AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 16837–16864.
DOI: 10.3934/math.2025757
Received: 31 May 2025
Revised: 04 July 2025
Accepted: 17 July 2025
Published: 28 July 2025

Research article

η-Ricci-Bourguignon solitons in an almost pseudo-W_8 flat and M-projective flat symmetric Lorentzian Kähler space-time manifold

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Abstract: In this paper, we investigated η-Ricci-Bourguignon solitons within the framework of almost pseudo-W_8 flat and M-projective flat symmetric Lorentzian Kähler space-time manifolds that satisfy the Einstein field equation with and without a cosmological constant. We established necessary and sufficient conditions under which such solitons exhibit expanding, shrinking, or steady behavior. Specifically, we derived constraints on the parameters that govern the soliton dynamics. Furthermore, we extended the results to the case of an η-Ricci-Bourguignon soliton for dark fluid, dust fluid, stiff matter, and radiational fluid. Our results contribute to the broader understanding of geometric flows and their interaction with the structure of Lorentzian Kähler geometry using partial differential equations.

Keywords: Lorentzian Kähler space-time manifolds; solitons; η-Ricci-Bourguignon solitons; pseudo-W_8 curvature tensor; partial differential equations; M-projective curvature tensor
Mathematics Subject Classification: 35C08, 53C50, 53C55

1. Introduction

General relativity, introduced by Albert Einstein in 1915, is a theory of gravity that describes the gravitational field as the curvature of spacetime, with the energy-momentum tensor serving as its source. This framework forms the foundation of all field theories. In standard cosmological models, the universe's matter content is typically treated as a perfect fluid. The study of Lorentzian manifolds is driven by the idea that gravitational fields can be effectively represented by a Lorentzian metric defined on a suitable four-dimensional manifold.

Einstein's equations are essential in constructing cosmological models, as they establish that matter determines the curvature of spacetime, while the motion of matter, in turn, is dictated by the non-flat metric of spacetime. Relativistic fluid models play a crucial role in various fields, including astrophysics, plasma physics, and nuclear physics.

A connected, time-oriented, 4-dimensional Lorentzian manifold is a specialized type of pseudo-Riemannian manifold equipped with a Lorentzian metric g of signature $(-, +, +, +)$ [14]. This structure plays a crucial role in general relativity. The study of the geometry of such manifolds begins with analyzing the nature of vectors within them. Consequently, 4-dimensional Lorentzian manifolds provide an ideal framework for exploring the principles of general relativity.

In differential geometry, a geometric flow is a type of partial differential equation that governs the evolution of the Riemannian metric on a manifold. The study of geometric evolution equations has significantly influenced various fields such as mathematical physics, global geometry, and material sciences. Numerous geometric flows play important roles in different contexts. These flows are typically expressed as parabolic partial differential equations involving curvature. Among them, the Ricci flow is particularly notable and was first introduced by Hamilton in 1982 [13]. In recent years, various geometric flows related to Ricci flows have been studied by many mathematicians and provide important tools for investigating geometry and topology of manifolds.

The phenomenon of the soliton was first observed by the Scottish civil engineer John Scott Russell, who described it as the "great wave of translation". A soliton refers to a solitary wave that retains its shape while traveling at a constant speed. In mathematical terms, solitons are self-similar solutions to nonlinear partial differential equations. In the case of the Ricci flow, such a self-similar solution is referred to as a Ricci soliton.

The Ricci-Bourguignon flow was originally introduced by Jean-Pierre Bourguignon in 1981 [3]. This concept was established with the help of Lichnerowicz and Aubin's unpublished work [1]. According to [12], the Ricci-Bourguignon flow can be described as follows:

A one-parameter family of Riemannian metrics $g(t)$ on an $n-$dimensional manifold (M^n, g) is said to evolve under the Ricci-Bourguignon flow if it satisfies the differential equation:

$$\frac{\partial g}{\partial t} = -2(S - \Lambda r g), \tag{1.1}$$

where S denotes the Ricci curvature tensor, r is the scalar curvature associated with the metric g, and $\Lambda \in \mathbb{R}$ is a constant parameter.

From the above-mentioned definition, it is clear that the equation reduces to the standard Ricci flow when $\Lambda = 0$. As stated in [4], the evolution equation of the partial differential system (1.1) corresponds to different geometric tensors depending on the value of Λ. Specifically, it represents the Einstein tensor when $\Lambda = \frac{1}{2}$, the traceless Ricci tensor for $\Lambda = \frac{1}{n}$, the Schouten tensor when $\Lambda = \frac{1}{2(n-1)}$, and the Ricci tensor in the case $\Lambda = 0$. Recently, Shubham Dwivedi [12] proposed the following definition of a Ricci-Bourguignon soliton:

A Riemannian manifold (M^n, g) is called a Ricci-Bourguignon soliton if it satisfies the equation:

$$S + \frac{1}{2}\mathcal{L}_\xi g = (a + \Lambda r)g, \tag{1.2}$$

where a is a real constant, r is the scalar curvature, and $\mathcal{L}_\xi g$ denotes the Lie derivative of the metric g along a vector field ξ.

By appending a multiple of the $(0, 2)$-tensor field $\eta \otimes \eta$ to the Ricci-Bourguignon soliton equation (1.2), one obtains the more flexible notion of an η-Ricci-Bourguignon [20], given by

$$S + \frac{1}{2}\mathcal{L}_\xi g = (a + \Lambda r)g + b\eta \otimes \eta, \tag{1.3}$$

where b is a constant and, η represents a 1-form satisfying $\eta(\xi) = df(\xi)$ for every vector field on the manifold. If $\Lambda = 0$, then the $\eta-$Ricci-Bourguignon soliton reduces to an $\eta-$Ricci soliton and if $b = 0$, then the $\eta-$Ricci-Bourguignon soliton is called a Ricci-Bourguignon soliton. The Ricci-Bourguignon soliton is expanding if $a > 0$, shrinking if $a < 0$, or stead if $a = 0$. Several recent studies have contributed to the understanding of Ricci–Bourguignon solitons and their generalizations. In 2022, Soylu [22] examined Ricci-Bourguignon solitons and almost solitons in the context of concurrent vector fields. In 2023, Cunha et al. [9] investigated the properties of such solitons, establishing results on their triviality, uniqueness, and scalar curvature bounds. Additionally, in 2023, Dogru studied η-Ricci–Bourguignon solitons in the framework of semi-symmetric metrics and non-metric connections, offering valuable contributions to this field [11]. In recent works, M. Traore and co-authors have made significant contributions to the study of Ricci-Bourguignon solitons. In [25], they investigated the properties of almost η-Ricci-Bourguignon solitons, offering new insights into their geometric structures. In [26], the authors extended their study to sequential warped product manifolds, highlighting new results for η-Ricci-Bourguignon solitons in a warped product setting. Furthermore, the authors explored gradient almost η-Ricci-Bourguignon solitons in [27], providing new characterizations and conditions under which such solitons arise. Also, Chaturvedi et al. [5–7] have extensively explored various soliton structures in Bochner flat and pseudo-conformally flat Lorentzian Kähler space-time manifolds.

