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Research article

Slicing method for nonlinear integral inequalities related to critical nonlinear wave equations

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Abstract: This paper is devoted to a simple and short proof on the sharp upper bound of lifespan of classical solutions to wave equations with the critical power nonlinearities of spatial derivatives of the unknown function. Such a proof is so-called "slicing method", which may help us to extend the result for various equations and systems.

Keywords: nonlinear wave equation; blow-up; critical power; ordinary differential inequality; lifespan; slicing method
Mathematics Subject Classification: 35B44, 35L71

1. Introduction

In this paper, we are focusing on the following system of integral inequalities for unknown functions $H \in C([R,T))$:

$$\begin{cases} H(t) \geq At^a(\log t)^{-b}\left(\log \dfrac{t}{R}\right)^c, & t \in [R,T), \\ H(t) \geq B(\log t)^x \displaystyle\int_R^t ds \int_R^s r^y \left(\log \frac{r}{R}\right)^z |H(r)|^p dr, & t \in [R,T), \end{cases} \tag{1.1}$$

where all $A > 0$, $B > 0$, and $T > R > 1$ are constants. We assume that exponents a, b, c, x, y, z, p satisfy

$$\begin{cases} p > 1,\ a \leq 1,\ b \geq \max\left\{0, \dfrac{x}{p-1}\right\}, \\ y + pa = -1,\ z + cp > -1,\ z + cp \geq c - 1. \end{cases} \tag{1.2}$$

When

$$a = 1,\ b = 0,\ c = 1,\ x = -p,\ y = -p - 1,\ z = 1,$$

(1.1) can be found in Shao, Takamura, and Wang [1, Lemma 3.1], which leads to blow-up, as well as the optimal upper bound of the lifespan, of classical solutions to wave equations with critical power nonlinearities of spatial derivative-type;

$$\begin{cases} u_{tt} - \Delta_x u = |\nabla_x u|^{(n+1)/(n-1)} & \text{in } \mathbf{R}^n \times (0, T), \\ u(x, 0) = \varepsilon f(x),\ u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3}$$

where $n \geq 2$, $f, g \in C_0^\infty(\mathbf{R}^n)$ and $0 < \varepsilon \ll 1$. See the introduction of [1] and references therein for the background and criticality of the problem (1.3). The lifespan $T(\varepsilon)$ is defined as the maximal existence time of the solution, and [1] shows

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-2/(n-1)}\right), \tag{1.4}$$

where C is a positive constant independent of ε. We remark that there is no critical case in one space dimension; see Sasaki, Takamatsu, and Takamura [2] on the equation $u_{tt} - u_{xx} = |u_x|^p$ $(p > 1)$, and see Haruyama and Takamura [3] for its application to quasilinear versions. If $\nabla_x u$ in the nonlinear term in (1.3) is replaced with u_t, the proof of the blow-up is completely different and is reduced to a comparison with one first-order ordinary differential equation; see Zhou [4] for its details, which are available in all space dimensions.

The purpose of this paper is not only to give a short and simple proof of Lemma 3.1 in [1], which is the key lemma to show (1.4), but also to extend the results under a more general setting on the exponents as in (1.2) by making use of the so-called "slicing method", which enables us to prove blow-up of H in (1.1) by an iteration argument with logarithmic terms. We note that the proof of Lemma 3.1 in [1] has 5 pages with relatively complicated arguments. But one can see below that we have 2 pages with simple computations even for more general cases. The iteration argument was first introduced to the point-wise estimate of the solution to semilinear wave equations $u_{tt} - \Delta_x u = |u|^p$ $(p > 1)$ in three space dimensions by John [5] to show blow-up in the sub-critical cases. And it was first combined with the functional method by Lai and Takamura [6] to show the sharp upper bound of the lifespan in the sub-critical case of energy solutions to semilinear damped wave equations in the scattering case $u_{tt} - \Delta_x u + \mu u_t/(1 + t)^\beta = |u|^p$ $(p > 1, \mu > 0, \beta > 1)$ in all space dimensions greater than 1. The critical cases cannot be covered by these methods due to the specialty of this nonlinear term $|u|^p$. See Ikeda, Sobajima, and Wakasa [7], including references therein, for the history and the latest proof of the critical case of semilinear wave equations, and see Wakasa and Yordanov [8] for those of damped wave equations. Roughly speaking, as shown in the table below, we can summarize critical exponents for nonlinear wave equations with small data, including some open parts though.

