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Research article

Geometric analysis of translation surfaces based on special curves in the modified orthogonal frame

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Abstract: The geometry of translation surfaces generated by curves related by the modified orthogonal frame in Euclidean 3-space was investigated in this paper. We described the geometric features leading to minimality and developability by taking into account particular pairs as Bertrand, Mannheim, and involute-evolute curves. Furthermore, we revealed new connections between surface behavior and curvature requirements by incorporating adjoint curves into the building process. Understanding translation surfaces in the wider context of differential geometry has advanced considerably due to the examples and theorems that have been presented.

Keywords: translation surfaces; adjoint curve; modified orthogonal frame; Bertrand partner curves; Mannheim partner curves; involute-evolute curves
Mathematics Subject Classification: 53A04, 53A05

1. Introduction

The study of curves and surfaces holds a central role in differential geometry, providing the foundational tools for understanding the intrinsic and extrinsic properties of geometric objects. Early works by Bertrand [1] and Mannheim [2] introduced key ideas regarding the behavior of curves and their associated surfaces, while do Carmo's seminal textbook [3] systematically organized the fundamental theories that continue to influence modern research. Foundational contributions by Struik [4] and O'Neill [5], have further enriched the theoretical landscape, offering robust frameworks for both classical and contemporary explorations in geometry.

Translation surfaces, generated by translating one curve along another, have been extensively studied due to their relative simplicity and diverse applications. Liu [6] analyzed surfaces with constant mean curvature, and Yoon [7] extended investigations to Minkowski space via Gauss map properties. Further contributions include Yuan and Liu [8, 9], Çetin et al. [10, 11], and Atalay et al. [12], who introduced new construction methods such as using the Frenet and Bishop frames. Later, Abd-

Ellah [13], Ali et al. [14], and López and Perdomo [15] explored curvature properties and surface evolutions, expanding the scope to Lorentzian geometries. Aydın et al. [16] studied translation surfaces in Galilean space, while Lone et al. [17, 18] and Nurkan et al. [19] introduced classifications based on specialized curve frames. Recent studies by Özel and Bektaş [20], Has and Yılmaz [21], Ali [22], and Ismoilov et al. [23] have further refined the understanding of translation surfaces under broader geometric structures and partner curve relationships.

The Frenet–Serret frame, consisting of orthonormal vectors (t, n, b), is an effective tool for analyzing spatial curves. Nonetheless, it may become undefined or unstable under degenerate scenarios, such as when curvature approaches zero. The modified orthogonal frame (MOF) has been created as a more adaptable option to address these restrictions. The issue of undefined behavior at inflection points has been addressed in early work by Sasai [24], who proposed a modified framing approach to ensure the continuity of geometric descriptions even when curvature vanishes, and later developed by Bükcü and Karacan [25], offers a consistent and smooth alternative. This framework maintains orthogonality while scaling the normal and binormal vectors as $N = \kappa n$ and $B = \kappa b$, yielding a more fluid and smooth representation of curves, particularly in proximity to singularities. The MOF avoids the discontinuities typically observed in the Frenet–Serret framework and facilitates continuous analysis, even when curvature or torsion approaches zero. The MOF is more appropriate for contemporary differential geometry issues due to its flexibility and adaptability. It streamlines the analytical formulations of curvature and torsion, offering an effective structure for the construction and examination of intricate curves and surfaces, especially in cases where the classical framework does not provide smooth behavior. Eren and Kosal [26] studied the evolution of space curves using the MOF and emphasized its effectiveness especially near singularities, where the classical Frenet frame fails to produce a consistent representation of the curve geometry.

Initial explorations by Bükcü and Karacan [25, 27] laid the foundation, later expanded by Lone et al. [17,18] through the study of special curves such as Mannheim curves. Damar et al. [28], Eren and Ersoy [29], and Atalay [30] contributed by introducing AW(k)-type curves, Smarandache curves, and special surface families. Baek et al. [31] and Damar et al. [32] focused on ruled and tubular surfaces based on adjoint curves, while Akyigit et al. [33] further studied tubular surfaces using the MOF. Extending the framework to non-Euclidean settings, Elsharkawy et al. [34] examined involute-evolute curves in Galilean space, demonstrating the adaptability of the MOF beyond classical environments.

In this study, we analyze translation surfaces constructed within the framework of the modified orthogonal frame. The second section gives the essential definitions and theorems. The third section creates and analyzes, in relation to the modified orthogonal frame, translation surfaces produced respectively by Bertrand curve pairs, Mannheim curve pairs, involute-evolute curves, and the adjoint curve of a specified curve. The requirements for each surface to be developable or minimum are examined, and its asymptotic and geodesic curves are identified. Illustrative examples are shown. The fourth section concentrates on the study's findings. Throughout the paper, u and v denote independent variables. Any geometric coupling between α and β (Bertrand, Mannheim, involute–evolute) affects their Frenet data but not their parametrization independence.

