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Research article

Invariant Jordan curves in the Julia sets of rational maps

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Abstract: We give a way to construct invariant Jordan curves in the Julia sets of rational maps that are not boundaries of Fatou components.

Keywords: Julia sets; invariant Jordan curves; mating; rational maps; combinatorial equivalence
Mathematics Subject Classification: 37F10, 37F20

1. Introduction

The existence and regularity of invariant curves in the Julia sets of rational maps have been studied by many people. The restrictions of rational functions on such invariant curves can be seen as decompositions of the dynamical systems on the Riemann sphere $\widehat{\mathbb{C}}$, and this viewpoint was passed down from Fatou [1]. There are mainly two types of such curves. The *first type* consists of those ones so that the restrictions of the rational maps on the curves are homeomorphisms, and the *second type* consists of those curves so that the restrictions of the rational maps on the curves are endomorphisms of degree at least two.

The first type of invariant Jordan curves can appear as the boundaries of Siegel disks (where foundationally Douady [2] and Herman [3] established existence criteria, Petersen-Zakeri [4] extended to quadratic polynomials; regarding boundary regularity, Avila-Buff-Chéritat [5] constructed smooth boundaries and Zhang [6] proved quasi-disks for bounded-type; in recent advances, Shishikura-Yang [7] resolved Jordan domains for high-type, and Cheraghi [8] explored attractor topology) and Herman rings (see [9] for Siegel-to-Herman quasiconformal surgery). For non-linear polynomials, the bounded immediate attracting and parabolic basins are Jordan domains [10], and their boundaries provide the second type of invariant curves. In [11], Eremenko constructed the second type of invariant Jordan curves under Lattés maps (whose Julia sets are the whole Riemann sphere). Note that Blaschke products can produce both types invariant Jordan curves (actually circles) in the Julia sets.

Recently, as the first type of invariant Jordan curves in Julia sets, the degenerate Herman rings have been constructed in [12, 13], and they are not obtained by quasiconformal deformations of Blaschke products. The main goal of this paper is to provide a way to construct the second type of invariant Jordan curves in the Julia sets of rational maps, and our main result is:

Main Theorem. *There exist rational maps that are neither Lattés maps nor quasiconformal deformations of Blaschke products, such that their Julia sets contain a second type of invariant Jordan curve that is not the boundary of any Fatou component.*

The general idea of the construction is as follows. Consider two post-critically finite (see Definition 1) non-conjugate and non-unicritical polynomials f and g of degrees d_f and d_g, respectively, such that both of them have the same degree $d_0 \geqslant 2$ in their fixed bounded immediate super-attracting basins D_f and D_g. We assume that for both f and g, the other critical orbits do not intersect with D_f and D_g (D_f and D_g are Jordan domains). By gluing f and g along the boundaries of D_f and D_g such that $f : D_f \to D_f$ is replaced by $g : \widehat{\mathbb{C}} \setminus \overline{D}_g \to \widehat{\mathbb{C}}$ (see §2.1 for details), one can obtain a 2-sphere on which a branched covering $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d_f + d_g - d_0$ is well-defined. It was proved in [14] that F is always *combinatorially equivalent* (see §2.1 for the precise definition) to a rational map $G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d_f + d_g - d_0$, and this procedure is called *Jordan mating* *.

By gluing ∂D_g and ∂D_f, this produces an invariant Jordan curve of F. However, this does not guarantee that G also has an invariant Jordan curve automatically. In fact, for the classical mating of quadratic polynomials [15], there are no invariant Jordan curves after mating in general. In this paper, we prove that such an invariant Jordan curve is retained after Jordan mating and that F is actually conjugate to G. Since the dynamics of f in D_f is replaced by that of g in the complement of $\overline{D}_g$, it follows that G contains an invariant Jordan curve satisfying the Main Theorem.

As an example, let f and g be two cubic polynomials, both of which contain a super-attracting fixed point, and moreover, assume that the other critical point c of f belongs to the boundary of the immediate super-attracting basin D_f with $f^2(c) = f(c)$ and the other critical point of g is of period two. According to [14] and the Main Theorem, by gluing f and g along ∂D_f and ∂D_g, one can obtain a quartic rational map G containing a second type invariant Jordan curve. See Figure 1.

The proposed framework requires a Jordan boundary condition (dependent on super-attracting basins), thereby excluding irrational neutral basins. Currently, the algorithm supports only Jordan matings of two polynomial maps (with at least one polynomial). When mating two such rational maps, the resulting mating may not conjugate to a genuine rational map, as Thurston obstructions can arise in this construction, thus making the existence of second-type invariant curves even less likely. These issues remain unresolved.