Equation	Critical exponent
$u_{tt} - \Delta_x u = \lvert u_t\rvert^p$	$p_G(n)$
$u_{tt} - \Delta_x u = \lvert\nabla_x u\rvert^p$	$p_G(n)$
$u_{tt} - \Delta_x u = \lvert u\rvert^p$	$p_S(n)$

Here, the so-called Glassey exponent $p_G(n)$ and Strauss exponent $p_S(n)$ are defined by

$$p_G(n) := \frac{n+1}{n-1}, \quad p_S(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n-1)} \quad (n \geq 2).$$

Since the blow-up results for critical and sub-critical exponents are established, the estimate of the lifespan $T(\varepsilon)$ becomes our primary concern.

Finally, we point out that the slicing method was first introduced by Agemi, Kurokawa, and Takamura [9] to obtain the optimal upper bound of the lifespan of the solutions to weakly coupled systems of semilinear wave equations in three space dimensions. Later, this method has been applied to various equations. For example, see Wakasa and Yordanov [10] for the variable coefficient case and Kitamura, Takamura, and Wakasa [11] for weighted nonlinearities in one dimension. Therefore, one can expect to extend our result to more various nonlinearities than (1.3) and also to weakly coupled systems easily.

This paper is organized as the theorem and its proof are in the next section. The concluding remarks are added at the end of this paper. This work was carried out when the first author had employed in a cross-appointment system between Tohoku Univ. and Musashino Univ. on Apr. 2020 – Mar. 2025, and during the second author's stay in Tohoku Univ. on Nov. 2024 – Apr. 2025.

2. Theorem and proof

We shall prove the following theorem.

Theorem 2.1. *Let $H \in C([R,T))$ be a solution to* (1.1)*. Under assumption* (1.2)*, T has to satisfies*

$$T \leq \exp\left(\max\left\{2\log R_\infty, (A^{-1}D)^{\frac{p-1}{x+z+1+(c-b)(p-1)}}\right\}\right), \tag{2.1}$$

where

$$R_\infty = R\prod_{k=1}^{\infty}(1+2^{-k}),$$
$$D = 2^{\left(c+\frac{p+(z+1)(p-1)}{(p-1)^2}\right)}p^{\frac{p}{(p-1)^2}}\left(\frac{1}{B}\max\left\{c+\frac{z+1}{p-1}, c+\frac{z+1}{p}\right\}\right)^{\frac{1}{p-1}} > 0.$$

Remark 2.1. *When $a=1$, $b=0$, $c=1$, $x=-p$, $y=-p-1$, $z=1$, the estimate* (2.1) *becomes*

$$T \leq \exp\left(\max\left\{2\log R_\infty, (A^{-1}D)^{p-1}\right\}\right)$$

and yields the result in Lemma 3.1 of [1]. Then, using modified Rammaha's functionals, we can obtain estimates (1.4) *for the solutions to Eq* (1.3)*; see [1] for details.*

Proof of Theorem 2.1. Substituting H in the right-hand side of the first line in (1.1) into the right-hand side of the second line in (1.1), we have that for $t \geq R$

$$\begin{aligned} H(t) &\geq A^pB(\log t)^x\int_R^t ds\int_R^s r^{y+pa}(\log r)^{-pb}\left(\log\frac{r}{R}\right)^{pc+z}dr \\ &\geq A^pB(\log t)^{-(pb-x)}\int_R^t ds\int_R^s r^{-1}\left(\log\frac{r}{R}\right)^{pc+z}dr \\ &= \frac{A^pB}{pc+z+1}(\log t)^{-(pb-x)}\int_R^t\left(\log\frac{s}{R}\right)^{pc+z+1}ds, \end{aligned}$$