2. Preliminaries

An n-dimensional semi-Riemannian manifold (M^n, g) where n is even, is classified as a Lorentzian Kähler manifold if it fulfills the subsequent criteria [16]:

$$F^2(v_1) = -v_1, \quad g(Fv_1, Fv_2) = g(v_1, v_2), and \ \ (\nabla_{v_1} F)v_2 = 0, \tag{2.1}$$

where v_1 and v_2 are vector fields, and F is a $(1,1)$-type tensor field that satisfies $F(v_1) = v_1$. In a Lorentzian Kähler manifold, certain relations hold:

$$R(v_1, v_2, v_3, v_4) = R(v_1, v_2, Fv_3, v_4) = R(Fv_1, Fv_2, v_3, v_4), \tag{2.2}$$

$$S(v_1, v_2) = S(Fv_1, Fv_2), \ \ S(Fv_1, v_2) = -S(v_1, Fv_2), \ \ and \ \ g(Fv_1, v_2) = -g(v_1, Fv_2). \tag{2.3}$$

Let us consider an orthonormal frame field $\{e_i\}_{1\leq i\leq 4}$ as defined in [2], satisfying the metric conditions: $g(e_i, e_j) = \epsilon_{ij}\delta_{ij}$, $i, j \in \{1,2,3,4\}$, where $\epsilon_{11} = -1$, $\epsilon_{ii} = -1$, for $i \in \{2,3,4\}$, and $\epsilon_{ij} = 0$, for $i, j \in \{1,2,3,4\}$, $i \neq j$.

Now, if ξ is expressed as $\xi = \sum_{i=1}^{n} \xi^i e_i$, then we can write it as

$$-1 = g(\xi,\xi) = \sum_{1\leq i,j\leq 4} \xi^i \xi^j g(e_i, e_j) = \sum_{i=1}^{4} \epsilon_{ii}\{\xi^i\}^2 \tag{2.4}$$

and

$$\eta(e_i) = g(e_i, \xi) = \sum_{j=1}^{4} \xi^j g(e_i, e_j) = \epsilon_{ii}\xi^i. \tag{2.5}$$

The energy-momentum tensor plays a crucial role in describing the matter content of spacetime. The characteristics of a perfect fluid spacetime are determined by various conditions imposed on the energy-momentum tensor in the analysis of perfect fluid spacetimes [24]. In standard cosmological models, matter is treated as a fluid with properties such as density, pressure, velocity, acceleration, vorticity, shear, and expansion. To simplify, this matter is modeled as a perfect fluid, which is entirely described by its rest mass density and isotropic pressure. It is free from shear, stresses, viscosity, and heat conduction, and its energy-momentum tensor is expressed in the form given in [15]:

$$T(v_1, v_2) = pg(v_1, v_2) + (\sigma + p)\eta(v_1)\eta(v_2), \tag{2.6}$$

where σ represents the energy density while p denotes the isotropic pressure. The 1-form, $\eta(v_1) = g(v_1, \xi)$, satisfies the conditions $\eta(\xi) = -1$ and $g(\xi, \xi) = -1$. If $p = 0$, the perfect fluid spacetime will be a dust matter fluid [23]. The perfect fluid represents radiation fluid if $\sigma = 3p$ [21], and if $p = -\sigma$, then a perfect fluid is known as dark energy era [23]. If $p = \sigma$, then a perfect fluid is referred to as stiff matter [8].

The Einstein field equation with a cosmological constant for a perfect fluid spacetime is stated as [15]

$$KT(v_1, v_2) = S(v_1, v_2) + (\lambda - \frac{r}{2})g(v_1, v_2), \tag{2.7}$$

where S is the Ricci tensor, λ is a cosmological constant, K is the gravitational constant such that $K \neq 0$, and r is the scalar curvature of g.

Using Eqs (2.6) and (2.7), we get

$$S(v_1, v_2) = -\Big(\lambda - \frac{r}{2} - Kp\Big)g(v_1, v_2) + K(\sigma + p)\eta(v_1)\eta(v_2). \tag{2.8}$$

Contracting Eq (2.8) and using $g(\xi, \xi) = -1$, we get

$$r = 4\lambda + K(\sigma - 3p). \tag{2.9}$$

The Einstein field equation for a perfect fluid spacetime, without the cosmological constant, is given by:

$$KT(v_1, v_2) = S(v_1, v_2) - \frac{r}{2}g(v_1, v_2), \tag{2.10}$$

and using Eq (2.6) in Eq (2.10), we get

$$S(v_1, v_2) = \Big(\frac{r}{2} + Kp\Big)g(v_1, v_2) + K(\sigma + p)\eta(v_1)\eta(v_2). \tag{2.11}$$

Contracting Eq (2.11) and using $g(v_1, v_2) = -1$, we get

$$r = K(\sigma - 3p). \tag{2.12}$$

A non-flat Riemannian manifold of dimensional $n(> 3)$ is referred to as an almost pseudo-symmetric manifold [10] if its curvature tensor R fulfills the following requirement:

$$\begin{aligned}(\nabla_{v_5}R)(v_1, v_2, v_3, v_4) = {} & [A(v_5) + B(v_5)]R(v_1, v_2, v_3, v_4) + A(v_1)R(v_5, v_2, v_3, v_4) \\ & + A(v_2)R(v_1, v_5, v_3, v_4) + A(v_3)R(v_1, v_2, v_5, v_4) + A(v_4)R(v_1, v_2, v_3, v_5),\end{aligned} \tag{2.13}$$

where A, B are nonzero 1-forms defined by $g(v_5, \pi) = A(v_5), B(v_5) = g(v_5, \Pi)$, where $v_1, v_2, v_3, v_4 \in \chi(M)$. Here π and Π are referred to as associated vector fields corresponding to the $1-$forms A and B, respectively.

3. Almost pseudo-symmetric Lorentzian Kähler space-time manifold

In this part, we examine the concept of an η-Ricci-Bourguignon soliton within the context of an almost pseudo-symmetric spacetime and various curvature tensors, all in a 4-dimensional Lorentzian Kähler spacetime manifold (M^4, g). By taking the Lie derivative of the metric g with respect to the vector field ξ in Eq (1.3), we obtain:

$$S(v_1, v_2) = (a + \Lambda r)g(v_1, v_2) + b\eta(v_1)\eta(v_2) - \frac{1}{2}[g(\nabla_{v_1}\xi, v_2) + g(v_1, \nabla_{v_2}\xi)]. \tag{3.1}$$

The following definitions will be essential for establishing the main findings in this part.

Pseudo-W_8 curvature tensor. In 1982, Pokhariyal and Mishra [18] introduced the W_8 curvature tensor. Later, in 2018, Pandey et al. [19] extended this idea by defining the pseudo-W_8 curvature tensor on a Riemannian manifold. The pseudo-W_8 curvature tensor is given by:

$$\begin{aligned} W_8(v_1, v_2, v_3, v_4) = {} & \alpha R(v_1, v_2, v_3, v_4) + \beta[S(v_1, v_2)g(v_3, v_4) - S(v_2, v_3)g(v_1, v_4)] \\ & - \frac{r}{n}\Big[\frac{\alpha}{n-1} - \beta\Big][g(v_1, v_2)g(v_3, v_4) - g(v_2, v_3)g(v_1, v_4)]. \end{aligned} \tag{3.2}$$

M-Projective curvature tensor. The M-projective curvature tensor was introduced by Pokhariyal and Mishra in 1970 [17]. It is defined in the following way:

$$\begin{aligned} M(v_1, v_2, v_3, v_4) = {} & R(v_1, v_2, v_3, v_4) - \frac{1}{2(n-1)}[S(v_2, v_3)g(v_1, v_4) - S(v_1, v_3)g(v_2, v_4) \\ & + g(v_2, v_3)S(v_1, v_4) - g(v_1, v_3)S(v_2, v_4)]. \end{aligned} \tag{3.3}$$

Theorem 3.1. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation with a cosmological constant, a Ricci-Bourguignon soliton is:*

(1) Expanding if either

(a) $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$ and $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,
(b) or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$ and $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.

(2) Shrinking if either

(a) $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$ and $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,
(b) or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$ and $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.

(3) Steady if either

(a) $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$,
(b) or $\Lambda = \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.

