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Research article

On the number of zeros of Abelian integrals arising from perturbed quadratic reversible centers

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Abstract: Hilbert's 16th problem has been a significant topic in mathematics and its applications, with Arnold proposing a weakened version focusing on differential equations. Although considerable progress has been made in studying Hamiltonian systems, integrable non-Hamiltonian systems have received comparatively less attention. Recently, there has been a growing interest in quadratic reversible systems within this framework, leading to notable advancements. This study, which is grounded in qualitative analysis theory, investigates the upper bound on the number of zeros of Abelian integrals for a specific class of quadratic reversible systems under polynomial perturbations of degree n. By employing the Picard–Fuchs and the Riccati equation methods, we establish that for $n \geq 4$, the upper bound for the number of zeros of the Abelian integrals is $3n - 4$. To achieve this result, we first transform the first integral of the quadratic reversible system into a standard form using numerical methods. Then, by integrating the Picard–Fuchs and the Riccati equation approaches, we derive explicit representations of the Abelian integrals and estimate their maximum number of zeros using relevant theoretical results. These findings provide an upper bound for the number of limit cycles in the system, demonstrating that when the degree of the polynomial perturbation is sufficiently large (specifically $n \geq 4$), these analytical techniques effectively determine the maximum number of zeros of the Abelian integrals.

Keywords: quadratic reversible systems; polynomial perturbations; Picard–Fuchs equation; Riccati equation; upper bound; Abelian integrals
Mathematics Subject Classification: 34A05, 34A30, 34B05

1. Introduction

Since its formulation in 1900, Hilbert's 16th problem has been a central challenge in studying dynamical systems and differential equations. This problem investigates the number and distribution of limit cycles in polynomial systems, significantly influencing the development of mathematical theory and its applications across interdisciplinary fields such as physics and biology. A weakened version of the problem proposed by Arnold focuses on bifurcations of limit cycles in perturbed systems using Abelian integrals. This has connected the classical problem with the nonlinear analysis of integrable systems, creating a new paradigm for theoretical exploration. While there has been significant progress in Hamiltonian systems regarding limit cycle construction and stability analysis, research on integrable non-Hamiltonian systems remains notably underdeveloped, especially for quadratic reversible systems with symmetry. These systems exhibit unique dynamical properties, such as the reversibility of periodic orbits, making them ideal models for analyzing complex oscillatory phenomena, including the regulation of biological rhythms and the dynamic evolution of tumor microenvironments. Recent studies on the upper bounds of limit cycles in quadratic reversible systems have gained traction, but systematic conclusions about their global behavior under polynomial perturbations are still elusive.

This paper analyzes the number of zeros of Abelian integrals for a class of quadratic reversible center systems under n-degree polynomial perturbations to refine existing theoretical estimates of upper bounds on limit cycles. By utilizing the Picard–Fuchs equation method, which describes the analytic evolution of integrals with parameters, alongside the Riccati equation method, which addresses integrability conditions for nonlinear terms, we rigorously prove that when the perturbation degree n is greater than or equal to 4, the strict upper bound for the number of zeros of Abelian integrals is $3n-4$. To achieve this, we normalize the first integral of the integrable quadratic reversible system into a standard form using numerical techniques. This establishes a direct relationship between the perturbation parameters and the zeros of Abelian integrals. Our theoretical derivations demonstrate that the combined use of these methods effectively decouples the nonlinear interactions of high-degree perturbation terms, thereby characterizing the generation mechanisms of limit cycles with precision. This result not only provides a new theoretical limit for the maximum number of limit cycles in quadratic reversible systems but also extends its methodological framework to analyze oscillatory behaviors in biological dynamical systems, such as proliferation-dormancy cycle models of lung cancer cells, offering valuable mathematical tools for interdisciplinary research.

The significance of this study is twofold:

(1) It addresses the theoretical gap in determining upper bounds for limit cycles in integrable non-Hamiltonian systems under polynomial perturbations;
(2) Methodological innovation (synergistic application of Picard–Fuchs and Riccati equations) enhances the universality of zero-point estimation for Abelian integrals.

Before delving into the detailed study, it is essential to systematically outline the historical context, research significance, and key contributions in the field.