because $b \geq 0$, $y + pa = -1$, and $pc + z + 1 > 0$ by assumption (1.2). Then, for $\delta > 0$ and $t \geq (1+\delta)R$, $H(t)$ can be estimated from below as

$$\begin{aligned} H(t) &\geq \frac{A^p B}{pc+z+1}(\log t)^{-(pb-x)} \int_{\frac{t}{1+\delta}}^{t} \left(\log \frac{s}{R}\right)^{pc+z+1} ds \\ &\geq \frac{\delta A^p B}{(1+\delta)(pc+z+1)} t (\log t)^{-(pb-x)} \left(\log \frac{t}{(1+\delta)R}\right)^{pc+z+1} \\ &\geq \frac{\delta A^p B}{(1+\delta)(pc+z+1)} t^a (\log t)^{-(pb-x)} \left(\log \frac{t}{(1+\delta)R}\right)^{pc+z+1}, \end{aligned} \tag{2.2}$$

since $a \leq 1$.

Now, set $\delta_j := 2^{-j}$ for $j \in \mathbf{N}$ and define sequences $\{b_j\}$, $\{c_j\}$, $\{R_j\}$, and $\{A_j\}$ by

$$\begin{cases} b_{j+1} = pb_j - x, & b_0 = b, \\ c_{j+1} = pc_j + z + 1, & c_0 = c, \\ R_{j+1} = (1+\delta_j)R_j, & R_0 = R, \\ A_{j+1} = \dfrac{\delta_j A_j^p B}{(1+\delta_j)(pc_j + z + 1)}, & A_0 = A. \end{cases}$$

Then, by direct calculations, it follows that for $j \in \mathbf{N}$

$$\begin{aligned} & b_j = p^j\left(b - \frac{x}{p-1}\right) + \frac{x}{p-1}, \quad c_j = p^j\left(c + \frac{z+1}{p-1}\right) - \frac{z+1}{p-1}, \\ & R_{j+1} = R\prod_{k=0}^{j}(1+2^{-k}). \end{aligned}$$

Since

$$b \geq \max\left\{0, \frac{x}{p-1}\right\}, \ z + cp > -1, \ \text{and } z + cp \geq c - 1,$$

we have $b_j \geq 0$ and $c_{j+1} > 0$ for all $j \in \mathbf{N}$. Thus, according to the calculation in (2.2), we deduce that

$$H(t) \geq A_j t^a (\log t)^{-b_j} \left(\log \frac{t}{R_j}\right)^{c_j} \quad \text{for } t \geq R_j$$

implies

$$H(t) \geq A_{j+1} t^a (\log t)^{-b_{j+1}} \left(\log \frac{t}{R_{j+1}}\right)^{c_{j+1}} \quad \text{for } t \geq R_{j+1}.$$

Note that

$$\begin{aligned} \log A_{j+1} &= p \log A_j + \log B - \log(1+2^j) - \log c_{j+1} \\ &\geq p \log A_j - (j+1)\log(2p) - \log C, \end{aligned} \tag{2.3}$$

where

$$C := \frac{1}{B}\max\left\{c + \frac{z+1}{p-1}, c + \frac{z+1}{p}\right\} > 0.$$

Then, inequality (2.3) is equivalent to

$$\begin{aligned}&\log A_{j+1}-(j+1)\frac{\log(2p)}{p-1}-\frac{\log\left((2p)^{\frac{p}{p-1}}C\right)}{p-1}\\&\geq p\left(\log A_j-j\frac{\log(2p)}{p-1}-\frac{\log\left((2p)^{\frac{p}{p-1}}C\right)}{p-1}\right).\end{aligned}$$

Hence, we obtain that

$$\log A_j\geq p^j\log\left(\frac{A}{(2p)^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}}\right)+j\frac{\log(2p)}{p-1}+\log\left((2p)^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}\right)$$

for $j\in\mathbf{N}$.