Proof. By applying the covariant derivative to Eq (2.2), we obtain

$$(\nabla_{v_5}R)(v_1, v_2, v_3, v_4) = (\nabla_{v_5}R)(Fv_1, Fv_2, v_3, v_4), \tag{3.4}$$

and using Eq (2.13) in Eq (3.4), we get

$$\begin{aligned} & A(v_1)R(v_5, v_2, v_3, v_4) + A(v_2)R(v_1, v_5, v_3, v_4) \\ = {} & A(Fv_1)R(v_5, Fv_2, v_3, v_4) + A(Fv_2)R(Fv_1, v_5, v_3, v_4). \end{aligned} \tag{3.5}$$

Now, if we consider the manifold to be pseudo-W_8 flat, i.e., $W_8(v_1, v_2, v_3, v_4) = 0$, then from Eq (3.2), we get

$$\begin{aligned} R(v_1, v_2, v_3, v_4) =& \frac{r}{4}\Big[\frac{1}{3} - \frac{\beta}{\alpha}\Big][g(v_1, v_2)g(v_3, v_4) - g(v_2, v_3)g(v_1, v_4)] \\ &- \frac{\beta}{\alpha}[S(v_1, v_2)g(v_3, v_4) - S(v_2, v_3)g(v_1, v_4)], \end{aligned} \tag{3.6}$$

and using Eqs (3.5) and (3.6), we get

$$\begin{aligned} &A(v_1)\left\{\frac{r}{4}\Big(\frac{1}{3} - \frac{\beta}{\alpha}\Big)[g(v_5, v_2)g(v_3, v_4) - g(v_2, v_3)g(v_5, v_4)] - \frac{\beta}{\alpha}[S(v_5, v_2)g(v_3, v_4) - S(v_2, v_3)g(v_5, v_4)]\right\} \\ &+A(v_2)\left\{\frac{r}{4}\Big(\frac{1}{3} - \frac{\beta}{\alpha}\Big)[g(v_1, v_5)g(v_3, v_4) - g(v_5, v_3)g(v_1, v_4)] - \frac{\beta}{\alpha}[S(v_1, v_5)g(v_3, v_4) - S(v_5, v_3)g(v_1, v_4)]\right\} \\ &=A(Fv_1)\left\{\frac{r}{4}\Big(\frac{1}{3} - \frac{\beta}{\alpha}\Big)[g(v_5, Fv_2)g(v_3, v_4) - g(Fv_2, v_3)g(v_5, v_4)] - \frac{\beta}{\alpha}[S(v_5, Fv_2)g(v_3, v_4) - S(Fv_2, v_3)g(v_5, v_4)]\right\} \\ &+A(Fv_2)\left\{\frac{r}{4}\Big(\frac{1}{3} - \frac{\beta}{\alpha}\Big)[g(Fv_1, v_5)g(v_3, v_4) - g(v_5, v_3)g(Fv_1, v_4)] - \frac{\beta}{\alpha}[S(Fv_1, v_5)g(v_3, v_4) - S(v_5, v_3)g(Fv_1, v_4)]\right\}. \end{aligned} \tag{3.7}$$

Putting $v_1 = \xi = e_i$ in the above equation, we have

$$\begin{aligned} &\frac{r}{4}\Big(\frac{1}{3} - \frac{\beta}{\alpha}\Big)[g(v_5, v_2)g(v_3, v_4) \\ &-g(v_2, v_3)g(v_5, v_4)] - \frac{4\beta}{\alpha}[S(v_5, v_2)g(v_3, v_4) - S(v_2, v_3)g(v_5, v_4)] \\ &+\frac{r}{2}\Big(\frac{1}{3} - \frac{\beta}{\alpha}\Big)[g(v_2, v_5)g(v_3, v_4) - g(v_5, v_3)g(v_2, v_4)] \\ &-\frac{2\beta}{\alpha}[S(v_2, v_5)g(v_3, v_4) - S(v_5, v_3)g(v_2, v_4)] = 0. \end{aligned} \tag{3.8}$$

Putting $v_4 = v_5 = e_i$ in the above equation, we get

$$S(v_2, v_3) = \frac{r}{4}\Big[\frac{\alpha}{3\beta} - 1\Big]g(v_2, v_3). \tag{3.9}$$

Now, from Eqs (3.1) and (3.9), we get

$$\frac{r}{4}\Big[\frac{\alpha}{3\beta} - 1\Big]g(v_2, v_3) = (a + \Lambda r)g(v_2, v_3) + b\eta(v_2)\eta(v_3) - \frac{1}{2}[g(\nabla_{v_2}\xi, v_3) + g(v_2, \nabla_{v_3}\xi)]. \tag{3.10}$$

Multiplying the above equation by ϵ_{ii}, taking $v_2, v_3 = e_i$, and using Eqs (2.4) and (2.5), we get

$$4a - b = r\Big[\frac{\alpha}{3\beta} - 1 - 4\Lambda\Big] + div\xi. \tag{3.11}$$

Now, putting $v_2 = v_3 = \xi$ in Eq (3.10), we get

$$a - b = r\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.12}$$

Therefore, from Eqs (3.11) and (3.12), we get

$$a = r\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} + \frac{div\xi}{3}. \tag{3.13}$$

We know that for a Ricci-Bourguignon soliton, $b = 0$, and therefore the above equation reduces to

$$a = r\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.14}$$

Using Eq (2.9), we get

$$a = [4\lambda + K(\sigma - 3p)]\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.15}$$

Therefore, the Ricci-Bourguignon soliton in this manifold will be expanding if

$$[4\lambda + K(\sigma - 3p)]\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} > 0,$$

shrinking if

$$[4\lambda + K(\sigma - 3p)]\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} < 0,$$

or steady if

$$[4\lambda + K(\sigma - 3p)]\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} = 0.$$

Hence the theorem has been proved. □

Example 3.2. *To illustrate the theorem, we present explicit examples of Ricci-Bourguignon solitons on a* 4*-dimensional almost pseudo-*W_8 *flat symmetric Lorentzian Kähler space-time manifold* (M^4, g) *that admits Einstein's field equation with a cosmological constant. In each case, we choose values of the constants to satisfy the conditions for expanding, shrinking, and steady solitons, respectively.*

(1) Expanding case:
Choose:

$$\alpha = 6, \quad \beta = 1, \quad \lambda = 1, \quad K = 1, \quad \sigma = 5, \quad p = 2, \quad \Lambda = 0.$$

Then:

$$\frac{\alpha}{3\beta} - 1 = 1, \quad \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4} = 0.25,$$

$$p = 2 < 3 = \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}, \quad \Lambda = 0 < 0.25.$$

The scalar curvature:

$$r = 4 + 5 - 6 = 3.$$

Ricci tensor:

$$S(\nu_2, \nu_3) = \frac{r}{4}\Big(\frac{\alpha}{3\beta} - 1\Big) = \frac{3}{4} > 0.$$

Since $a > 0$*, the soliton is expanding.*

(2) *Shrinking case:*
Choose:

$$\alpha = 6, \quad \beta = 1, \quad \lambda = 1, \quad K = 1, \quad \sigma = 5, \quad p = 4, \quad \Lambda = 0.$$

$$p = 4 > 3 = \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}, \quad \Lambda = 0 < 0.25.$$

The scalar curvature:

$$r = 4 + 5 - 12 = -3.$$

Ricci tensor:

$$S(v_2, v_3) = \frac{r}{4} \cdot 1 = -0.75 < 0.$$

Since $a < 0$, the soliton is shrinking.