The following differential equations describe the perturbed Hamiltonian system:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H(x,y)}{\partial y} + \mu q(x,y), \\ \dfrac{dy}{dt} = -\dfrac{\partial H(x,y)}{\partial x} + \mu p(x,y), \end{cases} \tag{1.1}$$

where $0 < |\mu| \ll 1$ (perturbation parameter), $H(x, y)$ denotes a bivariate real polynomial of degree $n+1$ in x and y, and $f(x, y)$, $g(x, y)$ represent real polynomial perturbations of degree at most n. For the unperturbed system ($\varepsilon = 0$), let Γ_h denote the family of periodic orbits generated by the system (1.1), with $\Gamma_h(D) \subset \mathbb{R}^2$ specifying their maximal existence domain defined as follows:

$$\Gamma_h(D) = \left\{(x, y) \in R^2 \mid H(x, y) = h\right\}. \tag{1.2}$$

The associated Abelian integral is expressed as:

$$I(h) = \oint_{\Gamma_h} [f(x, y)dy - g(x, y)dx], \tag{1.3}$$

where $Z(m, n)$ denotes the minimal upper bound for the number of isolated zeros of $I(h)$. The weak Hilbert's 16th problem (also known as the Hilbert–Arnold problem; see [1]) remains a vibrant frontier in mathematical research. While Khovansky and Varchenko independently established the finiteness of Z(m,n) (see [2, 3]), the quest for explicit upper bounds has long eluded researchers. Notably, Li and Zhang derived the exact upper bound on the number of zeros of Abelian integrals for all quadratic polynomial one-forms over closed orbits of generic quadratic Hamiltonian systems featuring both a saddle loop and a cusp point (see [4]). Currently, research efforts predominantly focus on perturbed Hamiltonian systems, yet non-Hamiltonian integrable systems continue to pose profound theoretical challenges.

1.1. System classification

Following the classification in [5], quadratic systems with centers are categorized into five types:

A. Hamiltonian type (Q_3^H): $\dot{z} = -iz - z^2 + 2|z|^2 + (b + ci)\bar{z}^2$;
B. Reversible type (Q_3^R): $\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2$;
C. Codimension-4 type (Q_4): $\dot{z} = -iz + 4z^2 + 2|z|^2 + (b + ci)\bar{z}^2 \quad (|b + ci| = 2)$;
D. Generalized Lotka-Volterra type (Q_3^{LV}): $\dot{z} = -iz + z^2 + (b + ci)\bar{z}^2$;
E. Hamiltonian triangle type: $\dot{z} = -iz + \bar{z}^2$.

Consider the system parameters satisfying $a, b, c \in \mathbb{R}$. Among quadratic reversible systems, the Type B (reversible) systems have received significant attention. However, the lack of effective analytical tools has presented considerable research challenges. To address this methodological gap, this paper employs a dual approach combining the Picard–Fuchs equation method and the Riccati equation technique. Our investigation specifically focuses on determining the number of zeros for a class of Abelian integrals associated with Type B reversible systems.

1.2. Analytical context

Studies on cubic vector fields [6, 7] have advanced Abelian integral analysis, while the connection between soliton solutions and limit cycles [8–10] provides complementary perspectives. In the research on estimating compact upper bounds for zeros of Abelian integrals in high-degree polynomial perturbed Hamiltonian systems, the Picard–Fuchs equation method and the Riccati equation method serve as core analytical tools. The relevant pioneering analysis on the distribution of complex zeros of elliptic integrals has laid the mathematical foundation for the compactness theory of high-degree systems [11, 12]. For quartic systems, the research has evolved three breakthroughs. The linear estimation method first established an explicit compact upper bound, verifying the theoretical boundary in the weak Hilbert problem [13]. The zero upper bound was significantly compressed based on the symmetric constraint mechanism of the double-center configuration [14]. In the extension to high-degree systems ($n \geq 5$), the constructed periodic ring bifurcation model revealed the asymptotic growth law between the number of limit cycles $B(n)$ and the perturbation order n, extending the compactness research to polynomial systems of arbitrary order [15]. The symmetry restriction was broken through the singularity order separation and hierarchical topological decomposition of perturbation terms, achieving theoretical closure [16, 17]. The above results collectively confirm that the Picard–Fuchs equation method and the Riccati equation method are effective and have stability and universality in systems with $n \geq 3$.

When the degree of the perturbing polynomial in the system is relatively low, the combined method of detection functions and numerical exploration can be employed to investigate the number and location of limit cycles. Researchers have applied this approach to perturbed quintic Hamiltonian systems, validating its practical utility in limit cycle analysis [18]. In studies of cubic Hamiltonian systems, this method has yielded significant achievements: research utilizing this method revealed that five distinct perturbed cubic Hamiltonian systems exhibit identical limit cycle distributions [19]. For Hamiltonian systems approaching the principal deformation of Z_4-field, this method established bifurcation sets and limit cycle distributions [20]. The method was leveraged to investigate the formation of compound-eye-shaped limit cycles and their bifurcation sets [21]. Furthermore, researchers analyzed bifurcation sets and limit cycle distributions for cubic Hamiltonian systems with higher-order perturbation terms [22] and discovered 14 limit cycles in a cubic Hamiltonian system with a ninth-order perturbation term [23]. It is precisely through these technical means that researchers have rigorously determined the existence and fundamental properties of limit cycles in such systems.

