1)(2h^2\mu(\mu+\epsilon)-1)\\ &+\beta(\mu(2h\mu-3)(2h^2\mu^2-1)+h\epsilon^2(2h\mu(2h\mu-1)-1)+\epsilon(h\mu(8h\mu(h\mu-1)-3)+1))\\ &+\epsilon^2(h\mu-1)(2h^2\mu^2-1)+\mu\epsilon(2h\mu-3)(2h^2\mu^2-1)+\mu^2(h\mu-1)(2h\mu(h\mu-1)-3))\\ &-\mathcal{L}_2-\sqrt{\mathcal{L}_1}\Big)\neq 0,\end{aligned}$$

and finally,

$$\begin{aligned}1-0.5P_\Lambda(1)\frac{\Delta_0^-(\Lambda)}{\Delta_1^+(\Lambda)}=&1-\frac{1+\varphi_1+\varphi_2+\varphi_3}{2(1+\varphi_3)}\\ =&\frac{h^2(\mu+\beta)(\mu+\epsilon)(\mu(\mu+\beta)(\mu+\epsilon)-\alpha\Lambda\epsilon)}{\left\{\begin{array}{c}2(\beta(\epsilon^2(h^2(\alpha\Lambda-2\mu^2)+1)-2h^2\mu^4+\epsilon(h\mu-1)(\alpha h\Lambda-4h\mu^2-4\mu)\\ +3\mu^2)+\beta^2(-(\mu+\epsilon))(h^2\mu(\mu+\epsilon)-1)-\mu(\mu+\epsilon)\\ (h^2\mu^3+h^2\mu^2\epsilon-2\mu-\epsilon)+\alpha\Lambda\epsilon(h\mu-1)(h(\mu+\epsilon)-1))\end{array}\right\}}+1\neq\cos\frac{2\pi}{l},\end{aligned}$$

with $l = 3, 4, \cdots$. □

Theorem 4.2. *If*

$$\Lambda > \max\left\{\frac{\mathcal{L}_2 + \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}, \frac{\mathcal{L}_2 - \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}\right\}, \tag{4.11}$$

$$\Lambda > \max\left\{\frac{\mathcal{L}_4 + \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}, \frac{\mathcal{L}_4 - \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\}, \tag{4.12}$$

$$\Lambda > \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}, \tag{4.13}$$

$$\Lambda = \frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon)+8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} = \Lambda_0, \tag{4.14}$$

$$\begin{aligned}\Lambda > \max\Bigg\{&\frac{(\beta+\mu)(\mu+\epsilon)(\beta(h^2\mu(\mu+\epsilon)-1)+h^2\mu^2(\mu+\epsilon)-2\mu-\epsilon)}{\alpha\epsilon(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)},\\ &\frac{(\beta+\mu)(\mu+\epsilon)(h^3\mu(\beta+\mu)(\mu+\epsilon)-h(\beta+2\mu+\epsilon)+2)}{\alpha h\epsilon(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)}\Bigg\},\end{aligned} \tag{4.15}$$

$$\Lambda \neq \frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon)+7\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)}, \tag{4.16}$$

where

$$\begin{cases}\mathcal{L}_6 = & h^3\mu(\mu+\epsilon)(\mu^2(\mu+\epsilon)-\alpha\Lambda\epsilon)+2\alpha h^2\Lambda\epsilon(2\mu+\epsilon)+\beta^2h(\mu+\epsilon)(h^2\mu(\mu+\epsilon)-4)\\ & +\beta(\mu(2h^3\mu^3-12h\mu+7)-h\epsilon^2(h^2(\alpha\Lambda-2\mu^2)+4)+\epsilon(h(h\mu-2)(-\alpha h\Lambda+4h\\ & \mu^2+8\mu)+7))-4h(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))+7\mu(\mu+\epsilon),\\ \mathcal{L}_7 = & 2\mu(\epsilon(-\alpha h^2\Lambda+2h\epsilon-6)+2\beta^2h+\beta(8h\epsilon-6))-\alpha h\Lambda\epsilon(h(\beta+\epsilon)-4)+12\mu^2(h\\ & (\beta+\epsilon)-1)+4\beta\epsilon(h(\beta+\epsilon)-3)+8h\mu^3,\end{cases} \tag{4.17}$$

then $\Lambda = \Lambda_0$, the tuberculosis model (1.11) undergoes flip bifurcation.

Proof. By using Lemma 4.2 for $t = 3$, one gets:

$$\begin{aligned}\Delta_2^-(\Lambda) =& 1-\varphi_2+\varphi_3(\varphi_1-\varphi_3)\\ =& h^3\Big(\frac{1}{(\beta+\mu)^2(\mu+\epsilon)^2}\Big(\alpha^2\Lambda^2\epsilon^2(\beta+2\mu+\epsilon)+\alpha\Lambda\epsilon(\beta+\mu)(\mu+\epsilon)(\beta^2+3\mu(\beta+\epsilon)+\beta\epsilon+3\mu^2+\epsilon^2)\\ &+\mu(\mu+\beta)^3(\mu+\epsilon)^3\Big)-h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))^2\\ &-\frac{2\alpha h^2\Lambda\epsilon}{(\mu+\beta)(\mu+\epsilon)}((\beta+2\mu+\epsilon)(\mu(\beta+\mu)(\mu+\epsilon)-\alpha\Lambda\epsilon))\\ &-h(\frac{\alpha^2\Lambda^2\epsilon^2}{(\mu+\beta)^2(\mu+\epsilon)^2}((\beta^2+5\mu(\beta+\epsilon)+3\beta\epsilon+5\mu^2+\epsilon^2))\\ &-\alpha\Lambda\epsilon(\beta+\mu+\epsilon)+\mu(\mu+\beta)(\mu+\epsilon)(\beta+2\mu+\epsilon))\Big)\\ >&0\end{aligned}$$

implies

$$\Lambda > \max\left\{\frac{\mathcal{L}_2 + \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}, \frac{\mathcal{L}_2 - \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}\right\},$$

$$\begin{aligned}\Delta_2^+(\Lambda) =& 1 + \varphi_2 - \varphi_3(\varphi_1 + \varphi_3)\\ =& \frac{1}{(\mu+\beta)(\mu+\epsilon)}(\beta(\epsilon(\alpha h^2\Lambda - 8h\mu + 3) + 3\mu(1 - 2h\mu) - 2h\epsilon^2) + \alpha h^2\Lambda\epsilon(2\mu + \epsilon)\\ &- 2h(\alpha\Lambda\epsilon + \mu(\mu+\epsilon)(2\mu+\epsilon)) - 2\beta^2 h(\mu+\epsilon) + 3\mu(\mu+\epsilon)) - (h^3(\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon))\\ &- \frac{\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}(\beta + 2\mu + \epsilon) + \frac{2h}{(\beta+\mu)(\mu+\epsilon)}(\alpha\Lambda\epsilon + \beta^2(\mu+\epsilon) + \beta(\mu+\epsilon)(3\mu+\epsilon)\\ &+ \mu(\mu+\epsilon)(2\mu+\epsilon)) - 4)(h^3(\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon)) - \frac{\alpha h^2}{(\beta+\mu)(\mu+\epsilon)}\Lambda\epsilon(\beta + 2\mu + \epsilon)\\ &+ h(\frac{\epsilon}{(\beta+\mu)(\mu+\epsilon)}(\alpha\Lambda + (\beta+\mu)(\mu+\epsilon)) + \beta + 2\mu) - 1) + 1 > 0\end{aligned}$$

from these calculations one gets

$$\Lambda > \max\left\{\frac{\mathcal{L}_4 + \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}, \frac{\mathcal{L}_4 - \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\},$$

$$P_\Lambda(1) = 1 + \varphi_1 + \varphi_2 + \varphi_3 = h^3(\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon)) > 0$$

implies

$$\Lambda > \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon},$$

$$\begin{aligned}P_{\Lambda_0}(-1) =& 1 - \varphi_1 + \varphi_2 - \varphi_3\\ =& h^3(\mu(\beta+\mu)(\mu+\epsilon) - \alpha\Lambda\epsilon) + \frac{2\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}(\beta + 2\mu + \epsilon)\\ &- h(\frac{4\epsilon}{(\beta+\mu)(\mu+\epsilon)}(\alpha\Lambda + (\beta+\mu)(\mu+\epsilon)) - 4\beta - 8\mu) + 8 = 0\end{aligned}$$

implies

$$\Lambda = \frac{(\beta+\mu)(\mu+\epsilon)(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta + 2\mu + \epsilon) + 8)}{\alpha h\epsilon(h(\beta+\mu) - 2)(h(\mu+\epsilon) - 2)},$$