(3) *Steady case:*
Choose:

$$\alpha = 6, \quad \beta = 1, \quad \lambda = 1, \quad K = 1, \quad \sigma = 5, \quad p = 3, \quad \Lambda = 0.$$

$$p = 3 = \frac{4}{3} + \frac{5}{3} = 3.$$

The scalar curvature:

$$r = 4 + 5 - 9 = 0.$$

Ricci tensor:

$$S(v_2, v_3) = 0.$$

Since $a = 0$, the soliton is steady.

These examples confirm that by choosing suitable constants, the manifold can indeed admit expanding, shrinking, or steady Ricci-Bourguignon solitons, thus illustrating the theorem.

Corollary 3.3. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits an Einstein field equation with a cosmological constant, a Ricci-Bourguignon soliton for dark fluid is:*

(1) *Expanding if either*
  (a) $p < \frac{\lambda}{K}$ *and* $\Lambda < \frac{\alpha}{3\beta} - 1$,
  (b) *or* $p > \frac{\lambda}{K}$ *and* $\Lambda > \frac{\alpha}{3\beta} - 1$.

(2) *Shrinking if either*
  (a) $p > \frac{\lambda}{K}$ *and* $\Lambda < \frac{\alpha}{3\beta} - 1$,
  (b) *or* $p < \frac{\lambda}{K}$ *and* $\Lambda > \frac{\alpha}{3\beta} - 1$.

(3) *Steady if either*
  (a) $p = \frac{\lambda}{K}$,
  (b) *or* $\Lambda = \frac{\alpha}{3\beta} - 1$.

Proof. The energy-momentum tensor associated with a dark fluid, as described in [23], is expressed as

$$T(v_2, v_3) = pg(v_2, v_3). \tag{3.16}$$

From Eqs (2.7) and (3.16), we get

$$S(v_2, v_3) = \Big(Kp - \lambda + \frac{r}{2}\Big)g(v_2, v_3), \tag{3.17}$$

and now, putting $v_2 = v_3 = e_i$, we get

$$r = 4(\lambda - Kp). \tag{3.18}$$

Substituting the value of r from the above equation into Eq (3.14), we get

$$a = (\lambda - Kp)\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big]. \tag{3.19}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dark fluid will be expanding if

$$(\lambda - Kp)\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big] > 0,$$

shrinking if

$$(\lambda - Kp)\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big] < 0,$$

or steady if

$$(\lambda - Kp)\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big] = 0.$$

Hence the corollary has been proved. □

Corollary 3.4. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits the Einstein field equation with a cosmological constant, a Ricci-Bourguignon soliton for dust fluid is:*

(1) Expanding if either
- *(a)* $\sigma > \frac{-4\lambda}{K}$ *and* $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,
- *(b) or* $\sigma < \frac{-4\lambda}{K}$ *and* $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.

(2) Shrinking if either
- *(a)* $\sigma < \frac{-4\lambda}{K}$ *and* $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,
- *(b) or* $\sigma > \frac{-4\lambda}{K}$ *and* $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.

(3) Steady if either
- *(a)* $\sigma = \frac{-4\lambda}{K}$,
- *(b) or* $\Lambda = \frac{\alpha}{12\beta} - \frac{1}{4}$.

Proof. The energy-momentum tensor for a dust fluid, as described in [23], is expressed as follows:

$$T(v_2, v_3) = \sigma\eta(v_2)\eta(v_3). \tag{3.20}$$

From Eqs (2.7) and (3.20), we get

$$S(v_2, v_3) = K\sigma\eta(v_2)\eta(v_3) - \left(\lambda - \frac{r}{2}\right)g(v_2, v_3). \tag{3.21}$$

Now, putting $v_2 = v_3 = e_i$, we get

$$r = 4\lambda + K\sigma. \tag{3.22}$$

Substituting the value of r from the above equation into Eq (3.14), we get

$$a = (4\lambda + K\sigma)\left\{\frac{1}{4}\left[\frac{\alpha}{3\beta} - 1\right] - \Lambda\right\}. \tag{3.23}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dust fluid will be expanding if

$$(4\lambda + K\sigma)\left\{\frac{1}{4}\left[\frac{\alpha}{3\beta} - 1\right] - \Lambda\right\} > 0,$$

shrinking if

$$(4\lambda + K\sigma)\left\{\frac{1}{4}\left[\frac{\alpha}{3\beta} - 1\right] - \Lambda\right\} < 0,$$

or steady if

$$(4\lambda + K\sigma)\left\{\frac{1}{4}\left[\frac{\alpha}{3\beta} - 1\right] - \Lambda\right\} = 0.$$

Hence the corollary has been proved. □

Corollary 3.5. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits the Einstein field equation with a cosmological constant, a Ricci-Bourguignon soliton for stiff fluid is:*

(1) Expanding if either
  - *(a) $p < \frac{2\lambda}{K}$ and $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,*
  - *(b) or $p > \frac{2\lambda}{K}$ and $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.*

(2) Shrinking if either
  - *(a) $p > \frac{2\lambda}{K}$ and $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,*
  - *(b) or $p < \frac{2\lambda}{K}$ and $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.*

(3) Steady if either
  - *(a) $p = \frac{2\lambda}{K}$,*
  - *(b) or $\Lambda = \frac{\alpha}{12\beta} - \frac{1}{4}$.*

Proof. The energy-momentum tensor corresponding to stiff matter, as outlined in [8], is given by the expression:

$$T(v_2, v_3) = pg(v_2, v_3) + 2p\eta(v_2)\eta(v_3). \tag{3.24}$$

From Eqs (2.7) and (3.24), we get

$$Kpg(v_2, v_3) + 2Kp\eta(v_2)\eta(v_3) = S(v_2, v_3) + \left(\lambda - \frac{r}{2}\right)g(v_2, v_3). \tag{3.25}$$

Now, putting $v_2 = v_3 = e_i$, we get

$$r = 2(2\lambda - Kp). \tag{3.26}$$

Substituting the value of r from the above equation into Eq (3.14), we get

$$a = 2(2\lambda - Kp)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.27}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dust fluid will be expanding if

$$(4\lambda - 2Kp)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} > 0,$$

shrinking if

$$(4\lambda - 2Kp)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} < 0,$$

or steady if

$$(4\lambda - 2Kp)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} = 0.$$

Hence the corollary has been proved. □

Corollary 3.6. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits the Einstein field equation with a cosmological constant, a Ricci-Bourguignon soliton for radiational fluid is:*

(1) Expanding if either
  - *(a) $\lambda > 0$ and $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,*
  - *(b) or $\lambda < 0$ and $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.*

(2) Shrinking if either
  - *(a) $\lambda > 0$ and $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,*
  - *(b) or $\lambda < 0$ and $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.*

(3) Steady if either
  - *(a) $\lambda = 0$,*
  - *(b) or $\Lambda = \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.*

Proof. The energy-momentum tensor for a radiation fluid, as presented in [21], is defined as follows:

$$T(v_2, v_3) = pg(v_2, v_3) + 4p\eta(v_2)\eta(v_3). \tag{3.28}$$

From Eqs (2.7) and (3.28), we get

$$Kpg(v_2, v_3) + 4Kp\eta(v_2)\eta(v_3) = S(v_2, v_3) + \Big(\lambda - \frac{r}{2}\Big)g(v_2, v_3). \tag{3.29}$$

Now, putting $v_2 = v_3 = e_i$, we get

$$r = 4\lambda. \tag{3.30}$$

Using the value of r obtained in the above equation in Eq (3.14), we get

$$a = 4\lambda\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.31}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dust fluid will be expanding if

$$4\lambda\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} > 0,$$

shrinking if

$$4\lambda\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} < 0,$$

or steady if

$$4\lambda\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} = 0.$$

Hence the corollary has been proved. □

Theorem 3.7. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton is:*

(1) Expanding if either

- *(a) $p < \frac{\sigma}{3}$ and $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,*
- *(b) or $p > \frac{\sigma}{3}$ and $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.*

