Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 16642–16663.
DOI: 10.3934/math.2025746
Received: 28 April 2025
Revised: 26 June 2025
Accepted: 04 July 2025
Published: 24 July 2025

Research article

On Mittag-Leffler-Gegenbauer polynomials arising by the convolution of Mittag-Leffler function and Hermite polynomials

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Abstract: Gegenbauer polynomials hold a significant role in the constructive theory of spherical functions, while the Mittag-Leffler function is widely used in fractional calculus. In this paper, we introduce a new class of Mittag-Leffler-Gegenbauer polynomials (MLGPs) by convolutionally combining the classical Hermite polynomials with the Mittag-Leffler function of three parameters. We explore some of its aspects, such as symbolical identities, recurrence relations, differential equations, generating functions, integral representations, finite summations, and Rodrigues-type and orthogonal formulas. Additionally, we demonstrate the relevance of the MLGPs by developing and solving a fractional kinetic equation associated with the MLGPs in the kernel. Finally, employing Saigo fractional-type operators, we establish fractional integrals and derivatives formulae for our innovative MLGPs. We conclude by proposing an open question regarding the Hermite numbers and their umbral calculus for further discussion in the field of this study.

Keywords: Mittag-Leffer function; Gegenbauer polynomials; Hermite polynomials; recurrence relations; integrals; fractional calculus; kinetic equation; orthogonality; symbolic operator
Mathematics Subject Classification: 33C45, 33E12, 26A33

1. Introduction and basic notations

Analogously to the efficient usage of Legendre polynomials in the theory of the well-known 3-dimensional spherical harmonics, Gegenbauer polynomials play a crucial role in the theory of hyper-spherical harmonics [1]. Notably, Legendre polynomials can be regarded as a specific case of Gegenbauer polynomials. The Mittag-Leffler function is an important function that is widely used in the subject of fractional calculus. Indeed, solutions of diverse differential and integral equations which involve fractional derivatives can be extracted from Mittag-Leffler functions. Moreover, the Mittag-Leffler function is effectively employed in the solution of the framework of the boundary value problems (see [12, 15, 16]). For $x \in \mathbb{C}$, the Mittag-Leffler function [20] is defined by:

$$E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)}, \quad Re(\alpha) > 0, \tag{1.1}$$

where as usual $\Gamma(\lambda)$ is the Gamma function, and $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, n \geq 0, \lambda \neq 0, -1, -2, \ldots$, is the Pochhammer symbol [31]. The generalized form of (1.1)

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, \quad Re(\alpha), Re(\beta) > 0 \tag{1.2}$$

was defined and studied by Wiman [33]. An interesting related function to the Mittag-Leffler function stated in (1.2) is the Wright function $W(\alpha, \beta; x)$, which is defined by [35]:

$$W(\alpha, \beta; x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\,\Gamma(\alpha n + \beta)}. \tag{1.3}$$

Another generalization of (1.2) was introduced in terms of the following series representation by Prabhakar [22]:

$$E^{\delta}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\alpha n + \beta)} \frac{x^n}{n!}, \tag{1.4}$$

where $\alpha, \beta, \delta \in \mathbb{C}; Re(\alpha), Re(\beta), Re(\delta) > 0$.

In [28], the authors investigated a modified generalization of the Mittag-Leffler function in (1.4) defined as follows:

$$E^{(\delta)}_{\alpha,\beta,\gamma}(x, y) = \sum_{m,n=0}^{\infty} \frac{(\delta)_{m+n}\, x^m\, y^n}{m!\, n!\, \Gamma(\alpha m + \beta n + \gamma)}. \tag{1.5}$$

The symbolic method offers a practical and efficient approach for presenting and analyzing both new and existing special functions. The work gave rise to the symbolic technique in [2]. In [3], Babusci et al. presented a novel symbolic method for investigating special functions using the derivation of particular operators referred to as symbolic operators. Dattoli et al. [6] deduced the symbolic operator $\hat{d}_{(\alpha,\beta)}$. The following equations describe how this operator functions on the vacuum function φ_0:

$$\hat{d}^{k}_{(\alpha,\beta)}\varphi_0 = \frac{\Gamma(k+1)}{\Gamma(\alpha k + \beta)} \ \left(\alpha, \beta \in \mathbb{R}^+, k \in \mathbb{R}\right), \tag{1.6}$$

and

$$\hat{d}^{\delta+k-1}_{(\alpha,\alpha(1-\delta)+\beta)}\varphi_0 = \frac{\Gamma(\delta + k)}{\Gamma(\alpha k + \beta)} \quad \left(\alpha, \beta, \delta \in \mathbb{R}^+, |\delta| \leq 1, k \in \mathbb{R}\right). \tag{1.7}$$

Notably, when $k = 0$, then Eq (1.6) yields

$$\varphi_0 = \frac{1}{\Gamma(\beta)}.$$

Observe that Eq (1.6) satisfies the properties (see [6]):

$$\hat{d}^{k}_{(\alpha,\beta)} \times \hat{d}^{m}_{(\alpha,\beta)} = \hat{d}^{k+m}_{(\alpha,\beta)} \ and \ \left(\hat{d}^{k}_{(\alpha,\beta)}\right)^r = \hat{d}^{rk}_{(\alpha,\beta)}.$$

The symbolic definition of Mittag-Leffler function $E_{\alpha,\beta}(x)$ in terms of $\hat{d}_{\alpha,\beta}$ can be given as [6]

$$E_{\alpha,\beta}(x) = e^{x\hat{d}_{(\alpha,\beta)}}\varphi_0. \tag{1.8}$$

By considering Eqs (1.6) and (1.7), the symbolic representations of the Mittag-Leffler functions in (1.4) and (1.5) are given as follows.

Lemma 1.1. *The following symbolic representations hold:*

$$E^{\delta}_{\alpha,\beta}(x) = e^{\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{1.9}$$

$$E^{\delta}_{\alpha,\beta}(x) = {}_1F_1\left[\delta;1;x\hat{d}_{(\alpha,\beta)}\right]\varphi_0, \tag{1.10}$$

$$E^{(\delta)}_{\alpha,\alpha,\beta}(x,y) = e^{\left((x+y)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{1.11}$$

where ${}_1F_1$ denotes the confluent hypergeometric function defined by [31]:

$${}_1F_1[\alpha,\beta;x] = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!}.$$

Proof. To verify (1.9), we have

$$e^{\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\} = \sum_{n=0}^{\infty}\frac{x^n}{n!\,\Gamma(\delta)}\hat{d}^{\delta+n-1}_{(\alpha,\alpha(1-\delta)+\beta)}\varphi_0 = \sum_{n=0}^{\infty}\frac{(\delta)_n\,x^n}{n!\,\Gamma(\alpha n+\beta)},$$

which implies the desired result. Moreover,

$${}_1F_1\left[\delta;1;x\hat{d}_{(\alpha,\beta)}\right]\varphi_0 = \sum_{n=0}^{\infty}\frac{(\delta)_n\,x^n\,\hat{d}^n_{(\alpha,\beta)}\varphi_0}{(n!)^2} = \sum_{n=0}^{\infty}\frac{(\delta)_n\,x^n}{n!\,\Gamma(\alpha n+\beta)},$$

which gives (1.10). The assertions (1.11) run parallel to the proofs of (1.9), and thus we skip the details. □

