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Research article

Seidel Laplacian energy of bipolar fuzzy graphs and enhanced score functions for decision-making applications

Sivaranjani Krishnaraj[1,2], **Shanmuga Sundaram O.V.**[2], **Prasantha Bharathi Dhandapani**[1] **and Taha Radwan**[3,*]

[1] Department of Mathematics, Sri Eshwar College of Engineering, Coimbatore 641202, Tamil Nadu, India

[2] Department of Mathematics, Sree Saraswathi Thyagaraja College, Pollachi, Coimbatore 642107, Tamil Nadu, India

[3] Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: t.radwan@qu.edu.sa.

Abstract: Bipolar fuzzy sets (BPFs) provide a suitable framework for knowledge representation if some data contains imprecise and ambiguous information. In this manuscript, the lower and upper bounds of the Seidel Laplacian energy of a bipolar fuzzy graph were examined with suitable illustrative examples. Moreover, the energy of a bipolar fuzzy graph, the Laplacian energy of a bipolar fuzzy graph, and the Seidel Laplacian energy of a bipolar fuzzy graph were examined. Furthermore, to address complex multi-criteria decision-making (MCDM) problems involving uncertainty and bipolar information, we proposed novel score functions: The score function, improved score function, and double improved score function. These functions were demonstrated through examples to effectively handle ambiguity and duality in decision-makers' inputs represented via bipolar fuzzy sets.

Keywords: bipolar fuzzy graph; fuzzy logic; energy of a graph; Seidel Laplacian energy; symmetry; computational analysis; decision making
Mathematics Subject Classification: 68R10, 90B10, 05C72

1. Introduction

The application of mathematics is particularly prominent in the fields of engineering, technology, and medicine. Key areas of mathematical research include differential equations, topology, automata theory, queuing theory, operations research, and graph theory, among many others. Graph theory, a branch of discrete mathematics, focuses on the study of graphs. A graph is a mathematical structure made of points, called vertices, which are connected by lines, known as edges. Graphs are used to represent relationships or connections such as a map of cities connected by roads or a social network of people linked by friendships [1]. It establishes a pairwise link between the things. Based on provided data or a set of points, graph theory was first proposed by renowned Swiss mathematician Leonhard Euler to answer a variety of mathematical issues [2]. This resource helps in learning graph theory and exploring its applications in diverse fields and real-world scenarios. By employing matrices connected to a graph, more types of graph energy are defined in an analogous manner. After 25 years, Ivan Gutman's 1978 introduction of graph energy received attention and produced a plethora of new findings in the field of mathematics. The energy of a graph is equal to the sum of the absolute values of its eigenvalues [3]. Numerous researchers have examined graph energies, including but not limited to distance energy, hub energy, Seidel energy, Laplacian energy, color energy, Harary energy, Kirchhoff energy, domination energy, Hermitian energy, and Seidel Laplacian energy. The energy of the graph is discussed in [4, 5]. Several researchers also explore different forms of energy applied in decision-making processes [6]. Various other fields related to energy [7], such as fuzzy multi-criteria decision-making and digital green innovation [8, 9], are also explored in some research studies [10]. If s is a fuzzy subset of a set S and t is a fuzzy relation on s, then the pair G: (s, t) is a fuzzy graph. It is helpful to take into account fuzzy relations that transfer from fuzzy sets contained in the universal sets into the unit interval in particular situations, such as in graph theory. Fuzzy graph theory is increasingly being applied across modern scientific and technological domains, including information theory, neural networks, expert systems, cluster analysis, medical diagnostics, and control theory. Fuzzy modeling plays a crucial role in a wide range of disciplines, including science, engineering, and medicine. L. A. Zadeh presented the idea of fuzzy sets and fuzzy relations in 1965, and was further explored. In 1975, Rosenfeld created the theory of fuzzy graphs by taking into account fuzzy relations on fuzzy sets.

The fuzzy set concept is advantageous since it addresses ambiguity and uncertainty that the Cantorian set was unable to handle. A single number between zero and one designates an element's membership in a fuzzy set, according to the fuzzy set theory. The degree of non-membership of an element in a fuzzy set may not necessarily equal 1 minus the membership degree though, as there may be some degree of doubt. Double-sided or bipolar judgemental thinking, which has positive and negative sides, is the foundation upon which human decision-making is formed. Think of competitiveness and cooperation: Enmity and hostility, competing and complementary interests, consequence and side effects, non-probability, and probability. Every real-world object has two or more facets, which are highly helpful to anyone who want to learn about the advantages and drawbacks of any given object. Moreover, frameworks are available on the market to help comprehend all these features mathematically, but first we look at the concept of bipolar soft theory, some theory-related operations, and attributes [11]. Computational interactions between the positive and negative sides of bipolar fuzzy variables should be possible with bipolar fuzzy logic, enabling bipolar thinking. A bipolar fuzzy logic variable requires a positive and a negative pole [12, 13]. Binary relations with bipolar fuzzy logic can be extended to bipolar crisp and fuzzy relations. The theoretical foundation for bipolar clustering is provided by the bipolar fuzzy set theory, which unifies polarity and fuzzy in a single model of cooperation as well as decision analysis. Zhang introduced the notion of bipolar fuzzy sets in 1994 as a fuzzy set extension. A bipolar fuzzy set is a fuzzy set extension with a membership degree range between -1 and 1. Furthermore, dealing with bipolar information is vital in many disciplines. Positive information, which is observed, represents what is deemed feasible, whereas negative information represents what is seen as unfeasible [14–16]. Automata theory [17], labeling [18, 19, 23], topological indices [20], and differential equations [21, 22] are also some of the major research concepts, and a wide range of fuzzy applications that often involve them are epidemic modeling, optimization of economic order quantities, and decision-making. The origin of the paper is the Seidel Laplacian Energy (SLE) of graphs [24]. The authors, Sivaranjani. K. et al., extend the SLE of graphs to fuzzy graphs and intuitionistic fuzzy graphs [25, 26]. Here, SLE is extended to bipolar fuzzy graphs.

The findings of this research offer new theoretical insights and computational tools for analyzing bipolar fuzzy graphs, with potential applications in decision-making processes under uncertainty and imprecision. Scalability Issues: For large-scale graphs (e.g., thousands of vertices), computing all eigenvalues can be computationally expensive. Techniques such as sparse matrix representations or iterative eigen-solvers can mitigate this issue for traditional and Seidel-based measures and Parallelization or approximation techniques (e.g., Lanczos algorithm) can also be applied similarly across traditional and proposed measures. The outline of this article is as follows: Sections 1 and 2 contain a synopsis, background data, and notes for clarity. In Section 3, we provide an analysis of the Laplacian energy of a bipolar fuzzy graph along with several examples. In Section 4, we discuss the lower and upper bounds on the Seidel Laplacian energy with examples. Last, the seidel laplacian energy is used to examine a real-world decision-making situation in Section 5. Furthermore, the Appendix section contains C programming codes to determine the energy of a graph.

2. Preliminaries

To fully profit from this article, it is necessary to cover some basic ideas, which are reviewed in this section.

