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Research article

The normalizer problem for finite groups with prescribed 2-subgroups

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Abstract: Suppose that X is a finite group with prescribed 2-subgroups. Under certain conditions, it is shown that the normalizer property holds for X. In particular, let X be a semidirect product of a normal 2-complement $O_{2'}(X)$ by a Sylow 2-subgroup P. If m^3 is conjugate to m or m^{-1}, for all $m \in P$, then X has the normalizer property. Our result generalizes a result due to Mazur, which states that the normalizer property holds for finite groups that have the Sylow 2-subgroup of order 2.

Keywords: central units; normalizer property; the integral group ring; class-preserving Coleman automorphisms
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1. Introduction

Let X be a finite group and $\mathrm{U}(\mathbb{Z}X)$ be the group of units of the integral group ring $\mathbb{Z}X$. We use $\mathrm{Z}(\mathrm{U}(\mathbb{Z}X))$ to denote the center of $\mathrm{U}(\mathbb{Z}X)$. The normalizer problem (Problem 43 in Sehgal [1]) of integral group rings asks whether $N_{\mathrm{U}(\mathbb{Z}X)}(X) = X\,\mathrm{Z}(\mathrm{U}(\mathbb{Z}X))$ for any finite group X, where $N_{\mathrm{U}(\mathbb{Z}X)}(X)$ is the normalizer of X in $\mathrm{U}(\mathbb{Z}X)$. If this equality is satisfied, we say that the normalizer property holds for X. Historically, the first positive result regarding this question was established by Coleman [2], who demonstrated that the normalizer property holds for finite nilpotent groups. Jackowski and Marciniak [3] proved that the finite group having a normal Sylow 2-subgroup has the normalizer property. In particular, the normalizer property holds for groups of odd order. Although Hertweck [4] constructed the first counterexample to the normalizer problem, it still remains of interest to investigate for which groups have the normalizer property. Recently, a number of related works on the normalizer problem have been published; see [5–9].

Let $\mathrm{Aut}(X)$ be the automorphism group of X. By $\mathrm{conj}(x)$ we denote the inner conjugation $g \mapsto x^{-1}gx$ on X. In order to investigate the normalizer problem, as in [7], several special automorphisms of finite group X are defined as follows:

Let $\beta \in \mathrm{Aut}(X)$. If for any $y \in X$, there exists an $x \in X$ such that $y^\beta = x^{-1}yx$, then β is called a class-preserving automorphism. Denote by $\mathrm{Aut_c}(X)$ the class-preserving automorphism group of X.

Let $\beta \in \mathrm{Aut}(X)$. If for any prime $p \| |X|$ and any Sylow p-subgroup R of X, there exists an $x \in X$ such that $\beta|_R = \mathrm{conj}(x)|_R$, then β is said to be a Coleman automorphism. Denote by $\mathrm{Aut_{Col}}(X)$ the Coleman automorphism group of X.

Let $u \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$, $g^{\theta_u} = u^{-1}gu$ for all $g \in X$. Then $\theta_u \in \mathrm{Aut}(X)$. Write $\mathrm{Aut}_{\mathbb{Z}}(X) = \{\theta_u \in \mathrm{Aut}(X) \mid x^{\theta_u} = u^{-1}xu, u \in N_{\mathrm{U}(\mathbb{Z}X)}(X), x \in X\}$. Obviously, $\mathrm{Aut}_{\mathbb{Z}}(X) \le \mathrm{Aut}(X)$.

We set

$$\mathrm{Out}_c(X) = \mathrm{Aut}_c(X)/\mathrm{Inn}(X),$$

$$\mathrm{Out_{Col}}(X) = \mathrm{Aut_{Col}}(X)/\mathrm{Inn}(X)$$

and

$$\mathrm{Out}_{\mathbb{Z}}(X) = \mathrm{Aut}_{\mathbb{Z}}(X)/\mathrm{Inn}(X).$$

Jackowski and Marciniak [3] proved that $N_{\mathrm{U}(\mathbb{Z}X)}(X) = X\,\mathrm{Z}(\mathrm{U}(\mathbb{Z}X))$ if and only if $\mathrm{Out}_{\mathbb{Z}}(X) = 1$. In addition, $\mathrm{Out}_{\mathbb{Z}}(X) \le \mathrm{Out}_c(X) \cap \mathrm{Out_{Col}}(X)$ and $\mathrm{Out}_{\mathbb{Z}}(X)$ is an elementary abelian 2-group(see [1]). Hence, if it can be shown that $|\mathrm{Out}_c(X) \cap \mathrm{Out_{Col}}(X)|$ is an odd number, then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$. In particular, the normalizer property holds for X.