The Hermite polynomials [24]

$$H_n(x) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2x)^{n-2k}}{k!\,(n-2k)!}, \tag{1.12}$$

can be written by the generating function:

$$\exp\left(2xt-t^2\right) = \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}. \tag{1.13}$$

The following symbolic representation of the Hermite polynomials $H_n(x)$ can be readily derived using the derivative formula $\partial_z^n\,\{z^\lambda\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}\,z^{\lambda-n}$ $(n = 0,1,2,\cdots)$:

$$(-(1/2)\partial_x + 2x)^n = \sum_{k=0}^{n}\frac{(-1)^k n! 2^{2-2k}}{k!(n-k)!}\partial_x^k\,x^{n-k} = H_n(x). \tag{1.14}$$

The Hermite-Kampé de Fériet polynomials of two variables are determined by the representation (see e.g., [7]):

$$H_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-2k} y^k}{k!\,(n-2k)!}, \tag{1.15}$$

and the generating function is:

$$\exp\left(xt + yt^2\right) = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{1.16}$$

The Chebyshev polynomials of the second kind [10] and Legendre polynomials [24] are defined in the following series forms:

$$U_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (n-k)!\,(2x)^{n-2k}}{k!\,(n-2k)!}, \tag{1.17}$$

and

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n\, k!\,(n-k)!\,(n-2k)!}. \tag{1.18}$$

As an extended version of Chebyshev polynomials and Legendre polynomials, the so-called Gegenbauer polynomials are defined by the series [24]:

$$C_n^{\delta}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (\delta)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!}. \tag{1.19}$$

Clearly, by (1.17)–(1.19), we have

$$C_n^{\frac{1}{2}}(x) = P_n(x) \text{ and } C_n^{1}(x) = U_n(x). \tag{1.20}$$

For this work, we recall the following formula of the Fox-Wright function as introduced in [31]:

$${}_p\Psi_q \left[\begin{array}{cc} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); & \\ & z \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(\alpha_l + A_l n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}. \tag{1.21}$$

The current work, which is based on symbolic operators, attempts to establish and investigate a new class of the Mittag-Leffler-Gegenbauer polynomials (MLGPs), represented by ${}_EC_n^{\delta}(x;\alpha,\beta)$. The layout of the paper is as follows. In Section 2, we present the ${}_EC_n^{\delta}(x;\alpha,\beta)$ polynomials and deduce some essential characterizations, including the generating function, series representations, and symbolic rules. Section 3 derives recurrence relations and differential equations. Section 4 is devoted to deriving some integral representations. Section 5 focuses on establishing some finite summations. In Section 6, we derive the Rodrigues-type and orthogonal formulas. Section 7 demonstrates the significance relevance of the ${}_EC_n^{\delta}(x;\alpha,\beta)$ polynomials by formulating a fractional kinetic equation with ${}_EC_n^{\delta}(x;\alpha,\beta)$ polynomials in the kernel. In Section 8, we present fractional calculus formulas via the Saigo operator approach [19]. We conclude by summarizing key results of the current study.

2. Mittag-Leffler-Gegenbauer polynomials

Recently, several authors including Konhause [17], Krt and Özarslan [18, 21], Shahwan and Bin-Saad [29], Miomir et al. [30], Raza and Zainab [25], Subuhi Khan [32], and Konhause and Rainville [17, 24], studied various properties of Mittag-Leffler function in connections with certain polynomials such as Laguerre polynomials, Hermite polynomials, Sheffer-type polynomials, and Konhauser polynomials.

Motivated by the previously mentioned contributions, now we by making convolution of the Mittag-Leffler function $E^{\delta}_{\alpha,\beta}(x)$ (1.4) with the Hermite polynomials $H_n(x)$ (1.12), we introduce the MLGPs ${}_EC^{\delta}_n(x;\alpha,\beta)$. In view of Eqs (1.9) and (1.14), the symbolic definition of the MLGPs ${}_EC^{\delta}_n(x;\alpha,\beta)$ is derived as follows:

$$ {}_EC^{\delta}_n(x;\alpha,\beta) = \left(-(1/2)\partial_x + 2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)^n \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{2.1} $$

Alternatively, we can show that

$$ {}_EC^{\delta}_n(x;\alpha,\beta) = H_n\left(x\hat{d}^{\frac{1}{2}}_{(\alpha,\alpha(1-\delta)+\beta)}\right) \hat{d}^{\delta+\frac{1}{2}n-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{2.2} $$

For the series definition of the polynomials ${}_EC^{\delta}_n(x;\alpha,\beta)$, we prove the following theorem.

Theorem 2.1. *The MLGPs ${}_EC^{\delta}_n(x;\alpha,\beta)$ satisfy*

$$ {}_EC^{\delta}_n(x;\alpha,\beta) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\,(\delta)_{n-k}\,(2x)^{n-2k}}{k!\,(n-2k)!\,\Gamma(\alpha n - \alpha k + \beta)}. \tag{2.3} $$

Proof. Using (2.2) and (1.12), we get

$$ {}_EC^{\delta}_n(x;\alpha,\beta) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2x)^{n-2k}}{k!\,(n-2k)!} \hat{d}^{\delta+n-k-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, $$

which, by (1.7), gives the desired result (2.3). □

Remark 2.1. *For a particular case of the series representation (2.3), we have*

$$ {}_EC^{1}_n(x;1,1) = H_n(x). $$

Formula (2.3) and the connections in (1.20) suggest we define Mittag-Leffler-Legendre polynomials ${}_EP_n(x;\alpha,\beta)$ and Mittag-Leffler-Chebyshev polynomials ${}_EU_n(x;\alpha,\beta)$ in the series representations:

$$ {}_EP_n(x;\alpha,\beta) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\left(\frac{1}{2}\right)_{n-k}(2x)^{n-2k}}{\Gamma(\alpha n - \alpha k + \beta)(n-2k)!\,k!}, $$

and

$$ {}_EU_n(x;\alpha,\beta) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\,(n-k)!\,(2x)^{n-2k}}{\Gamma(\alpha n - \alpha k + \beta)(n-2k)!\,k!}. $$

Moreover, owing to (2.2), the symbolic definitions of the Mittag-Leffler-Legendre polynomials ${}_EP_n(x;\alpha,\beta)$ and Mittag-Leffler-Chebyshev polynomials ${}_EU_n(x;\alpha,\beta)$ are respectively given by

$$\begin{aligned}
{}_EP_n(x;\alpha,\beta) &= H_n\left(x\hat{d}^{\frac{1}{2}}_{\left(\alpha,\frac{1}{2}\alpha+\beta\right)}\right)\hat{d}^{\frac{1}{2}n-\frac{1}{2}}_{\left(\alpha,\frac{1}{2}\alpha+\beta\right)}\left\{\frac{\varphi_0}{\Gamma\left(\frac{1}{2}\right)}\right\},\\
{}_EU_n(x;\alpha,\beta) &= H_n\left(x\hat{d}^{\frac{1}{2}}_{(\alpha,\beta)}\right)\hat{d}^{\frac{1}{2}n}_{(\alpha,\beta)}\,\varphi_0.
\end{aligned}$$

Next, we establish some symbolic images for the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$.

Theorem 2.2. *For the MLGP ${}_EC_n^{\delta}(x;\alpha,\beta)$, we have the following symbolic Burchnalls-type formula:*

$$ {}_EC_n^{\delta}(x;\alpha,\beta) = \hat{D}_x^{\delta-1}\left(-\frac{1}{2}\hat{D}_x + 2x\hat{d}_{(\alpha,\beta)}\right)^n\left\{\frac{x^{\delta-1}\varphi_0}{\Gamma(\delta)}\right\}, \tag{2.4}$$

where $\hat{D}_x = \frac{\partial}{\partial x}$.