Let R_∞ denote $R\prod_{k=1}^{\infty}(1+2^{-k})$. It follows that

$$\begin{aligned}H(t)&\geq A_jt^a(\log t)^{-b_j}\left(\log\frac{t}{R_j}\right)^{c_j}\\&\geq(2p)^{\frac{p+(p-1)j}{(p-1)^2}}C^{\frac{1}{p-1}}\left(\frac{A}{(2p)^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}}\right)^{p^j}t^a(\log t)^{-b_j}\left(\log\frac{t}{R_\infty}\right)^{c_j}\end{aligned}$$

for $t\geq R_\infty$ and $j\in\mathbf{N}$. Then, for $t\geq R_\infty^2$ and $j\in\mathbf{N}$, we have

$$\begin{aligned}H(t)&\geq(2p)^{\frac{p+(p-1)j}{(p-1)^2}}C^{\frac{1}{p-1}}\left(\frac{A}{(2p)^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}}\right)^{p^j}t^a(\log t)^{-b_j}\left(\frac{1}{2}\log t\right)^{c_j}\\&=2^{\frac{z+1}{p-1}}(2p)^{\frac{p+(p-1)j}{(p-1)^2}}C^{\frac{1}{p-1}}t^a(\log t)^{-\frac{x+z+1}{p-1}}\left(\frac{A(\log t)^{\left(c-b+\frac{x+z+1}{p-1}\right)}}{2^{\left(c+\frac{z+1}{p-1}\right)}(2p)^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}}\right)^{p^j}.\end{aligned}\tag{2.4}$$

If T would satisfy

$$\frac{A(\log T)^{\left(c-b+\frac{x+z+1}{p-1}\right)}}{2^{\left(c+\frac{z+1}{p-1}\right)}(2p)^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}}>1.$$

$H(T)$ cannot be finite as $j\to\infty$. Therefore, T should satisfy

$$T\leq\exp\left(\max\left\{2\log R_\infty,(A^{-1}D)^{\frac{p-1}{x+z+1+(c-b)(p-1)}}\right\}\right),$$

where

$$D:=2^{\left(c+\frac{p+(z+1)(p-1)}{(p-1)^2}\right)}p^{\frac{p}{(p-1)^2}}C^{\frac{1}{p-1}}>0.$$

□

3. Conclusions

We have shown a simple proof to obtain the sharp upper bound of the lifespan of the solution to (1.3). "Slicing method" may help us to have an application to weakly coupled systems such as

$$\begin{cases} u_{tt} - \Delta u = |\nabla_x v|^p, \\ v_{tt} - \Delta v = |\nabla_x u|^q, \end{cases} \quad (p, q > 1)$$

with critical exponents (p, q) as well as its damped version. For references on the nonlinearities in which $\nabla_x u$ is replaced with u_t, see Ikeda, Sobajima, and Wakasa [7]. Also, it is interesting to see the "combined effect" by the new model,

$$u_{tt} - \Delta u = |\nabla_x u|^p + |u|^q \quad (p, q > 1).$$

See Introduction of Kido, Sasaki, Takamatsu, and Takamura [12] for all the references on the combined effect for nonlinear wave equations.

In a recent hot topic in this research area, one may have an application of our work to "modulus continuity", such as

$$u_{tt} - \Delta u = |\nabla_x u|^{p_G(n)} \mu(|\nabla_x u|),$$

where μ is some function that is weaker than any power but contributes to the total integrability of a function $|s|^{p_G(n)}\mu(s)$. For references on the modulus continuity, see Chen and Reissig [13] or Wang and Zhang [14], on which $\nabla_x u$ and $p_G(n)$ are replaced with u and $p_S(n)$, and see Chen and Palmieri [15] or Shao [16], on which $\nabla_x u$ is replaced with u_t. In this way, we have many possibilities to apply our result.

Author contributions

Takiko Sasaki: Conceptualization, methodology, writing – original draft, writing – review & editing; Kerun Shao: Validation, writing – original draft, writing – review & editing; Hiroyuki Takamura: Conceptualization, methodology, supervision, funding acquisition, writing – original draft, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests in this paper.

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