$$\begin{aligned}\Delta_j^\pm =& 1 \pm \varphi_3\\ =& 1 \pm h^3(\alpha\Lambda\epsilon - \mu(\beta+\mu)(\mu+\epsilon)) - \frac{\alpha h^2\Lambda\epsilon(\beta + 2\mu + \epsilon)}{(\beta+\mu)(\mu+\epsilon)} + h\left(\frac{\epsilon(\alpha\Lambda + (\beta+\mu)(\mu+\epsilon))}{(\beta+\mu)(\mu+\epsilon)} + \beta + 2\mu\right) - 1\\ >& 0,\end{aligned}$$

the simplification of calculations implies

$$\Lambda > \max\left\{\frac{(\beta+\mu)(\mu+\epsilon)(\beta(h^2\mu(\mu+\epsilon) - 1) + h^2\mu^2(\mu+\epsilon) - 2\mu - \epsilon)}{\alpha\epsilon(h(\beta+\mu) - 1)(h(\mu+\epsilon) - 1)}, \frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon) - h(\beta + 2\mu + \epsilon) + 2\right)}{\alpha h\epsilon(h(\beta+\mu) - 1)(h(\mu+\epsilon) - 1)}\right\},$$

and finally,

$$\frac{\sum_{i=1}^{3}(-1)^{3-i}\varphi_i^{'}}{\sum_{i=1}^{3}(-1)^{3-i}(3-i+1)\varphi_{i-1}} = \frac{\varphi_1^{'}-\varphi_2^{'}+\varphi_3^{'}}{3-2\varphi_1+\varphi_2} = \frac{\mathcal{L}_6}{\mathcal{L}_7} \neq 0,$$

where straightforward calculation yields that

$$\Lambda \neq \frac{(\beta+\mu)(\mu+\epsilon)\Big(h^3\mu(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+7\Big)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)}.$$

□

5. Chaos control

In order to get stable trajectories, feedback control strategy is utilized based on an existing theory (see, [39, 40]), where the tuberculosis model (1.11) takes the form:

$$\begin{aligned} s_{t+1} &= h\Lambda + (1-\mu h)s_t - \alpha h s_t i_t + \sigma(s_t - s),\\ e_{t+1} &= (1-\mu h-\epsilon h)e_t + \alpha h s_t i_t + \sigma(e_t - e),\\ i_{t+1} &= (1-h\mu-h\beta)i_t + h\epsilon e_t + \sigma(i_t - i), \end{aligned} \tag{5.1}$$

with σ as a control parameter. The $V^C|_{\Omega_2}$ of the controlled system (5.1) is

$$V^C|_{\Omega_2} = \begin{pmatrix} 1-h\Big(\mu+\frac{\alpha\epsilon\Lambda-\mu(\beta+\mu)(\epsilon+\mu)}{(\epsilon+\mu)(\beta+\mu)}\Big)+\sigma & 0 & -\frac{h(\beta+\mu)(\epsilon+\mu)}{\epsilon}\\ \frac{h(\alpha\epsilon\Lambda-\mu(\beta+\mu)(\epsilon+\mu))}{(\epsilon+\mu)(\beta+\mu)} & -h(\epsilon+\mu)+1+\sigma & \frac{h(\beta+\mu)(\epsilon+\mu)}{\epsilon}\\ 0 & h\epsilon & -h(\beta+\mu)+1+\sigma \end{pmatrix}. \tag{5.2}$$

The characteristic polynomial of $V^C|_{\Omega_2}$ about endemic fixed point Ω_2 is

$$P(\lambda) = \lambda^3 + \varphi_1^*\lambda^2 + \varphi_2^*\lambda + \varphi_3^* = 0, \tag{5.3}$$

where

$$\begin{cases} \varphi_1^* = h\Big(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu+\epsilon\Big)-3(\sigma+1),\\ \varphi_2^* = \dfrac{\left\{\begin{array}{c}\beta\Big(\epsilon\Big(\alpha h^2\Lambda-8h\mu(\sigma+1)+3(\sigma+1)^2\Big)-3\mu(\sigma+1)\\ (2h\mu-\sigma-1)-2h(\sigma+1)\epsilon^2\Big)+\alpha h^2\Lambda\epsilon(2\mu+\epsilon)-\\ 2h(\sigma+1)(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))-\\ 2\beta^2h(\sigma+1)(\mu+\epsilon)+3\mu(\sigma+1)^2(\mu+\epsilon)\end{array}\right\}}{(\beta+\mu)(\mu+\epsilon)},\\ \varphi_3^* = h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))+\dfrac{\left\{\begin{array}{c}h(\sigma+1)^2\Big(\alpha\lambda\epsilon+\beta^2(\mu+\epsilon)\\ +\beta(\mu+\epsilon)(3\mu+\epsilon)+\\ \mu(\mu+\epsilon)(2\mu+\epsilon)\Big)\end{array}\right\}}{(\beta+\mu)(\mu+\epsilon)}\\ \quad -\frac{\alpha h^2\Lambda(\sigma+1)\epsilon(\beta+2\mu+\epsilon)}{(\beta+\mu)(\mu+\epsilon)}-(\sigma+1)^3. \end{cases} \tag{5.4}$$

Theorem 5.1. Ω_2 *of the controlled tuberculosis model (5.1) is a sink if*

$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma-2)(h(\mu+\epsilon)-\sigma-2)} < \Lambda < \frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma)(h(\mu+\epsilon)-\sigma)}, \tag{5.5}$$

$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9}{\alpha h\epsilon\mathcal{L}_{10}} < \Lambda < \frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9}{\alpha h\epsilon\mathcal{L}_{10}}, \tag{5.6}$$

and

$$\Lambda < \min\left\{\frac{\zeta-\sqrt{\eta}}{\mathcal{L}_{11}}, \frac{\zeta+\sqrt{\eta}}{\mathcal{L}_{11}}\right\}, \tag{5.7}$$

where

$$\begin{cases}
\mathcal{L}_8 = & h^3\mu(\beta+\mu)(\mu+\epsilon) - h(\sigma+2)^2(\beta+2\mu+\epsilon) + (\sigma+2)^3,\\
\mathcal{L}_9 = & 3h^3\mu(\beta+\mu)(\mu+\epsilon) - h(\sigma+2)(3\sigma+2)(\beta+2\mu+\epsilon) + 3\sigma(\sigma+2)^2,\\
\mathcal{L}_9^* = & 3h^3\mu(\beta+\mu)(\mu+\epsilon) - h(\sigma+2)(3\sigma+4)(\beta+2\mu+\epsilon) + 3\sigma^2(\sigma+2),\\
\mathcal{L}_{10} = & 3h^2(\beta+\mu)(\mu+\epsilon) - h(3\sigma+4)(\beta+2\mu+\epsilon) + (\sigma+2)(3\sigma+2),\\
\mathcal{L}_{11} = & 2\alpha^2h^2\epsilon^2(h(\beta+\mu)-\sigma-1)(h(\mu+\epsilon)-\sigma-1)(h^2(\beta+\mu)(\mu+\epsilon)-\\
& h(\sigma+1)(\beta+2\mu+\epsilon)+\sigma(\sigma+2)),\\
\zeta = & \alpha h\epsilon(\beta+\mu)(\mu+\epsilon)(2h^5\mu(\beta+\mu)^2(\mu+\epsilon)^2 - 2h^4\mu(\sigma+1)(\beta+\mu)(\mu+\epsilon)\\
& (\beta+2\mu+\epsilon) - h^3(2\sigma(\sigma+2)+1)(\beta+\mu)(\mu+\epsilon)(\beta+\mu+\epsilon) + h^2(\sigma+1)\\
& (\beta^2(2\sigma(\sigma+2)+1) + \beta(\mu(10\sigma(\sigma+2)+3) + (6\sigma(\sigma+2)+1)\epsilon) + \mu^2(10\\
& \sigma(\sigma+2)+3) + (2\sigma(\sigma+2)+1)\epsilon^2 + \mu(10\sigma(\sigma+2)+3)\epsilon) - h\sigma(\sigma+2)\\
& (2\sigma+1)(2\sigma+3)(\beta+2\mu+\epsilon) + 2\sigma^2(\sigma+1)(\sigma+2)^2),\\
\eta = & \alpha^2h^4\epsilon^2(\beta+\mu)^2(\mu+\epsilon)^2(\sigma^2(\sigma+2)^2(\beta-\epsilon)^2 + h^4(\beta+\mu)^2(\mu+\epsilon)^2(\beta+\mu+\epsilon)^2\\
& -2h^3(\sigma+1)(\beta+\mu)(\mu+\epsilon)(\beta^3+2\beta^2(2\mu+\epsilon)+\beta(4\mu^2+2\epsilon^2+5\mu\epsilon)+(\mu+\epsilon)\\
& (\mu^2+\epsilon^2+3\mu\epsilon)) + h^2(\beta^4(\sigma+1)^2 + 2\beta^3(\mu(4\sigma(\sigma+2)+3)+(2\sigma(\sigma+2)+1)\epsilon)\\
& +\beta^2(\mu^2(15\sigma(\sigma+2)+11)+3(\sigma+1)^2\epsilon^2+4\mu(3\sigma(\sigma+2)+2)\epsilon)+2\beta(\mu^2+\epsilon^2\\
& +\mu\epsilon)(\mu(4\sigma(\sigma+2)+3)+(2\sigma(\sigma+2)+1)\epsilon)+\mu^4(\sigma+1)^2+(\sigma+1)^2\epsilon^4+2\mu\\
& (4\sigma(\sigma+2)+3)\epsilon^3+\mu^2(15\sigma(\sigma+2)+11)\epsilon^2+2\mu^3(4\sigma(\sigma+2)+3)\epsilon)-2h\sigma\\
& (\sigma+1)(\sigma+2)(\beta^3+3\beta^2\mu+\beta\mu(\mu-2\epsilon)+\epsilon^3+\mu\epsilon(\mu+3\epsilon))).
\end{cases} \tag{5.8}$$