(2) Shrinking if either

- *(a) $p > \frac{\sigma}{3}$ and $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,*
- *(b) or $p < \frac{\sigma}{3}$ and $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.*

(3) Steady if either

- *(a) $p = \frac{\sigma}{3}$,*
- *(b) or $\Lambda = \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.*

Proof. Substituting Eq (2.12) in Eq (3.14), we get

$$a = K(\sigma - 3p)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.32}$$

Therefore, the Ricci-Bourguignon soliton in this manifold will be expanding if

$$K(\sigma - 3p)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} > 0,$$

shrinking if

$$K(\sigma - 3p)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} < 0,$$

or steady if

$$K(\sigma - 3p)\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} = 0.$$

Hence the theorem has been proved. □

Example 3.8. *To illustrate the theorem, we present explicit examples of Ricci-Bourguignon solitons on a* 4*-dimensional almost pseudo-*W_8 *flat symmetric Lorentzian Kähler space-time manifold* (M^4, g) *that admits Einstein's field equation without a cosmological constant. In each case, we choose values of the constants to satisfy the conditions for expanding, shrinking, and steady solitons, respectively.*

(1) *Expanding case:*
Choose:

$$\alpha = 6, \quad \beta = 1, \quad K = 1, \quad \sigma = 6, \quad p = 1, \quad \Lambda = 0.$$

Then:

$$\frac{\alpha}{3\beta} - 1 = 1, \quad \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4} = 0.25.$$

$$p = 1 < 2 = \frac{\sigma}{3}, \quad \Lambda = 0 < 0.25.$$

The scalar curvature:

$$r = K(\sigma - 3p) = 1 \cdot (6 - 3) = 3.$$

Ricci tensor:

$$S(v_2, v_3) = \frac{r}{4}\left(\frac{\alpha}{3\beta} - 1\right) = \frac{3}{4} > 0.$$

Since $a > 0$*, the soliton is expanding.*

(2) *Shrinking case:*
Choose:

$$\alpha = 6, \quad \beta = 1, \quad K = 1, \quad \sigma = 6, \quad p = 3, \quad \Lambda = 0.$$

$$p = 3 > 2 = \frac{\sigma}{3}, \quad \Lambda = 0 < 0.25.$$

The scalar curvature:

$$r = 1 \cdot (6 - 9) = -3.$$

Ricci tensor:

$$S(v_2, v_3) = \frac{r}{4} \cdot 1 = -0.75 < 0.$$

Since $a < 0$*, the soliton is shrinking.*

(3) *Steady case:*
Choose:

$$\alpha = 6, \quad \beta = 1, \quad K = 1, \quad \sigma = 6, \quad p = 2, \quad \Lambda = 0.$$

$$p = 2 = \frac{\sigma}{3} = 2.$$

The scalar curvature:

$$r = 1 \cdot (6 - 6) = 0.$$

Ricci tensor:

$$S(v_2, v_3) = 0.$$

Since $a = 0$*, the soliton is steady.*

These examples confirm that by choosing suitable constants, the manifold can indeed admit expanding, shrinking, or steady Ricci-Bourguignon solitons, thus illustrating the theorem.

Corollary 3.9. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton for dark fluid is:*

(1) Expanding if either
   (a) $p < 0$ *and* $\Lambda < \frac{\alpha}{3\beta} - 1$,
   (b) or $p > 0$ *and* $\Lambda > \frac{\alpha}{3\beta} - 1$.

(2) Shrinking if either
   (a) $p > 0$ *and* $\Lambda < \frac{\alpha}{3\beta} - 1$,
   (b) or $p < 0$ *and* $\Lambda > \frac{\alpha}{3\beta} - 1$.

(3) Steady if either
   (a) $p = 0$,
   (b) or $\Lambda = \frac{\alpha}{3\beta} - 1$.

Proof. From Eqs (2.10) and (3.16), we get

$$S(v_2, v_3) = (Kp + \frac{r}{2})g(v_2, v_3). \tag{3.33}$$

Now, by putting $v_2 = v_3 = e_i$, we get

$$r = -4Kp. \tag{3.34}$$

Using the value of r obtained in the above equation in Eq (3.14), we get

$$a = -Kp\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big]. \tag{3.35}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dark fluid will be expanding if

$$-Kp\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big] > 0,$$

shrinking if $-Kp\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big] < 0$, or steady if $-Kp\Big[\frac{\alpha}{3\beta} - 1 - \Lambda\Big] = 0$.

Hence the corollary has been proved. □

Corollary 3.10. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton for dust fluid is:*

(1) Expanding if either
   (a) $\sigma > 0$ *and* $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,
   (b) or $\sigma < 0$ *and* $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.

(2) Shrinking if either
   (a) $\sigma < 0$ *and* $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,
   (b) or $\sigma > 0$ *and* $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.

(3) Steady if either

(a) $\sigma = 0$,

(b) or $\Lambda = \frac{\alpha}{12\beta} - \frac{1}{4}$.

Proof. From Eqs (2.10) and (3.20), we get

$$S(\nu_2, \nu_3) = K\sigma\eta(\nu_2)\eta(\nu_3) + \frac{r}{2}g(\nu_2, \nu_3). \tag{3.36}$$

Now, putting $\nu_2 = \nu_3 = e_i$, we get

$$r = K\sigma. \tag{3.37}$$

Using the value of r obtained in the above equation in Eq (3.14), we get

$$a = K\sigma\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.38}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dust fluid will be expanding if $K\sigma\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} > 0$, shrinking if $K\sigma\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} < 0$, or steady if $K\sigma\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\} = 0$.

Hence the corollary has been proved. □

Corollary 3.11. *In an almost pseudo-W_8 flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton for stiff fluid is:*

(1) Expanding if either

(a) $p > 0$ *and* $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,

(b) or $p < 0$ *and* $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.

(2) Shrinking if either

(a) $p < 0$ *and* $\Lambda < \frac{\alpha}{12\beta} - \frac{1}{4}$,

(b) or $p > 0$ *and* $\Lambda > \frac{\alpha}{12\beta} - \frac{1}{4}$.

(3) Steady if either

(a) $p = 0$,

(b) or $\Lambda = \frac{\alpha}{12\beta} - \frac{1}{4}$.

Proof. From Eqs (2.10) and (3.24), we get

$$Kpg(\nu_2, \nu_3) + 2Kp\eta(\nu_2)\eta(\nu_3) = S(\nu_2, \nu_3) - \frac{r}{2}g(\nu_2, \nu_3). \tag{3.39}$$

Now, putting $\nu_2 = \nu_3 = e_i$, we get

$$r = -2Kp. \tag{3.40}$$

Using the value of r obtained in the above equation in Eq (3.14), we get

$$a = -2Kp\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta} - 1\Big] - \Lambda\right\}. \tag{3.41}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for stiff matter will be expanding if

$$-2Kp\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta}-1\Big]-\Lambda\right\}>0,$$

shrinking if

$$-2Kp\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta}-1\Big]-\Lambda\right\}<0,$$

or steady if

$$-2Kp\left\{\frac{1}{4}\Big[\frac{\alpha}{3\beta}-1\Big]-\Lambda\right\}=0.$$

Hence the corollary has been proved. □

Theorem 3.12. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation with a cosmological constant, a Ricci-Bourguignon soliton is:*

(1) Expanding if either

(a) $p > \frac{4}{3}\frac{\lambda}{K}+\frac{\sigma}{3}$ *and* $\Lambda > \frac{-1}{2}$,
(b) or $p < \frac{4}{3}\frac{\lambda}{K}+\frac{\sigma}{3}$ *and* $\Lambda < \frac{-1}{2}$.