Our work establishes the upper bound $Z(2, n) \leq 3n - 4$ for quadratic reversible systems under degree n perturbations ($n \geq 4$). This extends previous Hamiltonian-focused results to non-Hamiltonian contexts, demonstrating novel methodological integration,

$$Z(2, n) \leq 3n - 4 \quad (n \geq 4)\{(Main\ Result)\}. \tag{1.4}$$

2. Analytical framework

2.1. Linear Picard–Fuchs framework

For the linear Picard–Fuchs equation framework, consider system (1.1), where the Hamiltonian H is a polynomial of degree $n + 1$ whose homogeneous part decomposes into the product of $n + 1$ linear factors. Define a set of Abelian integrals associated with the double integrals [24]:

$$\left\{ \iint_{\Gamma_h(D)} x^i y^j dxdy \right\} \quad (i, j \geq 0),$$

denoted as $\{I_1(h), I_2(h), \cdots, I_k(h)\}$ with cardinality $k = n^2$. The column vector formed by these Abelian integrals:

$$X(h) = \begin{pmatrix} I_1(h) \\ I_2(h) \\ \vdots \\ I_k(h) \end{pmatrix},$$

constitutes the fundamental solution set of the differential equation:

$$X'(h) = A(h)X(h), \tag{2.1}$$

which defines the linear Picard–Fuchs equation. Key properties of this equation include:

- $A(h)$ is an $n \times n$ matrix whose elements are rational functions in h;
- The singularities of $A(h)$ correspond to the critical values of H.

Generally, the original Abelian integral $I(h)$ admits a linear decomposition:

$$I(h) = \sum_{\ell=1}^{k} c_\ell I_\ell(h), \quad c_\ell \in \mathbb{R}. \tag{2.2}$$

2.2. Quadratic reversible systems

For the Q_3^R-type reversible system, we first express it in the real plane using the substitution $z = x+yi$, yielding:

$$\begin{cases} \dot{x} = (a+b+2)x^2 - (a+b-2)y^2 + y, \\ \dot{y} = -x[1-2(a-b)y]. \end{cases} \tag{2.3}$$

To simplify analysis, we apply the canonical transformation defined by new coordinates (X, Y, τ):

$$\begin{aligned} X &= 1 - 2(a-b)y, \\ Y &= x, \\ d\tau &= -2(a-b)dt, \end{aligned} \tag{2.4}$$

which transforms system (2.3) into the following decoupled form:

$$\begin{cases} \dot{X} = -XY, \\ \dot{Y} = -\dfrac{a+b+2}{2(a-b)}Y^2 + \dfrac{a+b-2}{8(a-b)^3}X^2 - \dfrac{b-1}{2(a-b)^3}X - \dfrac{a-3b+2}{8(a-b)^3}. \end{cases} \tag{2.5}$$

This transformation reveals the system's intrinsic geometric structure while preserving its dynamical properties.

2.3. Integrable structure and perturbation analysis

The first integral for system (2.5) is given by (see [25]):

$$H(X,Y) = X^\lambda \left[\frac{1}{2}Y^2 + \frac{1}{8(a-b)^2}\left(\frac{a+b-2}{a-3b-2}X^2 + 2\frac{b-1}{b+1}X + \frac{a-3b+2}{a+b+2}\right)\right], \tag{2.6}$$

where the exponent λ satisfies $\lambda = -\frac{a+b+2}{a-b}$. The corresponding n-degree perturbed system takes the form:

$$\begin{cases} \dot{X} = -XY + \mu f(X,Y), \\ \dot{Y} = -\dfrac{a+b+2}{2(a-b)}Y^2 + \dfrac{a+b-2}{8(a-b)^3}X^2 - \dfrac{b-1}{2(a-b)^3}X - \dfrac{a-3b+2}{8(a-b)^3} + \mu g(X,Y), \end{cases} \tag{2.7}$$

with perturbation polynomials $f, g \in \mathbb{R}[X,Y]_{\leq n}$.