Proof. By N & S conditions, one has

$$\begin{aligned}
|\varphi_1^*+\varphi_3^*| &= \Big|h\Big(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu+\epsilon\Big) - 3(\sigma+1) + h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))\\
&+ \frac{h(\sigma+1)^2\Big(\alpha\lambda\epsilon+\beta^2(\mu+\epsilon)+\beta(\mu+\epsilon)(3\mu+\epsilon)+\mu(\mu+\epsilon)(2\mu+\epsilon)\Big)}{(\beta+\mu)(\mu+\epsilon)}\\
&- \frac{\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}\Big((\sigma+1)(\beta+2\mu+\epsilon)) - (\sigma+1)^3\Big| < 1+\varphi_2^*\\
&= \frac{1}{(\beta+\mu)(\mu+\epsilon)}(\beta\Big(\epsilon\Big(\alpha h^2\Lambda - 8h\mu(\sigma+1)+3(\sigma+1)^2\Big) - 3\mu(\sigma+1)(2h\mu-\sigma-1) - 2h(\sigma+1)\epsilon^2\Big)\\
&+ \alpha h^2\Lambda\epsilon(2\mu+\epsilon) - 2h(\sigma+1)(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon)) - 2\beta^2h(\sigma+1)(\mu+\epsilon) + 3\mu(\sigma+1)^2(\mu+\epsilon)),
\end{aligned}$$

i.e.,

$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma-2)(h(\mu+\epsilon)-\sigma-2)}<\Lambda<\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma)(h(\mu+\epsilon)-\sigma)},$$

$$\begin{aligned}
&|\varphi_1^*-3\varphi_3^*|\\
=&\Big|h\Big(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu+\epsilon\Big)-3(\sigma+1)-3(h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))\\
&+\frac{h(\sigma+1)^2}{(\beta+\mu)(\mu+\epsilon)}\Big(\alpha\lambda\epsilon+\beta^2(\mu+\epsilon)+\beta(\mu+\epsilon)(3\mu+\epsilon)+\mu(\mu+\epsilon)(2\mu+\epsilon)\Big)\\
&-\frac{\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}\Big((\sigma+1)(\beta+2\mu+\epsilon))-(\sigma+1)^3)\Big|<3-\varphi_2^*\\
=&3-\frac{1}{(\beta+\mu)(\mu+\epsilon)}(\beta\Big(\epsilon\Big(\alpha h^2\Lambda-8h\mu(\sigma+1)+3(\sigma+1)^2\Big)-3\mu(\sigma+1)(2h\mu-\sigma-1)-2h(\sigma+1)\epsilon^2\Big)\\
&+\alpha h^2\Lambda\epsilon(2\mu+\epsilon)-2h(\sigma+1)(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))-2\beta^2h(\sigma+1)(\mu+\epsilon)+3\mu(\sigma+1)^2(\mu+\epsilon)),
\end{aligned}$$

so, one has

$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9}{\alpha h\epsilon\mathcal{L}_{10}}<\Lambda<\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9}{\alpha h\epsilon\mathcal{L}_{10}},$$

and

$$\begin{aligned}
&\varphi_{3^*}^2+\varphi_2^*-\varphi_3^*\varphi_1^*\\
=&\Big(h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))+\frac{h(\sigma+1)^2}{(\beta+\mu)(\mu+\epsilon)}\Big(\alpha\lambda\epsilon+\beta^2(\mu+\epsilon)+\beta(\mu+\epsilon)(3\mu+\epsilon)\\
&+\mu(\mu+\epsilon)(2\mu+\epsilon)\Big)-\frac{\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}\Big((\sigma+1)(\beta+2\mu+\epsilon))-(\sigma+1)^3\Big)^2\\
&+\frac{1}{(\beta+\mu)(\mu+\epsilon)}(\beta\Big(\epsilon\Big(\alpha h^2\Lambda-8h\mu(\sigma+1)+3(\sigma+1)^2\Big)-3\mu(\sigma+1)(2h\mu-\sigma-1)\\
&-2h(\sigma+1)\epsilon^2\Big)+\alpha h^2\Lambda\epsilon(2\mu+\epsilon)-2h(\sigma+1)(\alpha\Lambda\epsilon+\mu(\mu+\epsilon)(2\mu+\epsilon))\\
&-2\beta^2h(\sigma+1)(\mu+\epsilon)+3\mu(\sigma+1)^2(\mu+\epsilon))-\Big((h^3(\alpha\Lambda\epsilon-\mu(\beta+\mu)(\mu+\epsilon))\\
&+\frac{h(\sigma+1)^2}{(\beta+\mu)(\mu+\epsilon)}\Big(\alpha\lambda\epsilon+\beta^2(\mu+\epsilon)+\beta(\mu+\epsilon)(3\mu+\epsilon)+\mu(\mu+\epsilon)(2\mu+\epsilon)\Big)\\
&-\frac{\alpha h^2\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}\Big((\sigma+1)(\beta+2\mu+\epsilon))-(\sigma+1)^3)\Big)\Big(h\Big(\frac{\alpha\Lambda\epsilon}{(\beta+\mu)(\mu+\epsilon)}+\beta+2\mu+\epsilon\Big)-3(\sigma+1)\Big)\\
<&1
\end{aligned}$$

gives that $\Lambda<\min\left\{\frac{\zeta-\sqrt{\eta}}{\mathcal{L}_{11}},\frac{\zeta+\sqrt{\eta}}{\mathcal{L}_{11}}\right\}$ is the desired computable N and S parametric condition under which characteristics roots of (5.3) are inside the open unit disc. □

6. Numerical simulations

In this section, we verify the theoretical findings for both appropriate and real epidemiological data numerically.

6.1. Numerical simulations for appropriate data

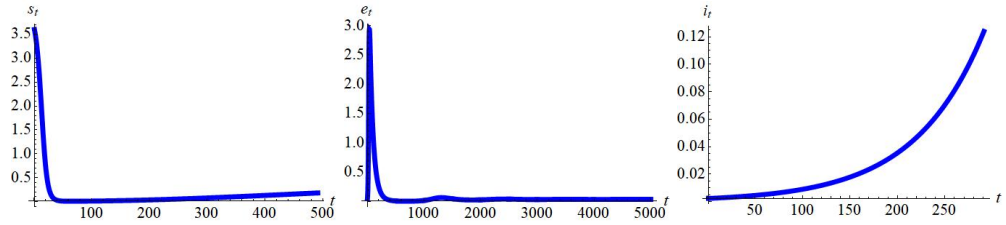
Example 7. *If* $\mu = 0.009$, $\beta = 0.01$, $\epsilon = 0.19$, $\alpha = 0.7$, $h = 0.081$ *and* $\Lambda = 0.01$ *then from (3.21)–(3.23) one gets*

$$\begin{aligned}
&\frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)}\\
&=-0.7008561965209708<\Lambda\\
&=0.01<\frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}=0.051157962406015035,\\
&\frac{(\beta+\mu)(\mu+\epsilon)\left(\beta\left(3h^2\mu(\mu+\epsilon)-4\right)+3h^2\mu^2(\mu+\epsilon)-4(2\mu+\epsilon)\right)}{\alpha\epsilon\left(3h^2(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+4\right)}\\
&=-0.006308705558149988<\Lambda\\
&=0.01<\frac{3h\mu(\beta+\mu)^2(\mu+\epsilon)^2}{\alpha\epsilon(\beta(3h(\mu+\epsilon)-2)+3h\mu(\mu+\epsilon)-2(2\mu+\epsilon))}=0.01747209338094942,
\end{aligned}$$

and

$$\begin{aligned}
\Lambda=&0.01<\min\left\{\frac{\xi-\sqrt{\chi}}{\mathcal{L}_5},\frac{\xi+\sqrt{\chi}}{\mathcal{L}_5}\right\}=\{0.021498689765038194,0.015730565329161603\}\\
=&0.015730565329161603,
\end{aligned}$$

which implies that $\Omega_2 = (0.02842857142857143, 0.04896554199569274, 0.4896554199569274)$ *of a tuberculosis model (1.11) is a sink (see Figure 8).*



(a) **(b)** **(c)**

Figure 8. Stability plots at $\Omega_2 = (0.02842857142857143, 0.04896554199569274, 0.4896554199569274)$ of the discrete tuberculosis model (1.11).