In this paper, the normalizer problem of finite groups with prescribed 2-subgroups is investigated. Mazur [10] conjectured that finite groups with abelian Sylow 2-subgroups have the normalizer property. He proved that the conjecture holds if the Sylow 2-subgroups of finite groups are of order 2. This result was generalized by Hertweck [8]; he proved that X has the normalizer property, provided that X has a normal 2-complement and X has a cyclic Sylow 2-subgroup or an abelian of exponent at most 4. Marciniak and Roggenkamp [11] constructed a group $X = (\mathrm{C}_2^4 \times \mathrm{C}_3) \rtimes \mathrm{C}_2^3$ such that $|\mathrm{Out_c}(X) \cap \mathrm{Out_{Col}}(X)|$ is an even number. This example shows that if the Sylow 2-subgroup of X is non-abelian, then, in general, $|\mathrm{Out}_c(X) \cap \mathrm{Out_{Col}}(X)|$ is not necessarily odd. In [6], Van Antwerpen proved that if a finite group X possesses a self-centralizing normal 2-subgroup, then X does not have any non-inner Coleman automorphisms. In particular, the normalizer property holds for X. Inspired by these findings, we are able to establish the following results.

Theorem 1.1. Let $P \in \mathrm{Syl}_2(X)$ and $F(X)$ be the Fitting subgroup of X. Assume that any chief factor of $X/F(X)$ is not isomorphic to C_2, and K is a maximal subgroup of P satisfying $K \trianglelefteq X$. Then $|\mathrm{Out}_c(X) \cap \mathrm{Out_{Col}}(X)|$ is an odd number; that is, X has the normalizer property.

Theorem 1.2. Let $X = O_{2'}(X) \rtimes P$ be a semidirect product of a normal 2-complement $O_{2'}(X)$ by a Sylow 2-subgroup P. If m^3 is conjugate to m or m^{-1}, for all $m \in P$, then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$, that is, X has the normalizer property.

Theorem 1.3. Let X be an extension of a centerless finite group A by a 2-group P, where $\mathrm{Aut_{Col}}(A) = \mathrm{Inn}(A)$. If m^3 is conjugate to m or m^{-1}, for all $m \in P$, then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$.

Throughout, X is a finite group, and C_p denotes a cyclic group of order p. Let $H \le X$ or $H \trianglelefteq X$, and $\vartheta \in \mathrm{Aut}(X)$. We denote $\vartheta|_H$ or $\vartheta|_{X/H}$, respectively, if ϑ fixes H or X/H. For a $y \in X$, $\mathrm{conj}(y)$ means $g^{\mathrm{conj}(y)} = g^y$ for all $g \in X$. For any $p \| |X|$, we denote $O_p(X)$ to be the largest normal p-subgroup of X and $O_{p'}(X)$ the largest normal p'-subgroup of X. Other notations are standard; refer to [7, 12].

2. Preliminaries

Lemma 2.1. [7] Assume that $H \trianglelefteq X$ and X/H is a p'-group. Then we have the following statements.

(1) If $\vartheta \in \mathrm{Aut}_c(X)$ is of p-power order, then $\vartheta|_H \in \mathrm{Aut}_c(H)$;
(2) If $\vartheta \in \mathrm{Aut}_{\mathrm{Col}}(X)$ is of p-power order, then $\vartheta|_H \in \mathrm{Aut}_{\mathrm{Col}}(H)$;
(3) If $\mathrm{Out}_c(H)$ or $\mathrm{Out}_{\mathrm{Col}}(H)$ is a p'-group, then so is $\mathrm{Out}_c(X)$ or $\mathrm{Out}_{\mathrm{Col}}(X)$.

Lemma 2.2. [8] $|\mathrm{Out}_c(X) \cap \mathrm{Out}_{\mathrm{Col}}(X)|$ is an odd number if there exists a cyclic Sylow 2-subgroup of X.

Lemma 2.3. Suppose that $H \trianglelefteq X$ and $\vartheta \in \mathrm{Aut}(X)$. Then we have the following statements.
(1) If $\vartheta \in \mathrm{Aut}_c(X)$, then $\vartheta|_H \in \mathrm{Aut}(H)$ and $\vartheta|_{X/H} \in \mathrm{Aut}_c(X/H)$.
(2) If $\vartheta \in \mathrm{Aut}_{\mathrm{Col}}(X)$, then $\vartheta|_H \in \mathrm{Aut}(H)$ and $\vartheta|_{X/H} \in \mathrm{Aut}_{\mathrm{Col}}(X/H)$.

Proof. These proofs are obvious, so we omit them. □

Lemma 2.4. Suppose that $H \trianglelefteq X$ and $\vartheta \in \mathrm{Aut}(X) \setminus \mathrm{Inn}(X)$ is a p-element. If $\vartheta|_{X/H} \in \mathrm{Inn}(X/H)$, then there exists a $\gamma \in \mathrm{Inn}(X)$ satisfying $\gamma\vartheta|_{X/H} = \mathrm{id}|_{X/H}$, and $\gamma\vartheta \in \mathrm{Aut}(X) \setminus \mathrm{Inn}(X)$ remains a p-element.