Proof. Denote the right side of (2.4) by I. Then, in view of the binomial theorem, we have

$$\begin{aligned}
I &= \sum_{k=0}^{n}(-1)^k\,2^{n-2k}\binom{n}{k}\hat{D}_x^{\delta+k-1}\left\{x^{n-k+\delta-1}\right\}\hat{d}^{n-k}_{(\alpha,\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\},\\
&= n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\,(\delta)_{n-k}\,(2x)^{n-2k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)},
\end{aligned}$$

which gives the left side of the desired result (2.4). □

Theorem 2.3. *A symbolic connection between the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$ and the Hermite Kampé de Fériet polynomial $H_n(x,y)$ is given by*

$$ {}_EC_n^{\delta}(x;\alpha,\beta) = H_n\Big(2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}, -\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\Big)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{2.5}$$

Proof. Denote the *right side* of (2.5) by I. Then, by using (1.15), we have

$$ I = n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2x)^{n-2k}}{k!\,(n-2k)!}\hat{d}^{\delta+n-k-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}.$$

Hence, by applying (1.7), the result follows. □

Theorem 2.4. *The following generating function holds:*

$$ \sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = E^{(\delta)}_{\alpha,\alpha,\beta}\left(2xt,-t^2\right). \tag{2.6}$$

Proof. We have

$$ \sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\,(\delta)_{n-k}\,(2x)^{n-2k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)}\,t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\,(\delta)_{n+k}\,(2xt)^n\,t^{2k}}{k!\,n!\,\Gamma(\alpha n+\alpha k+\beta)},$$

which, by (1.5), gives (2.6). □

However, another generating functions for the polynomials ${}_EC_n^{\delta}(x;\alpha,\beta)$ would accrue when we employ the function $E_{\alpha,\beta}^{\delta}(x)$ as provided in the following theorems.

Theorem 2.5. *If* $\{Re(\alpha), Re(\beta), Re(\delta)\} > 0$, *then the following generating function holds true:*

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = E_{\alpha,\beta}^{\delta}\left(2xt - t^2\right), \tag{2.7}$$

or alternatively

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = {}_1F_1\left[\delta; 1; \left(2xt - t^2\right)\hat{d}_{(\alpha,\beta)}\right]\varphi_0. \tag{2.8}$$

Proof. We have

$$E_{\alpha,\beta}^{\delta}\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{n!\,\Gamma(\alpha n+\beta)}\left(2xt - t^2\right)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(\delta)_n(2x)^{n-k}}{k!\,(n-k)!\,\Gamma(\alpha n+\beta)}\,t^{n+k},$$

which by letting $n = n - k$ and considering definition (2.3), we obtain the desired result (2.7). By using (1.10) in (2.7), we arrive at the result (2.8). □

Remark 2.2. *Letting* $\delta = -m$, $m \in \mathbb{N}$ *in (2.8) and using the relation [1, p.395]:*

$$L_m(x) = {}_1F_1[-m; 1; x], \tag{2.9}$$

we reach an interesting relationship,

$$\sum_{n=0}^{\infty} {}_EC_n^{-m}(x;\alpha,\beta)\frac{t^n}{n!} = L_m\left(\left(2xt - t^2\right)\hat{d}_{(\alpha,\beta)}\right)\varphi_0, \tag{2.10}$$

where $L_m(x)$ *are the Laguerre polynomials [24, p.200].*

Theorem 2.6. *The following generating functions hold:*

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = e^{\left((2xt-t^2)\ \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = e^{2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}t}\ E_{\alpha,\beta}^{\delta}\left(-t^2\right). \tag{2.12}$$

Proof. Using (2.5), it follows that

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n\left(2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}, -\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\frac{t^n}{n!}, \tag{2.13}$$

which, upon using (1.16) in the *right side* of the above equation, we get (2.11).

Regarding the Lie bracket $[\hat{A}, \hat{B}]$ defined by $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, it follows that

$$\left[2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}t, -\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}t^2\right] = 0.$$

Consequently, using the Weyl decoupling identity [8],

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{\frac{-k}{2}},\ k = \left[\hat{A}, \hat{B}\right],\ k \in \mathbb{C}. \tag{2.14}$$

We obtain an assertion (2.12) by utilizing (2.14) on the *right side* of (2.11) and (1.9) in the resulting equation. □

Theorem 2.7. *The following generating functions hold:*

$$\sum_{n=0}^{\infty} {}_{E}C_n^{\delta}(x;\alpha,\beta)\,\frac{t^n}{n!} = \left[1 - \hat{d}_{(\alpha,\beta)}\left(2t - x^{-1}\,t^2\right)\hat{D}_x^{-1}\right]^{-\delta}\varphi_0. \tag{2.15}$$

In particular, we have

$$\sum_{n=0}^{\infty} {}_{E}P_n(x;\alpha,\beta)\,\frac{t^n}{n!} = \left[1 - \hat{d}_{(\alpha,\beta)}\left(2t - x^{-1}\,t^2\right)\hat{D}_x^{-1}\right]^{-\frac{1}{2}}\varphi_0, \tag{2.16}$$

$$\sum_{n=0}^{\infty} {}_{E}U_n(x;\alpha,\beta)\,\frac{t^n}{n!} = \left[1 - \hat{d}_{(\alpha,\beta)}\left(2t - x^{-1}\,t^2\right)\hat{D}_x^{-1}\right]^{-1}\varphi_0. \tag{2.17}$$

Proof. Denote the right side of the assertion (2.15) by I. Then

$$\begin{aligned}
I &= \left[1 - \hat{d}_{(\alpha,\beta)}\left(2t - x^{-1}\,t^2\right)\hat{D}_x^{-1}\right]^{-\delta}\varphi_0\\
&= \sum_{n=0}^{\infty}\frac{(\delta)_n\left(2t - x^{-1}\,t^2\right)^n\,\hat{D}_x^{-n}\hat{d}^{n}_{(\alpha,\beta)}\varphi_0}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k\,(\delta)_n\,x^{-k}\,2^{n-k}\,\hat{D}_x^{-n}\,\hat{d}^{n}_{(\alpha,\beta)}\,\varphi_0}{k!\,(n-k)!}t^{n+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\,(\delta)_{n-k}\,x^{-k}\,2^{n-2k}\,\hat{D}_x^{-(n-k)}\,\hat{d}^{n-k}_{(\alpha,\beta)}\,\varphi_0}{k!\,(n-2k)!}\,t^n.
\end{aligned}$$

Thus, by using (1.6) and (2.3), and by taking into account that $\hat{D}_x^{-\varrho}\{1\} = \frac{x^{\varrho}}{\varrho!}$, we arrive at the desired result (2.15). The results (2.16) and (2.17) follow from assertion (2.15) by employing the identities (1.20). □

For the polynomials ${}_{E}C_n^{\delta}(x;\alpha,\beta)$, we can derive another symbolic generating function by using the symbolic operator $\hat{c}^{\alpha}\,\varphi_0$ (see [9]) in the following form.