Theorem 1 (Zeros of Abelian integrals)**.** *For parameter values* $(a,b) = (7,3)$ *with integrating factor* $M(X,Y) = X^{-4}$*, the Abelian integral*

$$I(h) = \oint_{\Gamma_h} X^{-4}[g\,dX - f\,dY], \tag{2.8}$$

satisfies the upper bound estimate:

$$\mathcal{Z}(2,n) \leq 3n-4. \tag{2.9}$$

Proof. The proof follows from the genus-one algebraic curve structure of the level sets $H(X,Y) = h$, permitting the application of elliptic function theory. This approach fundamentally differs from previous analyses based on rational curves by utilizing modular function properties and periodic lattice characteristics. □

2.4. Concrete system realization

For the specific parameter pair $(a,b) = (7,3)$, the planar dynamical system (2.5) reduces to:

$$\begin{cases} \dot{X} = -XY, \\ \dot{Y} = -\dfrac{3}{2}Y^2 + \dfrac{1}{64}X^2 - \dfrac{1}{64}X. \end{cases} \tag{2.10}$$

Topological characteristics. Two essential topological features characterize this non-Hamiltonian integrable system:

- The coordinate axis $X = 0$ forms an invariant algebraic curve;
- An isolated central singularity at $(1,0)$ surrounded by a one-parameter family of nested closed orbits. $\{\Gamma_h\}$

The energy parameter h ranges through $(-2^{-7}, 0)$, corresponding to the shaded region in the phase portrait. The system admits the first integral (see Figure 1):

$$H(X,Y) = X^{-3}\left(\frac{1}{2}Y^2 - \frac{1}{64}X^2 + \frac{1}{128}X\right) = h. \tag{2.11}$$

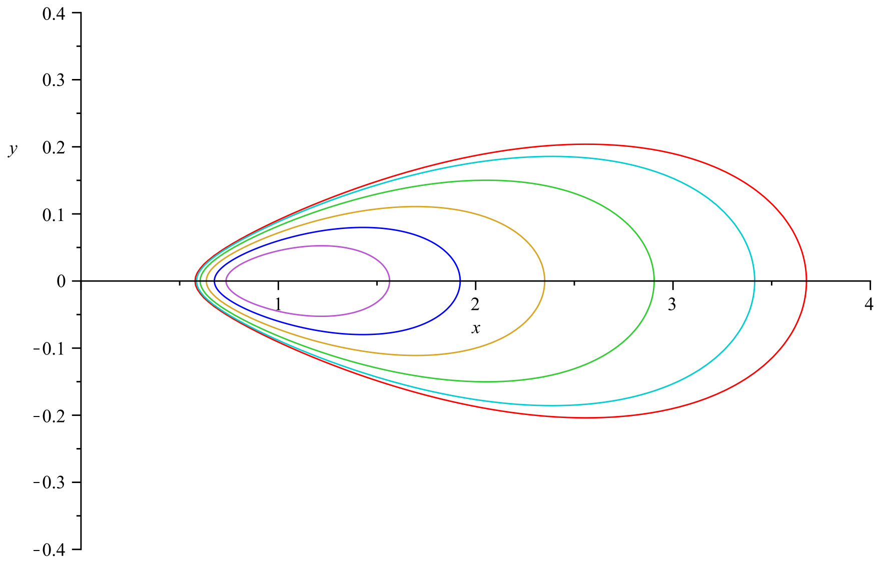

Figure 1. The closed orbits of the unperturbed system (2.10).

2.5. *Integral transformation and analysis*

Applying the integrating factor X^{-4} to the perturbed system yields the regularized dynamics:

$$\begin{cases} \dot{X} = -X^{-3}Y + \mu X^{-4} f(X,Y), \\ \dot{Y} = -\dfrac{3}{2}Y^2X^{-4} + \dfrac{1}{64}X^{-2} - \dfrac{1}{64}X^{-3} + \mu X^{-4} g(X,Y), \end{cases} \tag{2.12}$$

where $f(X,Y)$ and $g(X,Y)$ are n-th degree polynomials with coefficients a_{ij}, b_{ij} respectively.

Under the non-degeneracy condition (2.14), the zero-counting problem reduces to evaluating period integrals over genus-one closed orbits Γ_h. The Abelian integral in weighted coordinates takes the form:

$$I(h) = \oint_{\Gamma_h} \sum_{0\le i+j\le n} \left(b_{ij}X^{i-4}Y^j dX - a_{ij}X^{i-4}Y^j dY \right), \tag{2.13}$$

where:

- $h \in (-2^{-7}, 0)$ parametrizes the homology basis of toroidal phase curves Γ_h;
- Indices $i, j \in \{0, 1, \ldots, n\}$ correspond to polynomial coefficients of perturbations;
- The X^{-4} weighting stems from the integrating factor transformation.

The resolution of this problem leverages the algebraic-geometric properties inherent to the genus-one first integral structure, requiring explicit evaluation of these period integrals.