Proof. By $\vartheta|_{X/H} \in \mathrm{Inn}(X/H)$, so we suppose that $\vartheta|_{X/H} = \mathrm{conj}(x)|_{X/H}$ for some $x \in X$. Let i be a positive integer and $o(\vartheta) = p^i$. Denote $\beta = \mathrm{conj}(x)$. Hence, $\beta^{-1}\vartheta|_{X/H} = \mathrm{id}|_{X/H}$. Suppose that n is positive integer and $(n, p) = 1$. If $(\beta^{-1}\vartheta)^n$ is the p-part of $\beta^{-1}\vartheta$, then there exist $s, t \in \mathbb{Z}$ satisfying $sn + tp^i = 1$. It is clear that $o((\beta^{-1}\vartheta)^{sn})$ is a power of p, and $(\beta^{-1}\vartheta)^{sn}|_{X/H} = \mathrm{id}|_{X/H}$. By $\mathrm{Inn}(X) \trianglelefteq \mathrm{Aut}(X)$, there exists a $\gamma \in \mathrm{Inn}(X)$ satisfying $(\beta^{-1}\vartheta)^{sn} = \gamma\vartheta^{sn} = \gamma\vartheta^{1-tp^i} = \gamma\vartheta$. Therefore, $\gamma\vartheta|_{X/H} = (\beta^{-1}\vartheta)^{sn}|_{X/H} = \mathrm{id}|_{X/H}$. □

Lemma 2.5. Suppose that $N \leq X$ and $\vartheta \in \mathrm{Aut}(X)$ is a p-element. Let $\vartheta|_N \in \mathrm{Aut}(N)$ and $\vartheta|_N = \mathrm{conj}(x)|_N$ for some $x \in X$; then there exists a p-element $h \in X$ satisfying $\vartheta|_N = \mathrm{conj}(h)|_N$.

Proof. Let $o(\vartheta) = p^r$, $o(x) = p^s j$, where $r,\ s,\ j \in \mathbb{N}$ and $(p, j) = 1$. Set $i = max\{r, s\}$. By $(p^i, j) = 1$, then there exists $u, v \in \mathbb{Z}$, satisfying $up^i + vj = 1$. Let $h = x^{vj}$, so that h is a p-element. By $z = z^{\vartheta^{up^i}} = z^{x^{up^i}}$ for any $z \in N$, so that $z^\vartheta = z^x = z^{x^{up^i+vj}} = (z^{x^{up^i}})^{x^{vj}} = z^{x^{vj}} = z^h$. Hence, $\vartheta|_N = \mathrm{conj}(h)|_N$. □

Lemma 2.6. [13] Suppose that $\vartheta \in \mathrm{Aut}(X)$ is a p-element and any chief factor of $X/F^*(X)$ is not isomorphic to C_p. Assume that $H \trianglelefteq X$ with $H^\vartheta = H$. If $\vartheta|_{X/H} = \mathrm{id}|_{X/H}$ and there is an $x \in X$ satisfying $\vartheta|_R = \mathrm{conj}(x)|_R$, where $R \in \mathrm{Syl}(H)$, then we have $y \in O_p(X)H$ and $\vartheta|_R = \mathrm{conj}(y)|_R$.

Lemma 2.7. [13] Let $\vartheta \in \mathrm{Aut}(X)$ be a p-element. Suppose that $H \trianglelefteq X$ satisfying $\vartheta|_H = \mathrm{id}|_H$ and $\vartheta|_{X/H} = \mathrm{id}|_{X/H}$. Then $\vartheta|_{X/O_p(\mathrm{Z}(H))} = \mathrm{id}|_{X/O_p(\mathrm{Z}(H))}$. Moreover, we have $\vartheta \in \mathrm{Inn}(X)$ if $\vartheta|_P = \mathrm{id}|_P$, where $P \in \mathrm{Syl}_p(X)$.

Lemma 2.8. [14] Suppose that $v \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$. Then y is conjugate to $v^{-1}yv$ for any $y \in X$.

Lemma 2.9. [1] All central units of $\mathbb{Z}(X)$ are trivial if and only if for every $x \in X$, any generator of $< x >$ is conjugate to either x or x^{-1}.

Lemma 2.10. [5] Let $w \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$, $H \trianglelefteq X$, and let $R \leq X$ be a p-subgroup. Assume that $w^\beta = Hx \in X/H$ for some $x \in X$, where $\beta : \mathbb{Z}X \to \mathbb{Z}(X/H)$ is the natural homomorphism. Then there exists an $h \in H$ satisfying $w^{-1}yw = (hx)^{-1}y(hx)$ for every $y \in R$.

Lemma 2.11. [1] Suppose that $v \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$, and $\psi_v \in \mathrm{Aut}_{\mathbb{Z}}(X)$. Then $\psi_v^2 \in \mathrm{Inn}(X)$.