However, such second-type invariant Jordan curves expose the complex topological connectivity of rational map Julia sets under non-standard conditions, directly addressing the absence of invariant curves beyond Fatou component boundaries. Our mating ensures the geometric richness of non-degenerate curves and enables the construction of infinitely many instances exhibiting second-type invariant Jordan curves. This provides a foundational construct for other complex dynamical systems. It offers a paradigm shift for future research: mating can efficiently produce infinitely many second-type curves without requiring restrictive dynamical conditions (e.g., Diophantine constraints or linearizability).

*Distinguished from classical mating by omission of external ray equivalence identifications

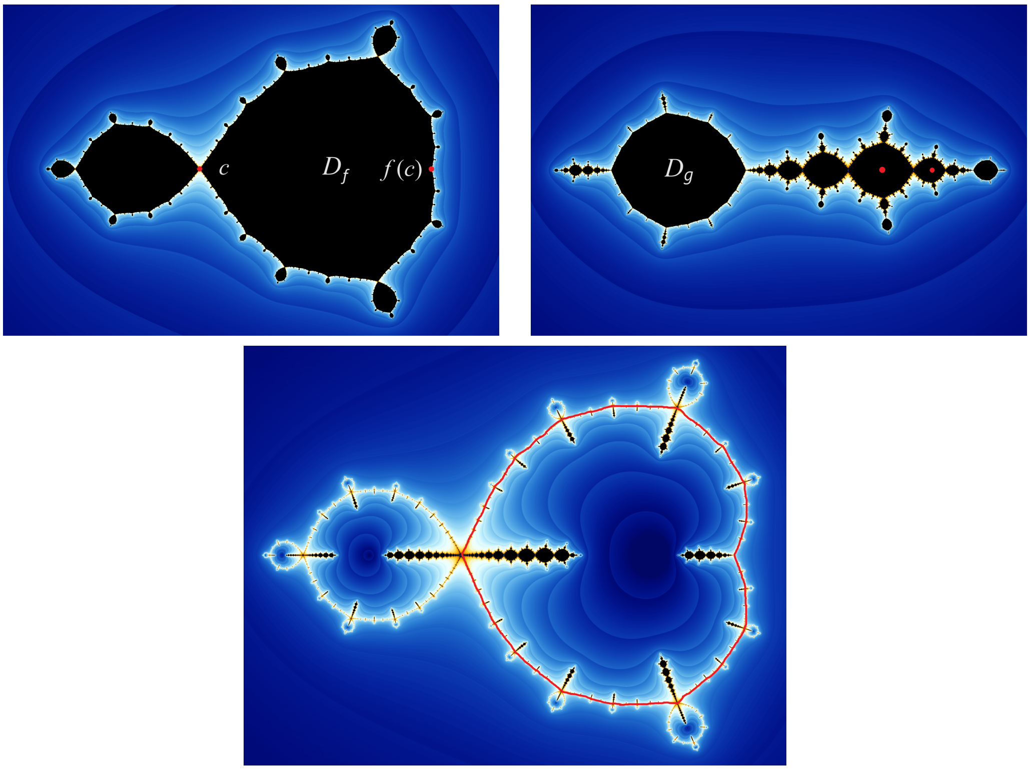

Figure 1. After Jordan mating of the cubic polynomials f and g, i.e., gluing f and g such that $f|_{D_f}$ is replaced by $g|_{\widehat{\mathbb{C}}\setminus\overline{D}_g}$, one can obtain a quartic rational map G having an invariant Jordan curve (colored red) on which the degree of G is two.

2. Materials and methods

2.1. Jordan mating and combinatorial equivalence

Throughout the following we use $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{T}$ to denote the Riemann sphere, the complex plane, the unit disk, and the unit circle, respectively.

Definition 1. A rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is post-critically finite if its post-critical set $P_f = \bigcup_{n\geq 1} f^n(C_f)$ is finite, where the critical set C_f consists of points where f fails to be locally injective, and P_f denotes the union of all forward orbits of critical points under iteration of f.

Let f and g be two post-critically finite, non-conjugate, and non-unicritical polynomials of degrees d_f and d_g, respectively, such that both of them have the same degree $d_0 \geqslant 2$ in their fixed bounded immediate super-attracting basins D_f and D_g, whose complements in $\widehat{\mathbb{C}}$ are denoted by D_f^c, D_g^c, respectively. We assume that for both f and g, the other critical orbits do not intersect with D_f and D_g. Then there exist two conformal maps, $\phi : D_f \to \mathbb{D}$ and $\psi : D_g \to \mathbb{D}$, which conjugate f and g to $z \mapsto z^{d_0}$ in D_f and D_g respectively. Since D_f and D_g are Jordan domains, ϕ and ψ can be extended to homeomorphisms $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ respectively. Define

$$\Phi := \phi^{-1} \circ \tau \circ \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \tag{2.1}$$

where $\tau(z) = e^{2\pi i \frac{j}{d_0-1}}/z$ with $0 \leqslant j \leqslant d_0 - 2$. Then $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism satisfying