2.6. *Algebraic structure of the Abelian integral $I(h)$*

To analyze the algebraic structure of the transformed Abelian integral (2.13), we introduce the parametric family of period integrals for the original system (2.12):

$$I_{i,j}(h) = \oint_{\Gamma_h} X^{i-4}Y^j \, dX, \quad h \in \left(-\frac{1}{2^7}, 0\right), \tag{2.14}$$

where $i \in \{-1, 0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, n\}$. For the special case $j = 1$, we adopt the notational convention:

$$I_{i,1}(h) = \oint_{\Gamma_h} X^{i-4} Y \,\mathrm{d}X = J_i, \quad i \in \{-1, 0, 1, \ldots, n\}. \tag{2.15}$$

Applying Green's theorem on the genus-one torus Γ_h, we derive the differential identity:

$$\oint_{\Gamma_h} X^{i-4} Y^j \,\mathrm{d}Y = -\frac{i-4}{j+1} \oint_{\Gamma_h} X^{i-5} Y^{j+1} \,\mathrm{d}X, \tag{2.16}$$

establishing a recursive relation between the dY-component and dX-component integrals. Let $\deg(\alpha(h))$ denote the degree of polynomial $\alpha(h)$, and introduce the meromorphic parameter $\hbar = h^{-1}$ for Laurent series analysis near the singular fiber $h = 0$.

Lemma 1 (Meromorphic representation)**.** *The algebraic-geometric structure of the transformed Abelian integral* (2.13) *admits the meromorphic decomposition:*

$$J(h) = \hbar I(h) = \alpha(\hbar) J_0(h) + \beta(\hbar) J_1(h), \tag{2.17}$$

where $\hbar = \frac{1}{h}$ *serves as the local parameter near* $h = 0$. *The polynomial coefficients satisfy* $\deg(\alpha) \leq n-2$ *and* $\deg(\beta) \leq n-2$ *for* $n \geq 4$. *This decomposition reveals the hierarchical structure of period integrals, with* $J_0(h)$ *and* $J_1(h)$ *forming a homology basis for the genus-one system.*

Proof. The complete derivation incorporating differentiation, weighting, and integration operations yields the unified expression:

$$\oint_{\Gamma_h} \left[X^{i-3} Y^{j-1} \,dY - \frac{3}{2} X^{i-4} Y^j + \frac{1}{64} (X^{i-2} - X^{i-3}) Y^{j-2} \right] dX = 0, \tag{2.18}$$

which synthesizes three essential steps: implicit differentiation of the first integral (2.11), multiplication by the monomial weight $X^i Y^{j-2}$ to introduce perturbation indices, and path integration over the closed orbits Γ_h. This fundamental integral relation establishes the recursive computational foundation for Abelian integrals $I_{i,j}(h)$.

The application of the differential relation $\frac{\partial Y}{\partial X} \mathrm{d}X = \mathrm{d}Y$ to Eq (2.18) yields the fundamental recursion:

$$I_{i,j}(h) = \frac{j}{32(3j + 2i - 6)} \left(I_{i+2,j-2}(h) - I_{i+1,j-2}(h) \right), \tag{2.19}$$

derived through the intermediate relation

$$-\frac{3}{2} I_{i,j} - \frac{i-3}{j} I_{i,j} + \frac{1}{64} (I_{i+2,j-2} - I_{i+1,j-2}) = 0.$$

The parity-reduced representation of Abelian integrals takes the form:

$$\hbar^p I_{i,j}(h) = \sum_{k=0}^{\frac{j-1}{2}} c_{ik} \hbar^p J_{i+k+\frac{j-1}{2}}(h), \tag{2.20}$$

where $p \in \{0, 1\}$ controls the $\hbar$-scaling, and j must be odd due to the parity condition $I_{i,\text{even}}(h) \equiv 0$.

The Hamiltonian constraint

$$\frac{1}{2}Y^2 - \frac{1}{64}X^2 + \frac{1}{128}X = hX^3$$

(from Eq (2.11)) combined with contour integration over Γ_h under monomial weighting $X^{i-4}Y^{j-2}$ establishes the fundamental integral relation:

$$hI_{i+3,j-2} = \frac{1}{2}I_{i,j}(h) - \frac{1}{64}I_{i+2,j-2}(h) + \frac{1}{128}I_{i+1,j-2}(h), \tag{2.21}$$

providing a direct algebraic connection between the system energy h and Abelian integrals. □

Substituting Eq (2.19) into (2.21), we obtain the reduced form:

$$hI_{i+3,j-2}(h) = -\frac{1}{128}\frac{(4j+4i-12)I_{i+2,j-2} + (6-j-2i)I_{i+1,j-2}}{3j+2i-6}. \tag{2.22}$$