Lemma 2.12. [13] Assume that $\pi(X)$ and $\pi(\mathrm{Aut}_{\mathrm{Col}}(X))$ are the sets of prime divisors of $|X|$ and $|\mathrm{Aut}_{\mathrm{Col}}(X)|$, respectively. Then $\pi(\mathrm{Aut}_{\mathrm{Col}}(X)) \subseteq \pi(X)$.

Lemma 2.13. [13] Let X be a simple group. Then there exists a $q \mid |X|$ such that q-central automorphisms of X are inner automorphisms.

Lemma 2.14. Suppose that $H \leq X$ and $\vartheta \in \mathrm{Aut}(X)$ is a p-element. If $\vartheta|_H = \mathrm{conj}(x)|_H$ for some $x \in X$, then there exists a $\gamma \in \mathrm{Inn}(X)$ that satisfies $\gamma\vartheta|_H = \mathrm{id}|_H$ and $\gamma\vartheta$ remains a p-element.

Proof. Similar to the proof of Lemma 2.4, so we omit it. □

3. Proof of the theorems

Recall that $F^*(X)$ is said to be the generalized Fitting subgroup of X if $F^*(X)$ is a central product of its Fitting subgroup $F(X)$ and its layer $E = E(X)$, which is generated by the components, i.e., the subnormal quasisimple subgroups of X.

Theorem 3.1. Let $P \in \mathrm{Syl}_2(X)$ and $F(X)$ be the Fitting subgroup of X. Assume that any chief factor of $X/F(X)$ is not isomorphic to C_2, and K is a maximal subgroup of P satisfying $K \trianglelefteq X$. Then $|\mathrm{Out}_c(X) \cap \mathrm{Out}_{\mathrm{Col}}(X)|$ is an odd number; that is, X has the normalizer property.

Proof. Let $\vartheta \in \mathrm{Aut}_c(X) \cap \mathrm{Aut}_{\mathrm{Col}}(X)$ be a 2-element. Then we have to show that $\vartheta \in \mathrm{Inn}(X)$. If $P \trianglelefteq X$, then, by Lemma 2.1, the assertion follows. If $K = 1$, then P is a cyclic group of order 2. Hence, according to Lemma 2.2, this concludes the proof.

Henceforth, we suppose that $P \ntrianglelefteq X$ and $K \neq 1$. Under this assumption, we have $K = O_2(X) \neq P$. It follows that the Sylow 2-subgroups of $X/O_2(X)$ are cyclic groups of order 2. Then, by Lemma 2.2, $\mathrm{Out}_c(X/O_2(X)) \cap \mathrm{Out}_{\mathrm{Col}}(X/O_2(X))$ is of odd order. Further, by Lemma 2.3, we obtain that

$$\vartheta|_{X/O_2(X)} \in \mathrm{Aut}_c(X/O_2(X)) \cap \mathrm{Aut}_{\mathrm{Col}}(X/O_2(X)).$$

In addition, by assumption, ϑ is a 2-element. Consequently, $\vartheta|_{X/O_2(X)} \in \mathrm{Inn}(X/O_2(X))$. Therefore, by Lemma 2.4, without losing generality, we can assume the following:

$$\vartheta|_{X/O_2(X)} = \mathrm{id}|_{X/O_2(X)}.$$

Since $\vartheta \in \mathrm{Aut}_{\mathrm{Col}}(X)$, according to Lemma 2.5, there exists a 2-element $x \in X$ satisfying

$$\vartheta|_P = \mathrm{conj}(x)|_P.$$

We denote $F^*(X)$ as the generalized Fitting subgroup of X. Next we will show that $F^*(X) = F(X)$. Note that the Sylow 2-subgroups of $X/O_2(X)$ are cyclic groups of order 2. Hence, according to Burnside's theorem, there is a normal 2-complement of $X/O_2(X)$. By the Feit-Thompson theorem, which asserts that every group of odd order is solvable, X is solvable. Therefore, $F^*(X) = F(X)$.

Now, according to Lemma 2.6, there is $y \in O_2(X)$ satisfying

$$\vartheta|_{O_2(X)} = \mathrm{conj}(y)|_{O_2(X)}.$$

Moreover, given that $\vartheta|_P = \mathrm{conj}(x)|_P$ and $O_2(X) \leq P$, we conclude that

$$\vartheta|_{O_2(X)} = \mathrm{conj}(x)|_{O_2(X)}.$$

Consequently, $\mathrm{conj}(x)|_{O_2(X)} = \vartheta|_{O_2(X)} = \mathrm{conj}(y)|_{O_2(X)}$, which implies that

$$xy^{-1} \in C_X(O_2(X)).$$

Since $\vartheta|_{X/O_2(X)} = \mathrm{id}|_{X/O_2(X)}$, $P^\vartheta = P$. Additionally, by $\vartheta|_P = \mathrm{conj}(x)|_P$, we can have $P^\vartheta = P^x$. Accordingly, we have $P^x = P$, which yields $x \in N_X(P)$. Thus $x \in P$, as x is a 2-element. Note that $y \in O_2(X) \leq P$; we obtain that $xy^{-1} \in P$.