$\Phi(\overline{D}_g) = D_f^c$, $\Phi(D_g^c) = \overline{D}_f$, and $\Phi(\partial D_g) = \partial D_f$. Define†

$$F(z) := \begin{cases} f(z) & \text{if } z \in D_f^c, \\ \Phi \circ g \circ \Phi^{-1}(z) & \text{if } z \in D_f. \end{cases} \tag{2.2}$$

The dynamical behavior of the maps naturally partitions their domains: for any $z \in D_f^c$, its image $f(z)$ must lie in either D_f^c or D_f, and analogously for g with $z \in D_g^c$ mapping to D_g^c or D_g. This consistent dynamical separation guarantees that the combined mapping F is well-defined throughout its domain. Crucially, the conjugacy relation $f(z) = \Phi \circ g \circ \Phi^{-1}(z)$ holds precisely on the shared boundary $\partial D_f = \partial D_f^c$, as confirmed by direct computation. Since $D_f^c \cup D_f$ equals the Riemann sphere $\widehat{\mathbb{C}}$, the global map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering whose degree is $d_f + d_g - d_0$. In particular, F behaves like f in D_f^c, is conjugate to g in D_f, and is conjugate to $z \mapsto z^{d_0}$ on ∂D_f.

Recall that $P_F := \bigcup_{n\geqslant 1} F^n(C_F)$ is the post-critical set of F, and C_F is the critical set of F (i.e., the points with local degree at least 2). The following result is proved in [14].

Theorem 2.1. *There exists a rational map G of degree $d_f + d_g - d_0$ such that F is* combinatorially equivalent *to G, i.e., there exist two homeomorphisms $\phi_0, \phi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that ϕ_0 is isotopic to ϕ_1 on $\widehat{\mathbb{C}}$ relative to P_F and the following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\phi_1} & \widehat{\mathbb{C}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \widehat{\mathbb{C}} & \xrightarrow{\phi_0} & \widehat{\mathbb{C}}. \end{array} \tag{2.3}$$

The concept of combinatorial equivalence was introduced in [16]. To enhance readability for the audience, we restate this concept here in accordance with the original setting. Two branched mappings $f, g : S^2 \to S^2$ are equivalent if and only if there exist homeomorphisms $\theta, \theta' : \left(S^2, P_f\right) \to \left(S^2, P_g\right)$ such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\theta'} & (S^2, P_g) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (S^2, P_f) & \xrightarrow{\theta} & (S^2, P_g) \end{array}$$

commutes, and θ is isotopic to θ' relative to P_f. For further results, see [17–20] and the references therein.

Note that Theorem 2.1 holds when at least one of f or g is a polynomial and both belong to $\mathcal{R}_{d_0}$, where $\mathcal{R}_{d_0}$ denotes the family of all post-critically finite rational maps possessing a marked immediate super-attracting basin D that is a Jordan domain. Furthermore, the restriction of the map to D has degree exactly d_0, and all other critical orbits are disjoint from D.

Without loss of generality, we assume that both D_f and D_g contain the super-attracting fixed point 0 and that the homeomorphism Φ satisfies $\Phi(0) = \infty$ and $\Phi(\infty) = 0$. Replacing F by a new branched covering via topological conjugation, we identify ∂D_f as the unit circle $\mathbb{T}$, and $F : \mathbb{T} \to \mathbb{T}$ as $z \mapsto z^{d_0}$. Using the Böttcher conjugations of f near ∞ (to $z \mapsto z^{d_f}$) and g near 0 (to $z \mapsto z^{d_g}$), we define F's

†For simplicity, the definition of F here is slightly different from [14].

internal rays and *equipotential curves* by topologically transplanting these structures via Φ. This is well-defined without assuming F's analyticity, as Φ preserves landing angles on ∂D_f. The segments L_θ^f and L_θ^g are formally defined in (2.5).

To enhance the readability of the article, we will briefly describe the definitions of equipotential curve and internal ray. We know that an equipotential curve is a level set $\{z \in \mathbb{C} : G^*(z) = c\}$ for $c > 0$, where G^* is the Green's function for the filled Julia set K. It forms a closed curve surrounding K, mapped d-to-one onto another equipotential by F, where d is the degree of F. An internal ray is the continuous orthogonal trajectory to the family of equipotential curves, defined as the locus $\eta^{-1}(\{re^{i\theta} \mid 0 < r < 1\}), \theta \in [0, 2\pi]$, where η is the Böttcher isomorphism conjugating F to $w \mapsto w^d$.

