AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 16624–16641.
DOI:10.3934/math.2025745
Received: 01 April 2025
Revised: 02 July 2025
Accepted: 11 July 2025
Published: 24 July 2025

Research article

The well-posedness and regularity of mild solutions to the time-fractional Cable equation

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Abstract: This paper investigates the well-posedness of mild solutions for a linear time-fractional Cable equation on a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ with a C^2 boundary:

$$\begin{cases} \partial_t u = \partial_t^{1-\alpha} \Delta u - \partial_t^{1-\beta} u + f, & (t, x) \in (0, T) \times \Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $0 < \alpha, \beta < 1$, and $\partial_t^{1-\beta}$ and $\partial_t^{1-\alpha}$ denote the Riemann–Liouville fractional derivatives of orders $1-\beta$ and $1-\alpha$, respectively. By employing the eigenfunction expansion method, we constructed the mild solution and established its definition. Utilizing the Banach contraction mapping principle and properties of the Mittag-Leffler function, we derived the existence, uniqueness, and regularity of mild solutions for the linear problem. Furthermore, we introduced a weighted Hölder continuous function space and demonstrated the existence and uniqueness of mild solutions within this frameworks. The results obtained in this work contribute to the theoretical understanding of time-fractional Cable equations and serve as a foundation for further studies in fractional-order diffusion processes.

Keywords: time-fractional Cable equation; well-posedness; weighted Hölder space
Mathematics Subject Classification: 26A33, 34A12, 35R11

1. Introduction

The time-fractional Cable equation analyzed in this work stems from the time-fractional diffusion equation

$$\partial^\alpha u = \Delta u,$$

which provides a flexible framework for modeling anomalous transport phenomena in various physical settings through the adjustment of the fractional-order α. When $\alpha \in (0,1)$, the equation is frequently utilized to describe subdiffusion processes, such as the trapping effects of particles in porous media or the sluggish diffusion observed in biological tissues. The investigation of fractional diffusion equations is primarily driven by the necessity to accurately model and comprehend the anomalous diffusion behaviors encountered in natural and engineered systems. In contrast to classical integer-order diffusion equations, fractional-order models provide a refined and more comprehensive description of complex diffusion dynamics.

As a subclass of fractional subdiffusion equations, the fractional Cable equation serves as a mathematical model for elucidating the propagation of electrical signals in neuronal dendrites. By incorporating fractional calculus, it extends the classical Cable equation, enabling a more precise representation of dendritic structural complexity and heterogeneity.

In recent years, extensive research efforts have been devoted to the fractional Cable equation. For example, in [6], Langlands and Henry introduced fractional Nernst–Planck equations, formulated fractional Cable models, and derived solutions for infinite and semi-infinite Cable equations. They also provided fundamental solutions for two distinct types of fractional Cable equations in [7]. In [5], Li and Deng obtained analytical solutions to the time-space fractional Cable equation via the integral transform method. In [15], Zhuang and Liu applied the Galerkin method to simulate the fractional Cable equation and validated their theoretical findings. Then, Liu and Du developed a two-grid algorithm in conjunction with the finite element method to solve a nonlinear fractional Cable equation in [8]. Zhang and Yang [14] proposed a discrete-time orthogonal spline collocation method for solving the two-dimensional fractional Cable equation.

On the other hand, in [2], Bhrawy and Zaky introduced a spectral collocation method based on the shifted Jacobi collocation procedure combined with the shifted Jacobi operational matrix to solve one- and two-dimensional variable-order fractional nonlinear Cable equations. Abdelkawy and Alqahtani [1] employed the spectral collocation method to solve the one- and two-dimensional Stokes first problem for a heated generalized second-grade fluid.

Additionally, several other numerical approaches have been explored for solving the fractional Cable equation. For instance, Lin and Jiang [9] introduced an algorithm rooted in reproducing kernel theory to derive exact and numerical solutions for Stokes' first problem in a heated generalized second-grade fluid. Dehghan and Abbaszadeh [3] developed an error estimation framework for a numerical scheme based on the element-free Galerkin method for the fractional Cable equation with Dirichlet boundary conditions. In [13], Saxena and Tomovski investigated the well-posedness of a generalized time-space fractional Cable equation with a source term

$$\tau_\gamma D_t^\gamma V(x,t) = D_t^{1-\alpha}\left(\Delta^\mu V(x,t)\right) - \lambda^2 D_t^{1-\beta}(V(x,t)) + f(x,t).$$

Utilizing the Fourier-Laplace transform, they derived an infinite series representation for Green's function and examined the asymptotic behavior of the fundamental solution in both short and long time regimes. In [10], Ma and Chen developed a finite difference method for solving a two-dimensional generalized time-fractional Cable equation and validated its accuracy through numerical error analysis. In [4], Khan and Alias constructed a high-order implicit finite difference iterative scheme for discretizing the Caputo fractional derivative and implemented a fourth-order implicit scheme for spatial derivatives.