If $xy^{-1} \notin O_2(X)$, then $P =< xy^{-1}, O_2(X) >$ since $O_2(X)$ is a maximal subgroup of P. We derive that $xy^{-1} \in \mathrm{Z}(P)$ and thus

$$\vartheta\mathrm{conj}(y^{-1})|_P = \mathrm{conj}(xy^{-1})|_P = \mathrm{id}|_P.$$

Moreover, it is obvious that

$$\vartheta\mathrm{conj}(y^{-1})|_{O_2(X)} = \mathrm{id}|_{O_2(X)}$$

and

$$\vartheta\mathrm{conj}(y^{-1})|_{X/O_2(X)} = \mathrm{id}|_{X/O_2(X)}.$$

According to Lemma 2.7, we have $\vartheta\mathrm{conj}(y^{-1}) \in \mathrm{Inn}(X)$, which implies that $\vartheta \in \mathrm{Inn}(X)$.

If $xy^{-1} \in O_2(X)$, then $x \in O_2(X)$. As a result, we have

$$\vartheta\mathrm{conj}(x^{-1})|_{X/O_2(X)} = \mathrm{id}|_{X/O_2(X)}.$$

In addition, we can see that

$$\vartheta\mathrm{conj}(x^{-1})|_{O_2(X)} = \mathrm{id}|_{O_2(X)},$$

and

$$\vartheta\mathrm{conj}(x^{-1})|_P = \mathrm{id}|_P.$$

Using Lemma 2.7, we have $\vartheta\mathrm{conj}(x^{-1}) \in \mathrm{Inn}(X)$ and thus $\vartheta \in \mathrm{Inn}(X)$. Hence, in either case, we have $\vartheta \in \mathrm{Inn}(X)$. □

Remark 3.2. The requirement that $X/F(X)$ does not have a chief factor isomorphic to C_2 cannot be omitted. For instance, with the assumption that $\mathrm{Out_c}(X) \cap \mathrm{Out_{Col}}(X)$ is of even order, Marciniak and Roggenkamp [11] constructed a group $X = (\mathrm{C}_2^4 \times \mathrm{C}_3) \rtimes \mathrm{C}_2^3$.

Theorem 3.3. Let $X = O_{2'}(X) \rtimes P$ be a semidirect product of a normal 2-complement $O_{2'}(X)$ by a Sylow 2-subgroup P. If m^3 is conjugate to m or m^{-1}, for all $m \in P$, then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$, that is, X has the normalizer property.

Proof. To demonstrate that the assertion is true for X, we need only prove that $\mathrm{Aut}_{\mathbb{Z}}(X) \subseteq \mathrm{Inn}(X)$. Suppose that $\vartheta \in \mathrm{Aut}_{\mathbb{Z}}(X)$. It follows that for all $x \in X$, there exists $w \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$ satisfies $x^{\vartheta} = w^{-1}xw$. We suppose the augmentation map as follows:

$$\epsilon : \mathbb{Z}X \to \mathbb{Z}\,(\sum_{x\in X} r_x x \mapsto \sum_{x\in X} r_x),$$

where $r_x \in \mathbb{Z}$ for any $x \in X$. Then we have $\epsilon(w) = 1$ or -1 since $w \in \mathrm{U}(\mathbb{Z}X)$. It is evident that $\vartheta = \mathrm{conj}(w) = \mathrm{conj}(-w)$. Hence, we can assume that $\epsilon(w) = 1$.

For $X/O_{2'}(X)$, we use the bar notation for the elements and subgroups. Namely, we denote $\bar{x} := xO_{2'}(X)$, $\bar{U} := UO_{2'}(X)/O_{2'}(X)$, for any $x \in X$, $U \leq X$, respectively. Specifically, we have $\bar{X} := X/O_{2'}(X)$.

Denote

$$\rho : \mathbb{Z}X \to \mathbb{Z}\bar{X}\,(\sum_{x\in X} r_x x \mapsto \sum_{x\in X} r_x \bar{x})$$

the natural homomorphism for $\mathbb{Z}X$ to $\mathbb{Z}\bar{X}$.

Claim 1. Notation as above, there exists an $h \in X$ satisfying $\rho(w) = \bar{h}$ and $\vartheta|_{\bar{X}} = \mathrm{conj}(h)|_{\bar{X}}$.

