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Research article

Solvability of an infinite system of n-th order differential equations in a paranormed sequence space and its numerical solution

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Abstract: In this paper, we introduce a ball measure of noncompactness in the Banach sequence space bv_q $(1 \leq q < \infty)$ containing the space l_q. By applying the technique of measures of noncompactness and Meir–Keeler condensing mappings, we investigate the existence of solutions of an infinite system of differential equations of order n with boundary conditions. Some examples are provided to support our main results. A numerical spectral method based on Bernoulli polynomials is applied to find the approximate solution of an example.

Keywords: Bernoulli polynomials; infinite system; measure of noncompactness; Meir–Keeler condensing mapping; sequence spaces
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1. Introduction

The degree of noncompactness of a bounded set is described by means of functions called measures of noncompactness (MNC), which were first introduced by Kuratowski ([26, 27]). There are different types of MNC in metric and linear topological spaces, which have been effectively used to study the solvability of various types of boundary value problems for differential equations and their infinite systems in sequence spaces (see [34–38]). In this connection, Aghajani and Pourhadi [4] studied the infinite system of second-order differential equations in the l_1 space. Then, Banaś et al. [13] focused on this system in the sequence space l_p. Alotaibi et al. [5] discussed the solvability of infinite system of second-order differential equations in the sequence space $n(\phi)$. Recently, many scholars have been devoted to studying the solvability of infinite systems of fractional differential equations in weighted and tempered sequence spaces [8, 21, 33, 35, 39].

On the other hand, in [20] Edelstein considered locally contractive mappings and derived as a consequence that each strict contraction of a compact space has a unique fixed point. Meir and Keeler [32] provided a fixed point theorem, which is a generalization of the Banach contraction principle [11]. After that, Aghajani et al. [3] proved some coupled fixed point theorems for Meir–Keeler condensing mappings with the help of measure of noncompactness. Moreover, in [10, 16], Meir–Keeler condensing operators for self-mappings and nonself-mappings in a Banach space via an arbitrary measure of weak noncompactness have been studied.

Numerical illustrations are useful for verifying the solutions of integral and differential equations [6, 7, 9, 44, 45]. The Bernoulli polynomials, which are named in honor of Jacob Bernoulli, combine the binomial coefficients and Bernoulli numbers. They are applicable for the Euler–MacLaurin formula, and with series expansion of functions. For a Bernoulli polynomial, the number of crossings of the x-axis in the unit interval does not increase with the degree. In the limit of large degrees, they approach the cosine and sine functions. Bernoulli polynomials are not orthogonal functions; nevertheless, they have some interesting properties that make them important, and many researchers have used them in solving different types of mathematical problems. For example, these polynomials are used for solving fractional integral equations in [29], PDEs [18,23], fractional differential equations [40,49], or integral equations [22].

Now, by using the technique of measure of noncompactness, we study the solvability of the following infinite system of differential equations of order n with boundary conditions

$$\begin{cases} \backepsilon_i^{(n)}(\xi) + f_i(\xi, \backepsilon(\xi)) = 0, \quad i = 1, 2, \ldots, \\ \backepsilon_i(\sigma) = \backepsilon_i'(\sigma) = \ldots = \backepsilon_i^{(n-2)}(\sigma) = 0 \text{ and } \backepsilon_i(\varrho) = 0, \end{cases} \tag{1.1}$$

in the new sequence space bv_q $(1 \leq q < \infty)$, where $\backepsilon(\xi) = (\backepsilon_i(\xi))$ $(\xi \in [\sigma, \varrho])$.

Differential equations are used to describe heat transfer and flow fields [1, 50], the vibrations of systems, and the propagation of waves [28] in physics. In quantum physics, differential equations such as the Schrödinger equations are used to describe the behavior of microscopic particles such as electrons and photons [46]. Differential equations are also used as a tool to study and describe the behavior of complex systems, such as the collective behavior of humans in a population, or the collective behavior of humans and machines in complex networks [48]. More applications of this issue can be found in [30,41,42]. Many researchers studied the solvability of differential equations by using the Leray–Schauder nonlinear alternative theorem, the contraction mapping principle, Krasnoselskiis, the Schauder fixed point theorems, the Monch fixed point theorem, the nonlinear alternative for Kakutani maps, etc. But in this manuscript, we consider the solvability of an infinite system of differential equations by using Meir–Keeler contractive mapping associated with Hausdorff measure of noncompactness. Then, we apply a numerical spectral method based on Bernoulli polynomials to determine the approximate solutions of a special infinite system. The paper is organized as follows: In the next section, we present some auxiliary facts that we need further on. In Section 3, we explain the construction of the Banach sequence space bv_q $(1 \leq q < \infty)$ and define an MNC in this space. In Section 4, we investigate the solvability of the infinite system (1.1) in the sequence spaces bv_q and give some examples. In Section 5, we propose a numerical spectral method based on Bernoulli polynomials to solve an infinite system approximately.

2. Preliminaries

In this section, we give some relevant facts that we will need further on. Throughout the manuscript, unless otherwise specified, we consider Λ to be a real Banach space. We also consider
• $B(\vartheta, \zeta)$ is a ball with center ϑ and radius ζ.
• If $\emptyset \neq \mathfrak{L} \subseteq \Lambda$, then Conv$\mathfrak{L}$ is the closure of the convex hull of $\mathfrak{L}$.
• $\mathfrak{M}_\Lambda = \{\mathcal{W} : \emptyset \neq \mathcal{W} \subseteq \Lambda$ and it is bounded$\}$.
• $\mathfrak{N}_\Lambda = \{\mathcal{W} : \mathcal{W} \in \mathfrak{M}_\Lambda$ and it is relatively compact$\}$.