Example 8. *If one takes* $\mu = 0.8$, $\beta = 1.5$, $\epsilon = 4.8$, $\alpha = 5.2$, *and* $h = 0.95$ *then from (4.4) one gets*

$$\Lambda = 0.9643683178535903 = \Lambda_0.$$

Therefore, at $(\mu,\beta,\epsilon,\alpha,h,\Lambda) = (0.8, 1.5, 4.8, 5.2, 1.04, 0.9, 0.9643683178535903)$, *one has the EFP* $\Omega_2 = (0.516025641025641, 0.09849067947019226, 0.20554576585083603)$, *where from (3.19) one gets*

$$\lambda^3 + 6.280396083303131\lambda^2 - 2.236444561416 26\lambda + 6.759215629645393 = 0$$

with

$$|\lambda_{1,2}| = |0.23940977317112597 \pm 0.9709186168315815\iota| = 1,$$

and

$$\lambda_3 = -6.75921.$$

This implies that eigenvalues criterion for the existence of N-S bifurcation holds, and so, tuberculosis model (1.11) may undergo N-S bifurcation. Now, if $(\mu, \beta, \epsilon, \alpha, h, \Lambda) = (0.8, 1.5, 4.8, 5.2, 1.04, 0.9, 0.9643683178535903)$*, then from (4.5)–(4.8) one gets*

$$\begin{aligned}\Lambda =& 0.9643683178535903 > \max\left\{\frac{\mathcal{L}_4 + \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}, \frac{\mathcal{L}_4 - \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\}\\ &= \max\{0.4011439546196724, -0.20071062094866143\} = 0.4011439546196724,\end{aligned}$$

$$\Lambda = 0.9643683178535903 > \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon} = 0.4128205128205127,$$

$$\begin{aligned}\frac{(\beta+\mu)(\mu+\epsilon)(h^3\mu(\beta+\mu)(\mu+\epsilon) - 4h(\beta+2\mu+\epsilon)+8)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} =& -11.661059864801171 < \Lambda\\ =& 0.9643683178535903,\end{aligned}$$

$$\begin{aligned}&\frac{h^3}{(\beta+\mu)^2(\mu+\epsilon)^2}\Big(\alpha\epsilon(\beta+\mu)(\mu+\epsilon)(\beta^2(h(\mu+\epsilon)-1)(2h^2\mu(\mu+\epsilon)-1)+\beta(\mu(2h\mu-3)(2h^2\mu^2-1)\\ &+h\epsilon^2(2h\mu(2h\mu-1)-1)+\epsilon(h\mu(8h\mu(h\mu-1)-3)+1))+\epsilon^2(h\mu-1)(2h^2\mu^2-1)\\ &+\mu\epsilon(2h\mu-3)(2h^2\mu^2-1)+\mu^2(h\mu-1)(2h\mu(h\mu-1)-3))-\mathcal{L}_2-\sqrt{\mathcal{L}_1}\Big) = -65.90524702127566 \neq 0,\end{aligned}$$

respectively. Finally, from (4.9), the non-degenerate condition yields

$$\begin{aligned}\cos\frac{2\pi}{l} \neq& \frac{h^2(\beta+\mu)(\mu+\epsilon)(\mu(\beta+\mu)(\mu+\epsilon)-\alpha\Lambda\epsilon)}{\left\{\begin{array}{c}2(\beta(\epsilon^2(h^2(\alpha\Lambda-2\mu^2)+1)-2h^2\mu^4+\epsilon(h\mu-1)(\alpha h\Lambda-4h\mu^2-4\mu)\\ +3\mu^2)+\beta^2(-(\mu+\epsilon))(h^2\mu(\mu+\epsilon)-1)-\mu(\mu+\epsilon)\\ (h^2\mu^3+h^2\mu^2\epsilon-2\mu-\epsilon)+\alpha\Lambda\epsilon(h\mu-1)(h(\mu+\epsilon)-1))\end{array}\right\}}+1\\ =& 0.23940977317112644\end{aligned}$$

with $\cos\frac{2\pi}{l} = 0.23940977317112644$ *implies* $l = \pm 4.727617486361454$*. These straightforward calculations confirm the conclusion of Theorem 4.1, and hence, N-S bifurcation diagrams and the maximum Lypunov exponent are drawn in Figure 9 accordingly.*

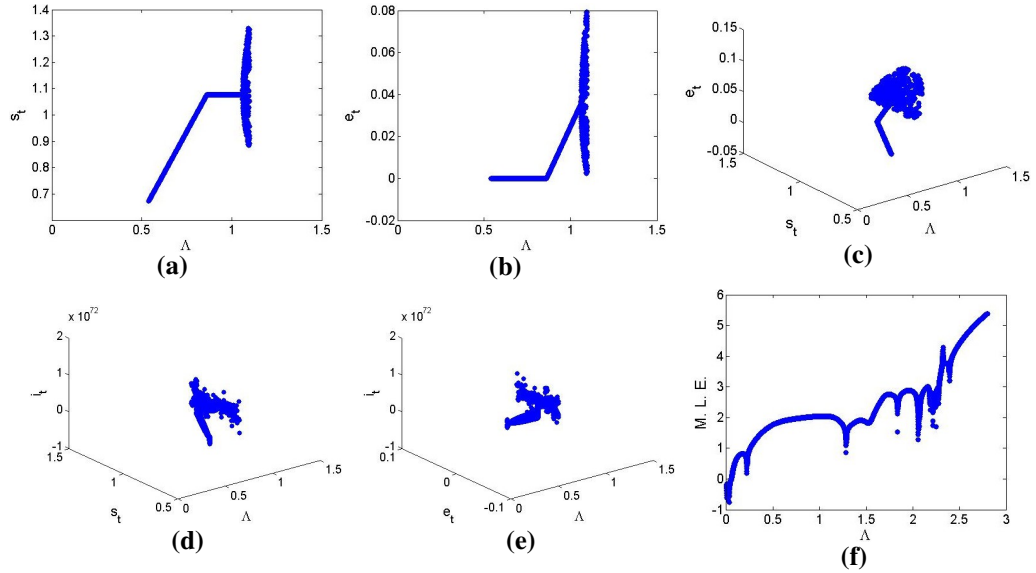


Figure 9. Bifurcation diagrams and MLE for the discrete tuberculosis model (1.11) if $(\mu, \beta, \epsilon, \alpha, h, \Lambda) = (0.8, 1.5, 4.8, 5.2, 1.04, 0.9, 0.9643683178535903)$.

Example 9. *If* $\mu = 0.009$, $\beta = 0.15$, $\epsilon = 0.19$, $\alpha = 0.7$, *and* $h = 0.09$ *then from (4.14), one gets*

$$\Lambda = 5.286372671498713 = \Lambda_0,$$

where model's EFP is $\Omega_2 = (0.23790225563909778, 26.553927392954574, 31.731108205417414)$. *Furthermore, if* $(\mu, \beta, \epsilon, \alpha, h, \Lambda) = (0.009, 0.15, 0.19, 0.7, 0.09, 5.286372671498713)$, *then from (3.19), one gets*

$$\lambda^3 - 0.9823545094174649\lambda^2 - 0.9999752020563639\lambda + 0.9823793073611001 = 0$$

with $\lambda_1 = -1$, *but* $\lambda_{2,3} = 0.9838941996602671, 0.9984603097571967 \neq 1$ *or* -1, *which satisfies the criterion of eigenvalues for the emergence of flip bifurcation. Therefore, for the indicated parametric values, from (4.11)–(4.13), (4.15) and (4.16), one gets*

$$\begin{aligned}
\Lambda =& 5.286372671498713 > \max\left\{\frac{\mathcal{L}_2 + \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}, \frac{\mathcal{L}_2 - \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}\right\} \\
&= \max\{-0.06536260734432897, -0.0007403317253933942\} = -0.0007403317253933942, \\
\Lambda =& 5.286372671498713 > \max\left\{\frac{\mathcal{L}_4 + \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}, \frac{\mathcal{L}_4 - \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\} \\
&= \max\{-2.636447951356862, 0.043875156751626385\} = 0.043875156751626385, \\
\Lambda =& 5.286372671498713 > \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon} = 0.0021411203007518803, \\
\Lambda =& 5.286372671498713 \\
&> \max\left\{\frac{(\beta+\mu)(\mu+\epsilon)(\beta(h^2\mu(\mu+\epsilon)-1)+h^2\mu^2(\mu+\epsilon)-2\mu-\epsilon)}{\alpha\epsilon(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)}, \right. \\
&\left. \frac{(\beta+\mu)(\mu+\epsilon)(h^3\mu(\beta+\mu)(\mu+\epsilon)-h(\beta+2\mu+\epsilon)+2)}{\alpha h\epsilon(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)}\right\} \\
&= \max\{-0.014254442285897097, -5.373298889372674\} = -0.014254442285897097,
\end{aligned}$$

and

$$\Lambda = 5.286372671498713 \neq \frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+7\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} = 4.61475636631299,$$

which confirm the conclusion of Theorem 4.2, and hence, flip bifurcation diagrams and maximum Lypunov exponent are drawn in Figure 10.