By $w \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$, then $\rho(w) \in N_{\mathrm{U}(\mathbb{Z}\bar{X})}(\bar{X})$. According to Lemma 2.8, $O_{2'}(X)^{\vartheta} = O_{2'}(X)$ and $\vartheta|_{\bar{X}} \in \mathrm{Aut}(\bar{X})$. Since for any $x \in X$, $x^{\vartheta} = w^{-1}xw$, so

$$\bar{x}^{\vartheta|_{\bar{X}}} = \overline{w^{-1}xw} = \rho(w^{-1}xw) = \rho(w)^{-1}\bar{x}\rho(w). \tag{3.1}$$

Therefore, $\vartheta|_{\bar{X}}$ is induced by $\rho(w)$ via conjugation, namely, $\vartheta|_{\bar{X}} \in \mathrm{Aut}_{\mathbb{Z}}(\bar{X})$. According to Lemma 2.3, we have $\vartheta|_{\bar{X}} \in \mathrm{Inn}(\bar{X})$. It follows that there is a $y \in X$ satisfying $\vartheta|_{\bar{X}} = \mathrm{conj}(\bar{y})|_{\bar{X}}$, which implies that $\bar{x}^{\vartheta|_{\bar{X}}} = \bar{x}^{\rho(w)} = \bar{x}^{\bar{y}}$ for any $\bar{x} \in \bar{X}$. Then, $\rho(w)\bar{y}^{-1} \in \mathrm{Z}(\mathrm{U}(\mathbb{Z}\bar{\mathrm{X}}))$. Moreover, for all $m \in P$, $\bar{X} \cong P$ and m^3 is conjugate to m or m^{-1}. According to Lemma 2.9, it is trivial for all central units of $\mathbb{Z}(\bar{X})$. Hence, there is a central element $\bar{z}$ of $\bar{X}$ satisfying $\rho(w)\bar{y}^{-1} = \bar{z}$. Denote by $\bar{h} = \overline{zy}$. Therefore, $\rho(w) = \bar{h}$. According to Eq (3.1), we obtain

$$\vartheta|_{\bar{X}} = \mathrm{conj}(h)|_{\bar{X}}. \tag{3.2}$$

Claim 2. $\vartheta\mathrm{conj}(h^{-1})|_{O_{2'}(X)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(X))$.

For any $k \in O_{2'}(X)$, we have $k^{\vartheta} = w^{-1}kw$. Since $w \in N_{\mathrm{U}(\mathbb{Z}X)}(X)$ and Lemma 2.8, this implies that $w^{-1}kw$ and k are conjugate in X. Therefore, there exists a $g \in X$ satisfying $k^{\vartheta} = k^{g}$. But $O_{2'}(X) \trianglelefteq X$, so we get that $k^{\vartheta} \in O_{2'}(X)$. Then, it follows that $\vartheta|_{O_{2'}(X)} \in \mathrm{Aut}(O_{2'}(X))$ and $\vartheta\mathrm{conj}(h^{-1})|_{O_{2'}(X)} \in \mathrm{Aut}(O_{2'}(X))$. Hence, we need to check that $\vartheta\mathrm{conj}(h^{-1})|_{O_{2'}(X)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(X))$. Suppose that $p \in \pi(O_{2'}(X))$ and $P \in \mathrm{Syl}_p(O_{2'}(X))$. Consequently, according to Lemma 2.10, there is a $b \in O_{2'}(X)$ satisfying

$$\vartheta\mathrm{conj}(h^{-1})|_{P} = \mathrm{conj}(b)|_{P} \tag{3.3}$$

Accordingly, $\vartheta\mathrm{conj}(h^{-1})|_{O_{2'}(X)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(X))$, since the Eq (3.3) holds.

Claim 3. $\vartheta \in \mathrm{Inn}(X)$.

Write $\psi := \vartheta\mathrm{conj}((bh)^{-1})$. By equation (3.2), we get that

$$\psi|_{X/O_{2'}(X)} = \mathrm{id}|_{X/O_{2'}(X)}. \tag{3.4}$$

Since $\vartheta \in \mathrm{Aut}_{\mathbb{Z}}(X)$, this implies that $\psi = \vartheta\mathrm{conj}((bh)^{-1}) \in \mathrm{Aut}_{\mathbb{Z}}(X)$. According to Lemma 2.11, $\psi^2 \in \mathrm{Inn}(X)$. We may suppose that ψ is a 2-element. So is $\psi|_{O_{2'}(X)}$. Note that

$$\psi|_{O_{2'}(X)} = \rho\mathrm{conj}(h^{-1})\mathrm{conj}(b^{-1})|_{O_{2'}(X)} \in \mathrm{Aut}_{\mathrm{Col}}(O_{2'}(X)).$$

By Lemma 2.12 and $\psi|_{O_{2'}(X)}$ being a 2-element, we deduce that

$$\psi|_{O_{2'}(X)} = \mathrm{id}|_{O_{2'}(X)}. \tag{3.5}$$

Now by Lemma 2.7, Eqs (3.4) and (3.5) yield that $\psi|_{X/O_2(\mathrm{Z}(O_{2'}(X)))} = \mathrm{id}|_{X/O_2(\mathrm{Z}(O_{2'}(X)))}$. Since $O_{2'}(X)$ is a $2'$-group, it follows that $\psi = \mathrm{id}$, that is, $\vartheta\mathrm{conj}((bh)^{-1}) = \mathrm{id}$. Hence $\vartheta \in \mathrm{Inn}(X)$. □

Corollary 3.4. Let X have a Sylow 2-subgroup of order 2. Then X has the normalizer property (see [10]).