2. Materials and methods

Let $\alpha(s)$ denote a regular curve in Euclidean 3-space $\mathbb{E}^3$, parameterized by arc length s. The modified orthogonal frame (MOF) associated to this curve is the orthonormal triple $\{T(s), N(s), B(s)\}$, formulated for use in degenerate cases as a zero curvature or torsion. When $\kappa(s) \neq 0$, the MOF relates to the standard Frenet frame $\{t, n, b\}$ via

$$T = t, \qquad N = \kappa n, \qquad B = \kappa b.$$

These frame vectors satisfy the inner product relations

$$< T, T >= 1, < N, N >=< B, B >= \kappa^2, \quad < T, N >=< T, B >=< N, B >= 0.$$

Differentiating the structure of the MOF $\{T, N, B\}$, one can derive the following system of differential equations:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.1}$$

where the torsion function $\tau(s)$ is defined as

$$\tau = \tau(s) = \frac{det(\alpha', \alpha'', \alpha''')}{\kappa^2}.$$

In [25], the differentiation formula for the modified orthogonal frame is represented by (2.1).

Definition 2.1. If α is a unit-speed curve in $\mathbb{E}^3$ whose torsion is non-zero, then its adjoint curve is defined by

$$\beta(s) = \int_{s_0}^{s} B(u)\, du,$$

where $B(u)$ is the binormal vector from the MOF [19, 35].

Let $\alpha(u)$ and $\beta(v)$ represent two unit-speed curves, each with corresponding MOF elements $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$, respectively. Various geometric relationships arise when these curves create certain pairs, such as [17, 18, 21].

(1) Bertrand curves: Θ represents the constant angle between the tangent vectors T_α and T_β of the curves α and β:

$$\begin{aligned} T_\alpha(u) &= \cos\Theta\, T_\beta(v) - \frac{\sin\Theta}{\kappa_\beta(v)} B_\beta(v), \\ \frac{N_\alpha(u)}{\kappa_\alpha(u)} &= \frac{N_\beta(v)}{\kappa_\beta(v)}, \\ \frac{B_\alpha(u)}{\kappa_\alpha(u)} &= \sin\Theta\, T_\beta(v) + \frac{\cos\Theta}{\kappa_\beta(v)} B_\beta(v), \end{aligned}$$

and

$$\kappa_\alpha(u) = \kappa_\beta(v)\cos\Theta + \tau_\beta(v)\sin\Theta, \tag{2.2}$$

$$\tau_\alpha(u) = -\kappa_\beta(v)\sin\Theta + \tau_\beta(v)\cos\Theta.$$

(2) Mannheim curves: Θ represents the constant angle between the tangent vectors T_α and T_β of the curves α and β:

$$\begin{aligned} T_\alpha(u) &= \cos\Theta\, T_\beta(v) + \frac{\sin\Theta}{\kappa_\beta(v)} N_\beta(v), \\ \frac{N_\alpha(u)}{\kappa_\alpha(u)} &= \frac{B_\beta(v)}{\kappa_\beta(v)}, \\ \frac{B_\alpha(u)}{\kappa_\alpha(u)} &= -\sin\Theta\, T_\beta(v) + \frac{\cos\Theta}{\kappa_\beta(v)} N_\beta(v), \end{aligned}$$

and

$$\begin{aligned} \kappa_\alpha(u) &= \tau_\beta(v)\sin\Theta\,\frac{dv}{du}, \\ \tau_\alpha(u) &= -\tau_\beta(v)\cos\Theta\,\frac{dv}{du}. \end{aligned}$$

(3) Involute-evolute curves: With the mutually tangent vectors of α and β being perpendicular,

$$\begin{aligned} T_\alpha(u) &= \frac{N_\beta(v)}{\kappa_\beta(v)}, \\ \frac{N_\alpha(u)}{\kappa_\alpha(u)} &= \cos\Theta\, T_\beta(v) + \frac{\sin\Theta}{\kappa_\beta(v)} B_\beta(v), \\ \frac{B_\alpha(u)}{\kappa_\alpha(u)} &= -\sin\Theta\, T_\beta(v) + \frac{\cos\Theta}{\kappa_\beta(v)} B_\beta(v), \end{aligned}$$

and

$$\kappa_\alpha(u) = \frac{\sqrt{\kappa_\beta(v)^2 + \tau_\beta(v)^2}}{(c-s)\,\kappa_\beta(v)}.$$