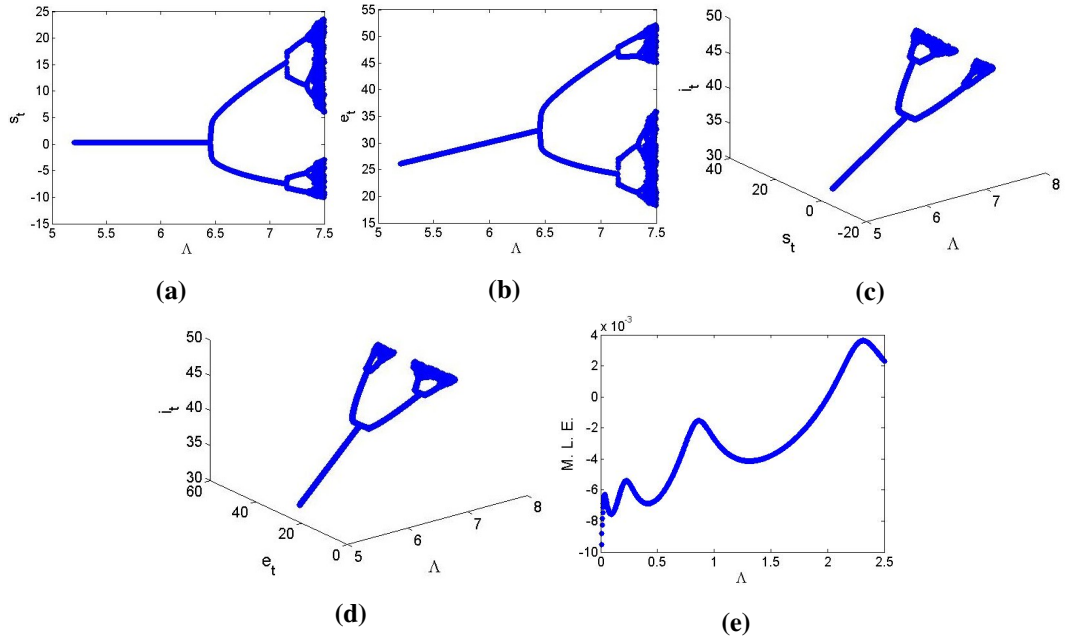


Figure 10. Flip bifurcation diagrams and MLE for tuberculosis model (1.11) if $(\mu, \beta, \epsilon, \alpha, h, \Lambda) = (0.009, 0.15, 0.19, 0.7, 0.09, 5.286372671498713)$.

Example 10. *If one has the data as in Example 9 with* $\sigma = 0.001$ *then from (5.5)–(5.7) one gets*

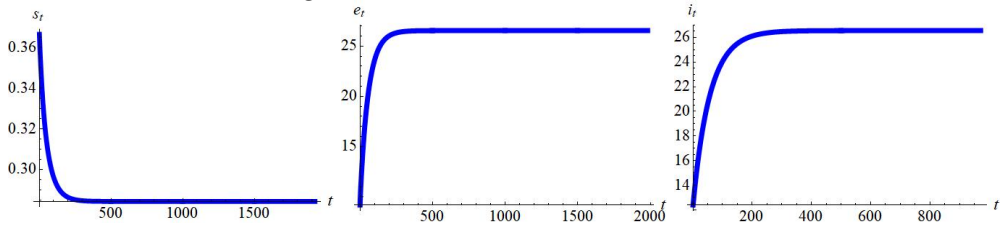
$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma-2)(h(\mu+\epsilon)-\sigma-2)} = 0.005145992347008056 < \Lambda$$
$$= 5.286372671498713 < \frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma)(h(\mu+\epsilon)-\sigma)} = 5.2890162045911895,$$
$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9}{\alpha h\epsilon\mathcal{L}_{10}} = -0.07979904740906814 < \Lambda$$
$$= 5.286372671498713 < \frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9^*}{\alpha h\epsilon\mathcal{L}_{10}} = 5.937396761502907,$$

and finally,

$$\Lambda = 5.286372671498713 < \min\left\{\frac{\zeta-\sqrt{\eta}}{\mathcal{L}_{11}}, \frac{\zeta+\sqrt{\eta}}{\mathcal{L}_{11}}\right\}$$
$$= \min\{6.601217184392636, 7.192942799106272\} = 6.601217184392636,$$

which satisfies conclusion of Theorem 5.1, and so, $\Omega_2 = (0.8799171842650103, 0.009606625258799172, 0.012997198879551822)$ *of the tuberculosis model*

(5.1) with $(1.9, 0.3, 0.1)$ *is a sink (see Figure 11). Furthermore, the regions of controllability for system (5.1) in 2D and 3D are drawn in Figures 12 and 13, respectively.*



(a) **(b)** **(c)**

Figure 11. Stability plots for the controlled tuberculosis model (5.1).

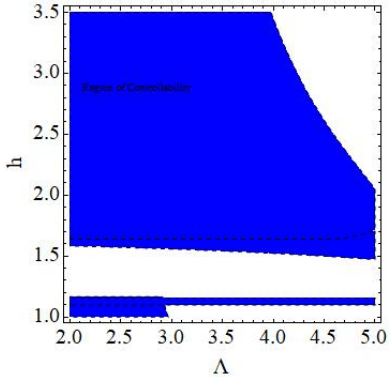


Figure 12. Region of controllability if $\mu = 0.9, \beta = 0.1, \epsilon = 0.9, \alpha = 0.7$ and $\sigma = 0.098$, and varying $\Lambda \in (2.0, 5.0)$ and $h \in (1.0, 3.5)$.

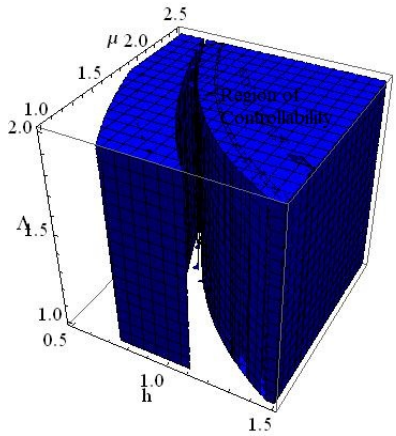


Figure 13. Region of controllability if $\beta = 0.65, \epsilon = 0.6, \alpha = 0.31$ and $\sigma = 0.3$, and varying $h \in (0.5, 1.5), \mu \in (1.0, 2.5)$ and $\Lambda \in (1.0, 2.0)$.

6.2. Numerical simulations for fitting real epidemiological data with $h = 1$

In this subsection, we utilize real epidemiological data from Turkey and the Ashanti region of Ghana to justify and calibrate the parameter settings used in our understudied discrete sei model, which is depicted in (1.11). This enhancement ensures that our theoretical findings are not merely abstract but are numerically consistent with actual TB transmission patterns in these regions. Furthermore, we demonstrate that the model retains its qualitative behavior such as the occurrence of bifurcations

and chaos under these realistic parameter values, thereby strengthening the biological relevance and applicability of obtained results.