Theorem 2.8. *The following generating functions hold:*

$$\sum_{n=0}^{\infty} {}_{E}C_n^{\delta}(x;\alpha,\beta)\,\frac{t^n}{n!} = \left[1 - \left(2xt - t^2\right)\hat{c}^{\alpha}\right]^{-\delta}\hat{c}^{\beta-1}\,\varphi_0, \tag{2.18}$$

where the symbolic operator $\hat{c}$ is defined by [9]:

$$\hat{c}^{\alpha}\,\varphi_0 = \frac{1}{\Gamma(\alpha+1)}. \tag{2.19}$$

In particular, we have

$$\sum_{n=0}^{\infty} {}_{E}P_n(x;\alpha,\beta)\,\frac{t^n}{n!} = \left[1 - \left(2xt - t^2\right)\hat{c}^{\alpha}\right]^{-\frac{1}{2}}\hat{c}^{\beta-1}\,\varphi_0, \tag{2.20}$$

$$\sum_{n=0}^{\infty} {}_{E}U_n(x;\alpha,\beta)\,\frac{t^n}{n!} = \left[1 - \left(2xt - t^2\right)\hat{c}^{\alpha}\right]^{-1}\hat{c}^{\beta-1}\,\varphi_0. \tag{2.21}$$

Proof. Denote the right side of the assertion (2.18) by I. Then

$$\begin{aligned} I &= \left[1-\left(2xt-t^2\right)\hat{c}^{\alpha}\right]^{-\delta}\hat{c}^{\beta-1}\varphi_0 \\ &= \sum_{n=0}^{\infty}\frac{(\delta)_n\left(2xt-t^2\right)^n\hat{c}^{\alpha n+\beta-1}\varphi_0}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(\delta)_n(2x)^{n-k}\hat{c}^{\alpha n+\beta-1}\varphi_0}{k!(n-k)!}t^{n+k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(\delta)_{n-k}(2x)^{n-k}\hat{c}^{\alpha n-\alpha k+\beta-1}\varphi_0}{k!(n-2k)!}t^n, \end{aligned}$$

which by using (2.19) and (2.3), we arrive at the desired result (2.18). The results (2.20) and (2.21) follow from assertion (2.18) by employing the identities (1.20). □

3. Recurrence relations and differential equation

The MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$, are rich in recurrence relations. Central to the development of many of these recurrence relations are the generating relations (2.7) and (2.11).

Theorem 3.1. *If* $\{Re(\alpha), Re(\beta), Re(\delta)\} > 0$, *then*

$${}_EC_n^{\delta}(x;\alpha,\beta) = 2\delta x\,{}_EC_{n-1}^{\delta+1}(x;\alpha,\alpha+\beta) - 2\delta(n-1)\,{}_EC_{n-2}^{\delta+1}(x;\alpha,\alpha+\beta), \tag{3.1}$$

$$x\hat{D}_x\,{}_EC_n^{\delta}(x;\alpha,\beta) = n\hat{D}_x\,{}_EC_{n-1}^{\delta}(x;\alpha,\beta) + n\,{}_EC_n^{\delta}(x;\alpha,\beta). \tag{3.2}$$

Proof. Differentiating (2.7) with respect to t, we get

$$2\delta(x-t)E_{\alpha,\alpha+\beta}^{\delta+1}\left(2xt-t^2\right) = \sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{n\,t^{n-1}}{n!}. \tag{3.3}$$

Using (2.7) in the left side of (3.3), in the resulting expression equate the coefficients $t^n/n!$ and let $n = n-1$ to obtain the result (3.1).

Differentiating (2.7) concerning x, we get

$$2\delta tE_{\alpha,\alpha+\beta}^{\delta+1}\left(2xt-t^2\right) = \sum_{n=0}^{\infty}\hat{D}_x\,{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}. \tag{3.4}$$

On multiplying (3.3) by t and (3.4) by $(x-t)$ and comparing the two resulting expressions, we find that

$$\sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{n\,t^n}{n!} = x\sum_{n=0}^{\infty}\hat{D}_x\,{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!} - \sum_{n=0}^{\infty}\hat{D}_x\,{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^{n+1}}{n!}. \tag{3.5}$$

Equating the coefficients of $t^n/n!$ in (3.5) leads to the result (3.2). □

Theorem 3.2. *Let $\alpha,\beta,\delta,x\in\mathbb{C}$ with $\{Re(\alpha),Re(\beta),Re(\delta)\}>0$. Then,*

$$\hat{D}_x\,{}_EC_n^{\delta}(x;\alpha,\beta)=2n\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\;{}_EC_{n-1}^{\delta}(x;\alpha,\beta), \tag{3.6}$$

$${}_EC_n^{\delta}(x;\alpha,\beta)=2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\;{}_EC_{n-1}^{\delta}(x;\alpha,\beta)-\hat{D}_x\,{}_EC_{n-1}^{\delta}(x;\alpha,\beta), \tag{3.7}$$

$${}_EC_{n+1}^{\delta}(x;\alpha,\beta)=2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\;{}_EC_{n}^{\delta}(x;\alpha,\beta)-2n\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}{}_EC_{n-1}^{\delta}(x;\alpha,\beta), \tag{3.8}$$

$${}_EC_n^{\delta}(x;\alpha,\beta)=\frac{x}{n}\hat{D}_x\;{}_EC_n^{\delta}(x;\alpha,\beta)-\hat{D}_x\;{}_EC_{n-1}^{\delta}(x;\alpha,\beta). \tag{3.9}$$

Proof. From (2.11), let

$$\Omega(\alpha,\beta,\delta;x,t)=e^{\left(2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t-\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t^2}\;\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}=\sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}. \tag{3.10}$$

Then,

$$\begin{cases}\hat{D}_x\,\Omega(\alpha,\beta,\delta;x,t)=2\,t\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\Omega(\alpha,\beta,\delta;x,t),\\ \hat{D}_t\,\Omega(\alpha,\beta,\delta;x,t)=2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}(x-t)\Omega(\alpha,\beta,\delta;x,t).\end{cases} \tag{3.11}$$

Comparing the two equations in (3.11), we infer that

$$(x-t)\hat{D}_x\,\Omega(\alpha,\beta,\delta;x,t)=t\hat{D}_t\,\Omega(\alpha,\beta,\delta;x,t). \tag{3.12}$$

Using (3.10) in the first equation of (3.11), we get

$$\sum_{n=0}^{\infty}\hat{D}_x\;{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}=2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^{n+1}}{n!}.$$

The desired result (3.6) is obtained by comparing the coefficients $t^n/n!$, on both sides of the aforementioned identity. In order to prove formula (3.7), we first multiply both sides of the assertion (3.6) by x, and then we combine the resultant expression with (3.2) to obtain the formula (3.7). Inserting (3.10) in the second equation of (3.11), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\;{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^{n-1}}{(n-1)!}\\ =&2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}-2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\sum_{n=0}^{\infty}n\,{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}.\end{aligned}$$

The result (3.8) is obtained by comparing the coefficients $t^n/n!$, on both sides of the aforementioned identity. Finally, using (3.10) in (3.12), we get

$$\sum_{n=0}^{\infty}\frac{x}{n}\hat{D}_x\;{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}-\sum_{n=0}^{\infty}\hat{D}_x\,{}_EC_{n-1}^{\delta}(x;\alpha,\beta)\frac{t^n}{(n-1)!}=\sum_{n=0}^{\infty}{}_EC_n^{\delta}(x;\alpha,\beta)\frac{t^n}{n!}.$$

The desired result (3.9) is obtained by comparing the coefficients $t^n/n!$, in both sides of the above assertion. □

Theorem 3.3. *For the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$, we have the following differential equation:*

$$\left(\hat{D}_x^2 - 2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\hat{D}_x + 2n\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right){}_EC_n^{\delta}(x;\alpha,\beta) = 0. \tag{3.13}$$

Proof. Owing to (3.6), we have

$$\hat{D}_x^2\,{}_EC_n^{\delta}(x;\alpha,\beta) = 2n\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\hat{D}_x\,{}_EC_{n-1}^{\delta}(x;\alpha,\beta). \tag{3.14}$$

Using (3.6) and (3.14) in (3.7), we arrive at the differential equation (3.13). □

Since the Mittag-Leffler-Gegenbauer polynomials are a part of the family of Gegenbauer polynomials, it is expected that the differential equation (3.13) satisfied by the polynomials ${}_EC_n^{\delta}(x;\alpha,\beta)$ arises naturally in mathematical physics and applied mathematics, particularly in problems involving, for example, spherical symmetry, higher dimensions, specific potential fields, and harmonic oscillators. Moreover, using umbral differential equations increases the applicability of the related polynomials.