Consider a smooth surface $\chi \in \mathbb{R}^3$ given locally by a parametrization $\chi(u, v)$. Essential concepts of the surface χ are described below. The unit normal vector n on surface χ is given by

$$n_\chi = \frac{\chi_u \times \chi_v}{\|\chi_u \times \chi_v\|}. \tag{2.3}$$

The first and second fundamental forms are defined by

$$I = E_\chi\, du^2 + 2F_\chi\, du\, dv + G_\chi\, dv^2, \quad II = e_\chi\, du^2 + 2f_\chi\, du\, dv + g_\chi\, dv^2,$$

where the coefficients are computed as

$$E_\chi = \langle \chi_u, \chi_u \rangle, \quad F_\chi = \langle \chi_u, \chi_v \rangle, \quad G_\chi = \langle \chi_v, \chi_v \rangle,$$

$$e_\chi = \langle \chi_{uu}, n_\chi \rangle, \quad f_\chi = \langle \chi_{uv}, n_\chi \rangle, \quad g_\chi = \langle \chi_{vv}, n_\chi \rangle.$$

The Gaussian and mean curvatures are as follows:

$$K_\chi = \frac{e_\chi g_\chi - f_\chi^2}{E_\chi G_\chi - F_\chi^2}, \quad H_\chi = \frac{E_\chi g_\chi + G_\chi e_\chi - 2F_\chi f_\chi}{2(E_\chi G_\chi - F_\chi^2)}. \tag{2.4}$$

Theorem 2.1. *Let χ denote a regular surface in $\mathbb{R}^3$. A surface χ is known as a developable surface if its Gaussian curvature is zero [5].*

Theorem 2.2. *Let χ denote a regular surface in $\mathbb{R}^3$. A surface χ is known as a minimal surface if its mean curvature is zero [5].*

A translation surface obtained from the MOF is formulated as follows:

$$\chi(u, v) = \alpha(u) + \beta(v), \tag{2.5}$$

where α and β represent generating curves. The partial derivatives of χ with respect to u and v yield

$$\chi_u = T_\alpha,\ \chi_v = T_\beta,\ \chi_{uu} = N_\alpha,\ \chi_{vv} = N_\beta,\ \chi_{uv} = 0, \tag{2.6}$$

where T_α, N_α and T_β, N_β are the modified orthogonal frame vectors of the curves α and β, respectively. The unit normal vector of the surface is expressed as

$$n_\chi = \frac{T_\alpha \times T_\beta}{||T_\alpha \times T_\beta||}.$$

Utilizing (2.6), the coefficients of the first and second fundamental form are given by

$$E_\chi = \langle \chi_u, \chi_u \rangle = 1,\ F_\chi = \langle \chi_u, \chi_v \rangle = \langle T_\alpha, T_\beta \rangle,\ G_\chi = \langle \chi_v, \chi_v \rangle = 1,$$

and

$$e_\chi = \langle \chi_{uu}, n_\chi \rangle = \frac{1}{||T_\alpha \times T_\beta||} \langle N_\alpha, T_\alpha \times T_\beta \rangle, \tag{2.7}$$

$$f_\chi = 0, \tag{2.8}$$

$$g_\chi = \langle \chi_{vv}, n_\chi \rangle = \frac{1}{||T_\alpha \times T_\beta||} \langle N_\beta, T_\alpha \times T_\beta \rangle. \tag{2.9}$$

3. Results

3.1. Translation surface generating curves α and β are Bertrand partner curves with the MOF

Let α and β be two Bertrand partner curves used to construct a translation surface via the MOF, as described via (2.5). Employing the relations given in (2.3) and (2.6), the unit normal vector field n of the resulting surface is computed as

$$n = -\frac{1}{\kappa_\beta} N_\beta. \tag{3.1}$$

Note that the negative sign in (3.1) arises from the geometric structure of Bertrand partner curves. Specifically, due to the constant angle between the tangent vectors T_α and T_β, the resulting surface normal vector aligns in the opposite direction of N_β. This expression is derived under the MOF. The first fundamental form coefficients simplify to

$$E = 1,\ F = \cos\Theta,\ G = 1. \tag{3.2}$$

Applying (2.7), (2.8), and (2.9), the components of the second fundamental form become

$$e = -\kappa_\alpha, \;\; f = 0, \;\; g = -\kappa_\beta. \tag{3.3}$$

The Gaussian and mean curvatures of the translation surface generated by the Bertrand partner curve pairs (α, β) are obtained by substituting the coefficients of fundamental forms given in (3.2) and (3.3) into the standard curvature formulas in (2.4):

$$K = \frac{\kappa_\alpha \kappa_\beta}{\sin^2 \Theta}, \;\; H = \frac{-\kappa_\alpha - \kappa_\beta}{2\sin^2 \Theta}. \tag{3.4}$$