Definition 2.1. *A mathematical structure known as a graph G = (V, E) is made up of a set of vertices (V) and a set of edges (E), where each edge is an unordered pair of different vertices.*

Definition 2.2. *The membership function defines a fuzzy set on a set V. The degree to which a point is a member of the fuzzy set is represented by the functions that are always equal to 0 and 1, respectively, which are the smallest and greatest elements.*

Definition 2.3. *The pair $G = (\alpha,\beta)$ represents a fuzzy graph with V as the underlying set. It consists of a fuzzy subset α:V→[0, 1] and a fuzzy relation β:V*V→[0, 1] on α such that $\beta(x, y) \leq \min(\alpha(x), \alpha(y))$.*

Definition 2.4. *Consider a non-empty set U. An object of bipolar fuzzy set A in U is of the form*

$$A = \{x, \beta^p(x), \beta^n(x), x \in U\},$$

where α^p:X→[0, 1] and α^n:X→[-1, 0].
The satisfaction level of an element x to the property corresponding to the bipolar fuzzy set A is shown by its positive membership degree, and the satisfaction level of an element x to an implicit counter property corresponding to a bipolar fuzzy set A is indicated by its negative membership degree.

Definition 2.5. *$G^* = (V, E)$ is a bipolar fuzzy graph(BPFG). A pair (A, B), where A is represented as (β^p_A, β^n_A) is a bipolar fuzzy set in V, and B is represented as (β^p_B, β^n_B) is a bipolar fuzzy set in V^2 such that $\beta^p_A(xy) \leq \min(\beta^p_A(x), \beta^p_A(y))$ and $\beta^n_A(xy) \geq \max(\beta^n_A(x), \beta^n_A(y))$ for all $xy \in V^2$.*

Basic notations of this manuscript are shown in Table 2.1.

Table 2.1. List of acronyms.

Acronym	Abbreviation
BPFs	Bipolar fuzzy set
BPFG	Bipolar fuzzy graph
S(G)	Seidel matrix
SL	Seidel Laplacian
L(G)	Laplacian matrix
LE	Laplacian Energy
SLE	Seidel Laplacian Energy
DMP	Decision Making Problem

Definition 2.6. *The energy of a graph is defined as a sum of all absolute values of latent values of a graph. The other names of eigenvalues are latent values, characteristic values, and proper values.*

Example 2.1. *Examine the example below. Suppose that G=(V,E) and that V={A,B,C} and E={AB,BC,AC}.*

The membership values of G=(X,Y), the BPFG of the graph of G in Figure 2.1, are recorded in Tables 2.2 and 2.3.

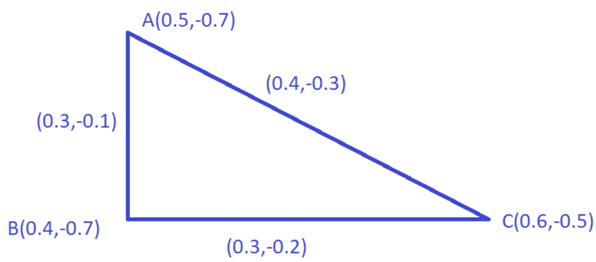


Figure 2.1. bipolar fuzzy graph.

Table 2.2. Representation of BPFG of X.

X	A	B	C
$\beta^p(x)$	0.5	0.4	0.6
$\beta^n(x)$	-0.7	-0.7	-0.5

Table 2.3. Representation of BPFG of Y.

Y	AB	BC	AC
$\beta^p(x)$	0.3	0.3	0.4
$\beta^n(x)$	-0.1	-0.2	-0.3

The adjacency matrix of a graph G is $A(G) = (a_{ij})$, where

$$A(G) = \begin{bmatrix} (0,0) & (0.3,-0.1) & (0.4,-0.3) \\ (0.3,-0.1) & (0,0) & (0.3,-0.2) \\ (0.4,-0.3) & (0.3,-0.2) & (0,0) \end{bmatrix}.$$

The positive membership values and negative membership values of an adjacency matrix $(A(\beta_A^p), A(\beta_A^n))$ or $(A_p(G), A_n(G))$ where

$$A_p(G) = \begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.3 & 0 & 0.3 \\ 0.4 & 0.3 & 0 \end{bmatrix},$$

and

$$A_n(G) = \begin{bmatrix} 0 & -0.1 & -0.3 \\ -0.1 & 0 & -0.2 \\ -0.3 & -0.2 & 0 \end{bmatrix}.$$

Its latent values are denoted as E_i and e_i.
The characteristic polynomial of $A(G)$ is $det(eI - A(G))$.
Its energy is

$$E(G) \quad = \quad (\sum_{i=1}^{n} |E_i|, \sum_{i=1}^{n} |e_i|).$$

For positive membership values, its characteristic equation is $-E^3 + (17/50)E + (9/125) = 0$ and its latent values are -0.4, -0.26904, and 0.66904. Its energy is 1.33808.
For negative membership values, its characteristic equation is $-e^3 + (7/50)e - (3/250) = 0$ and its latent values are -0.41131, 0.09112, and 0.32019. Its energy is 0.82262.

Definition 2.7. *Let G be a fuzzy graph, and its Laplacian matrix is defined as $L(G) = D(G) - A(G)$. The latent values of it are $b_1, b_2, ...b_n$. Consequently, a fuzzy graph G's Laplacian energy is defined as $LE(G) = \sum_{i=1}^{n} |b_i - \frac{2\sum(q_{ij})}{p}|$.*

Definition 2.8. *[18] A graph's Seidel matrix is its actual symmetric matrix $S(G) = (s_{ij})$, where*

$$(s_{ij}) = \begin{cases} 0, \text{ for nodes with } i = j, \\ 1, \text{ for non adjacent vertices}, \\ -1, \text{ for adjacent vertices.} \end{cases}$$

Definition 2.9. *[18] $DS(G) = diagonal(n-2d_1-1, n-2d_2-1, ...n-2d_n-1)$ where $d(G) = (d_1, d_2, ...d_n)$.*

Definition 2.10. *The Seidel Laplacian(SL) matrix of a fuzzy graph G is defined as SL(G)=DS(G)–S(G).*

Definition 2.11. *SL energy of a graph G is defined as follows: let $c_1, c_2, ...c_n$ be its latent values of SL(G) and $SLE(G) = \sum_{i=1}^{n} |c_i - \frac{(n(n-1)-4\sum(q_{ij})}{p}|$ where $1 \le i < j \le n$.*

3. Bipolar fuzzy graph's Laplacian energy

Definition 3.1. *[18] Laplacian matrix is defined as $l(G) = d(G) - A(G)$ for a bipolar fuzzy graph G.*

Let the latent values of this be $B_1, B_2, ...B_n$ and $b_1, b_2, ...b_n$ for positive membership and negative membership functions, respectively.
Thus, the LE of a BPFG is defined as

$$lE(G) = \sum_{i=1}^{n} |B_i - \frac{2\sum(q_{ij})}{p}|.$$

For positive membership function,

$$l_p(G) = d(G) - A_p(G) = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} - \begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.3 & 0 & 0.3 \\ 0.4 & 0.3 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3 & -0.4 \\ -0.3 & 0.6 & -0.3 \\ -0.4 & -0.3 & 0.7 \end{bmatrix}.$$

Its characteristic equation is $-B^3 + 2B^2 - (99/100)B = 0$ and its latent values are 0, 0.9, and 1.1. Its energy is

$$l_pE(G) = \sum_{i=1}^{3} |B_i - \frac{2*2}{3}| = |0 - 1.3333| + |0.4333 - 1.3333| + |0.6667 - 1.3333| = 1.3333 + 0.4333 + 0.6667 = 2.433.$$