For the special case $j = 3$, where the contour integral $\oint_{\Gamma_h} X^{i-4}YdX = J_i(h)$ holds, Eq (2.22) simplifies to:

$$hJ_{i+3} = -\frac{1}{128}\frac{4iJ_{i+2} + 3J_{i+1} - 2iJ_{i+1}}{3+2i}. \tag{2.23}$$

By reorganizing terms in Eq (2.23), we ultimately derive the recurrence relation:

$$\hbar J_i(h) = -\frac{1}{128}\left[\frac{4i-12}{2i-3}\hbar^2 J_{i-1}(h) + \frac{9-2i}{2i-3}\hbar^2 J_{i-2}(h)\right]. \tag{2.24}$$

From Eq (2.24), we establish two fundamental cases:

Case analysis

Case 1 ($i \geq 2$). For indices $i \geq 2$, the operator $\hbar J_i(h)$ admits a decomposition

$$\hbar J_i(h) = \alpha_{i,0}(\hbar)J_0(h) + \beta_{i,1}(\hbar)J_1(h), \tag{2.25}$$

where the coefficients satisfy:

- $\deg \alpha_{i,0}(\hbar) \leq i-1$ and $\deg \beta_{i,1}(\hbar) \leq i-1$ for $i \geq 4$;
- $\alpha_{3,0}(\hbar) \equiv 0$ with $\deg \beta_{3,1}(\hbar) = 2$;
- $\deg \alpha_{2,0}(\hbar) = \deg \beta_{2,1}(\hbar) = 2$.

This reduction to the basis $\{J_0(h), J_1(h)\}$ completes the verification of (2.25).

Case 2 ($i = 0, 1$). The boundary cases exhibit trivial decompositions:

$$\begin{aligned} \hbar J_0(h) &= \hbar J_0(h), \\ \hbar J_1(h) &= \hbar J_1(h), \end{aligned} \tag{2.26}$$

with coefficient degrees:

- For $i = 0$: $\deg \alpha_{0,0}(\hbar) = 1, \beta_{0,1}(\hbar) \equiv 0$;
- For $i = 1$: $\alpha_{1,0}(\hbar) \equiv 0, \deg \beta_{1,1}(\hbar) = 1$.

Integral representation

Combining Eqs (2.13) and (2.20) under the condition $p = 0$ yields the composite expression:

$$I(h) = \sum_{0\le i+j\le n} b_{ij} \sum_{k=0}^{\frac{j-1}{2}} c_{i,k} J_t + \frac{i-4}{j+1} \sum_{0\le i+j\le n} a_{ij} \sum_{k=0}^{\frac{j}{2}} c_{i-1,k} \hbar J_{t-\frac{1}{2}}, \tag{2.27}$$

where $t = i + k + \frac{j-1}{2}$. The transformed integral $J(h) = \hbar I(h)$ consequently satisfies:

$$J(h) = \hbar \sum_{0\le i+j\le n} b_{ij} \sum_{k=0}^{\frac{j-1}{2}} c_{i,k} \hbar J_t + \frac{i-4}{j+1} \sum_{0\le i+j\le n} a_{ij} \sum_{k=0}^{\frac{j}{2}} c_{i-1,k} \hbar J_{t-\frac{1}{2}}. \tag{2.28}$$

Degree analysis

Considering the index bounds in Eq (2.20) under the condition $p = 0$:

- For $j = 0$: $\max(i + k) = i + j - 1 = n - 1$;
- For $j = 1$: $\max(i - \frac{1}{2} + k + \frac{j-1}{2}) = n - 1$.

Equation (2.25) then implies for $n \ge 4$:

$$\deg \alpha(\hbar), \deg \beta(\hbar) \le n - 2. \tag{2.29}$$

Corollary 1 (Meromorphic representation)**.** *For $n \ge 4$, the transformed Abelian integral* (2.17) *admits the decomposition*

$$J(h) = \frac{1}{h^{n-1}} k(h), \quad k(h) = \alpha_1(h) J_0(h) + \beta_1(h) J_1(h), \tag{2.30}$$

where $\hbar = \frac{1}{h}$ parametrizes the neighborhood of the singular fiber $h = 0$, with polynomial coefficients satisfying $\deg \alpha_1(h), \deg \beta_1(h) \le n - 2$. This structure reflects the toroidal period hierarchy, where $\{J_0(h), J_1(h)\}$ form a homology basis and $\alpha_1(h)$, $\beta_1(h)$ encode polynomial perturbations constrained by the genus-one geometry.