Motivated by the aforementioned studies, this paper focuses on the following time-fractional Cable equation:

$$\begin{cases}\partial_t u = \partial_t^{1-\alpha}\Delta u - \partial_t^{1-\beta} u + f, & (t,x) \in (0,T)\times\Omega,\\ u(0,x) = u_0(x), & x \in \Omega,\\ u = 0, & x \in \partial\Omega,\end{cases} \tag{1.1}$$

where $0 < \alpha, \beta < 1$, and $\partial_t^{1-\beta}$ and $\partial_t^{1-\alpha}$ denote the Riemann–Liouville fractional derivatives. The domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded open set with a C^2 boundary, and the operator Δ is the Laplacian with respect to the spatial variable x. Given the initial condition $u(x,0) = u_0(x)$, the source term f is assumed to satisfy appropriate regularity conditions.

The structure of this paper is as follows. In Section 3, we derive an explicit representation of the mild solution using an eigenfunction expansion approach and establish estimates for the corresponding solution operator. In Section 4, by employing the Banach contraction mapping principle theorem and leveraging the properties of the Mittag-Leffler function, we prove the existence, uniqueness, and regularity of the mild solution. Furthermore, we introduce a weighted Hölder continuous function space and demonstrate the existence and uniqueness of the mild solution within this framework.

2. Preliminaries

Over the years, numerous definitions of fractional derivatives have been introduced [17]. In this study, we specifically focus on the Riemann–Liouville derivatives. This section presents the fundamental definitions of fractional derivatives and the key properties of the Mittag-Leffler function that will be employed throughout the paper. For omitted proofs, readers may refer to [17] and [11] or other standard references on fractional calculus.

As is customary, $\mathbb{R}^+$ denotes the set of positive real numbers, and $\mathbb{N}^+$ denotes the set of positive integers.

Let $(X, \|\cdot\|_X)$ be a Banach space, and denote by $L(X)$ the space of all bounded linear operators from X to itself, equipped with the operator norm $\|\cdot\|_{L(X)}$. We denote by $C(I,X)$ the space of bounded continuous functions from an interval I to X, endowed with the norm

$$\|u\|_{C(I,X)} = \sup_{t\in I} \|u(t)\|_X.$$

If A is a closed linear operator, we denote by $D(A^\gamma)$ for $\gamma > 0$ the fractional power spaces associated with the operator A.

Definition 2.1. *[17] (Riemann-Liouville fractional integrals).*

Let u be a function defined on $[a,b]$. The Riemann-Liouville fractional integrals of order γ for function u denoted by ${}_aD_t^{-\gamma}u(t)$, respectively, are defined by

$${}_aD_t^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}u(s)\mathrm{d}s, \quad t \in [a,b], \gamma > 0,$$

where $\Gamma > 0$ is the gamma function.

Definition 2.2. *[17] (Riemann-Liouville fractional derivatives).*

Let u be a function defined on $[a, b]$. *The Riemann-Liouville fractional derivatives of order* γ *for function u denoted by* ${}_aD_t^{\gamma}u(t)$*, respectively, are defined by*

$$ {}_aD_t^{\gamma}u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\, {}_aD_t^{\gamma-n}u(t) = \frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_a^t (t-s)^{n-\gamma-1}u(s)\mathrm{d}s\right), $$

where $t \in [a, b], n-1 \le \gamma < n$*, and* $n \in \mathbb{N}^+$.