Definition 2.1. *[2] A measure of noncompactness (MNC) in Λ is a map $\mu : \mathfrak{M}_\Lambda \to \mathbb{R}_+$ so that for any $\mathcal{W}, Q \in \mathfrak{M}_\Lambda$ we have:*

(1) $\mathfrak{N}_\Lambda \supseteq \ker \mu = \{\mathcal{W} \in \mathfrak{M}_\Lambda : \mu(\mathcal{W}) = 0\} \neq \emptyset$.
(2) $\mathcal{W} \subset Q \Rightarrow \mu(\mathcal{W}) \leq \mu(Q)$.
(3) $\mu(\mathcal{W}) = \mu(\overline{\mathcal{W}}) = \mu(Conv\mathcal{W})$.
(4) $\mu(\lambda\mathcal{W} + (1-\lambda)Q) \leq \lambda\mu(\mathcal{W}) + (1-\lambda)\mu(Q)$ *for each* $\lambda \in [0, 1]$.
(5) *Suppose that for each* $n \in \mathbb{N}$, $\overline{\mathcal{W}_n} = \mathcal{W}_n \subseteq \mathfrak{M}_\Lambda$, *and* $\mathcal{W}_{n+1} \subset \mathcal{W}_n$. *Then*

$$\lim_{n\to\infty} \mu(\mathcal{W}_n) = 0 \Rightarrow \emptyset \neq \mathcal{W}_\infty = \bigcap_{n=1}^{\infty} \mathcal{W}_n.$$

Definition 2.2. *[12] Let $(\mathcal{W}, d)$ be a metric space, let $\mathfrak{M}_\mathcal{W}$ be the family of all bounded subsets of $\mathcal{W}$, and let $\aleph \in \mathfrak{M}_\mathcal{W}$. Then the Kuratowski measure of noncompactness of $\aleph$ is*

$$\alpha(\aleph) = \inf \left\{ \xi > 0 : \aleph \subset \bigcup_{i=1}^{n} \ell_i,\ \ell_i \subset \mathcal{W},\ diam(\ell_i) < \xi,\ n \in \mathbb{N} \right\},$$

where $diam(\ell_i) = \sup\{d(w, u) : w, u \in \ell_i\}$ and the ball measure of noncompactness (Hausdorff MNC) is

$$\chi(\aleph) = \inf \left\{ \xi > 0 : \aleph \subset \bigcup_{i=1}^{n} B(w_i, \zeta_i),\ w_i \in \mathcal{W},\ \zeta_i < \xi,\ n \in \mathbb{N} \right\}.$$

Lemma 2.1. *[12] Assume that $\aleph, \aleph_1, \aleph_2 \in \mathfrak{M}_\mathcal{W}$. Then*

1) *$\aleph$ is totally bounded as a necessary and sufficient condition for $\chi(\aleph) = 0$,*
2) *$\aleph_1 \subset \aleph_2$ implies that $\chi(\aleph_1) \leq \chi(\aleph_2)$,*
3) $\chi(\overline{\aleph}) = \chi(\aleph)$,
4) $\chi(\aleph_1 \cup \aleph_2) = \max\{\chi(\aleph_1), \chi(\aleph_2)\}$.

Definition 2.3. *[3] Let $\emptyset \neq C \subseteq \Lambda$ and μ be an MNC in a metric space Λ. A mapping $K : C \to C$ is Meir–Keeler condensing if for any $\varepsilon > 0$, there is $\delta > 0$ in which*

$$\varepsilon \leq \mu(\mathcal{W}) < \varepsilon + \delta \ \Rightarrow\ \mu(K(\mathcal{W})) < \varepsilon$$

for any bounded subset $\mathcal{W}$ of C.

Theorem 2.1. *[3] Let $\emptyset \neq C \subseteq \Lambda$ be bounded, convex, and closed, and let μ be an MNC in Λ. If $K : C \to C$ is a continuous Meir–Keeler condensing mapping, then K has a fixed point, and the set of all fixed points of K is compact.*

Remark 2.1. *[12] If Γ is equicontinuous (i.e., for any $\varepsilon > 0$, there is $\delta > 0$ in which if for each $\varsigma, \varsigma' \in I$, $|\varsigma - \varsigma'| < \delta$, then $\|f(\varsigma) - f(\varsigma')\| < \varepsilon$ for each $f \in \Gamma$) and a bounded subset of $C(I, \Lambda)$, then $\mu(\Gamma(.))$ is continuous on I and*

$$\mu(\Gamma) = \sup_{\xi \in I} \mu(\Gamma(\xi)), \quad \mu\Big(\int_0^{\xi} \Gamma(\wp) d\wp\Big) \leq \int_0^{\xi} \mu(\Gamma(\wp)) d\wp.$$

We terminate this section with the description of Bernoulli polynomials.

Definition 2.4. *Bernoulli polynomials can be represented by means of monomials as follows:*

$$B_n(\xi) = \sum_{k=0}^{n} \binom{n}{k} \beta_k \xi^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} \xi^k,$$

where β_k, $k = 0, 1, \ldots, n$, is the Bernoulli number and $\beta_k = B_k(0)$. These polynomials can also be obtained by the use of the following recurrence relation:

$$B_{n+1}(\xi) = (\xi - \frac{1}{2}) B_n(\xi) - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_{n-k+1}(1)}{n-k+1} B_k(\xi).$$

The first five Bernoulli polynomials are as follows:

$$\begin{aligned} B_0(\xi) &= 1 \\ B_1(\xi) &= \xi - \frac{1}{2} \\ B_2(\xi) &= \xi^2 - \xi + \frac{1}{6} \\ B_3(\xi) &= \xi^3 - \frac{3}{2}\xi^2 + \frac{1}{2}\xi \\ B_4(\xi) &= \xi^4 - 2\xi^3 + \xi^2 - \frac{1}{30}. \end{aligned}$$

For more details about Bernoulli polynomials and their properties, error bounds, and convergence analysis, see [31].