For each $\theta \in [0, 2\pi]$, let $R^f(\theta)$ (starting from ∞) and $R^g(\theta)$ (starting from 0) be two rays of F landing at $e^{i\theta} \in \mathbb{T}$ (note that such rays are not necessarily unique). Here we assume that these rays include their end points. Define

$$R^F(\theta) := R^f(\theta) \cup R^g(\theta). \tag{2.4}$$

Let Γ_0 and Γ_∞ be two equipotential curves in a neighborhood of 0 and ∞, respectively, such that F is conformally conjugate to $z \mapsto z^{d_f}$ in the exterior[‡] of Γ_∞ and conformally conjugate to $z \mapsto z^{d_g}$ in the interior of Γ_0. Let L_θ be the curve segment of $R^F(\theta)$ that lies in the annular region enclosed by Γ_0 and Γ_∞. We denote (see Figure 2):

$$L_\theta^f := L_\theta \cap (\widehat{\mathbb{C}} \setminus \mathbb{D}) \quad \text{and} \quad L_\theta^g := L_\theta \cap \overline{\mathbb{D}}. \tag{2.5}$$

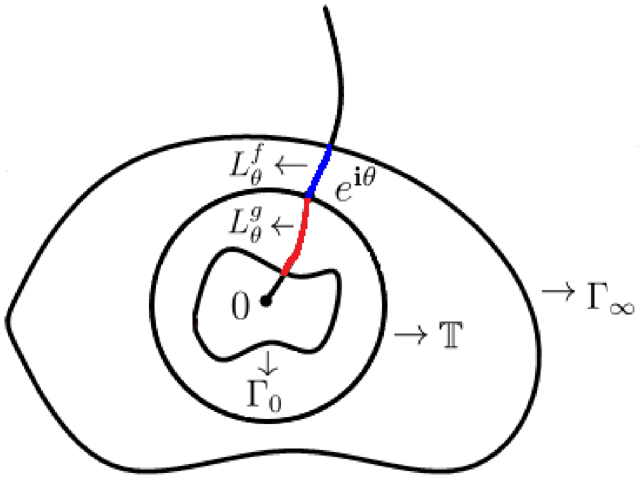

Figure 2. The ray segments L_θ^f and L_θ^g (colored blue and red respectively).

Remark. First, $R^F(\theta)$ denotes any external ray landing at the common point $e^{i\theta}$, Crucially, since the construction of L_θ relies exclusively on its landing endpoints and the specified point $e^{i\theta}$ on the unit circle (rather than on any other points along the ray), L_θ is fundamentally determined by these three points and therefore remains completely independent of the specific choice of rays connecting them.

Second, the aforementioned construction defines internal rays L_θ^f and L_θ^g for polynomials f and g. This framework extends to cases where at least one of f or g is a polynomial. Assuming $f \in R_{d_0}$

[‡]For a Jordan curve Γ in $\widehat{\mathbb{C}}$ which does not pass through ∞, the *exterior* of Γ denotes the component of $\widehat{\mathbb{C}} \setminus \Gamma$ containing ∞ and the *interior* of Γ denotes the other.

is a polynomial and $g \in R_{d_0}$ a rational function, we now define L^f_θ and L^g_θ. The ray $R^g(\theta)$ is defined with respect to the conformal center $a_g \in D_g$ (the super-attracting fixed point). Through the Riemann mapping $\psi : D_g \to \mathbb{D}$ with $\psi(a_g) = 0$, the ray system is conformally transplanted to originate from 0 in $\mathbb{D}$. The homeomorphism $\Phi = \phi^{-1} \circ \tau \circ \psi$ in (2.1) preserves this parameterized ray structure under the conjugacy. That is, $\Phi \circ (R^g(\theta))$ is topologically conjugate to a ray in D^c_f from ∞. This compatibility is topologically exact and ensures the ray segments L^f_θ, L^g_θ in (2.5) are well-defined for any $f, g \in \mathcal{R}_{d_0}$.

Let G be the rational map determined by Theorem 2.1. We may further assume that ϕ_0 and ϕ_1 are holomorphic and equal in the exterior of Γ_∞ and in the interior of Γ_0. Hence, we have the following result.

Lemma 2.2. *The map ϕ_0 maps $R^g(\theta) \setminus L^g_\theta$ to part of the internal ray of G starting from 0, and maps $R^f(\theta) \setminus L^f_\theta$ to part of an internal ray of G starting from ∞.*

By the definition of combinatorial equivalence (see Theorem 2.1) and the induction of lifts, we obtain a sequence of homeomorphisms $(\phi_k)_{k\geqslant 0}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\phi_{k+1}} & \widehat{\mathbb{C}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \widehat{\mathbb{C}} & \xrightarrow{\phi_k} & \widehat{\mathbb{C}}. \end{array} \tag{2.6}$$

In fact, $\phi_{k+1} = G^{-1} \circ \phi_k \circ F$.