Theorem 3.1. *Let $\chi(u, v)$ denote a translation surface formed by Bertrand partner curves α and β under the MOF. Then χ cannot be developable.*

Proof. Assume that $\chi(u, v)$ is developable. Then $K = 0$. Since $\kappa_\alpha, \kappa_\beta \neq 0$ according to the MOF, then $K \neq 0$. This is a contradiction. □

Theorem 3.2. *Let $\chi(u, v)$ denote a translation surface constructed from Bertrand partner curves α and β under the MOF. Then χ is minimal if and only if β is a helix.*

Proof. From the mean curvature (3.4) and using (2.2), we obtain

$$\kappa_\alpha = \kappa_\beta \cos\Theta + \tau_\alpha \sin\Theta,$$

$$H = 0 \Rightarrow \kappa_\alpha = -\kappa_\beta \Rightarrow \kappa_\beta(-1 - \cos\Theta) = \tau_\beta \sin\Theta \Rightarrow \frac{\tau_\beta}{\kappa_\beta} = \frac{-1 - \cos\Theta}{\sin\Theta} = \text{constant}.$$

Hence, β is a helix. Suppose that β is a helix. Then, $\frac{\tau_\beta}{\kappa_\beta} = c$ is a constant. From the Bertrand relation under the MOF, we get

$$\kappa_\alpha = \kappa_\beta \cos\Theta - \tau_\beta \sin\Theta = \kappa_\beta(\cos\Theta - c \sin\Theta). \tag{3.5}$$

Substituting (3.5) into the mean curvature formula,

$$H = \frac{\kappa_\beta(\cos\Theta + c\kappa_\beta \sin\Theta) - \kappa_\beta}{2\sin^2\Theta}.$$

Since $c = \dfrac{1 - \cos\Theta}{\sin\Theta}$, we get $H = 0$. Therefore, the surface χ is minimal. □

Theorem 3.3. *Consider the translation surface $\chi(u, v)$ formed by Bertrand partner curves α and β according to the MOF. It is not possible for either α or β to be asymptotic curves on χ.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.1), we obtain

$$\langle \alpha'', n \rangle = \langle N_\alpha, \frac{-1}{\kappa_\beta} N_\beta \rangle = \langle \frac{\kappa_\alpha}{\kappa_\beta} N_\beta, \frac{-1}{\kappa_\beta} N_\beta \rangle = -\kappa_\alpha.$$

Since $\kappa_\alpha \neq 0$, the curve α cannot be an asymptotic curve of the surface $\chi(u, v)$. One can see the same result for the curve β with the same steps. □

Theorem 3.4. *Consider the translation surface $\chi(u, v)$ generated by Bertrand partner curves α and β under the MOF. Then both α and β are geodesics on the surface χ.*

Proof. Since $\alpha'' = N_\alpha$, we obtain

$$\alpha'' \times n = N_\alpha \times \frac{-1}{\kappa_\beta} N_\beta = \frac{\kappa_\alpha}{\kappa_\beta} N_\beta \times \frac{-1}{\kappa_\beta} N_\beta = 0.$$

Hence, the curve α is the geodesic curve of the surface $\chi(u, v)$. One can see the same result for the curve β with the same step. □

Example 3.1. Let α and β be the Bertrand curve pairs parameterized by

$$\alpha(s) = \left(\frac{1}{2\sqrt{2}} \sin 2s, \frac{1}{2\sqrt{2}} \cos 2s, \frac{s}{\sqrt{2}}\right), \quad \beta(s) = \left(-\frac{1}{2\sqrt{2}} \sin 2s, -\frac{1}{2\sqrt{2}} \cos 2s, \frac{s}{\sqrt{2}}\right). \tag{3.6}$$

The modified orthogonal frame vectors for $\alpha(s)$ and β are given by

$$\begin{aligned} T_\alpha &= \left(\frac{1}{\sqrt{2}} \cos 2s,\ -\frac{1}{\sqrt{2}} \sin 2s,\ \frac{1}{\sqrt{2}}\right), & T_\beta &= \left(-\frac{1}{\sqrt{2}} \cos 2s,\ \frac{1}{\sqrt{2}} \sin 2s,\ \frac{1}{\sqrt{2}}\right), \\ N_\alpha &= \left(-\sqrt{2} \sin 2s,\ -\sqrt{2} \cos 2s,\ 0\right), & N_\beta &= \left(\sqrt{2} \sin 2s,\ \sqrt{2} \cos 2s,\ 0\right), \\ B_\alpha &= (\cos 2s,\ -\sin 2s,\ -1), & B_\beta &= (-\cos 2s,\ \sin 2s,\ -1). \end{aligned}$$