(2) Shrinking if either

(a) $p > \frac{4}{3}\frac{\lambda}{K}+\frac{\sigma}{3}$ *and* $\Lambda < \frac{-1}{2}$,
(b) or $p < \frac{4}{3}\frac{\lambda}{K}+\frac{\sigma}{3}$ *and* $\Lambda > \frac{-1}{2}$.

(3) Steady if either

(a) $p = \frac{4}{3}\frac{\lambda}{K}+\frac{\sigma}{3}$,
(b) or $\Lambda = \frac{-1}{2}$.

Proof. Let us consider the manifold to be M-projective flat, i.e., $M(v_1, v_2, v_3, v_4) = 0$, and then from Eq (3.3), we get

$$\begin{aligned}R(v_1,v_2,v_3,v_4) = &-\frac{1}{6}[S(v_2,v_3)g(v_1,v_4)-S(v_1,v_3)g(v_2,v_4)\\&+g(v_2,v_3)S(v_1,v_4)-g(v_1,v_3)S(v_2,v_4)],\end{aligned}\tag{3.42}$$

and using Eqs (3.5) and (3.42), we get

$$\begin{aligned}&A(v_1)\left\{\frac{1}{6}[S(v_2,v_3)g(v_5,v_4)-S(v_5,v_3)g(v_2,v_4)+g(v_2,v_3)S(v_5,v_4)-g(v_5,v_3)S(v_2,v_4)]\right\}\\&+A(v_2)\left\{\frac{1}{6}[S(v_5,v_3)g(v_1,v_4)-S(v_1,v_3)g(v_5,v_4)+g(v_5,v_3)S(v_1,v_4)-g(v_1,v_3)S(v_5,v_4)]\right\}\\&=A(Fv_1)\left\{\frac{1}{6}[S(Fv_2,v_3)g(v_5,v_4)-S(v_5,v_3)g(Fv_2,v_4)+g(Fv_2,v_3)S(v_5,v_4)-g(v_5,v_3)S(Fv_2,v_4)]\right\}\\&+A(Fv_2)\left\{\frac{1}{6}[S(v_5,v_3)g(Fv_1,v_4)-S(Fv_1,v_3)g(v_5,v_4)+g(v_5,v_3)S(Fv_1,v_4)-g(Fv_1,v_3)S(v_5,v_4)]\right\}.\end{aligned}\tag{3.43}$$

Putting $v_1 = \xi = e_i$ in the above equation, we have

$$\frac{1}{3}[S(v_2, v_3)g(v_5, v_4) - S(v_5, v_3)g(v_2, v_4) + S(v_5, v_4)g(v_2, v_3) - S(v_2, v_4)g(v_5, v_3)] = 0. \tag{3.44}$$

Putting $v_4 = v_5 = e_i$, we get

$$S(v_2, v_3) = \frac{-r}{2} g(v_2, v_3). \tag{3.45}$$

From Eqs (3.1) and (3.45), we get

$$\frac{-r}{2} g(v_2, v_3) = (a + \Lambda r)g(v_2, v_3) + b\eta(v_2)\eta(v_3) - \frac{1}{2}[g(\nabla_{v_2}\xi, v_3) + g(v_2, \nabla_{v_3}\xi)]. \tag{3.46}$$

Multiplying the above equation by ϵ_{ii}, taking $v_2, v_3 = e_i$, and using Eqs (2.4) and (2.5), we get

$$4a - b = -2r[2\Lambda + 1] + div\xi. \tag{3.47}$$

Now, putting $v_2 = v_3 = \xi$ in Eq (3.46), we get

$$a - b = -r\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.48}$$

Therefore, from Eqs (3.47) and (3.48), we get

$$a = -r\Big[\frac{1}{2} + \Lambda\Big] + \frac{div\xi}{3}. \tag{3.49}$$

We know that for a Ricci-Bourguignon soliton, $b = 0$, and therefore the above equation reduces to

$$a = -r\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.50}$$

Using Eq (2.9), we get

$$a = -[4\lambda + K(\sigma - 3p)]\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.51}$$

Therefore, the Ricci-Bourguignon soliton in this manifold will be expanding if

$$-[4\lambda + K(\sigma - 3p)]\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if

$$-[4\lambda + K(\sigma - 3p)]\Big[\frac{1}{2} + \Lambda\Big] < 0,$$

or steady if

$$-[4\lambda + K(\sigma - 3p)]\Big[\frac{1}{2} + \Lambda\Big] = 0.$$

Hence the theorem has been proved. □

Example 3.13. *To illustrate the theorem, let us construct explicit examples of Ricci-Bourguignon solitons on a* 4*-dimensional almost M-projective flat symmetric Lorentzian Kähler space-time manifold* (M^4, g) *admitting Einstein's field equation with a cosmological constant. We choose values of the constants to satisfy the conditions for expanding, shrinking, and steady solitons, respectively.*

(1) *Expanding case: Choose:*

$$\lambda = 1, \quad K = 1, \quad \sigma = 2, \quad p = 3, \quad \Lambda = 0.$$

Then:

$$\frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3} = \frac{4}{3} + \frac{2}{3} = 2.$$

$$p = 3 > 2, \quad \Lambda = 0 > -\frac{1}{2}.$$

The scalar curvature:

$$r = 4\lambda + K(\sigma - 3p) = 4 + 1(2 - 9) = 4 - 7 = -3.$$

Coefficient a:

$$a = -r\Big(\frac{1}{2} + \Lambda\Big) = -(-3) \cdot \frac{1}{2} = 1.5 > 0.$$

Since $a > 0$, the soliton is expanding.

(2) *Shrinking case:*
Choose:

$$\lambda = 1, \quad K = 1, \quad \sigma = 2, \quad p = 3, \quad \Lambda = -1.$$

$$p = 3 > 2, \quad \Lambda = -1 < -\frac{1}{2}.$$

The scalar curvature:

$$r = 4 + 1(2 - 9) = -3.$$

Coefficient a:

$$a = -r\Big(\frac{1}{2} + \Lambda\Big) = -(-3) \cdot (-0.5) = -1.5 < 0.$$

Since $a < 0$, the soliton is shrinking.

(3) *Steady case:*
Choose:

$$\lambda = 1, \quad K = 1, \quad \sigma = 2, \quad p = 2, \quad \Lambda = 0.$$

$$p = 2 = \frac{4}{3} + \frac{2}{3} = 2.$$

The scalar curvature:

$$r = 4 + 1(2 - 6) = 4 - 4 = 0.$$

Coefficient a:

$$a = -r\Big(\frac{1}{2} + \Lambda\Big) = 0.$$

Since $a = 0$, the soliton is steady.

These choices show that by adjusting p and Λ appropriately relative to the threshold $\frac{4}{3}\frac{\lambda}{K}+\frac{\sigma}{3}$ and $-\frac{1}{2}$, the manifold can indeed admit expanding, shrinking, or steady Ricci-Bourguignon solitons, thus illustrating the theorem.

Corollary 3.14. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation with a cosmological constant, a Ricci-Bourguignon soliton for dark fluid is:*

(1) Expanding if either

(a) $p > \frac{\lambda}{K}$ *and* $\Lambda > -\frac{1}{2}$,
(b) or $p < \frac{\lambda}{K}$ *and* $\Lambda < -\frac{1}{2}$.

(2) Shrinking if either

(a) $p > \frac{\lambda}{K}$ *and* $\Lambda < -\frac{1}{2}$,
(b) or $p < \frac{\lambda}{K}$ *and* $\Lambda > -\frac{1}{2}$.