3. The space bv_q $(1 \leq q < \infty)$.

In this section, we recall the Banach sequence space bv_q $(1 \leq q < \infty)$, and we define an MNC in this space. By $\hbar$ we mean the space of all complex sequences. By c and c_0 we denote the space of all convergent sequences, and null sequences, respectively. By $e^{(n)}$ $(n = 0, 1, \ldots)$ we mean the sequence so that $e^{(n)}_n = 1$ and $e^{(n)}_k = 0$ for $k \neq n$. For every sequence $w = (w_k)$, take $w^{[n]} = \sum_{k=0}^{n} w_k e^{(k)}$ is the n-section. A sequence space is BK or Banach coordinate if it is a linear metric space (induced by a norm) and complete so that convergence requires coordinatewise convergence. A BK space $\mathcal{W}$ has property AK if for any $w = (w_k) \in \mathcal{W}$, $x = \sum_{n=0}^{\infty} w_n e^{(n)}$. For example, c_0 and c are BK with $\|w\| = \sup_{k\in\mathbb{N}}|w_k|$. Also, for $1 \leq q < \infty$ the space $l_q = \{w = (w_k) \in \hbar : \sum_{k=0}^{\infty} |w_k|^q < \infty\}$ is BK (and has AK property) with the l_q-norm $\|w\| = (\sum_{k=0}^{\infty} |w_k|^q)^{\frac{1}{q}}$ [14, 25].

For sequence spaces $\mathcal{W}$ and $\mathcal{U}$, the class of all infinite matrices A that map $\mathcal{W}$ into $\mathcal{U}$ is denoted by $(\mathcal{W}, \mathcal{U})$. Now if $N = (b_{\iota k})_{\iota,k=0}^{\infty}$ is a complex matrix and N_ι is its ι-th row, then

$$N_\iota(w) = \sum_{k=0}^{\infty} b_{\iota k} w_k \text{ and } N(w) = (N_\iota(w))_{\iota=0}^{\infty}.$$

Also, $N \in (\mathcal{W}, \mathcal{U})$ if and only if $N_\iota(w)$ converges for each $w \in \mathcal{W}$ and ι and $N(w) \in \mathcal{U}$.
The matrix domain of N in $\mathcal{W}$, $\mathcal{W}_N$, [25] is defined by

$$\mathcal{W}_N = \{w \in \hbar : N(w) \in \mathcal{W}\}. \tag{3.1}$$

The difference space of the sequence space l_q $(1 \leq q < \infty)$ [15] is defined as follows:

$$bv_q = \{w = (w_\jmath) \in \hbar : \sum_{\jmath=0}^{\infty} |w_\jmath - w_{\jmath-1}|^q < \infty\}.$$

In view of (3.1), we may consider bv_q by $bv_q = \{l_q\}_\triangle$, where the matrix $\triangle = (\varepsilon_{mk})$ is defined by

$$(\varepsilon_{mk}) = \begin{cases} (-1)^{m-k}, & m-1 \leq k \leq m, \\ 0, & 0 \leq k < m-1, \text{ or } k > m. \end{cases}$$

The space bv_q is a Banach space for all values of q except the case $q = 2$ and $l_q \subseteq bv_q$. The continuous dual space of the sequence space bv_q and more properties of this space are completely studied in (see [17, 24] for more details). The formula for computing MNC in a given normed space or metric space is a rigorous task however in some normed spaces the exact formula is available for ball MNC. We are going to determine the ball MNC χ in the Banach space bv_q. For we need the subsequent result.

Lemma 3.1. *[34] Let $\Upsilon \in \mathfrak{M}_{\mathcal{W}}$, where $\mathcal{W} = l_q$ or c_0. Also, let $\mathfrak{R}_m : \mathcal{W} \to \mathcal{W}$ be the map given by $\mathfrak{R}_m((z_k)) = (z_0, z_1, \ldots, z_m, 0, 0, \ldots)$, then*

$$\chi(\Upsilon) = \lim_{m\to\infty} \{ \sup_{z\in\Upsilon} \|(I - \mathfrak{R}_m)z\|\}.$$

It is simple to verify that for $\Upsilon \in \mathfrak{M}_{l_q}$

$$\chi(\Upsilon) = \lim_{m\to\infty} \{ \sup_{m\in\Upsilon} \sum_{k\geq m} |z_k|^q\}.$$

Theorem 3.1. *[25] For normed sequence space $\mathcal{W}$, let χ_K and χ be the ball MNCs on $\mathfrak{M}_{\mathcal{W}_K}$ and $\mathfrak{M}_{\mathcal{W}}$, respectively. Then*

$$\chi_K(\Upsilon) = \chi(K(\Upsilon)),$$

for any $\Upsilon \in \mathfrak{M}_{\mathcal{W}_K}$.

Now, we are able to compute the ball MNC in the space bv_q.

Corollary 3.1. *[25] Let $K = \triangle$ and $\mathcal{W} = l_q$ for $1 \leq q < \infty$. Then*

$$\chi_K(\Upsilon) = \chi(K(\Upsilon)) = \lim_{m\to\infty} \sup_{w\in K(\Upsilon)} \|(I - \mathfrak{R}_m)(w)\| = \lim_{m\to\infty} \sup_{z\in\Upsilon} \|(I - \mathfrak{R}_m)(K(z))\|$$

for all $\Upsilon \in \mathfrak{M}_{\mathcal{W}_K}$, where

$$\|(I - \mathfrak{R}_m)(K(z))\| = (\sum_{k\geq m} |z_k - z_{k-1}|^p)^{\frac{1}{q}}.$$

Now, we can study the solvability of an infinite system of differential equations of order n with initial conditions (1.1).

4. Solvability results

In this section, we investigate the solvability of the system of differential equations of order $n \geq 2$ (1.1) in the space bv_q.