Lemma 2.1. *[11] Suppose that* $\lambda > 0$*,* $\xi, \mu, \nu > 0$*, and* $k \in \mathbb{N}$ *is a positive integer. Then*

$$ \frac{d^k}{dt^k}E_{\mu,1}(-\lambda t^{\mu}) = -\lambda\, t^{\mu-k}E_{\mu,\mu-k+1}(-\lambda t^{\mu}), \quad t > 0, $$

and

$$ \partial_t^{\xi}\Big(t^{\nu-1}E_{\mu,\nu}(-\lambda t^{\mu})\Big) = t^{\nu-\xi-1}E_{\mu,\nu-\xi}(-\lambda t^{\mu}), \quad t > 0. $$

In particular, by integrating the series term-by-term for the Mittag–Leffler function, one obtains

$$ \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\, s^{\beta-1}E_{\alpha,\beta}(\lambda s^{\alpha})\, ds = t^{\beta+\theta-1}E_{\alpha,\beta+\theta}(\lambda t^{\alpha}), \quad \theta > 0,\ \beta > 0,\ t > 0. $$

Lemma 2.2. *[11] If* $0 < \mu < 2$*,* $\nu \in \mathbb{R}$*, and* $\pi\mu/2 < \theta < \min(\pi, \pi\mu)$*, then*

$$ \left|E_{\mu,\nu}(z)\right| \le \frac{C}{1+|z|}, \quad z \in \mathbb{C}, \quad \theta \le |\arg z| \le \pi, $$

where the constant $C > 0$ *depends on* μ*,* ν*, and* θ.

Lemma 2.3. *[11] If* $\mu < 2$*,* $\nu \in \mathbb{R}$*, and for* θ *satisfying* $\pi\mu/2 < \theta < \min(\pi, \pi\mu)$*, let* C_1 *and* C_2 *be positive real constants. Then,*

$$ \left|E_{\mu,\nu}(z)\right| \le C_1\,(1+|z|)^{\frac{1-\nu}{\mu}}\exp\Big(\mathrm{Re}\big(z^{1/\mu}\big)\Big) + \frac{C_2}{1+|z|}, \quad z \in \mathbb{C}, \quad |\arg z| \le \theta. $$

3. Mild solution formulation

Let $L^2(\Omega)$ be the real Hilbert space equipped with the standard norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Given that the operator $A = -\Delta$ is self-adjoint in the Sobolev space $H_0^1(\Omega) \cap H^2(\Omega)$, then there exists an orthonormal basis $\{\lambda_k, e_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ consisting of eigenvalues λ_k and corresponding eigenfunctions e_k of A.

For any $\gamma \ge 0$, the fractional power operator A^{γ} is defined as

$$ A^{\gamma}u = \sum_{k=1}^{\infty}\lambda_k^{\gamma}(u, e_k)e_k, \quad u \in D(A^{\gamma}), $$

with the associated domain given by

$$ \mathcal{D}(A^{\gamma}) = \left\{ v \in L^2(\Omega) : \sum_{k=1}^{\infty}\lambda_k^{2\gamma}\,|(v, e_k)|^2 < \infty \right\}. $$

The space $D(A^\gamma)$ forms a Hilbert space under the norm

$$D(A^\gamma) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(u, e_k)|^2 < \infty \right\}.$$

Next, we will define the mild solution for Eq (1.1).

Assume that $u : [0, T] \times \Omega \to \mathbb{R}$ is a solution of Eq (1.1), and that $u(t, x)$ admits the decomposition

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t) e_k(x), \quad k = 1, 2, 3, \ldots, \tag{3.1}$$

where the coefficients are given by $u_k(t) = (u(t, x), e_k)$.

Given initial data $u_0 \in D(A^\gamma)$, we exploit the properties of the operator $A = -\Delta$ and project Eq (1.1) onto the eigenfunctions $\{e_k(x)\}$. This yields the following fractional ordinary differential equation:

$$u_k'(t) + \lambda_k \partial_t^{1-\alpha} u_k = f_k(t) - \partial_t^{1-\beta} u_k, \quad u_k(0) = u_{0k},$$

where $u_{0k} = (u_0, e_k)$ and $f_k(t) = (f(t, \cdot), e_k)$.

It follows from Lemma 2.2 in [16] and the uniqueness of the Laplace transform that the solution of this fractional ordinary differential equation can be expressed in terms of Mittag-Leffler operators. Specifically, we have

$$u_k(t) = E_{\alpha,1}(-\lambda_k t^\alpha) u_{0k} + \int_0^t E_{\alpha,1}(-\lambda_k (t-s)^\alpha) \left[f_k(s) - \partial_s^{1-\beta} u_k(s) \right] ds.$$

By combining with Eq (3.1), we obtain the representation of the mild solution to Eq (1.1):

$$u(t, x) = \sum_{k=1}^{\infty} \left[E_{\alpha,1}(-\lambda_k t^\alpha) u_{0k} + \int_0^t E_{\alpha,1}(-\lambda_k (t-s)^\alpha) \left(f_k(s) - \partial_s^{1-\beta} u_k(s) \right) ds \right] e_k(x). \tag{3.2}$$