For the negative membership function,

$$l_n(G) = d(G) - A_n(G) = \begin{bmatrix} -0.4 & 0 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & -0.5 \end{bmatrix} - \begin{bmatrix} 0 & -0.1 & -0.3 \\ -0.1 & 0 & -0.2 \\ -0.3 & -0.2 & 0 \end{bmatrix} = \begin{bmatrix} -0.4 & 0.1 & 0.3 \\ 0.1 & -0.3 & 0.2 \\ 0.3 & 0.2 & -0.5 \end{bmatrix}.$$

Its characteristic equation is $\frac{-100b^3-120b^2-33b}{100}$=0 and its latent values are 0, −0.77321, and −0.42679.
Its energy is

$$l_nE(G) = \sum_{i=1}^{3} |b_i - \frac{2*1.2}{3}| = |0+0.8| + |-0.77321+0.8| + |-0.42679+0.8| = 0.8+0.02679+0.37321 = 1.2.$$

4. Seidel Laplacian potential of a bipolar fuzzy network

Definition 4.1. *[18] The fuzzy graph G's Seidel Laplacian matrix is defined $Sl(G) = dS(G) - S(G)$.*

Definition 4.2. *[18] Define the SL energy of a bipolar graph G for its positive membership function by using $C_1, C_2, ...C_n$ as its latent values of Sl(G) is $SlE(G) = \sum_{i=1}^{n} |C_i - \frac{(n(n-1)-4\sum(q_{ij})}{p}|$ where $1 \le i < j \le n$.*

Remark 4.1. *For its negative membership function by using $c_1, c_2, ...c_n$ as its latent values.*

For positive membership function,

$$D_p(G) = \begin{bmatrix} 3-1-2*0.7 & 0 & 0 \\ 0 & 3-1-2*0.6 & 0 \\ 0 & 0 & 3-1-2*0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}.$$

Moreover, the Seidel matrix $S(G) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

$$S_pL(G) = D_pS(G) - S(G) = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 & 1 \\ 1 & 0.8 & 1 \\ 1 & 1 & 0.6 \end{bmatrix}.$$

Its characteristic equation is $-C^3 + 2C^2 + (42/25)C + (36/125) = 0$ and its latent values are -0.4, -0.26969, and 2.66969. Here, $\frac{n(n-1)-4\sum(q_{ij})}{p} = \frac{6-8}{3} = -0.6667$.
Its energy is

$$SL_pE(G) = |-0.4 + 0.6667| + |-0.26969 + 0.6667| + |2.66969 + 0.6667| = 0.2667 + 0.39701 + 3.33639 = 4.0001.$$

For the negative membership function,

$$D_n(G) = \begin{bmatrix} 3-1-2(-0.4) & 0 & 0 \\ 0 & 3-1-2(-0.3) & 0 \\ 0 & 0 & 3-1-2(-0.5) \end{bmatrix} = \begin{bmatrix} 2.8 & 0 & 0 \\ 0 & 2.6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$S_nL(G) = D_nS(G) - S(G) = \begin{bmatrix} 2.8 & 0 & 0 \\ 0 & 2.6 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2.8 & 1 & 1 \\ 1 & 2.6 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

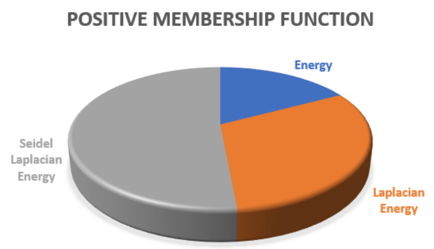
Its characteristic equation is $-c^3 + (42/5)c^2 - (512/25)c + (386/25) = 0$ and its latent values are 1.68018, 1.91095, and 4.80888. Here, $\frac{n(n-1)-4\sum(q_{ij})}{p} = \frac{6+4.8}{3} = 3.6$.
Its energy is

$$Sl_nE(G) = |1.68018-3.6| + |1.91095-3.6| + |4.80888-3.6| = 1.91982 + 1.68905 + 1.20888 = 4.81775.$$

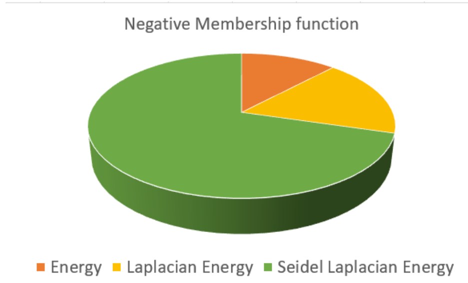
Various types of energy for Example 2.1 is analyzed and illustrated in Table 4.1 and Figures 4.1 and 4.2.

Table 4.1. Various energies of positive and negative membership function of Example 2.1.

Function	Energy	Laplacian Energy	Seidel Laplacian Energy
Membership	1.33808	2.433	4.0001
Non Membership	0.82262	1.2	4.81775

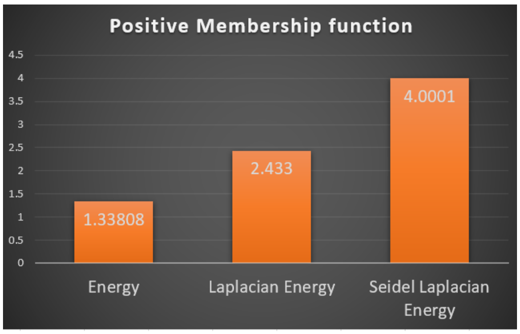


(**a**) positive membership function analysis

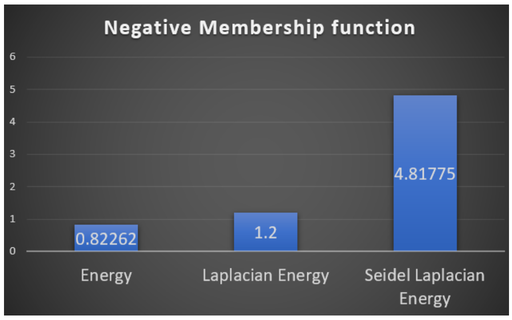


(**b**) negative membership function analysis

Figure 4.1. Various types of energy analysis using 3-D pie chart.



(**a**) positive membership function analysis



(**b**) negative membership function analysis

Figure 4.2. Various types of energy analysis using a bar chart.

Theorem 4.1. *Let G=(X,Y) be a BPFG and A(G) be its adjacency matrix, If* $E_1, E_2, ...E_n$ *and* $e_1, e_2, ...e_n$ *are the latent values of positive membership values of the adjacency matrix and negative membership values of adjacency matrix respectively. Then*
(i) $\sum_{i=1}^{n} E_i = 0$ *and* $\sum_{i=1}^{n} e_i = 0$,
(ii) $\sum_{i=1}^{p} E_i^2 = 2\sum_{i=1}^{n}(\beta_A^p)^2$ *and* $\sum_{i=1}^{p} e_i^2 = 2\sum_{i=1}^{n}(\beta_A^n)^2$.