Consider $I = [\sigma, \varrho]$. Assume that $C^n(I, \mathbb{R})$ is the set of all real-valued n-times continuously differentiable functions on I. Then $\eth \in C^n(I, \mathbb{R})$ is a solution of (1.1) if and only if $\eth \in C(I, \mathbb{R})$ and it is a solution of the infinite system of integral equations:

$$\eth_i(\xi) = \int_\sigma^\varrho G(\xi, \wp) f_i(\wp, \eth(\wp)) d\wp, \quad i = 1, 2, \ldots,$$

where $f_i(\xi, \eth(\xi)) \in C(I, \mathbb{R})$ $(\xi \in I)$ and the associated Green's function is defined by (see [19])

$$G(\xi, \wp) = \begin{cases} \frac{(\varrho-\wp)^{n-1}(\xi-\sigma)^{n-1} - (\varrho-\sigma)^{n-1}(\xi-\wp)^{n-1}}{(\varrho-\sigma)^{n-1}(n-1)!}, & \sigma \leq \wp \leq \xi \leq \varrho, \\ \frac{(\varrho-\wp)^{n-1}(\xi-\sigma)^{n-1}}{(\varrho-\sigma)^{n-1}(n-1)!}, & \sigma \leq \xi \leq \wp \leq \varrho. \end{cases}$$

It is routine to verify that $G(\xi, \wp) \leq \frac{2(\varrho-\sigma)^{n-1}}{n!}$ for all $(\xi, \wp) \in I^2$. Then, we have

$$\eth_i'(\xi) = \int_\sigma^\varrho \frac{\partial}{\partial \xi} G(\xi, \wp) f_i(\wp, \eth(\wp)) d\wp, \ldots, \eth_i^{(n-2)}(\xi) = \int_\sigma^\varrho \frac{\partial^{n-2}}{\partial \xi^{n-2}} G(\xi, \wp) f_i(\wp, \eth(\wp)) d\wp.$$

Here, using Green's function, we transform the system (1.1) into a system of integral equations and focus on the solvability of this new one.
Consider the subsequent hypotheses.
(i) Suppose that for any $i \in \mathbb{N}$, $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ is continuous and $f : I \times bv_q \to bv_q$ is given by

$$(\xi, \eth) \to (f\eth)(\xi) = (f_1(\xi, \eth(\xi)), f_2(\xi, \eth(\xi)), \ldots)$$

so that the family $((f\Game)(\xi))_{\xi\in I}$ is equicontinuous at any point of bv_q.
(ii) For any $\wp \in I$, $k \in \mathbb{N}$, and $\Game \in bv_q$, we have

$$|f_k(\wp, \Game(\wp)) - f_{k-1}(\wp, \Game(\wp))|^q \leq p_k(\wp) + q_k(\wp)|\Game_k(\wp) - \Game_{k-1}(\wp)|^q,$$

when $q_k, p_k \in C([\sigma, \varrho], \mathbb{R}_+)$. Also, $(q_k(\wp))$ is equibounded on interval I (i.e., there exists $M > 0$ such that $q_k(\wp) \leq M$ for any $k \in \mathbb{N}$ and each $\wp \in I$), $\sum_{k=1}^{\infty} p_k(\wp)$ uniformly converges on I, and $\limsup_{k\to\infty} q_k(\wp)$ is integrable on I, let

$$P = \sup_{\wp\in[\sigma,\varrho]} \sum_{k=1}^{\infty} p_k(\wp) < \infty, \text{ and } \aleph = \sup_{k\in\mathbb{N}} \sup_{\wp\in I} q_k(\wp).$$

Theorem 4.1. *If the conditions* (i) *and* (ii) *hold and* $(\frac{n!}{2})^q \frac{1}{(\varrho-\sigma)^{nq}} > \aleph$*, then the system* (1.1) *has a solution* $\Game = (\Game_i) \in C(I, bv_q)$.

Proof. Let $\Game = (\Game_i)$ be a sequence of continuous functions and be a solution of the system (1.1). The space $C(I, bv_q)$ is a normed space with

$$||\Game||_{C(I,bv_q)} = \sup_{\xi\in I} ||\Game(\xi)||_{bv_q}.$$

Using our assumptions, we obtain

$$\begin{aligned}
||(\Game)(\xi)||^q_{bv_q} &= \sum_{k=1}^{\infty} |\int_{\sigma}^{\varrho} G(\xi, \wp)(f_k(\wp, \Game(\wp)) - f_{k-1}(\wp, \Game(\wp)))d\wp|^q \\
&\leq \sum_{k=1}^{\infty} (\int_{\sigma}^{\varrho} |G(\xi, \wp)|^p d\wp)^{\frac{q}{p}} (\int_{\sigma}^{\varrho} |(f_k(\wp, \Game(\wp)) - f_{k-1}(\wp, \Game(\wp)))|^p d\wp) \\
&\leq \sum_{k=1}^{\infty} (\int_{\sigma}^{\varrho} |G(\xi, \wp)|^p d\wp)^{\frac{q}{p}} (\int_{\sigma}^{\varrho} p_k(\wp) + q_k(\wp)|\Game_k(\wp) - \Game_{k-1}(\wp)|^q d\wp) \\
&\leq \frac{2^q}{(n!)^q}(\varrho - \sigma)^{nq-1}(P(\varrho - \sigma) + \aleph \sum_{k=1}^{\infty} \int_{\sigma}^{\varrho} |\Game_k(\wp) - \Game_{k-1}(\wp)|^q d\wp) \\
&\leq \frac{2^q}{(n!)^q}(\varrho - \sigma)^{nq-1}(P(\varrho - \sigma) + \aleph ||\Game||^q_{C(I,bv_q)}(\varrho - \sigma)).
\end{aligned}$$

We infer that

$$||\Game||^p_{C(I,bv_q)} \leq (\frac{2}{(n!)})^q (\varrho - \sigma)^{nq-1}(P(\varrho - \sigma) + \aleph ||\Game||^q_{C(I,bv_q)}(\varrho - \sigma)).$$

It implies that

$$||\Game||^p_{C(I,bv_q)}\Big((\frac{n!}{2})^q \frac{1}{(\varrho - \sigma)^{nq-1}} - \aleph(\varrho - \sigma)\Big) \leq P(\varrho - \sigma).$$

The above inequality can be written as

$$||\Game||_{C(I,bv_q)} \leq \frac{P(\varrho - \sigma)^{nq}}{(\frac{n!}{2})^q - \aleph(\varrho - \sigma)^{nq}} = r_0.$$