The curvature and torsion of the curves α and β are given by

$$\kappa_\alpha = \sqrt{2}, \quad \tau_\alpha = -2, \quad \kappa_\beta = \sqrt{2}, \quad \tau_\beta = -2,$$

respectively. From T_α and T_β, it is found that $\Theta = \frac{\pi}{2}$ [17]. By the definition of translation surface (2.5), we have

$$\chi(u, v) = \left(\frac{1}{2\sqrt{2}} \sin 2u, \frac{1}{2\sqrt{2}} \cos 2u, \frac{u}{\sqrt{2}}\right) + \left(-\frac{1}{2\sqrt{2}} \sin 2v, -\frac{1}{2\sqrt{2}} \cos 2v, \frac{v}{\sqrt{2}}\right), \tag{3.7}$$

and its graph is illustrated in Figure 1.

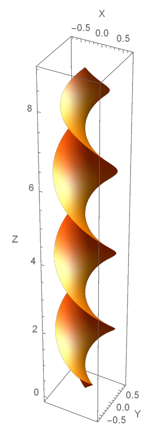

Figure 1. Translation surface (3.7) generated by the Bertrand partner curves (3.6) according to the MOF. When drawing in Mathematica, the parameters u and v are in $[0, 2\pi]$.

3.2. Translation surface generating curves α and β are Mannheim partner curves with the MOF

Let α and β, the generating curves of the translation surface as defined by the MOF in (2.5), be Mannheim partner curves. According to (2.3) and (2.6), the unit normal vector of the surface is given by

$$n = -\frac{1}{\kappa_\beta} B_\beta. \tag{3.8}$$

The coefficients of the first and second fundamental form are computed as

$$E = 1,\ F = \cos\Theta,\ G = 1, \tag{3.9}$$

and

$$e = -\kappa_\alpha,\ f = 0,\ g = 0, \tag{3.10}$$

respectively.

The Gaussian and mean curvatures of the translation surface generated by the Mannheim curve partner curve pairs (α,β) are obtained by substituting the coefficients of fundamental forms given in (3.9) and (3.10) into the standard curvature formulas in (2.4):

$$K = 0, \quad H = \frac{-\kappa_\alpha}{2\sin^2\Theta}. \tag{3.11}$$

Theorem 3.5. *Consider $\chi(u,v)$ is a translation surface generated by two Mannheim partner curves α and β according to the MOF. The surface χ is developable.*

Proof. It is obvious by (3.11). □

Theorem 3.6. *Consider a translation surface $\chi(u,v)$ produced from the Mannheim partner curves α and β according to the MOF. The surface χ cannot be minimal.*

Proof. According to the MOF, $\kappa_\alpha \neq 0$. Therefore by (3.11), $H \neq 0$. □

Theorem 3.7. *Consider $\chi(u,v)$ is a translation surface formed by the curves α and β, which are Mannheim partner curves according to the MOF. While β is asymptotic, the curve β cannot occur as an asymptotic curve on χ.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.8), we obtain

$$\langle \alpha'', n\rangle = \langle N_\alpha, -\frac{1}{\kappa_\beta} B_\beta\rangle = \langle \frac{\kappa_\alpha}{\kappa_\beta} B_\beta, -\frac{1}{\kappa_\beta} B_\beta\rangle = -\kappa_\alpha.$$

Since $\kappa_\alpha \neq 0$, the curve α cannot be an asymptotic curve of the surface $\chi(u,v)$. The same outcome can be observed with the curve β with the same increment. □

Theorem 3.8. *Consider $\chi(u,v)$ is a translation surface formed by the curves α and β, which are Mannheim partner curves according to the MOF. While the curve β is not geodesic, the curve α appear on χ as geodesic.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.8), we obtain

$$\alpha'' \times n = N_\alpha \times -\frac{1}{\kappa_\beta} B_\beta = \frac{\kappa_\alpha}{\kappa_\beta} B_\beta \times -\frac{1}{\kappa_\beta} B_\beta = 0.$$

Hence, the curve α is the geodesic curve of $\chi(u, v)$. The same outcome can be observed with the curve β with the same increment. □