Proof. (i) The sum of the main diagonal elements is zero because A(G) is the symmetric matrix for positive and negative case values. Therefore, the sum of the latent values is zero for both positive and negative cases.
i.e., $\sum_{i=1}^{n} E_i = 0$ and $\sum_{i=1}^{n} e_i = 0$.
(ii) Using matrix properties, the square of the sum of the main diagonal of the matrix = the sum of the square of the latent values. Additionally, the square of the sum of main diagonal elements
= square of the trace of the matrix
$= \beta_A^p(v_1v_2)^2 + ...\beta_A^p(v_1v_n)^2 + \beta_A^p(v_2v_1)^2 + ...+\beta_A^p(v_2v_n)^2 + ...\beta_A^p(v_nv_1)^2 + ...+\beta_A^p(v_nv_n)^2$
$= 2\sum_{i=1}^{n}(\beta_A^p(v_iv_j)^2$.
Therefore, $\sum_{i=1}^{p} E_i^2 = 2\sum_{i=1}^{n}(\beta_A^p)^2$ for the positive membership function and similarly for negative membership function $\sum_{i=1}^{p} e_i^2 = 2\sum_{i=1}^{n}(\beta_A^n)^2$. Checking the above theorem using Example 2.1,

i. $-0.4 - 0.26904 + 0.66904 = 0$, and $-0.41131 + 0.09112 + 0.32019 = 0$.

ii. $(-0.4) + (-0.26904^2) + (0.66904^2) = 2(0.3^2 + 0.4^2 + 0.3^2)$, and $(-0.41131^2) + (0.09112^2) + (0.32019^2) = 2((-0.1)^2 + (-0.3)^2 + (-0.3)^2)$.

That is 0.68 = 0.68 and 0.28 = 0.28. □

Theorem 4.2. *Let G=(X,Y) be a BPFG and A(G)=A(β_A^p, β_A^n) be the adjacency matrix of G. Then,*
(i) $\sqrt{\sum_{i=1}^{n}(\beta_A^p)^2 + n(n-1)det(A(\beta_A^p))} \leq (E(\beta_A^p) \leq \sqrt{n\sum_{i=1}^{n}(\beta_A^p)^2}$, *and*
(ii) $\sqrt{\sum_{i=1}^{n}(\beta_A^n)^2 + n(n-1)det(A(\beta_A^n))} \leq E(\beta_A^n) \leq \sqrt{n\sum_{i=1}^{n}(\beta_A^n)^2}$.

Proof. Using Cauchy inequality, refer to [13]

$$2\sum_{i,j}(E_iE_j) + \sum_{i=1}^{n}|E_i|^2 = \sum_{i=1}^{n}(|E_i|)^2.$$

Equating coefficient of E^{n-1} in the characteristic polynomial,

$$\sum_{i=1}^{n}(\beta_{A_i}^p)^2 = -\sum_{i,j}(E_iE_j).$$

Also

$$\sqrt{n}\sqrt{(\sum_{i=1}^{n}|E_i|^2)} \geq \sum_{i=1}^{n}(|E_i|)^2,$$

and

$$\sum_{i=1}^{n}(|E_i|)^2 \geq \sum_{i=1}^{n}|E_i| = E(\beta_A^p).$$

Therefore,

$$E(\beta_A^p) \leq \sqrt{n\sum_{i=1}^{n}(\beta_A^p)^2}.$$

Also

$$E(\beta_A^p)^2 = (\sum_{i=1}^{n}|E_i|)^2 = \sum_{i=1}^{n}|E_i|^2,$$

for $1 \leq i \leq j \leq n$, consider

$$|(E_1E_2)(E_2E_3).....(E_nE_{n-1})| = |E_1||E_2||E_3|......|E_n| = det(A(\beta_A^p)),$$

and $\sum_{i=1}^{n}(\beta_A^p)^2 + n(n-1)det(A(\beta_A^p)) \leq (E(\beta_A^p)^2$.
Therefore, $\sqrt{\sum_{i=1}^{n}(\beta_A^p)^2 + n(n-1)det(A(\beta_A^p))} \leq (E(\beta_A^p)$.
Hence, $\sqrt{\sum_{i=1}^{n}(\beta_A^p)^2 + n(n-1)det(A(\beta_A^p))} \leq (E(\beta_A^p) \leq \sqrt{n\sum_{i=1}^{n}(\beta_A^p)^2}$.
Similarly $\sqrt{\sum_{i=1}^{n}(\beta_A^n)^2 + n(n-1)det(A(\beta_A^n))} \leq E(\beta_A^n) \leq \sqrt{n\sum_{i=1}^{n}(\beta_A^n)^2}$.
Checking the above theorem using Example 2.1,

(i) $det(A(\beta_A^p)) = \begin{vmatrix} 0 & 0.3 & 0.4 \\ 0.3 & 0 & 0.3 \\ 0.4 & 0.3 & 0 \end{vmatrix} = 0.07.$

Also,

$$\sqrt{0.68+3(3-1)(0.07)} \leq 1.33808 \leq \sqrt{3(0.68)}\sqrt{0.68+0.42} \leq 1.33808 \leq \sqrt{2.04}1.0488 \leq 1.33808 \leq 1.4282,$$

and (ii) $det(A(\beta_A^n)) = \begin{vmatrix} 0 & -0.1 & -0.3 \\ -0.1 & 0 & -0.2 \\ -0.3 & -0.2 & 0 \end{vmatrix} = -0.01.$

Also

$$\sqrt{0.28+3(3-1)(0.01)} \leq 0.82262 \leq \sqrt{3(0.28)}\sqrt{0.28+0.06} \leq 0.82262 \leq \sqrt{0.84}0.5831 \leq 0.822628 \leq 0.9165.$$

□

Theorem 4.3. *Let G=(X,Y) be a BPFG. Then,*

i. $Sl_pE(G) \leq \frac{n(2+\sqrt{(n)})}{2}$,

ii. $Sl_nE(G) \leq \frac{n(2+\sqrt{(n)})}{2}$.

Proof. Using the result

$$E(B^p(v_iv_j) \leq \frac{2\sum_{i=1}^{n}(\beta_A^p(v_iv_j)^2}{n} + \sqrt{(n-1)(2\sum_{i=1}^{n}(\beta_A^p(v_iv_j)^2) - (\frac{2\sum_{i=1}^{n}(\beta_A^p(v_iv_j)^2}{n})^2},$$

refer [7].

By a result of calculus, it is proved that

$$g(z) = \sqrt{(2z-\frac{4z^2}{n^2})(n-1)} + \frac{2z}{n} = \frac{2z}{n} + \sqrt{2nz - \frac{2z}{n} - 2z + \frac{4z^2}{n^2}}.$$

It is maximum when $z = \frac{n(n+\sqrt{n})}{4}$. Therefore, $n + \frac{n\sqrt{(n)}}{2} \geq E(G)$ (refer to [7]).

Therefore, the following results are also valid.

i. $Sl_pE(G) \geq \frac{n(n+\sqrt{(n)})}{2}$,

ii. $Sl_nE(G) \geq \frac{n(n+\sqrt{(n)})}{2}$.

Checking the theorem using Example 2.1,

i. For positive membership function,

$$4.0001 \geq \frac{3(3+\sqrt{(3)})}{2},$$
$$4.0001 \geq 2.366.$$

ii. For negative membership function,

$$4.81775 \geq \frac{3(3+\sqrt{(3)}}{2},$$
$$4.81775 \geq 2.366.$$

□

Theorem 4.4. *$A(G)=A(\beta_A^p,\beta_A^n)$ is the adjacency matrix of G where G = (X,Y) be a BPFG. Then,*

i. $(Sl_pE(G))^2 + n^2 \leq (n + Sl_pE(G))^2$,

ii. $(Sl_nE(G))^2 + n^2 \leq (n + Sl_nE(G))^2$.

Proof. Consider the basic result, $x^2 + y^2 + 2xy = (x + y)^2$, which can be rephrased as $x^2 + y^2 \leq (x+y)^2$. Replacing $x = SlE(G)$ and $y = n$, then the following results are valid for positive and negative membership values.

i. $(Sl_pE(G))^2 + n^2 \leq (n + Sl_pE(G))^2$,

ii. $(Sl_nE(G))^2 + n^2 \leq (n + Sl_nE(G))^2$.