3. Results

Main Theorem. *There exist rational maps that are neither Lattés maps nor quasiconformal deformations of Blaschke products, such that their Julia sets contain a second type of invariant Jordan curve that is not the boundary of any Fatou component.*

3.1. Proof of the main theorem

An *orbifold metric* on a Riemann surface is a conformal metric $\rho(z)|dz|$ expressed in local uniformizing coordinates z that is smooth and nonzero except at a locally finite set of ramified points $\{a_j\}$, where for each a_j there exists an integer ramification index $\nu_j \geq 2$ such that under the branched covering $z(w) = a_j + w^{\nu_j}$, the induced metric $\rho(z(w))\left|\frac{dz}{dw}\right||dw|$ becomes smooth and nonsingular near $w = 0$. Since G is post-critically finite, one can define an orbifold metric $\rho(z)|dz|$ on $\widehat{\mathbb{C}}$ as in [21, §19]. Let $l_\rho(\cdot)$ be the length with respect to the metric ρ. The following fact that G expands the metric ρ in a neighborhood of its Julia set is proved in [21, Theorem 19.6]. To enhance the readability of the article, we will briefly describe the proof.

Lemma 3.1. *The rational map G strictly expands the orbifold metric ρ in the complement of a neighborhood of all the super-attracting cycles.*

Proof. Since G is post-critically finite, its critical orbits are finite. We endow the dynamical sphere with the canonical orbifold structure $(\mathcal{S}, \nu)$, where $\mathcal{S} = \widehat{\mathbb{C}} \setminus \{\text{attracting periodic orbits}\}$ and ν assigns a ramification index $\nu(a_j) \geq 2$ at each postcritical point a_j minimized via least common multiples along critical orbits to satisfy the liftability condition $\nu(G(z)) \mid \deg(G, z) \cdot \nu(z)$. The universal cover $\widetilde{\mathcal{S}}_\nu$ is simply connected, permitting a holomorphic lift $\widetilde{G} : \widetilde{\mathcal{S}}_\nu \to \widetilde{\mathcal{S}}_\nu$ of G. To confirm that $\widetilde{G}$ strictly contracts the path metric on $\widetilde{\mathcal{S}}_\nu$ in $W = \mathcal{S} \setminus \bigcup B_\epsilon$(superattracting points), we exhaustively address three mutually exclusive conformal types:

(1) Conformally Hyperbolic Case: When $\widetilde{\mathcal{S}}_\nu$ supports a hyperbolic metric, $\widetilde{G}$ cannot be an isometry since isometry preservation under G would force all periodic points to be indifferent, contradicting repelling cycles in $J(G)$. Consequently, $\widetilde{G}$ is metric-decreasing, so G is metric-increasing. Simple connectivity of $\widetilde{\mathcal{S}}_\nu$ ensures $\|\widetilde{G}'(w)\|_{\text{hyp}} < 1$ universally; projection to $\mathcal{S}$ yields $\|DG(z)\|_\rho > 1$ locally near $J(G)$.

(2) Conformally Euclidean Case: If $\widetilde{\mathcal{S}}_\nu \cong \mathbb{C}$, redefine ν to ν' by doubling ν on a periodic orbit in $\mathcal{S}$. This forces $\widetilde{\mathcal{S}}_{\nu'}$ to be hyperbolic as a nontrivial branched cover of $\mathbb{C}$. The induced lift $\widetilde{G}' : \widetilde{\mathcal{S}}_{\nu'} \to \widetilde{\mathcal{S}}_{\nu'}$ contracts: $\|(\widetilde{G}')'(w)\|_{\text{hyp}} < 1$. Since $\nu' \geq \nu$ and expansion depends locally on ramification, the original orbifold metric ρ expands identically in W away from the modified orbit.

(3) Spherical Case: $\widetilde{\mathcal{S}}_\nu \cong \widehat{\mathbb{C}}$ necessitates $\mathcal{S} = \widehat{\mathbb{C}}$, implying the projection $\pi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has $\deg \pi = 1$. However, G lifts to $\widetilde{G} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ requiring $\deg \widetilde{G} = \deg G \geq 2$, contradicting $\deg \pi = 1$. This case is vacuous.

The exclusion of ϵ-neighborhoods of superattracting points ensures ρ is defined throughout $W \supset J(G)$; in all admissible cases, orbifold expansion $\|DG(z)\|_\rho \geq \kappa > 1$ holds uniformly near $J(G)$. □

Since the post-critical sets P_F and P_G are finite, we clearly have the following:

Lemma 3.2. *(see Figure 3) There exists $0 < M < \infty$ such that for any $\theta \in [0, 2\pi]$, there is a curve $\widetilde{L}^g_\theta \subset \mathbb{D}$ that is homotopic to L^g_θ relative to the endpoints of L^g_θ such that*

$$l_\rho(\phi_0(\widetilde{L}^g_\theta)) \leqslant M. \tag{3.1}$$

The similar property holds for L^f_θ.