Example 11. *If one has the parametric values, which are depicted in Table 1, then from (3.21)–(3.23) one gets*

$$\frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} = 0.005047753470693089 < \Lambda$$
$$=0.01737 < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon} = 0.96012975333546,$$
$$\frac{(\beta+\mu)(\mu+\epsilon)\left(\beta\left(3h^2\mu(\mu+\epsilon)-4\right)+3h^2\mu^2(\mu+\epsilon)-4(2\mu+\epsilon)\right)}{\alpha\epsilon\left(3h^2(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+4\right)} = 0.004090529336689837 < \Lambda$$
$$=0.01737 < \frac{3h\mu(\beta+\mu)^2(\mu+\epsilon)^2}{\alpha\epsilon(\beta(3h(\mu+\epsilon)-2)+3h\mu(\mu+\epsilon)-2(2\mu+\epsilon))} = 6.809557614765738,$$

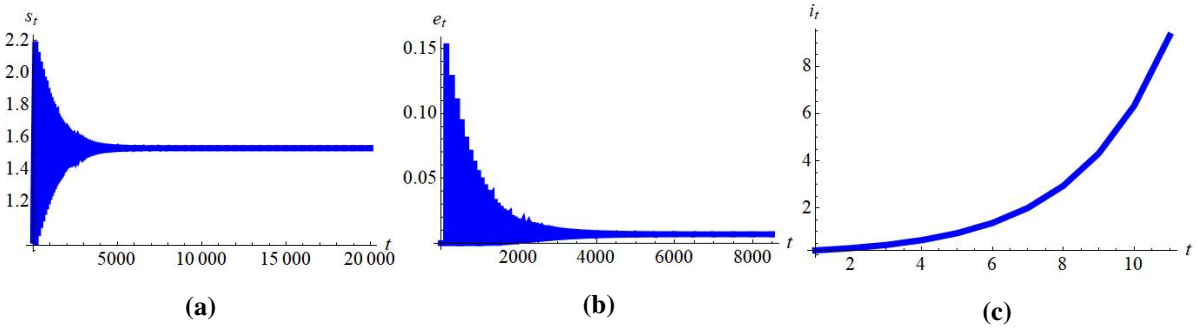
and

$$\Lambda = 0.01737 < \min\left\{\frac{\xi-\sqrt{\chi}}{\mathcal{L}_5}, \frac{\xi+\sqrt{\chi}}{\mathcal{L}_5}\right\}$$
$$= \min\{22809.742329480487, 23225.225974105397\} = 22809.742329480487,$$

which implies that $\Omega_2 = (1.030153769529202, 0.00859951603692296, 0.012793112720831326)$ *of a tuberculosis model (1.11) is a sink (see Figure 14).*

Table 1. Fitting real data for Turky in the discrete TB model (1.11).

Parameter	Value	Source
β	0.955	[41]
ϵ	1.428	[41,42]
α	0.935	[41]
μ	0.0049	[41,42]
Λ	0.01737	[41,42]



(a) **(b)** **(c)**

Figure 14. Stability plots at $\Omega_2 = (1.2885115659794872, 14.30663287363192, 4.7039757397129005)$ of the discrete tuberculosis model (1.11).

Example 12. *If one has the data from Table 2, then Eq (4.14) gives* $\Lambda = 2.4940817111117206 = \Lambda_0$, *where model's EFP is* $\Omega_2 = (1.2885115659794872, 14.30663287363192, 4.7039757397129005)$.

Furthermore, if $(\mu, \beta, \epsilon, \alpha, h, \Lambda) = (0.007, 0.5, 0.1667, 0.041, 1.0, 2.494081711117206)$*, then from (3.19), one gets*

$$\lambda^3 - 0.38366994671771015\lambda^2 - 0.9150767292953252\lambda + 0.46859321742238813 = 0,$$

with $\lambda_1 = -1$ *but* $\lambda_{2,3} = 0.7920468136775569, 0.5916231330401543 \neq 1$ *or* -1*, which satisfies the criterion of eigenvalues for the emergence of flip bifurcation. Therefore, for the indicated parametric values, from (4.11)–(4.13), (4.15), and (4.16), one gets*

$$\begin{aligned}
\Lambda =& 2.4940817111117206 > \max\left\{\frac{\mathcal{L}_2 + \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}, \frac{\mathcal{L}_2 - \sqrt{\mathcal{L}_1}}{\mathcal{L}_5}\right\} \\
&= \max\{-1.678193238965701, -0.0042442412380117\} = -0.0042442412380117, \\
\Lambda =& 2.4940817111117206 > \max\left\{\frac{\mathcal{L}_4 + \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}, \frac{\mathcal{L}_4 - \sqrt{\mathcal{L}_3}}{\mathcal{L}_5}\right\} \\
&= \max\{-5.991008560899027, -1.600101512111261\} = -1.600101512111261, \\
\Lambda =& 2.4940817111117206 > \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon} = 0.009019501984580961856 41, \\
\Lambda =& \Lambda_0 > \max\left\{\frac{(\beta+\mu)(\mu+\epsilon)(\beta(h^2\mu(\mu+\epsilon)-1)+h^2\mu^2(\mu+\epsilon)-2\mu-\epsilon)}{\alpha\epsilon(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)},\right. \\
&\left.\frac{(\beta+\mu)(\mu+\epsilon)(h^3\mu(\beta+\mu)(\mu+\epsilon)-h(\beta+2\mu+\epsilon)+2)}{\alpha h\epsilon(h(\beta+\mu)-1)(h(\mu+\epsilon)-1)}\right\} \\
&= \max\{1.0080602213371435, -4.174938615411271\} = 1.0080602213371435,
\end{aligned}$$

and

$$\Lambda = 2.4940817111117206 \neq \frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+7\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} = 2.0215223242137927,$$

which confirm the conclusion of Theorem 4.2, and hence, flip bifurcation diagrams and the maximum Lypunov exponent are drawn in Figure 15.

Table 2. Fitting real data for Ashanti Region of Gahana in the discrete TB model (1.11).

Parameter	**V**alue	**S**ource
β	0.5	[26, 43–45]
ϵ	0.1667	[26, 44]
α	0.041	[26]
μ	0.007	[26, 42]

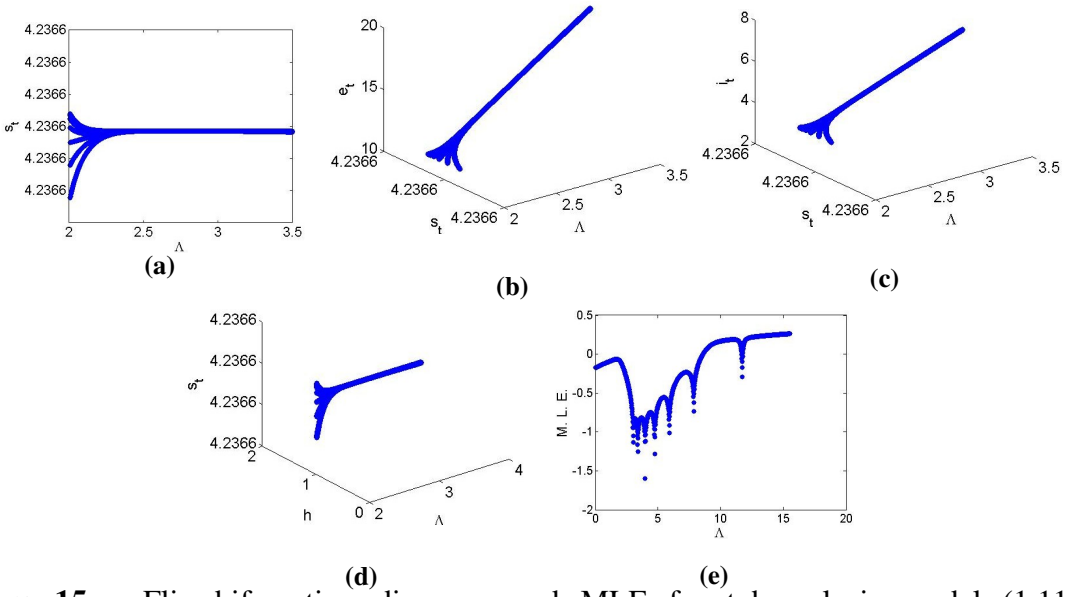


Figure 15. Flip bifurcation diagrams and MLE for tuberculosis model (1.11) if $(\mu, \beta, \epsilon, \alpha, h, \Lambda) = (0.007, 0.5, 0.1667, 0.41, 1.0, 2.4940817111117206)$.

Example 13. *If one has the data as in Example 12 with* $\sigma = 0.009$*, then, from (5.5)–(5.7), one gets*

$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma-2)(h(\mu+\epsilon)-\sigma-2)} = 0.026661133009721394 < \Lambda$$
$$= 2.4940817111117206 < \frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_8}{\alpha h\epsilon(h(\beta+\mu)-\sigma)(h(\mu+\epsilon)-\sigma)} = 2.5062091173052474,$$
$$\frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9}{\alpha h\epsilon\mathcal{L}_{10}} = -2.149443708262437 < \Lambda$$
$$= 2.4940817111117206 < \frac{(\beta+\mu)(\mu+\epsilon)\mathcal{L}_9^*}{\alpha h\epsilon\mathcal{L}_{10}} = 3.7564547003194324,$$

and finally,

$$\Lambda = 2.49408171117206 < \min\left\{\frac{\zeta-\sqrt{\eta}}{\mathcal{L}_{11}}, \frac{\zeta+\sqrt{\eta}}{\mathcal{L}_{11}}\right\}$$
$$= \min\{18.66011500807496, 4.665797326053959\} = 4.665797326053959,$$

which satisfies conclusion of Theorem 5.1, and so, $\Omega_2 = (1.2885115659794872, 14.30663287363192, 4.7039757397129005)$ *of the tuberculosis model (5.1) with* $(0.09, 30.0, 17.1)$ *is a sink (see Figure 16).*

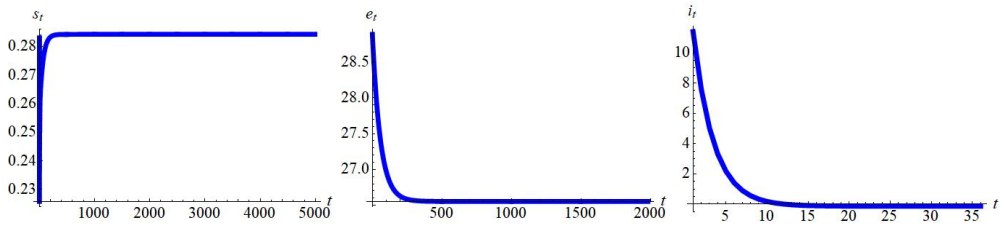


Figure 16. Stability plots for the controlled tuberculosis model (5.1).