Checking the theorem using Example 2.1,

i.

$$(4.0001)^2 + 3^2 \leq (3 + 4.0001)^2,$$
$$16.0008 + 9 \leq (7.0001)^2,$$
$$25.0008 \leq 49.0014.$$

Similarly,

ii.

$$(4.81775)^2 + 3^2 \leq (3 + 4.81775)^2,$$
$$23.2107 + 9 \leq (7.81775)^2,$$
$$32.2107 \leq 61.11721.$$

□

5. SLE of a BPFG in DMP

A crucial tool for determining a workable solution in the modern world is the decision-making problem (DMP). It is used in numerous fields, and many scholars are working with DMP in fuzzy set theory. Differentiating wise choices and wise results is crucial. It is possible to predict the best results from wise decisions. The Method for Order Preference by Similarity to an Ideal Solution (TOPSIS) approach, developed by Hwang and Yoon in 1981, is a well-liked multicriteria decision-making technique [29]. In 1996 [30], Triantaphyllou et al. created a fuzzy version of TOPSIS. The extension of the TOPSIS approach for fuzzy data decision-making problems was conducted by Jahanshahloo et al., 2006 [28]. TOPSIS was then extended to fuzzy group decision-making issues by Chen in 2000 [27]. Implementing the proposed score functions in real-world decision-making entails defining the problem, identifying decision criteria, selecting or designing appropriate score functions, collecting data, evaluating alternatives, and making decisions based on the resulting scores. These steps are critical to ensure effective application and improved decision outcomes. Readers can refer to [31]. Furthermore, online meal delivery is made possible by mobile apps, which are like our faithful companions in a society where smartphones are the norm. Furthermore, digital wallets and online prepaid ways are dominating the payment landscape, making up an astounding 80 of the whole financial feast. Apps for food delivery are online marketplaces that link customers with nearby eateries so they can place online orders and have food delivered to their door. With the help of these applications, consumers can easily and quickly sample a wide range of cuisines without having to leave their homes. The app makes it easy for users to explore menus, place orders, and pay, which improves the eating experience. With just a few smartphone touches, users can order food thanks to restaurant delivery applications. Furthermore, it appears that the nation has adopted the online meal-ordering trend in the last few years.

As a result, the different meal delivery apps on this list have been rated based on three factors: Pricing, the average number of restaurants that offer food in a particular area, and delivery speed.
(i) Pricing: The competitiveness and transparency of fees, such as base menu prices, service fees, delivery fees, and any relevant promotions or discounts, are considered in this criterion.
(ii) Average Number of Restaurants in a Service Area: The quantity and variety of eateries the app lists in each area or city is used to measure selection diversity.
(iii) Delivery Speed: The app's advertised service-level commitments and the average order fulfillment times as reported by users are used to calculate the delivery speed, and faster delivery improves customer satisfaction and the quality of the meal when it arrives, especially for people who value timeliness and convenience.
Impact on robustness of the results:
(i) Uncertainty is represented more accurately and steadily when membership and non-membership values are carefully chosen.
(ii) Negligent parameter selection may result in bias or noise that spreads to other calculations, such as energy calculations and centrality assessments.
(iii) Robustness is enhanced by consistency and repeatability in the selection process, which prevents small changes in input from causing excessive fluctuations in the result.

Here, we analyzed 6 apps such as Zomato(I), Swiggy(II), Domino's(III), Dunzo(IV), McDonald's(V), and KFC(VI). Three specialists were invited to take part in the selection process to choose the necessary prepared coalition partner. These professionals were from the financial division, HR office, designing dieticians in New Delhi, and the Department of Health and Family Welfare, respectively. Depending on what they have included, the masters evaluated each pair of replacements and made their own judgments with six different BPFG as follows in Figures 5.1–5.3, and the energy is represented in Tables 5.1 and 5.2. Its corresponding membership values are also given:

$$A(I) = \begin{bmatrix} (0,0) & (0.1,-0.4) & (0.1,-0.4) \\ (0.2,-0.5) & (0,0) & (0.2,-0.5) \\ (0.2,-0.4) & (0.3,-0.5) & (0,0) \end{bmatrix},$$

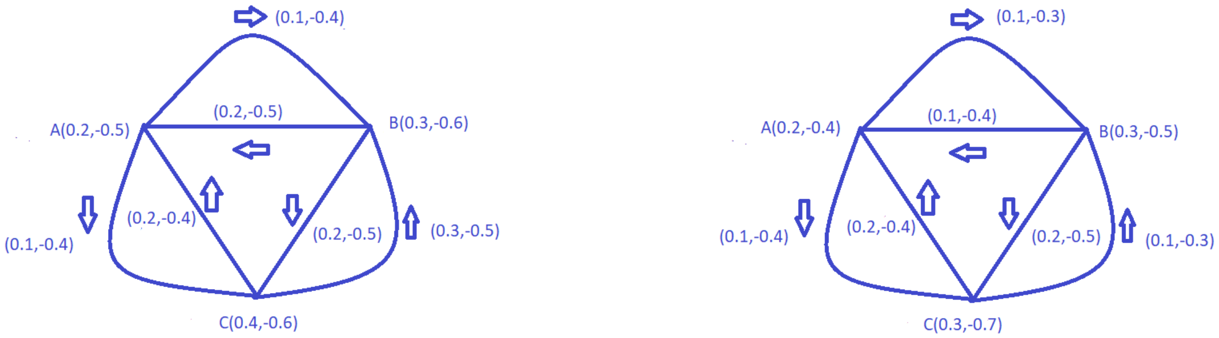
$$A(II) = \begin{bmatrix} (0,0) & (0.1,-0.3) & (0.1,-0.2) \\ (0.1,-0.4) & (0,0) & (0.2,-0.5) \\ (0.2,-0.4) & (0.1,-0.3) & (0,0) \end{bmatrix},$$

$$A(III) = \begin{bmatrix} (0,0) & (0.2,-0.4) & (0.2,-0.5) \\ (0.3,-0.5) & (0,0) & (0.2,-0.6) \\ (0.3,-0.4) & (0.3,-0.5) & (0,0) \end{bmatrix},$$

$$A(IV) = \begin{bmatrix} (0,0) & (0.1,-0.3) & (0.2,-0.4) \\ (0.1,-0.5) & (0,0) & (0.2,-0.5) \\ (0.1,-0.4) & (0.1,-0.4) & (0,0) \end{bmatrix},$$

$$A(V) = \begin{bmatrix} (0,0) & (0.2,-0.3) & (0.2,-0.7) \\ (0.3,-0.4) & (0,0) & (0.1,-0.4) \\ (0.2,-0.3) & (0.2,-0.5) & (0,0) \end{bmatrix},$$