Proof. The result follows directly from (2.28) combined with Lemma 1, completing the reduction of $J(h)$ to the fundamental period combination in (2.30). □

2.6.1. Picard–Fuchs equation

The fundamental periods $\{J_0(h), J_1(h)\}$ satisfy the matrix Picard–Fuchs equation:

$$\begin{pmatrix} J_0(h) \\ J_1(h) \end{pmatrix} = \begin{pmatrix} \frac{4}{5}h + \frac{1}{120} & \frac{4}{15}h \\ \frac{1}{96} & \frac{4}{3}h \end{pmatrix} \begin{pmatrix} J_0'(h) \\ J_1'(h) \end{pmatrix}. \tag{2.31}$$

Proof. Starting from the first integral

$$H(X, Y) = X^{-3} \left(\frac{1}{2} Y^2 - \frac{1}{64} X^2 + \frac{1}{128} X \right) = h,$$

we derive:

$$Y^2 = 2hX^3 + \frac{1}{32}X^2 - \frac{1}{64}X. \tag{2.32}$$

Differentiating (2.23) yields the partial derivatives:

$$\frac{\partial Y}{\partial h} = \frac{X^3}{Y}, \quad \frac{\partial Y}{\partial X} = \frac{3hX^2 + \frac{1}{32}X - \frac{1}{128}}{Y}. \tag{2.33}$$

For the period integrals $J_i(h) = \oint_{\Gamma_h} X^{i-4} Y \, \mathrm{d}X$, differentiation gives:

$$J_i'(h) = \oint_{\Gamma_h} \frac{X^{i-1}}{Y} \, \mathrm{d}X, \tag{2.34}$$

$$J_i(h) = \oint_{\Gamma_h} \frac{2hX^{i-1} + \frac{1}{32}X^{i-2} - \frac{1}{64}X^{i-3}}{Y} \, \mathrm{d}X = 2hJ_i'(h) + \frac{1}{32}J_{i-1}'(h) - \frac{1}{64}J_{i-2}'(h). \tag{2.35}$$

Using Stokes' theorem, we obtain the dual representation:

$$J_i(h) = -\frac{1}{i-3} \oint_{\Gamma_h} X^{i-3} \, \mathrm{d}Y. \tag{2.36}$$

Combining (2.35) and (2.36) through the chain rule $\mathrm{d}Y = \frac{\partial Y}{\partial X}\mathrm{d}X$ yields:

$$(i-3)J_i(h) = -3hJ_i'(h) - \frac{1}{32}J_{i-1}'(h) + \frac{1}{128}J_{i-2}'(h). \tag{2.37}$$

For specific indices, we derive the reduced relations:

$$(i-2)J_i(h) = -hJ_i'(h) - \frac{1}{128}J_{i-2}'(h), \tag{2.38}$$

$$(i - \frac{5}{2})J_i(h) = -2hJ_i'(h) - \frac{1}{64}J_{i-1}'(h). \tag{2.39}$$

Specializing to $i = 0, 1$ produces the coupled system:

$$J_1(h) = \frac{4}{3}hJ_1'(h) + \frac{1}{96}J_0'(h), \tag{2.40}$$

$$J_0(h) = \left(\frac{4}{5}h + \frac{1}{120}\right)J_0'(h) + \frac{4}{15}hJ_1'(h). \tag{2.41}$$

The matrix form (2.31) follows directly from (2.39) and (2.41). □

Corollary 2 (Vanishing properties)**.** *The periods satisfy* $J_0(-\frac{1}{2^7}) = J_1(-\frac{1}{2^7}) = 0$ *with positive derivatives* $J_0'(h), J_1'(h) > 0$ *for* $h \in (-\frac{1}{2^7}, 0)$.

2.6.2. Riccati equation

The period ratio $V(h) = \frac{J_0(h)}{J_1(h)}$ satisfies the Riccati equation:

$$B(h)V'(h) = 5V^2(h) + A(h)V(h) - 128h, \tag{2.42}$$

here $B(h) = 4h(1 + 128h)$ and $A(h) = 256h - 4$.

Proof. Differentiating $V(h)$ gives:

$$V'(h) = \frac{J_0'(h)J_1(h) - J_0(h)J_1'(h)}{J_1^2(h)}. \tag{2.43}$$

Substituting the Picard–Fuchs system (2.31) into (2.43) and clearing denominators yields (2.42). □

2.7. The number of zeros of Abelian integrals

Lemma 2 (Riccati structure). *For $n \geq 4$, the quotient $W(h) = \dfrac{k(h)}{J_1(h)}$ satisfies the Riccati-type equation:*

$$B(h)\alpha_1(h)W'(h) = 5W^2(h) + F(h)W(h) + G(h), \tag{2.44}$$

with coefficient polynomials:

$$F(h) = B(h)\alpha_1'(h) - 10\beta_1(h) + A(h)\alpha_1(h),$$
$$G(h) = B(h)\alpha_1(h)\beta_1'(h) - B(h)\alpha_1'(h)\beta_1(h) + 5\beta_1^2(h) - A(h)\alpha_1(h)\beta_1(h) - 128h\alpha_1^2(h),$$

satisfying degree bounds $\deg F(h) \leq n - 1$ *and* $\deg G(h) \leq 2n - 3$.