Proof. Let $K = \overline{D}_g \cap \{z : \delta_0 \leq |z| \leq 1\}$ be compact with $\Gamma_0 \subset K \setminus \{0\}$ and $K \cap (C_g \setminus \{0\}) = \emptyset$, noting that the curve $\widetilde{L}^g_\theta$ lies entirely in K. Denote the finite boundary critical points by $C = C_g \cap \partial D_g = \{c_1, \ldots, c_m\}$, each with index $\nu_j \geq 2$. Fix $\varepsilon > 0$ sufficiently small so that neighborhoods $B_\varepsilon(c_j)$ are pairwise disjoint. For the non-singular part $K_\varepsilon = K \setminus \bigcup_j B_\varepsilon(c_j)$, observe first that $\phi_0(K_\varepsilon)$ is compact since ϕ_0 is a homeomorphism on K, and ρ is continuous here as $\phi_0(K_\varepsilon)$ avoids the critical points of G. Consequently, there exists $M_\varepsilon > 0$ such that $\sup_{w \in \phi_0(K_\varepsilon)} \rho(w) \leq M_\varepsilon$. Furthermore, the Euclidean length of $\widetilde{L}^g_\theta \cap K_\varepsilon$ is bounded uniformly in θ by A_ε because it lies in a compact set with ρ_g equivalent to the Euclidean metric on K_ε. Since ϕ_0 maps compact sets to compact sets continuously, $\phi_0(\widetilde{L}^g_\theta \cap K_\varepsilon)$ has a uniform Euclidean length bound B_ε. We thus obtain:

$$l_\rho(\phi_0(\widetilde{L}^g_\theta \cap K_\varepsilon)) \leq M_\varepsilon \cdot B_\varepsilon \equiv C_1(\varepsilon).$$

Near each critical point in $B_\varepsilon(c_j)$, let $\gamma_{\theta,j} = \widetilde{L}^g_\theta \cap B_\varepsilon(c_j)$. The continuity of ϕ_0 on K ensures that as $\varepsilon \to 0$, the diameter $\text{diam}_{\text{eu}}(\phi_0(\gamma_{\theta,j})) \to 0$ uniformly in θ. Meanwhile, the metric ρ satisfies $\rho(w) \leq C_j|w - \phi_0(c_j)|^{1/\nu_j - 1}$ near the critical value $\phi_0(c_j)$. Choose $\delta_1(\varepsilon) > 0$ such that $\sup_\theta \text{diam}_{\text{eu}}(\phi_0(\gamma_{\theta,j})) \leq \delta_1(\varepsilon)$ with $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, which gives the length estimate:

$$l_\rho(\phi_0(\gamma_{\theta,j})) \leq \int_0^{\delta_1(\varepsilon)} C_j r^{1/\nu_j - 1} dr = C_j \nu_j \delta_1(\varepsilon)^{1/\nu_j} \equiv C_2^{(j)}(\varepsilon).$$

Finally, selecting ε sufficiently small so that $C_2^{(j)}(\varepsilon) < 1$ for all $j = 1, \ldots, m$, we combine the estimates to conclude:

$$\sup_{\theta\in[0,2\pi]} l_\rho(\phi_0(\widetilde{L}_\theta^g)) \le C_1(\varepsilon) + \sum_{j=1}^{m} C_2^{(j)}(\varepsilon) \equiv M < \infty.$$

□

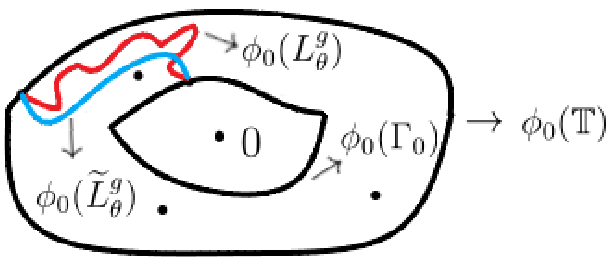

Figure 3. The homotopic curves $\phi_0(L_\theta^g)$ and $\phi_0(\widetilde{L}_\theta^g)$.

Let $(\phi_k)_{k\geqslant 0}$ be the sequence of homeomorphisms determined by (2.6).