4. Integral representations

In the current section, we established two integral representations for ${}_EC_n^{\delta}(x;\alpha,\beta)$. First, with the aid of Euler's integral [31],

$$\Gamma(z) = \int_0^{\infty} u^z e^{-u}du,\ \ Re(z) > 0, \tag{4.1}$$

we can deduce the following theorem.

Theorem 4.1. *Let $\alpha,\beta,\delta,u \in \mathbb{C}$ with $\{Re(\alpha), Re(\beta), Re(\delta)\} > 0$. Then,*

$${}_EC_n^{\delta}(x;\alpha,\beta) = \frac{(2x)^n}{\Gamma(\delta)}\int_0^{\infty} u^{\delta+n-1}e^{-u}\,W(\alpha,\alpha n+\beta;-u)du. \tag{4.2}$$

Proof. Starting from (2.3) and using the result:

$$(\delta)_m = \frac{1}{\Gamma(\delta)}\int_0^{\infty} u^{\delta+m-1}e^{-u}du,\ \ Re(\delta) > 0,\ m = 0,1,2,\cdots,$$

which follows immediately from (4.1), we obtain

$$\frac{{}_EC_n^{\delta}(x;\alpha,\beta)}{n!} = \sum_{k=0}^{[\frac{n}{2}]}\int_0^{\infty}\frac{(-1)^k\,(2x)^{n-2k}\,u^{\delta+n-k-1}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)\Gamma(\delta)}e^{-u}du.$$

Interchanging the order of integration and summation and then putting $n = n+2k$ on the right side, we get

$${}_EC_n^{\delta}(x;\alpha,\beta) = \frac{(2x)^n}{\Gamma(\delta)}\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,\Gamma(\alpha k+\alpha n+\beta)}u^{\delta+n-1}e^{-u}du,$$

which by using the definition of the Wright function (1.3), we obtain (4.2). □

Another integral representation for the polynomials ${}_EC_n^{\delta}(x;\alpha,\beta)$ is based on the complex integral defined by Hankel [34, Eq (137)] as follows:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0_+)} e^t\, t^{-z} dt, \quad (|\arg(z)|) \leq \pi. \tag{4.3}$$

Theorem 4.2. *Let $\alpha,\beta,\delta,t \in \mathbb{C}$ with $\{Re(\alpha), Re(\beta), Re(\delta)\} > 0$. Then,*

$${}_EC_n^{\delta}(x;\alpha,\beta) = \frac{n!}{2\pi i}\int_{-\infty}^{(0_+)} e^t\, t^{-\left(\beta+\frac{\alpha n}{2}\right)} C_n^{\delta}\left(x t^{\frac{-\alpha}{2}}\right) dt. \tag{4.4}$$

Proof. From (2.3) and the result (4.3), we get

$${}_EC_n^{\delta}(x;\alpha,\beta) = \frac{n!}{2\pi i}\sum_{k=0}^{[\frac{n}{2}]}\int_{-\infty}^{(0_+)} \frac{(-1)^k(2x)^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!}\, t^{-(\alpha n-\alpha k+\beta)} e^t\, dt,$$

which, by interchanging the order of integration and summation and using the definition of the Gegenbauer polynomials (1.19), we get the result (4.4). □

5. Finite summations

Now, we establish the following summation formulae for the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$.

Theorem 5.1. *The following summation formula for the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$ holds true:*

$${}_EC_n^{\delta}(x+v;\alpha,\beta) = \sum_{k=0}^{n}\binom{n}{k}(2v)^k\, \hat{d}^k_{(\alpha,\alpha(1-\delta)+\beta)}\, {}_EC_{n-k}^{\delta}(x;\alpha,\beta). \tag{5.1}$$

Proof. Replacing x by $x+v$ in (2.11), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x+v;\alpha,\beta)\frac{t^n}{n!} &= e^{\left(2(x+v)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t-\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t^2}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\\ &= e^{\left(2v\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t} e^{\left(2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t-\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)t^2}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}.\end{aligned}$$

Expanding the first exponential of the *right side* of the above equation and using (2.11), we have

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x+v;\alpha,\beta)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \hat{d}^k_{(\alpha,\alpha(1-\delta)+\beta)}\, {}_EC_n^{\delta}(x;\alpha,\beta)\frac{(2v)^k}{n!\,k!}t^{n+k}.$$

Using the series rearrangement formula gives

$$\sum_{n=0}^{\infty} {}_EC_n^{\delta}(x+v;\alpha,\beta)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(2v)^k\, \hat{d}^k_{(\alpha,\alpha(1-\delta)+\beta)}\, {}_EC_{n-k}^{\delta}(x;\alpha,\beta)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of like powers of $t^n/n!$ in the above equation, we get assertion (5.1). □

Theorem 5.2. *The following summation formula for the MLGPs polynomials ${}_EC_n^{\delta}(x;\alpha,\beta)$ holds true:*

$$ {}_EC_{k+l}^{\delta}(w;\alpha,\beta)=\sum_{n=0}^{k}\sum_{r=0}^{l}\binom{k}{n}\binom{l}{r}(2(w-x))^{n+r}\ \hat{d}^{n+r}_{(\alpha,\alpha(1-\delta)+\beta)}\,{}_EC_{k+l-n-r}^{\delta}(x;\alpha,\beta). \tag{5.2}$$

Proof. Replacing t by $t+u$ in (2.11), we get

$$\sum_{k=0}^{\infty}{}_EC_k^{\delta}(x;\alpha,\beta)\frac{(t+u)^k}{k!}=e^{\left(2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)-\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)^2}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}.$$

Using the formula in [31], we obtain that

$$\sum_{n=0}^{\infty}f(n)\frac{(x+y)^n}{n!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{5.3}$$

Therefore,

$$\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(x;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}=e^{\left(2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)-\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)^2}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{5.4}$$

which can be written as

$$e^{\left(-2x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)}\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(x;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}=e^{\left(-\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)^2}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}.$$

Multiplying both sides of the above equation with $\exp\left(2w\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}(t+u)\right)$, we find that

$$\begin{aligned}&\exp\left(2(w-x)(t+u)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(x;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}\\&=e^{\left(2w\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)-\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)(t+u)^2}\ \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\},\end{aligned}$$

which, by using (5.4) on the right side, it follows that

$$\sum_{n=0}^{\infty}\frac{2^n(w-x)^n(t+u)^n}{n!}\ \hat{d}^{n}_{(\alpha,\alpha(1-\delta)+\beta)}\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(x;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(w;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}.$$