7. Conclusions

In this work, we explore local dynamics at fixed points, evaluating the basic reproduction number, bifurcation, and chaos of a discrete tuberculosis epidemic model (1.11). Our findings reveal that for all given set of parameters h, α, Λ, μ, ϵ, and β, the model exhibits DFEP Ω_1 and EEP Ω_2 under certain conditions. Specifically, Ω_2 exists if $\Lambda > \frac{\mu(\mu+\epsilon)(\mu+\beta)}{\alpha\epsilon}$, where $\mathcal{R}_0 = \frac{\Lambda\alpha\epsilon}{\mu(\epsilon+\mu)(\beta+\mu)}$ acts as a basic reproduction number. Biologically, DFEP corresponds to a scenario where TB has been eradicated from the population, while EEP reflects a persistent presence of TB within the population at a stable level. Furthermore, medically, $\mathcal{R}_0$ represents the average number of new TB infections generated by one infectious individual in a wholly susceptible population. If $\mathcal{R}_0 < 1$, then TB will eventually die out, and the population will converge to the disease-free state. Conversely, if $\mathcal{R}_0 > 1$, then TB can sustain itself and possibly spread, leading to the emergence of an endemic state. Further, local dynamical characteristics at fixed points Ω_1 and Ω_2 of the discrete tuberculosis epidemic model (1.11) are explored. It is investigated that Ω_1 is a stable node if $0 < h < \min\left\{\frac{2}{\mu}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, which imply biologically that small initial infections do not propagate, and the population returns to a TB-free state, an unstable node if $h > \max\left\{\frac{2}{\mu}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, indicating that even minor infections can grown, and TB may spread in the population; saddle node if one has $\max\left\{\frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\} < h < \frac{2}{\mu}$ or the condition $\frac{2}{\mu} < h < \min\left\{\frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}, \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}\right\}$, which implies that while some trajectories (population states) may return to health, others may spiral toward a TB outbreak, depending on initial conditions, non-hyperbolic if $h = \frac{2}{\mu}$ or $h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)+\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}$ or $h = \frac{4\sqrt{\mu}}{\sqrt{\mu}(\beta+\epsilon+2\mu)-\sqrt{4\alpha\epsilon\Lambda+(\beta+\epsilon)^2\mu}}$ implies that the system's long-term behavior is highly sensitive and may shift under even slight parameter perturbations. We also derive the N and S conditions, that are, $\frac{\mu(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)}{\alpha h^2\epsilon} < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}$, $\frac{\mu\mathcal{H}_1}{\alpha h\epsilon(3h\mu-4)} < \Lambda < \frac{\mu\mathcal{H}_2}{\alpha\epsilon(3h\mu-2)}$, $\Lambda < \min\left\{\frac{\mu(\beta(h\mu-1)+\mu(h\mu-2))(\mu(h\mu-2)+\epsilon(h\mu-1))}{\alpha\epsilon(h\mu-1)^2}, \frac{\mu(\beta(h(\mu+\epsilon)-1)+h\mu(\mu+\epsilon)-2\mu-\epsilon)}{\alpha h\epsilon}\right\}$ and the sufficient condition $0 < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}$ under which Ω_1 of epidemic model (1.11) is a sink. Biologically, these constraints highlight the importance of controlling population inflow or birth rates (for example, through migration policy or health screening at borders) in maintaining a TB-free environment. Furthermore, we derive the N and S conditions, that are, $\frac{(\beta+\mu)(\mu+\epsilon)\left(h^3\mu(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+8\right)}{\alpha h\epsilon(h(\beta+\mu)-2)(h(\mu+\epsilon)-2)} < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}$, $\Lambda < \min\left\{\frac{\xi-\sqrt{\chi}}{\mathcal{L}_5}, \frac{\xi+\sqrt{\chi}}{\mathcal{L}_5}\right\}$, $\frac{(\beta+\mu)(\mu+\epsilon)\left(\beta\left(3h^2\mu(\mu+\epsilon)-4\right)+3h^2\mu^2(\mu+\epsilon)-4(2\mu+\epsilon)\right)}{\alpha\epsilon\left(3h^2(\beta+\mu)(\mu+\epsilon)-4h(\beta+2\mu+\epsilon)+4\right)} < \Lambda < \frac{3h\mu(\beta+\mu)^2(\mu+\epsilon)^2}{\alpha\epsilon(\beta(3h(\mu+\epsilon)-2)+3h\mu(\mu+\epsilon)-2(2\mu+\epsilon))}$, and S condition, that is, $0 < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}$ under which Ω_2 of the discrete tuberculosis epidemic model (1.11) is a sink. The sufficient condition $0 < \Lambda < \frac{\mu(\beta+\mu)(\mu+\epsilon)}{\alpha\epsilon}$ reflects a critical window where the population can harbor TB without experiencing chaotic dynamics or system-wide destabilization. Biologically, this suggests that, although eradication may not be achievable in this regime, effective disease management and containment strategies (such as treatment, isolation, and reduction in transmission) can keep TB under control. Further, it is proved that no flip bifurcation exists at Ω_1, reinforcing its robustness

under certain thresholds. By utilizing explicit criteria, we study one-parameter bifurcations at Ω_2, indicating biologically that the endemic TB dynamics can become more complex and oscillatory with parameter changes. Such behavior could correspond to seasonal or cyclic outbreaks observed in real-world TB data. We also study the chaos by the feedback control strategy. Finally, theoretical results are numerically validated, confirming the predictive strength of the discrete TB model in capturing essential features of disease spread, control thresholds, and intervention impacts. These findings not only enhance our understanding of TB dynamics from a mathematical perspective but also provide valuable insight for public health decision-making and strategic planning in TB control.

Author contributions

Abdul Qadeer Khan: Conceptualization, Formal Analysis, Investigation, Methodology, Resources, Software, Supervision, Validation, Visualization, Writing-original draft, Writing-review & editing; Raja Ramiz Ahmed Khan: Conceptualization, Formal Analysis, Investigation, Software, Validation, Visualization, Writing-original draft, Writing-review & editing; Saud Fahad Aldosary: Conceptualization, Funding acquisition, Resources, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2025/R/1446).

Conflict of interest

The authors declare no conflict of interest.

References

1. Y. Adebisi, I. Agumage, T. Sylvanus, I. Nawaila, W. Alison Ekwere, M. Nasiru, et al., Burden of tuberculosis and challenges facing its eradication in West Africa, *International Journal of Infection*, **6** (2019), e92250.
2. D. Tollefson, F. Ngari, M. Mwakala, D. Gethi, H. Kipruto, K. Cain, et al., Under-reporting of sputum smear-positive tuberculosis cases in Kenya. *International Journal of Tuberculosis and Lung Disease*, **20** (2016), 1334–1341. https://doi.org/10.5588/ijtld.16.0156
3. N. Becker, The uses of epidemic models, *Biometrics*, **35** (1979), 295–305. https://doi.org/10.2307/2529951