Take

$$B = B(\eth^0, r_0) = \{\eth = (\eth_i) \in C(I, bv_q) : \ \|\eth\|_{C(I,bv_q)} \leq r_0, \eth_i(0) = \eth_i^{'}(0) = \ldots = \eth_i^{(n-2)}(0) = 0, \ \forall i \in \mathbb{N}\},$$

where $\eth^0(\xi) = (\eth_i^0(\xi))$ and $\eth_i^0(\xi) = 0$ for all $\xi \in I$. Take the map $K : C(I, B) \to C(I, B)$ defined by

$$(K\eth)(\xi) = \int_\sigma^\varrho G(\xi, \wp) f(\wp, \eth(\wp)) d\wp.$$

Clearly, B is convex, closed, and bounded and the map K is bounded.
Now, we aim to verify that K is continuous. For, let $\eth_1 \in B$ and $\varepsilon > 0$. Owing to (i), there exists $\delta > 0$ such that for any $\eth_2 \in B$,

$$\|\eth_1 - \eth_2\|_{C(I,bv_q)} \leq \delta \Rightarrow \|(f\eth_1) - (f\eth_2)\|_{C(I,bv_q)} \leq \frac{\varepsilon n!}{2(\varrho - \sigma)^n}.$$

Hence, for each ξ in $[\sigma, \varrho]$, we have

$$\begin{aligned}
&\|(K\eth_1)(\xi) - (K\eth_2)(\xi)\|_{bv_q}^q \\
&= \sum_{k=1}^{\infty} \left| \int_\sigma^\varrho G(\xi, \wp)[f_k(\wp, \eth_1(\wp)) - f_k(\wp, \eth_2(\wp)) - (f_{k-1}(\wp, \eth_1(\wp)) - f_{k-1}(\wp, \eth_2(\wp)))] d\wp \right|^q \\
&\leq (\frac{2}{n!})^q (\varrho - \sigma)^{nq} \sup_{\xi \in [\sigma, \varrho]} \|f_k(\xi, \eth_1(\xi)) - f_k(\xi, \eth_2(\xi))\|_{bv_q}^q \\
&= (\frac{2}{n!})^q (\varrho - \sigma)^{nq} \|(f\eth_1) - (f\eth_2)\|_{C(I,bv_q)}^q \\
&\leq \varepsilon^q.
\end{aligned}$$

Accordingly, we obtain

$$\|(K\eth_1) - (K\eth_2)\|_{C(I,bv_q)}^q \leq \varepsilon^q,$$

which implies the continuity of K.
Next, we show $(K\eth)$ is continuous on (σ, ϱ). Take $\xi_1 \in (\sigma, \varrho)$ and $\varepsilon > 0$. Since $G(., \wp)$ is continuous, then there is $\delta = \delta(\xi_1, \varepsilon) > 0$ so that if $|\xi - \xi_1| < \delta$, then $|G(\xi, \wp) - G(\xi_1, \wp)| < \frac{\varepsilon}{(\varrho - \sigma)(P + \aleph \|\eth\|_{C(I,bv_q)}^q)^{\frac{1}{q}}}$. We can write

$$\begin{aligned}
\|(K\eth)(\xi) - (K\eth)(\xi_1)\|_{bv_q}^q &= \sum_{k=1}^{\infty} \left| \int_\sigma^\varrho (G(\xi, \wp) - G(\xi_1, \wp))(f_k(\wp, \eth(\wp)) - f_{k-1}(\wp, \eth(\wp))) d\wp \right|^q \\
&\leq \sum_{k=1}^{\infty} \Big(\int_\sigma^\varrho |(G(\xi, \wp) - G(\xi_1, \wp))|^p d\wp \Big)^{\frac{q}{p}} \int_\sigma^\varrho |f_k(\wp, \eth(\wp)) - f_{k-1}(\wp, \eth(\wp))|^q d\wp \\
&\leq \frac{\varepsilon^q}{(\varrho - \sigma)^q (P + \aleph \|\eth\|_{C(I,bv_q)}^q)} (\varrho - \sigma)^{\frac{q}{p}} (P(\varrho - \sigma) + \aleph \|\eth\|_{C(I,bv_q)}^q (\varrho - \sigma)) = \varepsilon^q.
\end{aligned}$$

Finally, we prove that K is a Meir–Keeler condensing map. Regarding Remark 2.1 and Corollary 3.1, the ball MNC in $B \subset C(I, bv_q)$ can be defined by

$$\chi_{C(I,bv_q)}(B) = \sup_{\xi \in I} \chi_{bv_q}(B(\xi)),$$

where $B(\xi) = \{\eth(\xi) : \eth \in B\}$. Hence, we obtain

$$\begin{aligned}
(\chi_{bv_q}(KB)(\xi))^q &= \lim_{n\to\infty}\{\sup_{\eth\in B}\sum_{k=n}^{\infty}|\int_{\sigma}^{\varrho} G(\xi,\wp)\big(f_k(\wp,\eth(\wp)) - f_{k-1}(\wp,\eth(\wp))\big)d\wp|^p\} \\
&\leq \lim_{n\to\infty}\{\sup_{\eth\in B}\sum_{k=n}^{\infty}((\int_{\sigma}^{\varrho}|G(\xi,\wp)|^p d\wp)^{\frac{q}{p}}(\int_{\sigma}^{\varrho}|f_k(\wp,\eth(\wp)) - f_{k-1}(\wp,\eth(\wp))|^q d\wp))\} \\
&\leq \frac{2^q(\varrho-\sigma)^{nq-1}}{(n!)^q}\lim_{n\to\infty}\{\sup_{\eth\in B}\sum_{k=n}^{\infty}\int_{\sigma}^{\varrho} p_k(\wp) + q_k(\wp)|\eth_k(\wp) - \eth_{k-1}(\wp)|^q d\wp\} \\
&\leq \frac{2^q(\varrho-\sigma)^{nq-1}}{(n!)^q}\aleph\lim_{n\to\infty}\{\sup_{\eth\in B}\sum_{k=n}^{\infty}\int_{\sigma}^{\varrho}|\eth_k(\wp) - \eth_{k-1}(\wp)|^q d\wp\}. \\
&\leq \frac{2^q(\varrho-\sigma)^{nq-1}}{(n!)^q}\aleph(\varrho-\sigma)\sup_{\xi\in I}\lim_{n\to\infty}\sup_{\eth\in B}\sum_{k=n}^{\infty}|\eth_k(\xi) - \eth_{k-1}(\xi)|^q \\
&\leq \frac{2^q(\varrho-\sigma)^{nq}}{(n!)^q}\aleph(\chi_{C(I,bv_q)}(B))^q.
\end{aligned}$$