Using (5.3) in the first summation on the left side leads to

$$\sum_{n,r=0}^{\infty}\frac{2^{n+r}(w-x)^{n+r}t^nu^r}{n!\,r!}\hat{d}^{n+r}_{(\alpha,\alpha(1-\delta)+\beta)}\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(x;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{k,l=0}^{\infty}{}_EC_{k+l}^{\delta}(w;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{5.5}$$

Now, replacing k by $k-n$, l by $l-r$, and using the identity [24]:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A(n,k)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}A(n,k-n), \tag{5.6}$$

on the left side of (5.5), we find

$$\sum_{k,l=0}^{\infty} {}_E C^{\delta}_{k+l}(w;\alpha,\beta)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty}\sum_{n=0}^{k}\sum_{r=0}^{l} \frac{2^{n+r}(w-x)^{n+r}t^k u^l}{n!\,r!\,(k-n)!\,(l-r)!}\hat{d}^{n+r}_{(\alpha,\alpha(1-\delta)+\beta)}\, {}_E C^{\delta}_{k+l-n-r}(x;\alpha,\beta). \tag{5.7}$$

Finally, comparing the coefficients of like powers $t^k/k!$ and $u^l/l!$ in (5.7), we get assertion (5.2). □

Remark 5.1. *For $l = 0$ and $w = x + v$, then (5.2) reduces to (5.1).*

6. Rodrigues-type and orthogonal-type formulas

We provide the Rodrigues-type formula for the MLGPs ${}_E C^{\delta}_n(x;\alpha,\beta)$, as illustrated in the following result.

Theorem 6.1. *The MLGPs ${}_E C^{\delta}_n(x;\alpha,\beta)$, satisfy the following Rodrigues-type formula:*

$$ {}_E C^{\delta}_n(x;\alpha,\beta) = (-1)^n\, e^{\left(x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, D^n_x\, e^{\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{6.1}$$

Proof. Let

$$f(t) = e^{\left((2xt-t^2)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}.$$

Then, by the assertion (2.11) and the Maclaurin's theorem:

$$f(t) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}f(t)\right]_{t=0}\frac{t^n}{n!},$$

we get

$$\sum_{n=0}^{\infty} {}_E C^{\delta}_n(x;\alpha,\beta)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\frac{\partial^n}{\partial t^n}e^{\left((2xt-t^2)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\right]_{t=0}\frac{t^n}{n!}.$$

Comparing the coefficients of like powers of $t^n/n!$ in the above equation, we find that

$$\begin{aligned} {}_E C^{\delta}_n(x;\alpha,\beta) &= \left[\frac{\partial^n}{\partial t^n}\, e^{\left((2xt-t^2)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\right]_{t=0} \\ &= \left[e^{\left(x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\,\frac{\partial^n}{\partial t^n}e^{-\left((x-t)^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\right]_{t=0}. \end{aligned} \tag{6.2}$$

Let

$$\Delta(\alpha,\beta,\delta;x,t) = e^{-\left((x-t)^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\, \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{6.3}$$

Differentiating (6.3) with respect to t and x implies

$$\hat{D}_t\,\Delta(\alpha,\beta,\delta;x,t) = 2(x-t)\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\,\Delta(\alpha,\beta,\delta;x,t)$$

and

$$\hat{D}_x\Delta(\alpha,\beta,\delta;x,t) = -2(x-t)\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\,\Delta(\alpha,\beta,\delta;x,t),$$

respectively.

From the above two assertions, we infer

$$\hat{D}_t\,\Delta(\alpha,\beta,\delta;x,t) = -\hat{D}_x\Delta(\alpha,\beta,\delta;x,t).$$

In general, we have

$$D_t^n\,\Delta(\alpha,\beta,\delta;x,t) = (-1)^n\,\hat{D}_x^n\,\Delta(\alpha,\beta,\delta;x,t). \tag{6.4}$$

Using Eq (6.4) on the right side of (6.2) and then putting $t = 0$, we get the desired result (6.1). □

Remark 6.1. *In light of Remark 2.1 and the result (1.12), formula (6.1) is reduced to the following famous Rodrigues formula (see [24, p.189]):*

$$H_n(x) = (-1)^n \exp\left(x^2\right) D_x^n \exp\left(-x^2\right).$$

Now, we prove an orthogonal formula for the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$.

Theorem 6.2. *The MLGPs* ${}_EC_n^{\delta}(x;\alpha,\beta)$ *form an orthogonal set over the interval* $(-\infty,\infty)$ *with weight function* $\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)$ *as follows:*

$$\int_{-\infty}^{\infty} \exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right){}_EC_n^{\delta}(x;\alpha,\beta){}_EC_m^{\delta}(x;\alpha,\beta)\,dx = 0,\ \ m \neq n. \tag{6.5}$$

Proof. Equation (3.13) can be written

$$\begin{aligned}&\hat{D}_x\left[\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{D}_{x\,E}C_n^{\delta}(x;\alpha,\beta)\right]\\ &\quad + 2n\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right){}_EC_n^{\delta}(x;\alpha,\beta) = 0.\end{aligned} \tag{6.6}$$

Along with (6.6), write

$$\begin{aligned}&\hat{D}_x\left[\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{D}_{x\,E}C_m^{\delta}(x;\alpha,\beta)\right]\\ &\quad + 2m\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right){}_EC_m^{\delta}(x;\alpha,\beta) = 0.\end{aligned} \tag{6.7}$$

Then, Eqs (6.6) and (6.7) are combined to yield

$$\begin{aligned}&2(n-m)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right){}_EC_n^{\delta}(x;\alpha,\beta){}_EC_m^{\delta}(x;\alpha,\beta)\\ =&\hat{D}_x\left[\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\left\{{}_EC_n^{\delta}(x;\alpha,\beta)\ \hat{D}_{x\,E}C_m^{\delta}(x;\alpha,\beta) - {}_EC_m^{\delta}(x;\alpha,\beta)\ \hat{D}_{x\,E}C_n^{\delta}(x;\alpha,\beta)\right\}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&2(n-m)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\int_a^b \exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right){}_EC_n^{\delta}(x;\alpha,\beta)\ {}_EC_m^{\delta}(x;\alpha,\beta)\,dx\\ &= \left[\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\left\{{}_EC_n^{\delta}(x;\alpha,\beta)\ \hat{D}_{x\,E}C_m^{\delta}(x;\alpha,\beta) - {}_EC_m^{\delta}(x;\alpha,\beta)\ \hat{D}_{x\,E}C_n^{\delta}(x;\alpha,\beta)\right\}\right]_a^b.\end{aligned}$$

Since the product of any polynomials in x by $\exp\left(-x^2\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right) \to 0$ as $x \to \infty$ or as $x \to -\infty$, we get the desired result (6.5). □

Remark 6.2. *If we set* $\alpha = \beta = \delta = 1$ *in (6.5), we get the following known result ([24, p.192]]:*

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x)\,H_m(x)dx = 0,\ \ m \neq n. \tag{6.8}$$