Proof. From Corollary 1 and the definition of $v(h) = \dfrac{J_0(h)}{J_1(h)}$, we express:

$$W(h) = \alpha_1(h)v(h) + \beta_1(h). \tag{2.45}$$

Differentiation yields:

$$W'(h) = \alpha_1'(h)v(h) + \alpha_1(h)v'(h) + \beta_1'(h). \tag{2.46}$$

Through algebraic manipulation of (2.46) using:

- The squared relation $W^2(h) = \alpha_1^2(h)v^2(h) + 2\alpha_1(h)\beta_1(h)v(h) + \beta_1^2(h)$;
- The Picard–Fuchs system from (2.31);
- Coefficient matching of $v'(h)$ terms.

We derive the Riccati structure (2.31). The degree bounds follow the polynomial degrees in Corollary 1 and the composition laws for polynomial operations. □

Lemma 3 (Zero counting bound). *The meromorphic function $W(h)$ satisfies:*

$$\mathcal{Z}(W) \leq \mathcal{Z}(B) + \mathcal{Z}(\alpha_1) + \mathcal{Z}(G) + 1, \tag{2.47}$$

where $\mathcal{Z}(\cdot)$ counts isolated zeros with multiplicity in $\left(-\frac{1}{128}, 0\right)$.

Proof. This follows from applying the argument principle to (2.42) combined with reference Lemma 5.3 from [26] on differential equations of Riccati type—the +1 term accounts for the pole contribution at $h = 0$. □

Theorem 2 (Zero cardinality). *For the Hamiltonian system* (2.12) *with* $n \geq 4$*, the Abelian integral satisfies:*

$$\mathcal{Z}(I) \leq 3n - 4.$$

Proof. The zero-counting equivalence chain:

$$\mathcal{Z}(I) = \mathcal{Z}(J) = \mathcal{Z}(k) = \mathcal{Z}(W) \tag{2.48}$$

combines with Lemma 3 and the estimates:

$$\begin{aligned} \mathcal{Z}(\alpha_1) &\leq \deg \alpha_1 \leq n - 2, \\ \mathcal{Z}(G) &\leq \deg G \leq 2n - 3, \\ \mathcal{Z}(B) &= 0 \quad (\text{since } B(h) \neq 0 \text{ on } (-\tfrac{1}{128}, 0)), \end{aligned}$$

yielding $\mathcal{Z}(I) \leq (n - 2) + (2n - 3) + 1 = 3n - 4$. □

3. Conclusions

This investigation establishes an upper bound for the number of zeros of Abelian integrals in quadratic reversible systems with polynomial perturbations, where the unperturbed trajectories foliate cubic curves. By leveraging analytical techniques rooted in Picard–Fuchs equation theory and Riccati-type transformations, we rigorously prove that the number of isolated zeros (counting multiplicity) of these integrals does not exceed $3n-4$ for perturbations of degree $n \geq 4$. For lower-degree perturbations $n < 4$, we develop a hybrid methodology combining algebraic discriminant analysis with numerical bifurcation techniques to characterize limit cycle configurations systematically. While effective within their operational domains, these methods reveal inherent limitations in existing analytical frameworks, underscoring the necessity for novel theoretical constructs to address higher complexity regimes.

Our results demonstrate a linear scaling relationship between Abelian integrals' perturbation degree and zero cardinality, extending prior work on hierarchical polynomial perturbations. In the context of Hilbert's 16th problem, this study advances the boundary of knowledge for specialized reversible systems but falls short of providing a universal resolution. Although potent for structured cases, the employed Picard–Fuchs and Riccati-based approaches lack the generality required to encompass the full scope of the problem's nonlinear complexity. A definitive solution to Hilbert's 16th problem will likely necessitate paradigm-shifting mathematical innovations that transcend current perturbative methodologies. These findings validate the efficacy of algebraic–geometric tools in constrained dynamical settings and highlight critical gaps in nonlinear systems theory, urging the development of transformative analytical frameworks.

Author contributions

Yanjie Wang: Investigation, Visualization, Writing - Original Draft, Funding Acquisition; Beibei Zhang: Methodology, Validation, Supervision, Writing - Review; Chun Tong: Formal Analysis, Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors state no conflicts of interest.

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