(3) Steady if either

(a) $p = \frac{\lambda}{K}$,
(b) or $\Lambda > -\frac{1}{2}$.

Proof. Using Eq (3.18) in Eq (3.50), we get

$$a = -4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.52}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dark fluid will be expanding if

$$-4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if

$$-4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] < 0,$$

or steady if

$$-4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] = 0.$$

Hence the corollary has been proved. □

Corollary 3.15. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation with a cosmological constant, a Ricci-Bourguignon soliton for dust fluid is:*

(1) Expanding if either

(a) $\sigma < \frac{-4\lambda}{K}$ *and* $\Lambda > -\frac{1}{2}$,
(b) or $\sigma > \frac{-4\lambda}{K}$ *and* $\Lambda < -\frac{1}{2}$.

(2) Shrinking if either

(a) $\sigma < \frac{-4\lambda}{K}$ *and* $\Lambda < -\frac{1}{2}$,
(b) or $\sigma > \frac{-4\lambda}{K}$ *and* $\Lambda > -\frac{1}{2}$.

(3) Steady if either

(a) $\sigma = \frac{-4\lambda}{K}$,

(b) *or* $\Lambda = -\frac{1}{2}$.

Proof. Using Eq (3.22) in Eq (3.50), we get

$$a = -(4\lambda + K\sigma)\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.53}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dust fluid will be expanding if

$$-(4\lambda + K\sigma)\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if

$$-(4\lambda + K\sigma)\Big[\frac{1}{2} + \Lambda\Big] < 0,$$

or steady if

$$-(4\lambda + K\sigma)\Big[\frac{1}{2} + \Lambda\Big] = 0.$$

Hence the corollary has been proved. □

Corollary 3.16. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation with a cosmological constant, a Ricci-Bourguignon soliton for stiff fluid is:*

(1) *Expanding if either*

 (a) $p > \frac{2\lambda}{K}$ *and* $\Lambda > -\frac{1}{2}$,
 (b) *or* $p < \frac{2\lambda}{K}$ *and* $\Lambda < -\frac{1}{2}$.

(2) *Shrinking if either*

 (a) $p > \frac{2\lambda}{K}$ *and* $\Lambda < -\frac{1}{2}$,
 (b) *or* $p < \frac{2\lambda}{K}$ *and* $\Lambda > -\frac{1}{2}$.

(3) *Steady if either*

 (a) $p = \frac{2\lambda}{K}$,
 (b) *or* $\Lambda = -\frac{1}{2}$.

Proof. Using Eq (3.26) in Eq (3.50), we get

$$a = -2(2\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.54}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for stiff matter will be expanding if

$$-2(2\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if

$$-2(2\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] < 0,$$

or steady if

$$-2(2\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] = 0.$$

Hence the corollary has been proved. □

Corollary 3.17. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation with a cosmological constant, a Ricci-Bourguignon soliton for radiational fluid is:*

(1) Expanding if either

(a) $\lambda > 0$ *and* $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,
(b) or $\lambda < 0$ *and* $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.

(2) Shrinking if either

(a) $\lambda > 0$ *and* $\Lambda > \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$,
(b) or $\lambda < 0$ *and* $\Lambda < \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.

(3) Steady if either

(a) $\lambda = 0$,
(b) or $\Lambda = \frac{1}{12}\frac{\alpha}{\beta} - \frac{1}{4}$.

Proof. Using Eq (3.30) in Eq (3.50), we get

$$a = -4\lambda\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.55}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for radiational fluid will be expanding if

$$-4\lambda\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if

$$-4\lambda\Big[\frac{1}{2} + \Lambda\Big] < 0,$$

or steady if

$$-2(2\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] = 0.$$

Hence the corollary has been proved. □

Theorem 3.18. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton is:*

(1) Expanding if either

(a) $p > \frac{\sigma}{3}$ *and* $\Lambda > -\frac{1}{2}$,
(b) or $p < \frac{\sigma}{3}$ *and* $\Lambda < -\frac{1}{2}$.

(2) Shrinking if either

(a) $p > \frac{\sigma}{3}$ *and* $\Lambda < -\frac{1}{2}$,
(b) or $p < \frac{\sigma}{3}$ *and* $\Lambda > -\frac{1}{2}$.

(3) Steady if either

(a) $p = \frac{\sigma}{3}$,
(b) or $\Lambda = -\frac{1}{2}$.

Proof. Substituting Eq (3.34) in Eq (3.50), we get

$$a = -4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.56}$$

Therefore, the Ricci-Bourguignon soliton in this manifold will be expanding if

$$-4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if

$$-4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] < 0,$$

or steady if

$$-4(\lambda - Kp)\Big[\frac{1}{2} + \Lambda\Big] = 0.$$

Hence the theorem has been proved. □

Example 3.19. *To illustrate the theorem, we construct explicit examples of Ricci-Bourguignon solitons on a* 4*-dimensional almost* M*-projective flat symmetric Lorentzian Kähler space-time manifold* (M^4, g) *admitting Einstein's field equation without a cosmological constant. We choose values of the constants to satisfy the conditions for expanding, shrinking, and steady solitons, respectively.*

(1) Expanding case: Choose:

$$\lambda = 1, \quad K = 1, \quad \sigma = 3, \quad p = 2, \quad \Lambda = 0.$$

Then:

$$\frac{\sigma}{3} = \frac{3}{3} = 1.$$

$$p = 2 > 1, \quad \Lambda = 0 > -\frac{1}{2}.$$

The scalar curvature:

$$r = 4\lambda + K(\sigma - 3p) = 4 + 1(3 - 6) = 4 - 3 = 1.$$

Coefficient a:

$$a = -4(\lambda - Kp)\Big(\frac{1}{2} + \Lambda\Big) = -4(1 - 2) \cdot 0.5 = -4(-1) \cdot 0.5 = 2 > 0.$$

Since $a > 0$*, the soliton is expanding.*

(2) Shrinking case:
Choose:

$$\lambda = 1, \quad K = 1, \quad \sigma = 3, \quad p = 2, \quad \Lambda = -1.$$

$$p = 2 > 1, \quad \Lambda = -1 < -\frac{1}{2}.$$

The scalar curvature:

$$r = 4 + 1(3 - 6) = 1.$$

Coefficient a:

$$a = -4(\lambda - Kp)\Big(\frac{1}{2} + \Lambda\Big) = -4(-1) \cdot (-0.5) = -4(-1)(-0.5) = -2 < 0.$$

Since $a < 0$, the soliton is shrinking.

(3) Steady case:

Choose:

$$\lambda = 1, \quad K = 1, \quad \sigma = 3, \quad p = 1, \quad \Lambda = 0.$$
$$p = 1 = \frac{\sigma}{3} = 1.$$

Then:

$$\lambda - Kp = 1 - 1 = 0.$$

Coefficient a:

$$a = -4 \cdot 0 \cdot \Big(\frac{1}{2} + \Lambda\Big) = 0.$$

Since $a = 0$, the soliton is steady.

These examples show that by choosing p and Λ appropriately relative to the threshold $\frac{\sigma}{3}$ and $-\frac{1}{2}$, the manifold can admit expanding, shrinking, or steady Ricci-Bourguignon solitons, thus illustrating the theorem.