Proof. According to Burnside's theorem, there is a normal 2-complement $O_{2'}(X)$ of X. Then the consequence is immediate from Theorem 3.3. □

Theorem 3.5. Let X be an extension of a centerless finite group A by a 2-group P, where $\mathrm{Aut}_{\mathrm{Col}}(A) = \mathrm{Inn}(A)$. If m^3 is conjugate to m or m^{-1}, for all $m \in P$, then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$.

Proof. Let $\vartheta \in \mathrm{Aut}_{\mathbb{Z}}(X)$ be a p-element; we will show that $\vartheta \in \mathrm{Inn}(X)$. Since $X/A \cong P$ has proofs similar to those of Claim 1 and Claim 2 in Theorem 3.3, then there exists some $h \in X$ such that $\vartheta|_{X/A} = \mathrm{conj}(h)|_{X/A}$ and $\vartheta\mathrm{conj}(h^{-1})|_A \in \mathrm{Aut}_{\mathrm{Col}}(A)$. Since $\mathrm{Aut}_{\mathrm{Col}}(A) = \mathrm{Inn}(A)$, we know that $\vartheta\mathrm{conj}(h^{-1})|_A \in \mathrm{Inn}(A)$. Thus there exists some $a \in A$ satisfying

$$\vartheta\mathrm{conj}(h^{-1})|_A = \mathrm{conj}(a)|_A. \tag{3.6}$$

Write $\psi := \vartheta\mathrm{conj}((ah)^{-1}$. By $\vartheta|_{X/A} = \mathrm{conj}(h)|_{X/A}$, we obtain

$$\psi|_{X/A} = \mathrm{id}|_{X/A}. \tag{3.7}$$

By Eq (3.6), we have

$$\psi|_A = \mathrm{id}|_A. \tag{3.8}$$

Now by Lemma 2.7, Eqs (3.7) and (3.8) yield that $\psi|_{X/O_p(\mathrm{Z}(A))} = \mathrm{id}|_{X/O_p(\mathrm{Z}(A))}$. Since $\mathrm{Z}(A) = 1$, it follows that $\psi = \mathrm{id}$, that is, $\vartheta\mathrm{conj}((ah)^{-1}) = \mathrm{id}$. Hence $\vartheta \in \mathrm{Inn}(X)$. We are done. □

Corollary 3.6. Let X be an extension of a finite complete group F by a cyclic group P of order 4 or a quaternion group P of order 8. Then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$.

Proof. Since F is a complete group, then $\mathrm{Z}(F) = 1$ and $\mathrm{Aut}(F) = \mathrm{Inn}(F)$. Since P is a cyclic group of order 4 or $P =< a, b|a^4 = 1, b^2 = a^2, b^{-1}ab = a^3 >$, which implies that m^3 is conjugate to m or m^{-1}, for all $m \in P$. Thus the result is immediate from Theorem 3.5. □

Corollary 3.7. Let X be an extension of an almost simple group H by a cyclic group P of order 4 or a quaternion group P of order 8. Then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$.

Proof. Since H is an almost simple group, then there exists some non-abelian simple group A satisfying $A \leq H \leq \mathrm{Aut}(A)$. Obviously $\mathrm{Z}(H) = 1$. Next we show that $\mathrm{Aut}_{\mathrm{Col}}(H) = \mathrm{Inn}(H)$. By Lemma 2.13, there exists a prime $q \mid |A|$ such that q-central automorphisms of A are inner. Let $\vartheta \in \mathrm{Aut}_{\mathrm{Col}}(H)$ and $Q \in \mathrm{Syl}_{\mathrm{q}}(H)$. By definition, then there exists some $h \in H$ satisfying $\vartheta|_Q = \mathrm{conj}(h)|_Q$. According to Lemma 2.14, we may assume that $\vartheta|_Q = \mathrm{id}|_Q$. Set $D = Q \cap A$; thus, $D \in \mathrm{Syl}_q(A)$ and $\vartheta|_D = \mathrm{id}|_D$. Note that $A \trianglelefteq H$. By Lemma 2.3, we deduce that $\vartheta|_A$ is a q-central automorphism of A. Hence, $\vartheta|_A \in \mathrm{Inn}(A)$, that is, $\vartheta|_A = \mathrm{conj}(a)|_A$ for some $a \in A$. Set $\psi = \vartheta\mathrm{conj}(a^{-1})$, then $\psi|_A = \mathrm{id}|_A$. Again note that A is a non-abelian simple group; we obtain $C_H(A) = C_{\mathrm{Aut}(A)}(A) \cap H = 1$ because A identifies with $\mathrm{Inn}(A)$. Thus, for any $y \in H$ and $x \in A$, we have $(y^{-1}xy)^{\psi} = (y^{-1})^{\psi}xy^{\psi} = y^{-1}xy$; this implies that $y^{\psi}y^{-1} \in C_H(A) = 1$, i.e., $\psi = \mathrm{id}$. Hence, $\vartheta \in \mathrm{Inn}(H)$. So this result is immediate from Theorem 3.5. □

Example 3.8. Let X be an extension of a simple group A by a cyclic group P of order 4 or a quaternion group P of order 8. Then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$.