7. Fractional kinetic equation

In this section, we justify the relevance of the MLGPs ${}_EC_n^{\delta}(x;\alpha,\beta)$ by developing a fractional kinetic equation along with the new ${}_EC_n^{\delta}(x;\alpha,\beta)$ polynomials in the kernel. Saxena and Kalla's fractional kinetic equations are defined by [26, 27]:

$$\mathbf{N}(\tau) - \mathbf{N}_0 f(\tau) = -\varepsilon^{\nu}{}_0\hat{D}_{\tau}^{-\nu}\mathbf{N}(\tau), \quad (Re(\nu) > 0), \tag{7.1}$$

where $\mathbf{N}(\tau)$ is the number density of a given species at time τ and ε is a constant. When $\tau = 0$, then $\mathbf{N}_0 = \mathbf{N}(0)$. Consider $f \in L(0,\infty)$ and the Riemann-Liouville integral operator ${}_0\hat{D}_{\tau}^{-\nu}$ (see [11, 23]) as follows:

$${}_0\hat{D}_{\tau}^{-\nu} f(\tau) = \frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-s)^{\nu-1}f(s)ds, \quad (Re(\nu) > 0). \tag{7.2}$$

Theorem 7.1. *If $\mu > 0$ and $\nu > 0$, then the equation*

$$\mathbf{N}(\tau) - \mathbf{N}_0\, \tau^{\mu+\frac{n\nu}{2}-1}{}_EC_n^{\delta}\left(\tau^{\frac{\nu}{2}};\nu,\mu\right) = -\omega^{\nu}{}_0\hat{D}_{\tau}^{-\nu}\mathbf{N}(\tau) \tag{7.3}$$

has the solution

$$\mathbf{N}(\tau) = \mathbf{N}_0\sum_{k=0}^{[\frac{n}{2}]}\frac{n!\,(-1)^k\,2^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!}\,\tau^{\nu n-\nu k+\mu-1}E_{\nu,\nu n-\nu k+\mu}\left(-(\omega\tau)^{\nu}\right), \tag{7.4}$$

where $E_{\nu,\mu}(x)$ is the Mittag-Leffler function (1.2).

Proof. The Laplace transform involving the Riemann-Liouville fractional integral operator as introduced in [19] has the form:

$$L\left[{}_0\hat{D}_{\tau}^{-\nu}f(\tau) : p\right] = p^{-\nu}F(p), \tag{7.5}$$

where

$$F(p) = \int_0^{\infty} e^{-p\tau}f(\tau)d\tau, \quad (Re(p) > 0). \tag{7.6}$$

Now, by applying the Laplace transform on (7.3), it follows that

$$\begin{aligned}\mathbf{N}(p) &= \mathbf{N}_0\sum_{k=0}^{[\frac{n}{2}]}\frac{n!\,(-1)^k\,2^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!}p^{\nu k-\nu n-\mu} - \omega^{\nu}p^{-\nu}\mathbf{N}(p),\\ &= \mathbf{N}_0\sum_{k=0}^{[\frac{n}{2}]}\frac{n!\,(-1)^k\,2^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!\,p^{\nu n-\nu k+\mu}}\left(1+\omega^{\nu}p^{-\nu}\right)^{-1}.\end{aligned} \tag{7.7}$$

Computing the Laplace inverse of (7.7) and using $L^{-1}\left[p^{-\nu} : \tau\right] = \frac{\tau^{\nu-1}}{\Gamma(\nu)}$, we obtain that

$$L^{-1}\{\mathbf{N}(p)\} = \mathbf{N}_0\sum_{k=0}^{[\frac{n}{2}]}\frac{n!\,(-1)^k\,2^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!}\sum_{r=0}^{\infty}(-1)^r\omega^{\nu r}L^{-1}\left\{p^{-(\nu n-\nu k+\nu r+\mu)}\right\}, \tag{7.8}$$

which can be rewritten as

$$\begin{aligned}\mathbf{N}(\tau) &= \mathbf{N}_0 \sum_{k=0}^{[\frac{n}{2}]} \frac{n!\,(-1)^k\,2^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!} \sum_{r=0}^{\infty} \frac{(-1)^r \omega^{\nu r} \tau^{\nu n-\nu k+\nu r+\mu-1}}{\Gamma(\nu n-\nu k+\nu r+\mu)}, \\ &= \mathbf{N}_0 \sum_{k=0}^{[\frac{n}{2}]} \frac{n!\,(-1)^k\,2^{n-2k}(\delta)_{n-k}}{k!\,(n-2k)!} \tau^{\nu n-\nu k+\mu-1} \sum_{r=0}^{\infty} \frac{(-1)^r \omega^{\nu r} \tau^{\nu r}}{\Gamma(\nu r+\nu n-\nu k+\mu)}, \end{aligned} \tag{7.9}$$

which, by using (1.2), we get the desired result (7.4). □

We end this section by concluding that the Saxena and Kalla fractional kinetic equation (7.1) is a type of fractional differential equation used to model phenomena like diffusion in porous media, reaction and relaxation processes, and anomalous diffusion, whereas our fractional kinetic equation (7.3) is a specific type of fractional kinetic equation used to model a reaction kinetic fractional equation involving extended polynomials. In solving the both cases in (7.1) and (7.3), the researcher uses the same theoretical technique, which are the Laplace transform, inverse Laplace transform, and the Sumudu transform. Very recently, a number of authors published papers involving the problems of fractional reaction-diffusion systems using modern theoretical analysis and numerical simulation, see, e.g., [4, 13, 14].

8. Fractional calculus via Saigo operators

This section establishes some fractional integrals and fractional derivatives relations for the polynomials ${}_E C_n^{\delta}(t;\alpha,\beta)$. First, we recall the fractional integral operators due to Saigo (see [19]):

$$\left(I_{0,x}^{\mu,\nu,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda-\nu+\eta)}{\Gamma(\lambda-\nu)\Gamma(\lambda+\mu+\eta)} x^{\lambda-\nu-1}, \tag{8.1}$$

and

$$\left(I_{x,\infty}^{\mu,\nu,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\nu-\lambda+1)\Gamma(\eta-\lambda+1)}{\Gamma(1-\lambda)\Gamma(\nu+\mu-\lambda+\eta+1)} x^{\lambda-\nu-1}. \tag{8.2}$$

If we take $\nu = -\mu$ in Eqs (8.1) and (8.2), we have

$$\left(\mathcal{R}_{0,x}^{\mu}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)} x^{\lambda+\mu-1}, \tag{8.3}$$

and

$$\left(\mathcal{W}_{x,\infty}^{\mu}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\mu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda+\mu-1}, \tag{8.4}$$

where $\mathcal{R}_{0,x}^{\mu}$ and $\mathcal{W}_{x,\infty}^{\mu}$ denote the Riemann-Liouville and the Erdélyi-Kober fractional integral operators (see [19]), and if we take $\nu = 0$ in Eqs (8.1) and (8.2), we have

$$\left(\mathcal{E}_{0,x}^{\mu,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda+\eta)}{\Gamma(\lambda+\mu+\eta)} x^{\lambda-1}, \tag{8.5}$$

and

$$\left(\mathcal{K}_{x,\infty}^{\mu,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\eta-\lambda+1)}{\Gamma(\mu-\lambda+\eta+1)} x^{\lambda-1}, \tag{8.6}$$

where $\mathcal{E}_{0,x}^{\mu,\eta}$ and $\mathcal{K}_{x,\infty}^{\mu,\eta}$ denote the Weyl type and the Erdélyi-Kober fractional integral operators (see [19]).