Example 3.2. Let $\alpha(u) = \left(\frac{1}{7}\cos u,\ \frac{1}{7}\sin u,\ \frac{\sqrt{48}}{7}u\right)$ be a unit speed curve. A Mannheim partner curve of α is given by

$$\beta(v) = \left(-\frac{1}{7}(\sin v + \cos v),\ \frac{1}{7}(\sin v + \cos v),\ \frac{\sqrt{48}}{7}v\right).$$

Then, the translation surface χ is

$$\chi(u, v) = \left(\frac{1}{7}\cos u - \frac{1}{7}(\sin v + \cos v),\ \frac{1}{7}\sin u + \frac{1}{7}(\sin v + \cos v),\ \frac{\sqrt{48}}{7}(u + v)\right), \tag{3.12}$$

and its graph is illustrated in Figure 2.

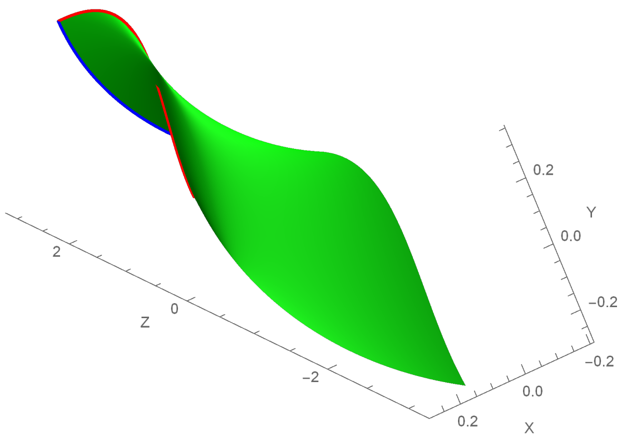

Figure 2. Translation surface (3.12) generated by the Mannheim partner curves α (blue) and β (red) according to the MOF. When drawing in Mathematica, the parameters u and v are in $[-\pi/2, \pi/2]$.

3.3. *Translation surface generating curves α and β are involute-evolute curves with the MOF*

Let α and β, the generating curves of the translation surface as defined by the modified orthogonal frame in (2.5), be involute-evolute partner curves. According to Eqs (2.3) and (2.6), the unit normal vector of the surface is

$$n = -\frac{1}{\kappa_\beta} B_\beta. \tag{3.13}$$

The coefficients of the first fundamental form are computed as

$$E = 1,\ F = 0,\ G = 1. \tag{3.14}$$

By using (2.7), (2.8), and (2.9), the second fundamental form coefficients are

$$e = -\kappa_\alpha \sin\Theta,\ \ f = 0,\ \ g = 0. \tag{3.15}$$

The Gaussian and mean curvatures of the translation surface generated by the involute-evolute partner curve pairs (α, β) are calculated by substituting the coefficients of fundamental forms given in (3.14) and (3.15) into the standard curvature formulas in (2.4):

$$K = 0, \quad H = -\frac{\kappa_\alpha \sin\Theta}{2}. \tag{3.16}$$

Theorem 3.9. *Let $\chi(u, v)$ be a translation surface generated, in the MOF, by an involute–evolute pair of curves α (the evolute) and β (the involute). Then χ is developable.*

Proof. It is obvious by (3.16). □

Theorem 3.10. *Let $\chi(u, v)$ be a translation surface whose generating curves α and β are involute-evolute partner curves according to the MOF. Then χ cannot be minimal.*

Proof. Since $\kappa_\alpha \neq 0$, (3.16) yields that $H \neq 0$. Hence, such a surface cannot be minimal. □

Theorem 3.11. *Let $\chi(u, v)$ be a translation surface whose generating curves α and β are involute-evolute partner curves according to the MOF. While β is an asymtotic, α cannot be the asymtotic curve of the translation surface.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.13), we obtain

$$\langle \alpha'', n\rangle = \langle N_\alpha, -\frac{1}{\kappa_\beta} B_\beta\rangle = \langle \kappa_\alpha \cos\Theta T_\beta + \frac{\kappa_\alpha}{\kappa_\beta}\sin\Theta B_\beta, -\frac{1}{\kappa_\beta} B_\beta\rangle = -\kappa_\alpha \sin\Theta.$$

Since $\kappa_\alpha \neq 0$, the curve α cannot be an asymptotic curve of the surface $\chi(u, v)$. One can see the same result for the curve β with the same steps. □