Lemma 3.3. *The sequence of $\phi_k(e^{i\theta})$ converges uniformly to $\phi(e^{i\theta})$ for $\theta \in [0, 2\pi]$.*

Proof. Since ϕ_0 is isotopic to ϕ_1 on $\widehat{\mathbb{C}}$ relative to P_F, there is a continuous map $H : \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$ such that $H(z,0) = \phi_0(z)$, $H(z,1) = \phi_1(z)$, $H(\cdot,t)|_{P_F} = H(\cdot,0)|_{P_F} = H(\cdot,1)|_{P_F}$, and $H(z,t)$ is a homeomorphism for all $z \in \widehat{\mathbb{C}}$ and $t \in [0,1]$. We can define $H : \widehat{\mathbb{C}} \times [0,\infty) \to \widehat{\mathbb{C}}$ by lifting $H : \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{H(z,t+1)} & \widehat{\mathbb{C}} \\ F\downarrow & & \downarrow G \\ \widehat{\mathbb{C}} & \xrightarrow{H(z,t)} & \widehat{\mathbb{C}}. \end{array} \tag{3.2}$$

For any $k \in \mathbb{N}$ and $\theta \in [0, 2\pi]$, there is a continuous curve $H(e^{i\theta}, t)$ connecting $\phi_k(e^{i\theta})$ and $\phi_{k+1}(e^{i\theta})$, $t \in [k, k+1]$. We denote it by Γ_k^θ. Denote $\gamma_k := \phi_k(\mathbb{T})$ for $k \geqslant 0$. Define

$$d(\gamma_k, \gamma_{k+1}) := \sup_{\theta\in[0,2\pi]} \inf_{\tau} l_\rho(\tau),$$

where inf is taken over all the smooth curves τ in the homotopy class of Γ_k^θ relative to P_G. By Lemma 3.1, there is a $0 < \delta < 1$ such that $d(\gamma_k, \gamma_{k+1}) < \delta^k d(\gamma_0, \gamma_1)$. This implies that $\phi_k(e^{i\theta})$ converges uniformly to some $\phi(e^{i\theta})$ for $\theta \in [0, 2\pi]$. □

Lemma 3.4. *The limit curve $\gamma = \{\phi(e^{i\theta}) : \theta \in [0, 2\pi]\}$ is a Jordan curve.*

Proof. The two components of $\widehat{\mathbb{C}} \setminus \gamma_k$ contain definite neighborhoods of 0 and ∞ respectively. So the spherical diameter of $(\gamma_k)_{k\geqslant 0}$ has a positive lower bound. Since γ_k converges to γ as $k \to \infty$, it follows that γ is not a singleton. To prove that γ is a Jordan curve, it suffices to prove that for any $z \in \gamma$, $\{\phi^{-1}(z)\}$ is a connected set.

Let us prove this by contradiction. Otherwise, there are four angles $\theta_1 < \theta_2 < \theta_3 < \theta_4$ such that

$$\phi(e^{i\theta_2}) \neq \phi(e^{i\theta_1}) = \phi(e^{i\theta_3}) \neq \phi(e^{i\theta_4}). \tag{3.3}$$

In the following, we assume that k is sufficiently large. Then the two points $\phi_k(e^{i\theta_1})$ and $\phi_k(e^{i\theta_3})$ divide the Jordan curve γ_k into two Jordan arcs, and the sum of their winding numbers around 0 is equal to 1. Since $\phi_k(e^{i\theta_1}) - \phi_k(e^{i\theta_3}) \to 0$ as $k \to \infty$, the two winding numbers around 0 are both close to integers. We denote the one with the larger winding number by α_k. Without loss of generality, we assume that $\phi_k(e^{i\theta_4}) \in \alpha_k$ and $\phi_k(e^{i\theta_2}) \in \gamma_k \setminus \alpha_k$. Denote by β_k the straight segment connecting $\phi_k(e^{i\theta_1})$ and $\phi_k(e^{i\theta_3})$. The set $\alpha_k \cup \beta_k$ can be parameterized as a continuous curve with a winding number around 0 greater than or equal to 1 (this is because the winding number of α_k around 0 is at least close to 1 and the winding number of β_k around 0 is close to 0). So it separates 0 and ∞. Let U_0 and U_∞ be the components of $\widehat{\mathbb{C}} \setminus (\alpha_k \cup \beta_k)$ containing 0 and ∞, respectively. Note that $\phi_k(e^{i\theta_2}) \in \widehat{\mathbb{C}} \setminus (\alpha_k \cup \beta_k)$. Then at least one of U_0 and U_∞ does not contain $\phi_k(e^{i\theta_2})$. See Figure 4.