We deduce that

$$(\chi_{C(I,bv_q)}(KB))^q \leq \frac{2^q(\varrho-\sigma)^{nq}}{(n!)^q}\aleph(\chi_{C(I,bv_q)}(B))^q.$$

Let $\varepsilon > 0$ be arbitrary. Now if

$$\chi_{C(I,bv_q)}(KB) \leq \frac{2(\varrho-\sigma)^n}{n!}\aleph^{\frac{1}{q}}\chi_{C(I,bv_q)}(B) < \varepsilon,$$

then

$$\chi_{C(I,bv_q)}(B) < \frac{1}{\frac{2(\varrho-\sigma)^n}{n!}\aleph^{\frac{1}{q}}}\varepsilon.$$

Put $\delta = \varepsilon(\frac{1}{\frac{2(\varrho-\sigma)^n}{n!}\aleph^{\frac{1}{q}}} - 1)$. One can check that K is a Meir–Keeler condensing map on $B \subset bv_q$. In virtue of Theorem 2.1, K possesses a fixed point in B, and so (1.1) is solvable in $C(I, bv_q)$. □

As we can see, the conditions (i) and (ii), together with the assumption $(\frac{n!}{2})^q\frac{1}{(\varrho-\sigma)^{nq}} > \aleph$, guarantee the existence of solutions of the system (1.1).

Example 4.1. *Consider the following system of third-order differential equations*

$$\eth_i^{(3)}(\xi) + \frac{\xi}{5}\sum_{k=1}^{i}\frac{k+2}{k+1} + \frac{\sin(\eth_k(\xi))\cos(2\xi+3)\tanh(\eth_k(\xi)+1)}{e^{\xi}} = 0,\ i = 1,2,\ldots \tag{4.1}$$

with the boundary conditions given by

$$\eth_i(1) = \eth_i'(1) = 0 \text{ and } \eth_i(1.7) = 0, \tag{4.2}$$

also

$$\eth_i^{(3)}(1) + \frac{1}{5}\sum_{k=1}^{i}\frac{k+2}{k+1} + \frac{\sin(\eth_k(1))\cos(5)\tanh(\eth_k(1)+1)}{e^1} = 0$$

and

$$v_i^{(3)}(1.7) + \frac{1.7}{5}\sum_{k=1}^{i}\frac{k+2}{k+1} + \frac{\sin(\supset_k(1.7))\cos(6.4)\tanh(\supset_k(1.7)+1)}{e^{1.7}} = 0$$

where $\xi \in [1, 1.7]$, *and* $i = 1, 2, \ldots$. *Equation (4.1) is clearly a special case of equation (1.1) when*

$$f_i(\xi, \supset(\xi)) = \frac{\xi}{5}\sum_{k=1}^{i}\frac{k+2}{k+1} + \frac{\sin(\supset_k(\xi))\cos(2\xi+3)\tanh(\supset_k(\xi)+1)}{e^{\xi}}.$$

We are going to show that the condition (ii) *of Theorem 4.1 is true. For, take* $\xi \in [1, 1.7]$ *and* $\supset \in bv_q$, *we have*

$$\begin{aligned}
&|f_i(\xi, \supset(\xi)) - f_{i-1}(\xi, \supset(\xi))|^q \\
\leq\ & 2^q\Big(\Big|\frac{\xi}{5}(\sum_{k=1}^{i}\frac{k+2}{k+1} - \sum_{k=i-1}^{i}\frac{k+2}{k+1})\Big|^q \\
&+2^q\Big|\frac{\sin(\supset_i(\xi))\cos(2\xi+3)\tanh(\supset_i(\xi)+1)}{e^{\xi}} - \frac{\sin(\supset_{i-1}(\xi))\cos(2\xi+3)\tanh(\supset_{i-1}(\xi)+1)}{e^{\xi}}\Big|^q\Big) \\
=\ & (\frac{4\xi}{5})^q(\frac{1}{i(i+1)})^q + 2^q|\cos(2\xi+3)|^q\Big|\sin(\supset_i(\xi)+1)(\tanh(\supset_i(\xi)+1) - \tanh(\supset_{i-1}(\xi)+1)) \\
&-\tanh(\supset_{i-1}(\xi)+1)(\sin(\supset_i(\xi)) - \sin(\supset_{i-1}(\xi)))\Big|^q \\
\leq\ & (\frac{4\xi}{5})^q\frac{1}{i^{2q}} + 2\times 4^q|\cos(2\xi+3)|^q|\supset_i(\xi) - \supset_{i-1}(\xi)|^q.
\end{aligned}$$

For any $i \in \mathbb{N}$, *let*

$$p_i(\xi) = (\frac{4\xi}{5})^q\frac{1}{i^{2q}},$$

and

$$q_i(\xi) = 2\times 4^q|\cos(2\xi+3)|^q.$$

It is easy to verify that $\sum_{i=1}^{\infty} p_i(\xi)$ *is uniformly convergent on interval* I, $(q_i(\xi))$ *is equibounded on* I *and* $\limsup_{i\to\infty} q_i(\xi)$ *is integrable on* I. *Moreover,* $P = (\frac{4\varrho}{5})^p \sum_{i=1}^{\infty}\frac{1}{i^{2q}}$ *and* $\aleph \leq 8^q$.

Taking into account Theorem 4.1, the infinite system of differential equations of order 3 *(4.1) has at least a solution in* $C(I, bv_q)$ *for all* $q \geq 1$.