Theorem 3.12. *Let $\chi(u, v)$ be a translation surface whose generating curves α and β are involute-evolute partner curves according to the MOF. α cannot be a geodesic curve of the translation surface.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.13), we obtain

$$\alpha'' \times n = N_\alpha \times -\frac{1}{\kappa_\beta} B_\beta = \kappa_\alpha \cos\Theta T_\beta + \frac{\kappa_\alpha}{\kappa_\beta}\sin\Theta B_\beta \times -\frac{1}{\kappa_\beta} B_\beta = \frac{\kappa_\alpha}{\kappa_\beta}\cos\Theta N_\beta.$$

Since $\kappa_\alpha \neq 0$, the curve α cannot be geodesic. □

Example 3.3. Let $\alpha(u) = \left(\frac{3}{5}\cos u,\ \frac{3}{5}\sin u,\ \frac{4}{5}u\right)$. An involute partner curve is given by

$$\beta(v) = \left(\frac{3}{5}\cos v + \frac{3}{5}v\sin v,\ \frac{3}{5}\sin v - \frac{3}{5}v\cos v,\ 0\right).$$

Then, the translation surface is defined by

$$\chi(u, v) = \left(\frac{3}{5}\cos u + \frac{3}{5}\cos v + \frac{3}{5}v\sin v, \frac{3}{5}\sin u + \frac{3}{5}\sin v - \frac{3}{5}v\cos v, \frac{4}{5}u\right), \tag{3.17}$$

and its graph is illustrated in Figure 3.

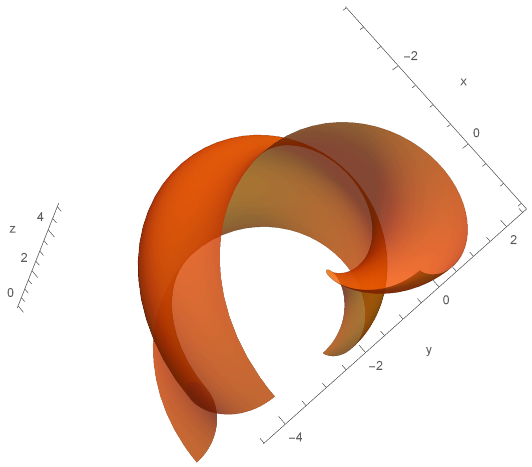

Figure 3. Translation surface (3.17) generated by the involute-evolute partner curves according to the MOF. When drawing in Mathematica, the parameters u and v are in $[0, 2\pi]$.

3.4. Translation surface generating curve α and its adjoint curve β with the MOF

Let α be a unit speed curve and β be its adjoint curve according to the MOF. The translation surface $\chi(u, v)$ which is generated by the curve α and its adjoint curve β with the MOF is given by (2.5). By using (2.3) and (2.6), the unit normal vector of the surface is

$$n = -\frac{1}{\kappa_\alpha} N_\alpha. \tag{3.18}$$

The coefficients of the first and second fundamental form are calculated as

$$E = 1, \ F = 0, \ G = 1, \tag{3.19}$$

and

$$e = -\kappa_\alpha, \ f = 0, \ g = \frac{\tau_\alpha}{\kappa_\alpha}, \tag{3.20}$$

respectively. The Gaussian and mean curvatures of the translation surface generated by the curves (α, β) are calculated by substituting the coefficients of fundamental forms given in (3.19) and (3.20) into the standard curvature formulas in (2.4):

$$K = -\tau_\alpha, \ H = \frac{-\kappa_\alpha^2 + \tau_\alpha}{2\kappa_\alpha}. \tag{3.21}$$

Theorem 3.13. *Let $\chi(u, v)$ be a translation surface with generating curve α and its adjoint curve β according to the MOF. Then χ is developable if and only if the curve α is a planar curve.*

Proof. From (3.21), if $K = 0$, then $-\tau_\alpha = 0$. This means that α is planar. □

Theorem 3.14. *Let $\chi(u, v)$ be a translation surface with generating curve α and its adjoint curve β according to the MOF. Then χ is minimal if and only if $\kappa_\alpha^2 = \tau_\alpha$.*

Proof. By using (3.21), if $H = 0$, then $\kappa_\alpha^2 = \tau_\alpha$. □

Theorem 3.15. *Let $\chi(u, v)$ be a translation surface with generating curve α and its adjoint curve β according to the MOF. Neither α nor β can be asymptotic curves of the surface $\chi(u, v)$.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.18), we obtain

$$\langle \alpha'', n \rangle = \langle N_\alpha, -\frac{1}{\kappa_\alpha} N_\alpha \rangle = -\kappa_\alpha.$$

Since $\kappa_\alpha \neq 0$, the curve α cannot be an asymptotic curve of the surface $\chi(u, v)$. The similar outcome for the curve β can be observed by following the same procedures. □