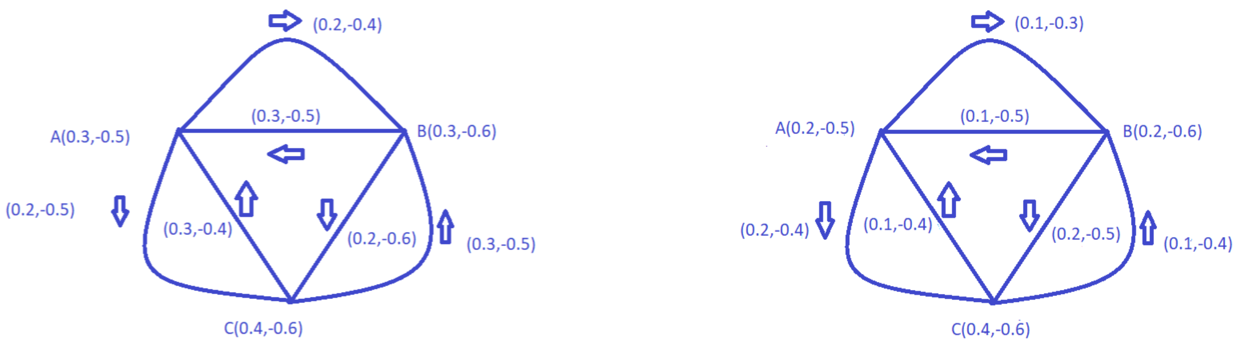
$$A(VI) = \begin{bmatrix} (0,0) & (0.2,-0.4) & (0.1,-0.3) \\ (0.2,-0.3) & (0,0) & (0.1,-0.2) \\ (0.1,-0.3) & (0.1,-0.3) & (0,0) \end{bmatrix}.$$



(a) BPF1

(b) BPF2

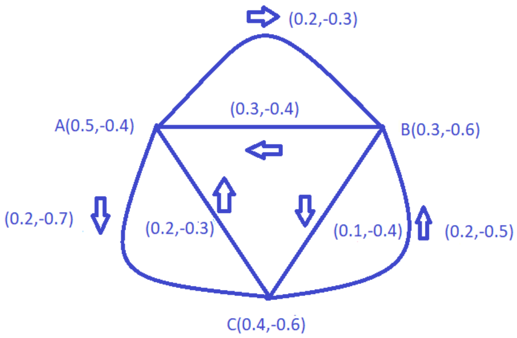
Figure 5.1. Representation of BPF1 & BPF2.



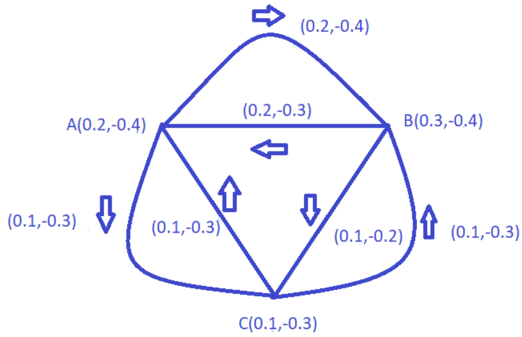
(a) BPF3

(b) BPF4

Figure 5.2. Representation of BPF3 & BPF4.



(a) BPF5



(b) BPF6

Figure 5.3. Representation of BPF5 & BPF6.

Table 5.1. Various energies of positive membership functions of BPFG.

Bipolar Fuzzy graph	BPFG1	BPFG2	BPFG3	BPFG4	BPFG5	BPFG6
Energy	0.71542	0.44112	0.98538	0.51232	1.05872	0.54642
Laplacian Energy	1.0999	0.7999	1.5	0.7999	1.2	0.7999
Seidel Laplacian Energy	3.30926	3.4727	3.23965	3.47268	3.2	3.47268

Table 5.2. Various Energies of negative membership functions of BPFG.

Bipolar Fuzzy graph	BPFG1	BPFG2	BPFG3	BPFG4	BPFG5	BPFG6
Energy	1.8	3.8639	1.92796	1.649	1.76359	1.19254
Laplacian Energy	2.7	2.1	2.8999	2.5004	2.6449	1.8
Seidel Laplacian Energy	5.81775	3.49589	5.95806	5.70926	5.75806	5.21775

Weights of a wide range of Seidel Laplacian energy are calculated using the formula below (similar to the Bayesian formula)

$$M_i = \left(\frac{SLE((G)_p)_i}{\sum_{i=1}^{n}((SLE)_p)_i}, \frac{SLE((G)_n)_i}{\sum_{i=1}^{n}((SLE)_n)_i}\right),$$

where i=1,2,3.

The comparison of the Weights of an SLE is shown in Table 5.3 and illustrated in Figure 5.4.

Table 5.3. Weights of SLE.

Membership function	BPFG1	BPFG2	BPFG3	BPFG4	BPFG5	BPFG6
Positive Case	0.16409	0.1722	0.16064	0.1722	0.15868	0.1722
Negative Case	0.18205	0.10939	0.18644	0.17866	0.18018	0.16328

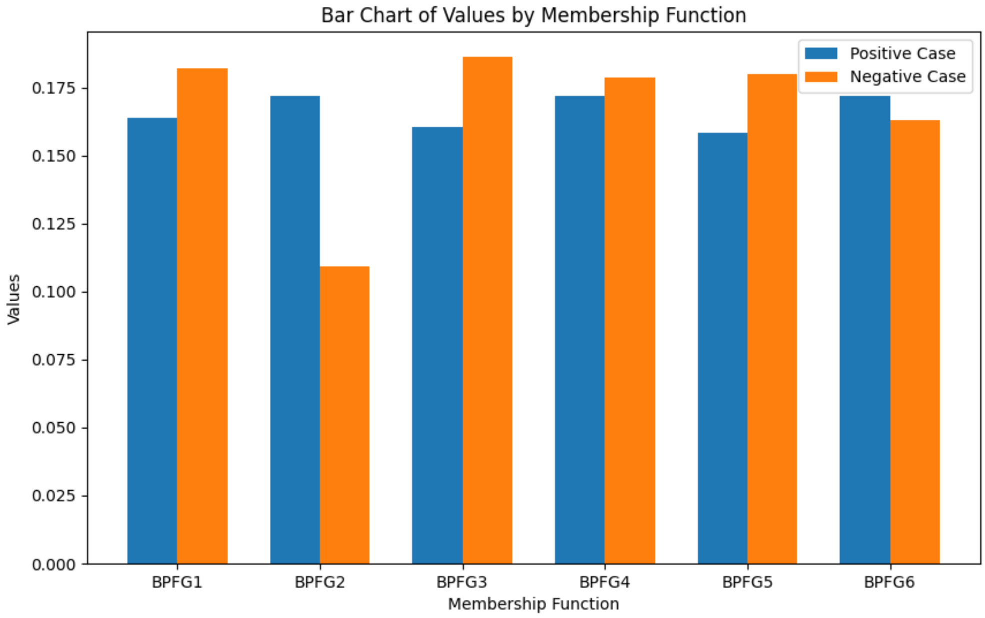


Figure 5.4. Comparision of membership function of positive and negative cases.

6. Score functions of BPFs

Definition 6.1. *A_i denotes bipolar fuzzy sets, and A_i has a value given by*

$$A_i = (\beta^p_{A_i}, \beta^n_{A_i}),$$

and its score function is defined as:

$$SC(A_i) = \beta^p_{A_i} + \beta^n_{A_i}.$$

Remark 6.1. *If A_1 and A_2 are two Bipolar fuzzy sets and $SC(A_1)$ and $SC(A_2)$ are the score functions of A_1 and A_2 respectively, then*

i. If $SC(A_1) < SC(A_2)$, then A_1 is smaller than A_2.

ii. If $SC(A_1) > SC(A_2)$, then A_1 is greater than A_2.

iii. If $SC(A_1) = SC(A_2)$, then it is not possible to decide either A_1 is greater than A_2 or A_2 is greater than A_1.