Example 4.2. *Consider the following system of differential equations of order* 5

$$\supset_i^{(5)}(\xi) + \frac{2\xi}{7}\sum_{k=1}^{i}\frac{1}{(k+1)(k+2)(k+3)} + \ln(1+|\supset_i(\xi)|)\sin(3\xi+1) = 0 \tag{4.3}$$

with the boundary conditions given by

$$\supset_i(2) = \supset_i'(2) = \supset_i''(2) = \supset_i'''(2) = 0 \text{ and } \supset_i(4.5) = 0,$$

also

$$\Game_i^{(5)}(2) + \frac{4}{7}\sum_{k=1}^{i}\frac{1}{(k+1)(k+2)(k+3)} + \ln(1+|\Game_i(2)|)\sin(7) = 0$$

and

$$\Game_i^{(5)}(4.5) + \frac{9}{7}\sum_{k=1}^{i}\frac{1}{(k+1)(k+2)(k+3)} + \ln(1+|\Game_i(4.5)|)\sin(14.5) = 0$$

where $\xi \in [2, 4.5]$*, and* $i = 1, 2, \ldots$*. Obviously, Eq (4.3) is a special case of Eq (1.1) when*

$$f_i(\xi, \Game(\xi)) = \frac{2\xi}{7}\sum_{k=1}^{i}\frac{1}{(k+1)(k+2)(k+3)} + \ln(1+|\Game_i(\xi)|)\sin(3\xi+1).$$

We are going to show that the condition (ii) *of Theorem 4.1 is true. For, take* $\wp \in [2, 4.5]$ *and* $\Game \in bv_q$*, we have*

$|f_i(\xi, \Game(\xi)) - f_{i-1}(\xi, \Game(\xi))|^q$

$$\begin{aligned}
&\leq 2^q\Big|\frac{2\xi}{7}\Big(\sum_{k=1}^{i}\frac{1}{(k+1)(k+2)(k+3)} - \sum_{k=1}^{i-1}\frac{1}{(k+1)(k+2)(k+3)}\Big)\Big|^q \\
&\quad + 2^q\Big|\sin(3\xi+1)(\ln(1+|\Game_i(\xi)|) - \ln(1+|\Game_{i-1}(\xi)|))\Big|^q \\
&= \Big(\frac{4\xi}{7}\Big)^q\Big(\frac{1}{(i+1)(i+2)(i+3)}\Big)^q + 2^q|\sin(3\xi+1)|^q|\ln\Big(\frac{1+|\Game_i(\xi)|}{1+|\Game_{i-1}(\xi)|}\Big)|^q \\
&\leq \Big(\frac{4\xi}{7}\Big)^q\Big(\frac{1}{i^{3p}}\Big) + 2^q|\sin(3\xi+1)|^q|\ln\Big(1+\frac{|\Game_i(\xi)| - |\Game_{i-1}(\xi)|}{1+|\Game_{i-1}(\xi)|}\Big)|^q \\
&\leq \Big(\frac{4\xi}{7}\Big)^q\Big(\frac{1}{i^{3p}}\Big) + 2^q|\sin(3\xi+1)|^q|\Game_i(\xi) - \Game_{i-1}(\xi)|^q.
\end{aligned}$$

For any $i \in \mathbb{N}$*, let*

$$p_i(\xi) = \Big(\frac{4\xi}{7}\Big)^q\frac{1}{i^{3q}},$$

and

$$q_i(\xi) = 2^q|\sin(3\xi+1)|^q.$$

It is easy to verify that $\sum_{i=1}^{\infty} p_i(\xi)$ *is uniformly convergent on interval* I, $(q_i(\xi))$ *is equibounded on* I *and* $\limsup_{i\to\infty} q_i(\xi)$ *is integrable on* I*. Moreover,* $P = (\frac{4\xi}{7})^q\sum_{i=1}^{\infty}\frac{1}{i^{3q}}$ *and* $\aleph \leq 2^q$*.*
Taking into account Theorem 4.1, the infinite system of differential equations of order 5 *(4.3) has at least a solution in* $C(I, bv_q)$ *for all* $q \geq 1$*.*

In the next section, we present a numerical method to solve Example 4.1 approximately.

5. Numerical results

In this section, we propose a numerical spectral method based on Bernoulli polynomials to solve system (4.1).

To solve the infinite system (4.1), we consider a finite Bernoulli spectral expansion approximation for each function $\eth_i(\xi)$ as follows:

$$\eth_i(\xi) \simeq \sum_{j=0}^{n} c_{i,j} B_j(\xi), \tag{5.1}$$

where $c_{i,j}$ ($i = 1, 2, \ldots,\ j = 0, 1, 2, \ldots$) are unknown constants.
Replacing (5.1) into (4.1), we have

$$\Big(\sum_{j=1}^{n} c_{i,j} B_j(\xi)\Big)''' = -\frac{\xi}{5} \sum_{k=1}^{i} \Big(\frac{k+2}{k+1} + \cos(2\xi + 3) e^{-\xi} \sin\big(\sum_{j=0}^{n} c_{k,j} B_j(\xi)\big) \tanh\big(\sum_{j=0}^{n} c_{k,j} B_j(\xi) + 1\big)\Big). \tag{5.2}$$

For Bernoulli polynomials we know that

$$B_n'(\xi) = n B_{n-1}(\xi).$$

Thus, the left-hand side of system (5.2) simplifies to

$$\Big(\sum_{j=0}^{n} c_{i,j} B_j(\xi)\Big)''' = \sum_{j=0}^{n} j(j-1)(j-2) c_{i,j} B_{j-3}(\xi).$$

Finally, the infinite system of 3-rd-order differential equations (5.2) can be rewritten as the following nonlinear algebraic system:

$$\sum_{j=0}^{n} j(j-1)(j-2) c_{i,j} B_{j-3}(\xi) = -\frac{\xi}{5} \sum_{k=1}^{i} \Big(\frac{k+2}{k+1} + \cos(2\xi + 3) e^{-\xi} \sin\big(\sum_{j=0}^{n} c_{k,j} B_j(\xi)\big) \tanh\big(\sum_{j=0}^{n} c_{k,j} B_j(\xi) + 1\big)\Big). \tag{5.3}$$