Theorem 3.16. *Let $\chi(u, v)$ be a translation surface with generating curve α and its adjoint curve β according to the MOF. Both α and β are the geodesic curves of the translation surface.*

Proof. Since $\alpha'' = N_\alpha$ and by (3.18), we obtain

$$\alpha'' \times n = N_\alpha \times \frac{1}{\kappa_\alpha} N_\alpha = 0.$$

Hence, the curve α is the geodesic curve of the surface $\chi(u, v)$. One can see the same result for the curve β with the same step. □

Example 3.4. Let us consider a curve parameterized as

$$\alpha(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right),$$

where $-1 < s < 1$ [5].

The modified orthogonal frame vectors of the curve α are

$$\begin{aligned} T(s) &= \frac{1}{2} \left(\sqrt{1+s}, -\sqrt{1-s}, \sqrt{2} \right), \\ N(s) &= \frac{1}{4} \left(\frac{1}{\sqrt{1+s}}, \frac{1}{\sqrt{1-s}}, 0 \right), \\ B(s) &= \frac{1}{4\sqrt{2}} \left(-\frac{1}{\sqrt{1-s}}, \frac{1}{\sqrt{1+s}}, \frac{\sqrt{2}}{\sqrt{1-s^2}} \right), \end{aligned}$$

where

$$\kappa = \tau = \frac{1}{\sqrt{8(1-s^2)}}.$$

The parametric equation of the adjoint curve of the curve α with a simple calculation is given by

$$\beta(s) = \frac{1}{4\sqrt{2}} \left(2\sqrt{1-s}, 2\sqrt{1+s}, \sqrt{2} \arcsin s \right).$$

Hence, the translation surface $\chi(u, v)$ generated by the curves α and β is given by

$$\chi(u, v) = \left(\frac{1}{2\sqrt{2}} \sqrt{1-u} + \frac{(1+v)^{3/2}}{3}, \frac{1}{2\sqrt{2}} \sqrt{1+u} + \frac{(1-v)^{3/2}}{3}, \sqrt{2} \arcsin u + \frac{v}{\sqrt{2}} \right), \tag{3.22}$$

and its graph is illustrated in Figure 4.

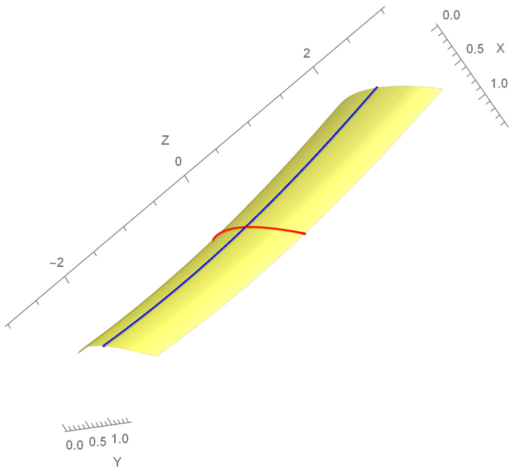

Figure 4. Translation surface (3.22) generated by the curve α (red) and its adjoint curve β (blue) according to the MOF. When drawing in Mathematica, the parameters u and v are in $[0, 2\pi]$.

4. Discussion and conclusions

This study examined translation surfaces generated by special curve pairs—Bertrand, Mannheim, involute-evolute, and adjoint curves—according to the modified orthogonal frame (MOF) in Euclidean 3-space. We established some results for minimality, developability, and the geodesic and asymptotic behavior of generating curves in terms of their curvature and torsion. The study shows that a translation surface generated from a Bertrand curve pair is not developable, whereas the surface obtained from Mannheim or involute curve pairs is developable. A surface generated with an adjoint curve becomes developable only when the curve α is planar. For a translation surface generated by a Bertrand pair to be minimal, the curve β must be a helix. In contrast, surfaces obtained from Mannheim or involute pairs can never be minimal.

To illustrate the theoretical findings, several examples were provided with explicit parametrizations and visual representations. These examples confirmed the validity of the analytical results and demonstrated the influence of curve geometry on the resulting surface structures.

Overall, the use of the modified orthogonal frame enables a deeper understanding of translation surfaces through intrinsic properties of space curves, offering a practical framework for both geometric modeling and theoretical exploration.

Author contributions

Burçin Saltık Baek: Conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing – original draft, writing – review and editing; Nural Yüksel: Conceptualization, data curation, funding acquisition, methodology, project administration, resources, supervision, validation, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

There is no conflict of interest.

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