Proof. According to the abelianity of the simple group, the proof splits into two cases.

(1) Let A be a non-abelian simple group. This is a direct consequence of Corollary 3.7.

(2) Let A be an abelian simple group. It is known that A is a cyclic group of order p, where p is a prime. If $p = 2$, then X is a 2-group. By the definition of Coleman automorphisms, we obtain that $\mathrm{Out}_{\mathrm{Col}}(X) = 1$; this implies that $\mathrm{Out}_{\mathbb{Z}}(X) = 1$. If $p \neq 2$, the assertion is a direct consequence of Theorem 3.3. □

Corollary 3.9. Let X be an extension of a symmetric group $\Sigma_i (i \geq 3)$ by a cyclic group P of order 4 or a quaternion group P of order 8. Then $\mathrm{Out}_{\mathbb{Z}}(X) = 1$.

Proof. If $i \geq 3$ and $i \neq 6$, then Σ_i is a complete group. Hence, the assertion is immediate from Corollary 3.6. If $i = 6$, then Σ_6 is an almost simple group. Hence, the assertion is immediate from Corollary 3.7. □

4. Conclusions

This paper continues the study of the normalizer problem of finite groups with prescribed 2-subgroups. We have proven that X has the normalizer property, if X is an extension of some centerless finite groups by 2-groups with trivial central units or X is a semidirect product of a finite group of odd order by a 2-group with trivial central units. Additionally, we have shown that under some conditions class-preserving Coleman automorphisms of 2-power order of some finite groups are inner. In particular, the normalizer property holds for these groups.

Author contributions

Liang Zhang: Conceptualization, Writing-original draft; Jinke Hai: Funding acquisition, Editing, Writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. S. K. Sehgal, *Units in integral group rings*, Longman Scientific & Technical, 1993.
2. D. B. Coleman, On the modular group ring of a p-group, *Proc. Amer. Math. Soc.*, **5** (1964), 511–514. https://doi.org/10.2307/2034735
3. S. Jackowski, Z. S. Marciniak, Group automorphisms inducing the identity map on cohomology, *J. Pure Appl. Algebra*, **44** (1987), 241–250. https://doi.org/10.1016/0022-4049(87)90028-4
4. M. Hertweck, A counterexample to the isomorphism problem for integral group rings, *Ann. Math.*, **154** (2001), 115–138. https://doi.org/10.2307/3062112

5. J. K. Hai, J. D. Guo, The normalizer property for the integral group ring of the wreath product of two symmetric groups S_k and S_n, *Commun. Algebra*, **45** (2017), 1278–1283. https://doi.org/10.1080/00927872.2016.1175613

6. A. Van Antwerpen, Coleman automorphisms of finite groups and their minimal normal subgroups, *J. Pure Appl. Algebra*, **222** (2018), 3379–3394. https://doi.org/10.1016/j.jpaa.2017.12.013

7. M. Hertweck, Class-preserving Coleman automorphisms of finite groups, *Monatsh. Math.*, **136** (2002), 1–7. https://doi.org/10.1007/s006050200029

8. M. Hertweck, Local analysis of the normalizer problem, *J. Pure Appl. Algebra*, **163** (2001), 259–276. https://doi.org/10.1016/S0022-4049(00)00167-5

9. M. Hertweck, E. Jespers, Class-preserving automorphisms and the normalizer property for Blackburn groups, *J. Group Theory*, **12** (2009), 157–169. https://doi.org/10.1515/JGT.2008.068

10. M. Mazur, Automorphisms of finite groups, *Commun. Algebra*, **22** (1994), 6259–6271. https://doi.org/10.1080/00927879408825187

11. Z. S. Marciniak, K. W. Roggenkamp, The normalizer of a finite group in its integral group ring and Čech cohomology, In: *Algebra-representation theory*, 2001, 159–188.

12. H. Kurzweil, B. Stellmacher, *The theory of finite groups: An introduction*, New York: Springer-Verlag, 2004. https://doi.org/10.1007/b97433

13. M. Hertweck, W. Kimmerle, Coleman automorphisms of finite groups, *Math. Z.*, **242** (2002), 203–215. https://doi.org/10.1007/s002090100318

14. T. Petit Lobão, C. Polcino Milies, The normalizer property for integral group rings of Frobenius groups, *J. Algebra*, **256** (2002), 1–6. https://doi.org/10.1016/S0021-8693(02)00156-4