4. A. Zumla, A. George, V. Sharma, R. Herbert, A. Oxley, M. Oliver, The WHO 2014 global tuberculosis report—further to go, *Lancet Glob. Health*, **3** (2015), e10–e12. https://doi.org/10.1016/S2214-109X(14)70361-4
5. L. Allen, A primer on stochastic epidemic models: formulation, numerical simulation, and analysis, *Infectious Disease Modelling*, **2** (2017), 128–142. https://doi.org/10.1016/j.idm.2017.03.001
6. S. Kyere, F. Boateng, G. Hoggar, P. Jonathan, The stochastic model, *Advances in Computer Sciences*, **1** (2018), 105.
7. T, Kurtz, Limit theorems for sequences of jump Markov processes approximating ordinary differential processes, *J. Appl. Probab.*, **8** (1971), 344–356. https://doi.org/10.2307/3211904
8. M. Roberts, V. Andreasen, A. Lloyd, L. Pellis, Nine challenges for deterministic epidemic models, *Epidemics*, **10** (2015), 49–53. https://doi.org/10.1016/j.epidem.2014.09.006
9. Z. Feng, C. Castillo-Chavez, A. Capurro, A model for tuberculosis with exogenous reinfection, *Theor. Popul. Biol.*, **57** (2000), 235–247. https://doi.org/10.1006/tpbi.2000.1451
10. I. Wangari, L. Stone, Backward bifurcation and hysteresis in models of recurrent tuberculosis, *PloS ONE*, **13** (2018), 0194256. https://doi.org/10.1371/journal.pone.0194256
11. T. Porco, S. Blower, Quantifying the intrinsic transmission dynamics of tuberculosis, *Theor. Popul. Biol.*, **54** (1998), 117–132. https://doi.org/10.1006/tpbi.1998.1366
12. S. Ullah, M. Khan, M. Farooq, Z. Hammouch, D. Baleanu, A fractional model for the dynamics of tuberculosis infection using Caputo-Fabrizio derivative, *Discrete Cont. Dyn.-S*, **13** (2020), 975–993. https://doi.org/10.3934/dcdss.2020057
13. S. Bowong, Optimal control of the transmission dynamics of tuberculosis, *Nonlinear Dyn.*, **61** (2010), 729–748. https://doi.org/10.1007/s11071-010-9683-9
14. Z. Hu, Z. Teng, L. Zhang, Stability and bifurcation analysis in a discrete SIR epidemic model, *Math. Comput. Simulat.*, **97** (2014), 80–93. https://doi.org/10.1016/j.matcom.2013.08.008
15. T. Parvin, A. Islam, P. Mondal, M. Biswas, Discrete type SIR epidemic model with nonlinear incidence rate in presence of immunity, *WSEAS Transactions on Biology and Biomedicine*, **17** (2020), 104–118. https://doi.org/10.37394/23208.2020.17.13
16. S. Elaydi, R. Lozi, Global dynamics of discrete mathematical models of tuberculosis, *J. Biol. Dynam.*, **18** (2024), 2323724. https://doi.org/10.1080/17513758.2024.2323724
17. P. Monisha, S. Devi, Analyzing dynamical behavior of tuberculosis transmission, *Contemp. Math.*, **4** (2023), 1331–1346. https://doi.org/10.37256/cm.4420232621
18. E. Ginting, D. Aldila, I. Febiriana, A deterministic compartment model for analyzing tuberculosis dynamics considering vaccination and reinfection, *Healthcare Analytics*, **5** (2024), 100341. https://doi.org/10.1016/j.health.2024.100341
19. A. Gemeda, L. Obsu, E. Gurmu, S. Degefa, Mathematical modeling of tuberculosis transmission dynamics with vaccination and two lines of treatments: a caputo fractional approach, *J. Appl. Math. Comput.*, **71** (2025), 2017–2049. https://doi.org/10.1007/s12190-024-02308-9
20. M. Yao, Y. Zhang, W. Wang, Bifurcation analysis for an in-host Mycobacterium tuberculosis model, *Discrete Cont. Dyn.-B*, **26** (2021), 2299–2322. https://doi.org/10.3934/dcdsb.2020324

21. B. Kotola, S. Teklu, Y. Abebaw, Bifurcation and optimal control analysis of HIV/AIDS and COVID-19 co-infection model with numerical simulation, *PLoS ONE*, **18** (2023), e0284759. https://doi.org/10.1371/journal.pone.0284759

22. H. Panigoro, E. Rahmi, S. Nasib, N. Gawa, O. Peter, Bifurcations on a discrete-time SIS-epidemic model with saturated infection rate, *Bulletin of Biomathematics*, **2** (2024), 182–197. https://doi.org/10.59292/bulletinbiomath.2024008

23. H. Toufga, L. Benahmadi, A. Sakkoum, M. Lhous, Optimal control of a spatiotemporal discrete tuberculosis model, *Int. J. Biomath.*, **18** (2025), 2350110. https://doi.org/10.1142/S1793524523501103

24. A. Jajarmi, B. Ghanbari, D. Baleanu, A new and efficient numerical method for the fractional modeling and optimal control of diabetes and tuberculosis co-existence, *Chaos*, **29** (2019), 093111. https://doi.org/10.1063/1.5112177

25. M. Choinski, M. Bodzioch, U. Forys, A non-standard discretized SIS model of epidemics, *Math. Biosci. Eng.*, **19** (2022), 115–133. https://doi.org/10.3934/mbe.2022006

26. F. Mettle, P. Osei Affi, C. Twumasi, Modelling the transmission dynamics of tuberculosis in the ashanti region of ghana, *Interdisciplinary Perspectives on Infectious Diseases*, **2020** (2020), 4513854. https://doi.org/10.1155/2020/4513854

27. S. Xiong, S. Cai, F. Wei, G. Chen, K. Zheng, J. Xie, Estimation methods of reproduction numbers for epidemics of varying strains of COVID-19, *Journal of Biosafety and Biosecurity*, **6** (2024), 265–270. https://doi.org/10.1016/j.jobb.2024.10.003

28. B. Abudunaibi, W. Liu, Z. Guo, Z. Zhao, J. Rui, W. Song, A comparative study on the three calculation methods for reproduction numbers of COVID-19, *Front. Med.*, **9** (2023), 1079842. https://doi.org/10.3389/fmed.2022.1079842

29. O. Onuorah Martins, M. Nasir, S. Ojo Moses, A. Ademu, A deterministic mathematical model for Ebola virus incorporating the vector population, *IJMTT*, **30** (2016), 8–15. https://doi.org/10.14445/22315373/IJMTT-V30P502

30. P. Van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48. https://doi.org/10.1016/S0025-5564(02)00108-6

31. E. Camouzis, G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*, New York: Chapman and Hall/CRC, 2007. https://doi.org/10.1201/9781584887669

32. E. Grove, G. Ladas, *Periodicities in nonlinear difference equations*, New York: Chapman and Hall/CRC, 2004. https://doi.org/10.1201/9781420037722

33. V. L. Kocic, G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Dordrecht: Springer, 1993. https://doi.org/10.1007/978-94-017-1703-8

34. H. Sedaghat, *Nonlinear difference equations: theory with applications to social science models*, Dordrecht: Springer, 2003. https://doi.org/10.1007/978-94-017-0417-5

35. G. Wen, Criterion to identify Hopf bifurcations in maps of arbitrary dimension, *Phys. Rev. E*, **72** (2005), 026201. https://doi.org/10.1103/PhysRevE.72.026201

36. S. Yao, New bifurcation critical criterion of Flip-Neimark-Sacker bifurcations for two-parameterized family of n-dimensional discrete systems, *Discrete Dyn. Nat. Soc.*, **2012** (2012), 264526. https://doi.org/10.1155/2012/264526

37. B. Xin, T. Chen, J. Ma, Neimark-Sacker bifurcation in a discrete-time financial system, *Discrete Dyn. Nat. Soc.*, **2010** (2010), 405639. https://doi.org/10.1155/2010/405639

38. G. Wen, S. Chen, Q. Jin, A new criterion of period-doubling bifurcation in maps and its application to an inertial impact shaker, *J. Sound Vib.*, **311** (2008), 212–223. https://doi.org/10.1016/j.jsv.2007.09.003

39. S. Elaydi, *An introduction to difference equations*, New York: Springer, 2005. https://doi.org/10.1007/0-387-27602-5

40. S. Lynch, *Dynamical systems with applications using Mathematica*, Cham: Birkhäuser, 2017. https://doi.org/10.1007/978-3-319-61485-4

41. Y. Ucakan, S. Gulen, K. Koklu, Analysing of tuberculosis in Turkey through SIR, SEIR and BSEIR mathematical models, *Math. Comp. Model. Dyn.*, **27** (2021), 179–202. https://doi.org/10.1080/13873954.2021.1881560

42. *Global Health Data Exchange, Health Statistics Yearbook 2010*, The Ministry of Health of Turkey, 2011. Available from: `https://ghdx.healthdata.org/record/turkey-health-statistics-yearbook-2010`.

43. *J. Jones, Notes on* R_0, Department of Anthropological Sciences, Stanford University, 2007. Available from: `https://web.stanford.edu/~jhj1/teachingdocs/Jones-on-R0.pdf`.

44. D. Rock, Tuberculosis: a global emergency, *Work*, 8 (1997), 93–105. https://doi.org/10.3233/WOR-1997-8111

45. E. Sarkodie, Modeling the spread of tuberculosis in central region using the susceptible-exposed-infected-susceptible (SEIS) mathematical model, MSc. Thesis, Kwame Nkrumah University of Scienceand Technology, 2014.