Definition 6.2. *The value of A_i for a Bipolar fuzzy set is provided by*

$$A_i = (\beta^p_{A_i}, \beta^n_{A_i}).$$

The improved score function is defined as:

$$IM(A_i) = (\beta^p_{A_i})^2 \cdot SC(A_i) + (\beta^n_{A_i})^2 \cdot SC(A_i) - (\beta^p_{A_i})(\beta^n_{A_i}) \cdot SC(A_i).$$

Definition 6.3. *Double Improved Score function is a real-valued function and is defined as $DI(A_i) = SC(A_i) + IM(A_i)$.*

Example 6.1. $A_1 = (0.5, -0.3)$, $A_2 = (0.2, -0.4)$, $B_1 = (0.8, -1)$, $B_2 = (0.5, -0.3)$, $C_1 = (0.8, -0.4)$, *and* $C_2 = (0.9, -0.5)$.

Its different types of score function are shown in Table 6.1.

Table 6.1. Comparision of score functions.

function	A_1	A_2	B_1	B_2	C_1	C_2
SC	0.2	−0.2	−0.2	0.4	0.4	0.4
IM	0.2	−0.024	−0.168	0.244	0.192	0.244
DI	0.4	−0.224	−0.368	0.644	0.592	0.644

Here, using all score function definitions,

i. $A_1 > A_2$,

ii. $B_2 > B_1$,

iii. $C_2 > C_1$, but it is not possible to conclude using the score function definition.

Now, using Weights of Seidel Laplacian energy, we find the Score function, Improved Score function, and Double Improved Score function.
Score function values are calculated below:

$$SC(A_i) = \beta^p(A_i) + \beta^n(A_i).$$

i. $SC(BPFG1) = 0.16409 + 0.18205 = 0.3461$,

ii. $SC(BPFG2) = 0.1722 + 0.10939 = 0.28159$,

iii. $SC(BPFG3) = 0, .16064 + 0.18644 = 0.34708$,

iv. $SC(BPFG4) = 0.1722 + 0.17866 = 0.35086$,

v. $SC(BPFG5) = 0.15868 + 0.18018 = 0.33886$,

vi. $SC(BPFG6) = 0.1722 + 0.16328 = 0.33548$.

Improved Score function values are calculated below:

i. $IM(BPFG1) = (0.16409)^2(0.34614) + (0.18209)^2(0.34614) – (0.16409)(0.18209)(0.34614) = 0.0932 + 0.011477 – 0.01034 = 0.010457$,

ii. $IM(BPFG2) = (0.1722)^2(0.28159) + (0.10939)^2(0.28159) – (0.1722)(0.28159)(0.10939) = 0.00835 + 0.00337 – 0.0053 = 0.00642$,

iii. $IM(BPFG3) = (0.16064)^2(0.34703) + (0.18644)^2(0.34708) - (0.16064)(0.18644)(0.34703) = 0.00897 + 0.01206 - 0.01039 = 0.01064$,

iv. $IM(BPFG4) = (0.1722)^2(0.35086) + (0.17866)^2(0.35086) – (0.1722)(0.17866)(0.35086) = 0.01040 + 0.011199 – 0.01079 = 0.010809$,

v. $IM(BPFG5) = (0.15868)^2(0.33886) + (0.18018)^2(0.33886) – (0.15868)(0.18018)(0.33886) = 0.00853 + 0.011001 – 0.00969 = 0.009841$,

vi. $IM(BPFG6) = (0.1722)^2(0.33548) + (0.16328)^2(0.33548) – (0.1722)(0.33548)(0.16328) = 0.00995 + 0.008944 – 0.00943 = 0.009464$.

Double Improved Score function values are calculated below:

i. $DI(BPFG1) = SC(BPFG1) + IM(BPFG1) = 0.34614 + 0.010457 = 0.356597$,

ii. $DI(BPFG2) = SC(BPFG2) + IM(BPFG2) = 0.28159 + 0.00642 = 0.28801$,

iii. $DI(BPFG3) = SC(BPFG3) + IM(BPFG3) = 0.34708 + 0.01064 = 0.35772$,

iv. $DI(BPFG4) = SC(BPFG4) + IM(BPFG4) = 0.35086 + 0.010809 = 0.361669$,

v. $DI(BPFG5) = SC(BPFG5) + IM(BPFG5) = 0.33886 + 0.009841 = 0.348701$,

vi. $DI(BPFG6) = SC(BPFG6) + IM(BPFG6) = 0.33548 + 0.009464 = 0.344944$,

score function values of BPFG are shown in Table 6.2 and illustrated in Figure 6.1. For score function values 2D, 3D, surface, and combo analysis.

Table 6.2. Score function values.

function	BPFG1	BPFG2	BPFG3	BPFG4	BPFG5	BPFG6
SC	0.34614	0.28159	0.34708	0.35086	0.33886	0.33548
IM	0.010457	0.00642	0.01064	0.010809	0.009841	0.009464
DI	0.356597	0.28801	0.35772	0.361669	0.348701	0.344944

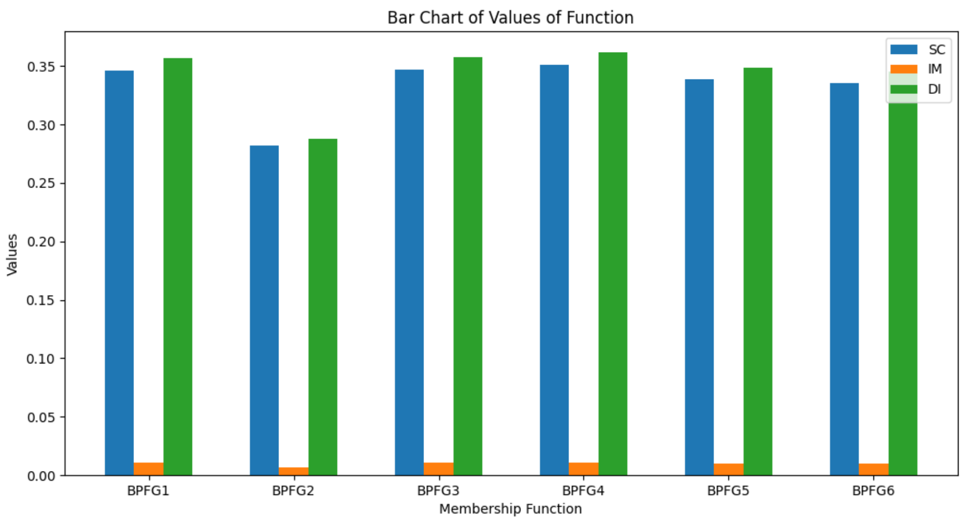


Figure 6.1. Comparision of Seidel Laplacian energy.

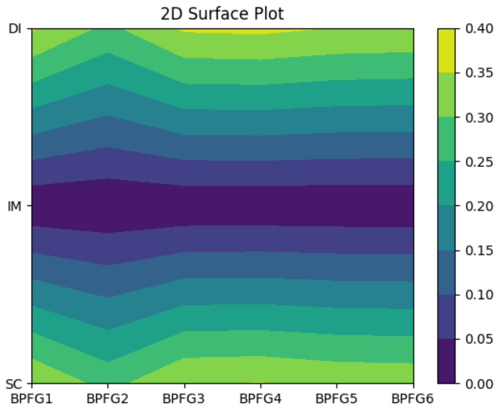
Best alternative is given by

i. $SC4 > SC3 > SC1 > SC4 > SC5 > SC2$,

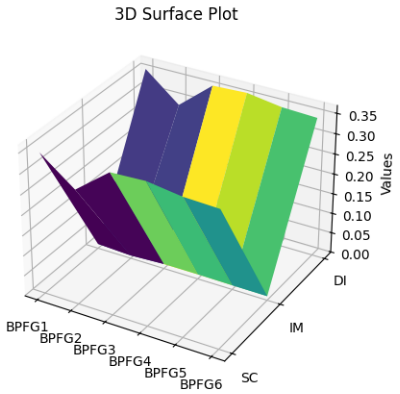
ii. $IM4 > IM3 > IM1 > IM4 > IM5 > IM2$,

iii. $DI4 > DI3 > DI1 > DI4 > DI5 > DI2$.