In order to solve the system (5.3) with the boundary conditions (4.2), we use the collocation method, i.e., we divide the interval [1, 1.7] into seven equally spaced subintervals and consider the points $\xi_m = 1 + 0.1m$ ($m = 1, 2, \ldots 6$) in this interval. At points ξ_0 and ξ_7 we use the boundary conditions (4.2), and at other points $\xi_1, \ldots, \xi_6$ we collocate the system (5.3). Hence, solving the system (5.3) changes to solving the following non-linear algebraic system with unknowns $c_{i,j}$:

$$\sum_{j=0}^{n} j(j-1)(j-2) c_{i,j} B_{j-3}(\xi_m)$$

$$= -\frac{\xi_m}{5} \sum_{k=1}^{i} \Big(\frac{k+2}{k+1} + \cos(2\xi_m + 3) e^{-\xi} \sin\big(\sum_{j=0}^{n} c_{k,j} B_j(\xi_m)\big) \tanh\big(\sum_{j=0}^{n} c_{k,j} B_j(\xi_m) + 1\big)\Big). \tag{5.4}$$

The numerical calculations are done with Maple 18 software. The function "fsolve" is used to solve the nonlinear system (5.3). Although the results of this function may lead to a local minimum, in the case of our example, we can see the good accuracy of the obtained solution. In fact, in the case of large non-linear systems, the proposed numerical algorithms mostly lead to a local minimum, not a global one.

In this paper, to be able to do the calculations, we consider only a finite system of third, order differential equations. Actually we consider the system (5.3) with just ten differential equations. i.e., $i = 1, 2, \ldots, 10$.

After solving the system (5.3), ten approximate functions $\widetilde{\eth_1}(\xi), \widetilde{\eth_2}(\xi), \ldots, \widetilde{\eth_{10}}(\xi)$ are obtained based on the truncated Bernoulli's expansion series (5.1). The graphs of these functions are plotted in Figure 1.

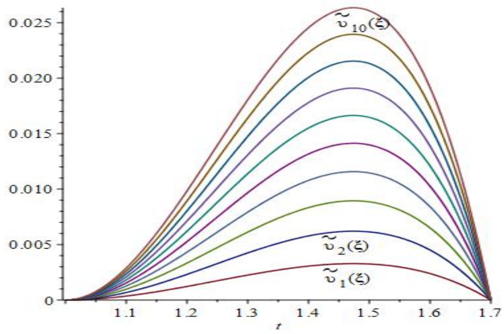

Figure 1. The graphs of approximate solutions $\eth_1(\xi)$,...,$\eth_{10}(\xi)$ obtained by showing the nonlinear system (4.1).

As one can see, the behavior of these functions is similar, and Figure 1 can give an insight to the solutions of an infinite system of differential equations. In order to verify the accuracy of our results, we define the following error functions based on the equation (4.1)

$$error_i(\xi) = \widetilde{\eth}_i^{(3)}(\xi) + \frac{\xi}{5}\sum_{k=1}^{i}\frac{k+2}{k+1} + \frac{\sin(\widetilde{\eth_k}(\xi))\cos(2\xi+3)\tanh(\widetilde{\eth_k}(\xi)+1)}{e^{\xi}}, \; i = 1, 2, \ldots$$

It is clear that if the $\widetilde{\eth_k}(\xi)$ functions are good approximation, then $error_i(\xi)$ for $i = 1, 2, \ldots, 10$ must tend to zero. The absolute error graph related to these functions are plotted in Figure 2, which shows the high accuracy of the obtained approximate solutions.

Instead of Bernoulli polynomials, we have also used Legendre and Chebyshev polynomials, and similar results are obtained. It is due to the fact that Legendre, Chebyshev, and Bernoulli functions are all polynomials. Meanwhile, the property (5.3) of Bernoulli functions helps us to find a simpler representation for the non-linear system (5.3). On the other hand, to use Legendre or Chebyshev functions in the interval $[1, 1.7]$, we have to use the shifted Legendre or Chebyshev functions to be able to use their orthogonality property, while in the case of Bernoulli polynomials, there is no need to shift them.

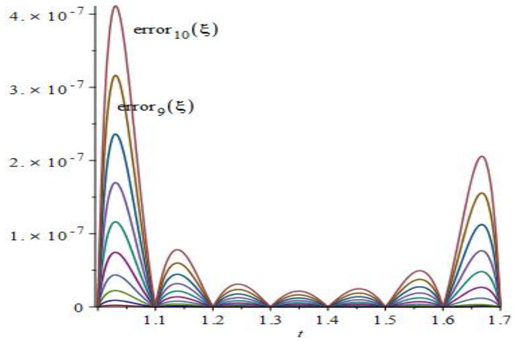

Figure 2. The absolute error graphs of the functions $error_i(\xi)$ $i = 1, 2, \ldots, 10$ (4.1).

6. Conclusions

In this paper, we have introduced a ball measure of noncompactness in the Banach sequence space bv_q $(1 \leq q < \infty)$ containing the space l_q. By applying the technique of measure of noncompactness, we investigated the existence of solutions of an infinite system of n-th order differential equations with boundary conditions in the Banach sequence space bv_q $(1 \leq q < \infty)$. Then, we proposed a numerical spectral method based on Bernoulli polynomials to find the solution of these systems. We applied this method for a finite system of differential equations to be able to plot the answers. For further works, we suggest solving various types of infinite systems of ordinary differential equations using the technique of measures of noncompactness in the new sequence spaces, such as n-tuple sequence spaces. Furthermore, one can apply other numerical methods to find the solutions of this system of equations.

Author contributions

Hojjatollah Amiri Kayvanloo: Writing-original draft, data curation; Mohammad Mehrabinezhad: Writing-original and review the numerical results; Mahnaz Khanehgir: Writing-review and editing; Reza Allahyari: Investigation, resources; Mohammad Mursaleen: Supervision, validation, editing. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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