Theorem 8.1. *Let $\alpha \in \mathbb{N}, \mu, \nu, \eta, \beta, \delta \in \mathbb{C}, x > 0$, and n be a non-negative integer. Then,*

$$\left(I_{0,x}^{\mu,\nu,\eta}\left[t^{\lambda-1}{}_{E}C_{n}^{\delta}(t;\alpha,\beta)\right]\right)(x) = \frac{2^{n}\,n!\,x^{\lambda-\nu+n-1}}{\Gamma(\delta)} \times {}_{3}\Psi_{4}\left[\begin{matrix}(\delta+n,-1),(\lambda+n,-2),(\lambda-\nu+\eta+n,-2);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(\lambda-\nu+n,-2),(\lambda+\mu+\eta+n,-2);\end{matrix}\ \frac{-1}{4x^{2}}\right]. \tag{8.7}$$

Proof. Seeking simplicity, we denote the *left side* of (8.7) by Ω. Then, by using (2.3), interchanging the order of integration and summation, we conclude

$$\Omega = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!\,2^{n-2k}(-1)^{k}(\delta)_{n-k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)}\left(I_{0,x}^{\mu,\nu,\eta}\,t^{\lambda+n-2k-1}\right)(x).$$

Employing relation (8.1), we arrive at

$$\Omega = x^{\lambda-\nu+n-1}\sum_{k=0}^{[\frac{n}{2}]}\frac{n!\,2^{n-2k}(-1)^{k}(\delta)_{n-k}\Gamma(\lambda+n-2k)\Gamma(\lambda-\nu+\eta+n-2k)x^{-2k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)\Gamma(\lambda-\nu+n-2k)\Gamma(\lambda+\mu+\eta+n-2k)},$$

$$= \frac{2^{n}\,n!\,x^{\lambda-\nu+n-1}}{\Gamma(\delta)} \times \sum_{k=0}^{[\frac{n}{2}]}\frac{\Gamma(\delta+n-k)\Gamma(\lambda+n-2k)\Gamma(\lambda-\nu+\eta+n-2k)\left(\frac{-1}{4x^{2}}\right)^{k}}{\Gamma(n+1-2k)\Gamma(\alpha n+\beta-\alpha k)\Gamma(\lambda-\nu+n-2k)\Gamma(\lambda+\mu+\eta+n-2k)k!}.$$

Finally, using definition (1.21) leads to formula (8.7). □

Corollary 8.1. *As a consequence of (8.3) and Theorem 8.1 with $\nu = -\mu$, we have*

$$\left(\mathcal{R}_{0,x}^{\mu}\left[t^{\lambda-1}{}_{E}C_{n}^{\delta}(t;\alpha,\beta)\right]\right)(x) = \frac{2^{n}\,n!\,x^{\lambda+\mu+n-1}}{\Gamma(\delta)} \times {}_{2}\Psi_{3}\left[\begin{matrix}(\delta+n,-1),(\lambda+n,-2);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(\lambda+\mu+n,-2);\end{matrix}\ \frac{-1}{4x^{2}}\right]. \tag{8.8}$$

Corollary 8.2. *As a consequence of (8.5) and Theorem 8.1 with $\nu = 0$, we have*

$$\left(\mathcal{E}_{0,x}^{\mu,\eta}\left[t^{\lambda-1}{}_{E}C_{n}^{\delta}(t;\alpha,\beta)\right]\right)(x) = \frac{2^{n}\,n!\,x^{\lambda+n-1}}{\Gamma(\delta)} \times {}_{2}\Psi_{3}\left[\begin{matrix}(\delta+n,-1),(\lambda+\eta+n,-2);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(\lambda+\mu+\eta+n,-2);\end{matrix}\ \frac{-1}{4x^{2}}\right]. \tag{8.9}$$

Theorem 8.2. *AAALet $\alpha \in \mathbb{N}, \mu, \nu, \eta, \beta, \delta \in \mathbb{C}, x > 0$ and n a non-negative integer. Then,*

$$\left(I_{x,\infty}^{\mu,\nu,\eta}\left[t^{\lambda-1}{}_{E}C_{n}^{\delta}\left(\frac{1}{t};\alpha,\beta\right)\right]\right)(x) = \frac{2^{n}\,n!\,x^{\lambda-\nu-n-1}}{\Gamma(\delta)} \times {}_{3}\Psi_{4}\left[\begin{matrix}(\delta+n,-1),(\nu-\lambda+n+1,-2),(\eta-\lambda+n+1,-2);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(1-\lambda+n,-2),(\nu+\mu-\lambda+\eta+n+1,-2);\end{matrix}\ \frac{-x^{2}}{4}\right]. \tag{8.10}$$

Proof. By considering the operator (8.2) and proceeding similarly as in the proof of Theorem 8.1, the result follows. □

Corollary 8.3. *As a consequence of (8.4) and Theorem 8.2 with $\nu = -\mu$, we have*

$$\left(\mathcal{W}^{\mu}_{x,\infty}\left[t^{\lambda-1}{}_{E}C^{\delta}_{n}\left(\frac{1}{t};\alpha,\beta\right)\right]\right)(x) = \frac{2^n\, n!\, x^{\lambda+\mu-n-1}}{\Gamma(\delta)} \times {}_2\Psi_3\left[\begin{array}{c}(\delta+n,-1),(1-\mu-\lambda+n,-2);\\ \\ (n+1,-2),(\alpha n+\beta,-\alpha),(1-\lambda+n,-2);\end{array}\ \frac{-x^2}{4}\right]. \tag{8.11}$$

Corollary 8.4. *As a consequence of (8.6) and Theorem 8.2 with $\nu = 0$, we have*

$$\left(\mathcal{K}^{\mu,\eta}_{x,\infty}\left[t^{\lambda-1}{}_{E}C^{\delta}_{n}\left(\frac{1}{t};\alpha,\beta\right)\right]\right)(x) = \frac{2^n\, n!\, x^{\lambda+n-1}}{\Gamma(\delta)} \times {}_2\Psi_3\left[\begin{array}{c}(\delta-n,-1),(\eta-\lambda+n+1,-2);\\ \\ (n+1,-2),(\alpha n+\beta,-\alpha),(\mu-\lambda+\eta+n+1,-2);\end{array}\ \frac{-x^2}{4}\right]. \tag{8.12}$$

9. Conclusions

In this study, we introduced and analyzed a new class of the MLGPs ${}_{E}C^{\delta}_{n}(x;\alpha,\beta)$ in the sense of a symbolic method. We established their fundamental properties, including generating functions, series representations, symbolic identities, recurrence relations, differential equations, integral representations, and finite summations. Also, we established a Rodrigues-type formula and orthogonal formulas for novel MLGPs. To illustrate their applicability, we developed a fractional kinetic equation incorporating these polynomials in the kernel. Furthermore, we explored their fractional calculus formulations via the Saigo operator approach. The results presented in this work not only extend the theory of special functions but also pave new ways for further research in fractional calculus, mathematical physics, and related applied fields. It is noted that Hermite numbers, within the context of umbral calculus, are related to Hermite polynomials, which can be viewed as Newton binomials and are useful for simplifying the study of special polynomials. Umbral calculus provides a framework to extend these polynomials [5]. In the upcoming publication, we will address the challenge of discussing the relations between Hermite numbers, umbral calculus, and the use of the umbral form of Hermite numbers.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

Mohra Zayed extends her appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through Large Research Project under grant number RGP2/181/46.

Conflict of interest

The authors declare no conflicts of interest.

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