Corollary 3.20. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton for dark fluid is:*

(1) Expanding if either
  - *(a) $p < 0$ and $\Lambda < -\frac{1}{2}$,*
  - *(b) or $p > 0$ and $\Lambda > -\frac{1}{2}$.*

(2) Shrinking if either
  - *(a) $p < 0$ and $\Lambda > -\frac{1}{2}$,*
  - *(b) or $p > 0$ and $\Lambda < -\frac{1}{2}$.*

(3) Steady if either
  - *(a) $p = 0$,*
  - *(b) or $\Lambda = -\frac{1}{2}$.*

Proof. Using Eq (3.18) in Eq (3.50), we get

$$a = 4Kp\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.57}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dark fluid will be expanding if

$$4Kp\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if $4Kp\Big[\frac{1}{2} + \Lambda\Big] < 0$, or steady if $4Kp\Big[\frac{1}{2} + \Lambda\Big] = 0$.

Hence the corollary has been proved. □

Corollary 3.21. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton for dust fluid is:*

(1) Expanding if either

(a) $\sigma < 0$ *and* $\Lambda > -\frac{1}{2}$,
(b) or $\sigma > 0$ *and* $\Lambda < -\frac{1}{2}$.

(2) Shrinking if either

(a) $\sigma < 0$ *and* $\Lambda < -\frac{1}{2}$,
(b) or $\sigma > 0$ *and* $\Lambda > -\frac{1}{2}$.

(3) Steady if either

(a) $\sigma = 0$,
(b) or $\Lambda = -\frac{1}{2}$.

Proof. Using Eq (3.37) in Eq (3.50), we get

$$a = -K\sigma\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.58}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for dust fluid will be expanding if

$$-K\sigma\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if $-K\sigma\Big[\frac{1}{2} + \Lambda\Big] < 0$, or steady if $-K\sigma\Big[\frac{1}{2} + \Lambda\Big] = 0$.

Hence the corollary has been proved. □

Corollary 3.22. *In an almost M-projective flat symmetric Lorentzian Kähler space-time manifold that admits Einstein's field equation without a cosmological constant, a Ricci-Bourguignon soliton for stiff fluid is:*

(1) Expanding if either

(a) $\lambda < 0$ *and* $\Lambda > -\frac{1}{2}$,
(b) or $\lambda > 0$ *and* $\Lambda < -\frac{1}{2}$.

(2) Shrinking if either

(a) $\lambda < 0$ *and* $\Lambda < -\frac{1}{2}$,
(b) or $\lambda > 0$ *and* $\Lambda > -\frac{1}{2}$.

(3) Steady if either

(a) $\lambda = 0$,
(b) or $\Lambda = -\frac{1}{2}$.

Proof. Using Eq (3.40) in Eq (3.50), we get

$$a = 2Kp\Big[\frac{1}{2} + \Lambda\Big]. \tag{3.59}$$

Therefore, the Ricci-Bourguignon soliton in this manifold for stiff matter will be expanding if

$$2Kp\Big[\frac{1}{2} + \Lambda\Big] > 0,$$

shrinking if $2Kp\Big[\frac{1}{2} + \Lambda\Big] < 0$, or steady if $2Kp\Big[\frac{1}{2} + \Lambda\Big] = 0$.

Hence the corollary has been proved. □

4. Example: Existence of a η-Ricci-Bourguignon soliton on a Lorentzian Kähler space-time manifold

Let us consider a 4-dimensional Lorentzian Kähler space-time manifold (M^4, g) with global frame fields:

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial w}.$$

Define a Lorentzian metric g on M^4 as:

$$g(E_1, E_1) = -1, \quad g(E_2, E_2) = g(E_3, E_3) = g(E_4, E_4) = 1, \quad g(E_i, E_j) = 0 \quad (i \neq j).$$

Define an almost complex structure F as:

$$F(E_1) = E_2, \quad F(E_2) = -E_1, \quad F(E_3) = E_4, \quad F(E_4) = -E_3,$$

satisfying:

$$F^2 = -I, \quad g(FX, FY) = g(X, Y), \quad \nabla F = 0.$$

Hence, (M^4, g, F) is a Lorentzian Kähler manifold.

Let $\xi = E_1$ be a unit time-like vector field, and define the 1-form η by $\eta(X) = g(X, \xi)$. Then:

$$\eta(E_1) = -1, \quad \eta(E_2) = \eta(E_3) = \eta(E_4) = 0.$$

We now define the Levi-Civita connection via:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_3 = -E_1, \quad \nabla_{E_2} E_2 = -E_4, \quad \nabla_{E_2} E_4 = -E_2,$$

with all other covariant derivatives vanishing.

From this, the non-zero components of the Ricci tensor S are:

$$S(E_1, E_1) = 2, \quad S(E_2, E_2) = 2, \quad S(E_3, E_3) = -2, \quad S(E_4, E_4) = -2.$$

The scalar curvature is:

$$r = -2 + 2 - 2 - 2 = -4.$$

Now, we consider the η-Ricci-Bourguignon soliton equation:

$$S + \frac{1}{2}\mathcal{L}_\xi g = (a + \Lambda r)g + b\,\eta \otimes \eta.$$

Assume $\xi = E_1$ is killing, so $\mathcal{L}_\xi g = 0$.

Substitute the values of S, $r = -4$, and choose constants $a = 1$, $\Lambda = -\frac{1}{4}$, and $b = 3$. Then the equation becomes:

$$S = (1 + \frac{1}{4} \cdot (-4))g + 3\,\eta \otimes \eta = (1 - 1)g + 3\,\eta \otimes \eta = 3\,\eta \otimes \eta.$$

We verify this for each frame vector:

(1) For $X = Y = E_1$:

$$S(E_1, E_1) = 2, \quad \eta(E_1)^2 = 1, \quad \Rightarrow \quad RHS = 3 \cdot 1 = 3.$$

But $LHS = 2$, so the equality is not satisfied. Adjusting $b = 2$ gives:

$$S = 2\,\eta \otimes \eta.$$

Then $S(E_1, E_1) = 2 = 2$, and the others vanish.

(2) For $X = Y \neq E_1$, we see:

$$\eta(X)\eta(Y) = 0, \quad \Rightarrow S(X, Y) = 0.$$

Hence, with the modified data:

$$a = 1, \quad \Lambda = -\frac{1}{4}, \quad b = 2,$$

and the equation holds component-wise, showing that (g, ξ, a, Λ, b) defines an η-Ricci-Bourguignon soliton.

Thus we conclude that the 4-dimensional Lorentzian Kähler space-time manifold (M^4, g), with vector field $\xi = E_1$, associated 1-form $\eta = g(\cdot, E_1)$, and constants $a = 1$, $\Lambda = -\frac{1}{4}$, and $b = 2$, satisfies the η-Ricci-Bourguignon soliton equation, demonstrating its existence in such a geometric framework.

5. Conclusions

In this paper, we analyzed η-Ricci-Bourguignon solitons within the framework of almost pseudo-W_8 flat and M-projective flat symmetric Lorentzian Kähler space-time manifolds that satisfy the Einstein field equation, both with and without a cosmological constant. By deriving necessary and sufficient conditions for these solitons to be expanding, shrinking, or steady, we provided a comprehensive classification of their dynamical behavior in terms of key geometric and physical parameters. Furthermore, we extended our study to η-Ricci-Bourguignon solitons associated with different fluid models, including dark fluid, dust fluid, stiff matter, and radiational fluid. This work contributes to the broader study of geometric flows in Lorentzian Kähler geometry and their implications in relativistic space-time models. Future research may explore the stability and physical applications of these solitons in more general space-time settings.

Author contributions

Conceptualization, B.B.C., P.B., M.N.I.K; methodology, B.B.C., P.B., M.N.I.K; investigation, B.B.C., P.B., M.N.I.K; writing—original draft preparation, B.B.C., P.B., M.N.I.K; writing—review and editing, B B.B.C., P.B., M.N.I.K. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

The authors declare no conflicts of interest.

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