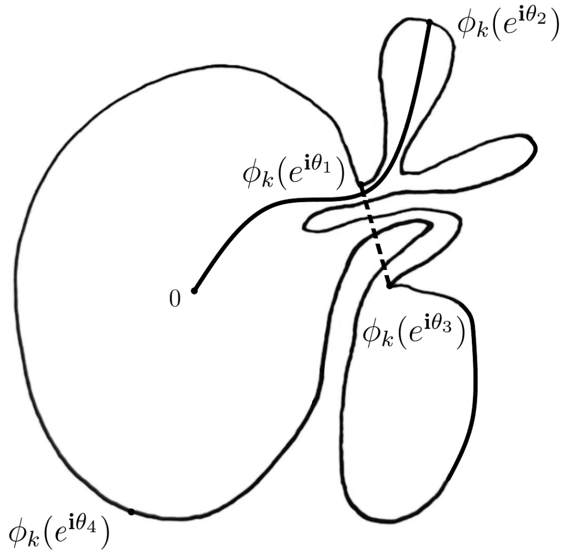

Figure 4. The Jordan curve γ_k marked with some special points.

Suppose $\phi_k(e^{i\theta_2}) \notin U_0$. We denote $\omega := d_0^k\theta_2$, where $d_0 \geqslant 2$ is the degree of $F : \mathbb{T} \to \mathbb{T}$. For each $\theta \in [0, 2\pi]$, denote

$$\widetilde{R}^g(\theta) := (R^g(\theta) \setminus L_\theta^g) \cup \widetilde{L}_\theta^g, \tag{3.4}$$

where L_θ^g is defined in (2.5) and $\widetilde{L}_\theta^g$ is introduced in Lemma 3.2. Note that $L := \phi_k(R^g(\theta_2))$ is the lift of $\phi_0(R^g(\omega))$ by G^k. Let $\widetilde{L}$ be the corresponding lift of $\phi_0(\widetilde{R}^g(\omega))$ by G^k. Then $\widetilde{L} \cap \beta_k \neq \emptyset$. We take $b_k \in \widetilde{L} \cap \beta_k$. Note that one can write $\widetilde{L}$ as $R \cup E$, where R and E are the lift of $\phi_0(R^g(\omega) \setminus L_\omega^g)$ and $\phi_0(\widetilde{L}_\omega^g)$ by G^k, respectively. By Lemma 2.2, R is part of an internal ray of G starting from 0, and as k is large enough, R is almost a whole ray. By Lemmas 3.1 and 3.2, the length $l_\rho(E)$ with respect to the metric $\rho(z)|dz|$ can be arbitrarily small if k is large enough.

By (3.3), there exists $\varepsilon_0 > 0$ such that the spherical distance $d(b_k, \phi_k(e^{i\theta_2})) > \varepsilon_0$ for all k large enough. Since $\phi_k(e^{i\theta_1}) - \phi_k(e^{i\theta_3}) \to 0$ as $k \to \infty$, $d(b_k, \phi_k(e^{i\theta_1})) \to 0$ as $k \to \infty$. Since G is post-critical finite and $\gamma_0 = \phi_0(\mathbb{T})$ does not contain a super-attracting point, it follows that $d(\gamma_k, J(G)) \to 0$ as $k \to \infty$, where $J(G)$ is the Julia set of G. So $\phi_k(e^{i\theta_1})$ and $\phi_k(e^{i\theta_2})$ can be arbitrarily close to $J(G)$

provided that k is large enough. Since $J(G)$ is locally connected ([22]), among all the points in an internal ray, only those near the end of the ray can be close to $J(G)$. Since R is almost a whole ray and E is small, both b_k and $\phi_k(e^{i\theta_2})$ must be near the end of some ray. This contradicts $d(b_k, \phi_k(e^{i\theta_2})) > \varepsilon_0$. The similar argument applies to the case that $\phi_k(e^{i\theta_2}) \notin U_\infty$. The proof is complete. □

Proof of the main theorem. By Lemma 3.4, we have $\phi \circ F = G \circ \phi$ and $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism. This implies that F is conjugate to G. By the construction of F in (2.2), we conclude that $\phi(\mathbb{T})$ is an invariant Jordan curve of G on which the degree of G is $d_0 \geqslant 2$ and that it is not the boundary of any Fatou component of G. Moreover, since f and g are not conjugate to each other, it implies that G is not a Blaschke product, and it is clearly not a Lattés map. □

4. Discussion

Note that the invariant Jordan curve is unique up to isotopy. Although the Jordan mating construction depends on the choice of the gluing map $\tau(z) = e^{2\pi i j/(d_0-1)}/z$ for $0 \le j \le d_0 - 2$, this rotational freedom yields curves in the same isotopy class relative to P_G. This uniformity occurs because the combinatorial equivalence in Theorem 2.1 fixes the isotopy class of ∂D_f (identified with $\mathbb{T}$) relative to P_F, while the limiting process in Lemmas 3.3 and 3.4 converges to a Jordan curve $\gamma = \phi(\mathbb{T})$ whose isotopy class is independent of initial conjugacies.

5. Conclusions

In this paper, we have proved the existence of invariant Jordan curves of the second type in the Julia sets of rational maps via Jordan mating.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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