Thus, among the meal delivery apps, the best app is Dunzo (IV) in a particular area. It is illustrated in Figures 6.2 and 6.3.

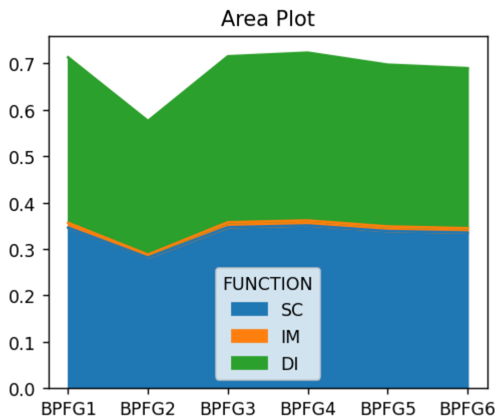


(a) 2D analysis

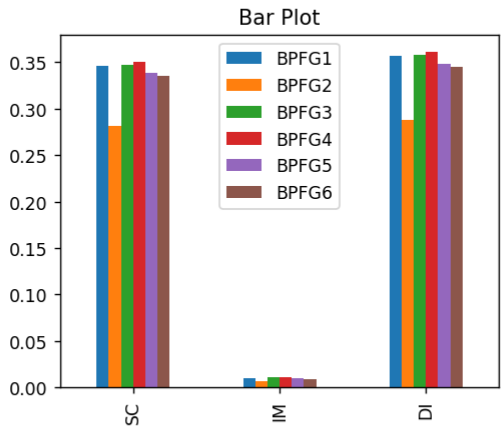


(b) 3D analysis

Figure 6.2. 2D and 3D analysis.



(a) Surface analysis



(b) Combo analysis

Figure 6.3. Surface and combo analysis.

7. Conclusions

In this study, we introduced and explored the concept of Seidel Laplacian energy in the context of bipolar fuzzy graphs. Theoretical analysis was conducted to establish upper and lower bounds for this energy measure, and are supported by relevant illustrative examples to validate the findings. Furthermore, to address complex multi-criteria decision-making (MCDM) problems involving uncertainty and bipolar information, we proposed novel score functions: The score function, improved score function, and double-improved score function. Overall, our findings offer new theoretical insights and computational tools to analyze bipolar fuzzy graphs, with potential applications in decision-making processes under uncertainty and imprecision. In future work, we intend to extend this study to explore the Seidel Laplacian energy of bipolar intuitionistic fuzzy graphs, Pythagorean fuzzy graphs, and Fermatean fuzzy graphs.

Author contributions

Sivaranjani Krishnaraj: Conceptualization, methodology, writing-original draft, formal analysis, investigation, software, visualization; Shanmuga Sundaram O. V.: Formal analysis, investigation, supervision, project administration; Prasantha Bharathi Dhandapani: Formal analysis, investigation, resources, data curation, writing-review & editing; Taha Radwan: Investigation, formal analysis, resources, project administration, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All the authors declare that they have no conflict of interest.

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Appendix: C programming code for the energy of a graph

The following C code helps readers calculate the energy of a graph.
For any $n * n$ matrix, the characteristic polynomial, latent (eigen) values, and the sum of the modulus of the latent values were calculated using the following C programming code. In this paper, the code was used to find the energy of a graph.

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>

double** create_matrix(int n);
double** allocate_matrix(int n);
void free_matrix(double** matrix, int n);
double determinant(double** matrix, int n);
void get_cofactor
(double** matrix, double** temp, int p, int q, int n);
double* characteristic_polynomial(double** matrix, int n);
double*solve_characteristic_polynomial
(double* coefficients, int n, int* num_solutions);
double sum_of_modulus(double* values, int n);
```

```
int main() {
    int n, i, j;

    printf("Enter the size of the square matrix (n): ");
    if (scanf("%d", &n) != 1 || n <= 0) {
        printf("Please enter a valid positive integer for the matrix size.\n");
        return 1;
    }

    double** matrix = create_matrix(n);
    if (!matrix) {
        printf("Memory allocation failed.\n");
        return 1;
    }

    double* char_poly = characteristic_polynomial(matrix, n);
    printf("Characteristic polynomial coefficients: ");
    for (i = 0; i <= n; ++i) {
        printf("%lf ", char_poly[i]);
 }
    printf("\n");

    int num_solutions;
    double* eigenvalues =
solve_characteristic_polynomial(char_poly, n, &num_solutions);
    printf("Eigenvalues: ");
    for (i = 0; i < num_solutions; ++i) {
        printf("%lf ", eigenvalues[i]);
    }
    printf("\n");

    double sum_modulus_eigenvalues = sum_of_modulus(eigenvalues, num_solutions);
    printf("Sum of Modulus of Eigenvalues: %lf\n", sum_modulus_eigenvalues);

    free_matrix(matrix, n);
    free(char_poly);
    free(eigenvalues);

    return 0;
}

double** create_matrix(int n) {
```

```
    double** matrix = allocate_matrix(n);
    if (!matrix) return NULL;

    for (int i = 0; i < n; ++i) {
 for (int j = 0; j < n; ++j) {
            printf("Enter element at position (%d, %d): ", i + 1, j + 1);
            if (scanf("%lf", &matrix[i][j]) != 1) {
                free_matrix(matrix, n);
                return NULL;
            }
        }
    }
    return matrix;
}

double** allocate_matrix(int n) {
    double** matrix = (double*)malloc(n * sizeof(double));
    if (!matrix) return NULL;
    for (int i = 0; i < n; ++i) {
        matrix[i] = (double*)malloc(n * sizeof(double));
        if (!matrix[i]) {
            free_matrix(matrix, i);
            return NULL;
        }
    }
    return matrix;
}

void free_matrix(double** matrix, int n) {
    for (int i = 0; i < n; ++i) {
      free(matrix[i]);
    }
    free(matrix);
}

double determinant(double** matrix, int n) {
    if (n == 1) return matrix[0][0];

    double det = 0;
    double** temp = allocate_matrix(n);
    if (!temp) return 0;

    int sign = 1;
```

```
    for (int f = 0; f < n; ++f) {
        get_cofactor(matrix, temp, 0, f, n);
        det += sign * matrix[0][f] * determinant(temp, n - 1);
        sign = -sign;
    }

    free_matrix(temp, n);
    return det;
}

void get_cofactor
(double** matrix, double** temp, int p, int q, int n) {
    int i = 0, j = 0;
    for (int row = 0; row < n; ++row) {
        for (int col = 0; col < n; ++col) {
          if (row != p && col != q) {
                temp[i][j++] = matrix[row][col];
                if (j == n - 1) {
                    j = 0;
                    i++;
                }
            }
        }
    }
}

double* characteristic_polynomial(double** matrix, int n) {
    double* coefficients = (double*)malloc((n + 1) * sizeof(double));
    if (!coefficients) return NULL;

    for (int i = 0; i <= n; ++i) {
        coefficients[i] = pow(-1, i) * determinant(matrix, i);
    }
    return coefficients;
}
```

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