On the basis of the relationship $\partial_t^{1-\beta} u(t) = I_t^\beta (\partial_t u(t))$ and applying Lemma 2.1 with interchange of integration order, we have

$$\int_0^t E_{\alpha,1}(-\lambda_k (t-s)^\alpha) \partial_s^{1-\beta} u_k(s) ds = \int_0^t (t-s)^\beta E_{\alpha,\beta+1}(-\lambda_k (t-s)^\alpha) \partial_s u_k(s) ds.$$

Using integration by parts with Lemma 2.1, this transforms to

$$\begin{aligned} \int_0^t (t-s)^\beta E_{\alpha,\beta+1}(-\lambda_k (t-s)^\alpha) \partial_s u_k(s) ds = & \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-\lambda_k (t-s)^\alpha) u_k(s) ds \\ & - t^\beta E_{\alpha,\beta+1}(-\lambda_k t^\alpha) u_k(0). \end{aligned}$$

We now define solution operators for $v \in L^2(\Omega)$ and a.e. $t > 0$:

$$\mathcal{S}_\alpha(t) v = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k t^\alpha)(v, e_k) e_k(x),$$

$$\mathcal{Q}_\alpha(t) v = \sum_{k=1}^{\infty} t^\beta E_{\alpha,\beta+1}(-\lambda_k t^\alpha)(v, e_k) e_k(x),$$

$$\mathcal{P}_\alpha(t) v = \sum_{k=1}^{\infty} t^{\beta-1} E_{\alpha,\beta}(-\lambda_k t^\alpha)(v, e_k) e_k(x).$$

According to Lemma 2.1, the solution operator has the following properties:

$$I_t^{\beta}\mathcal{S}_\alpha(t)v = \mathcal{Q}_\alpha(t)v, \quad \mathcal{Q}'_\alpha(t)v = \mathcal{P}_\alpha(t)v. \tag{3.3}$$

In the following, we focus exclusively on the time variable t, adopting the notational convention $u(t) = u(t,\cdot)$ and $f(t) = f(t,\cdot)$ for brevity. Under this formulation, Eq (1.1) can be equivalently expressed in its mild solution form as

$$u(t) = \mathcal{S}_\alpha(t)u_0 + \mathcal{Q}_\alpha(t)u_0 + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds - \int_0^t \mathcal{P}_\alpha(t-s)u(s)ds. \tag{3.4}$$

Remarkably, the term $\partial_t^{1-\beta}u(t,x)$ in the linear problem can be regarded as the nonlinear term in a nonlinear problem. In particular, when it is transformed into the fundamental solution, a significant amount of computation required to verify the properties of the solution can be avoided. We now introduce the definition of the mild solution for problem (1.1).

Next, we introduce the definition of the appropriate solution to problem (1.1).

Definition 3.1 (Mild solution). *A function $u \in C([0,T];L^2(\Omega))$ is called a **mild solution** to problem* (1.1) *if it satisfies the integral equation* (3.4).

Next, to establish the well-posedness and regularity of the solution, we present the following properties of the solution operator.

Lemma 3.1 (Operator estimates). *Let $0 \le \iota \le \gamma \le 1$ and $v \in D(A^\iota)$. For any $t > 0$, the following estimates hold:*

$$\|\mathcal{Q}_\alpha(t)v\|_{D(A^\gamma)} \le Mt^{\beta-\alpha\gamma+\alpha t}\|v\|_{D(A^\iota)},$$
$$\|\mathcal{P}_\alpha(t)v\|_{D(A^\gamma)} \le Mt^{(\beta-\alpha\gamma+\alpha t)-1}\|v\|_{D(A^\iota)},$$
$$\|\mathcal{S}_\alpha(t)v\|_{D(A^\gamma)} \le Mt^{\alpha(\iota-\gamma)}\|v\|_{D(A^\iota)}.$$

Proof. Assume that $0 \le \iota \le \gamma \le 1$. Then, for any $t > 0$ and $v \in D(A^\iota)$, we have

$$\begin{aligned}
\|\mathcal{Q}_\alpha(t)v\|^2_{D(A^\gamma)} &= \sum_{k=1}^{\infty} t^{2\beta}\lambda_k^{2(\gamma-\iota)}\left[E_{\alpha,\beta+1}(-\lambda_k t^\alpha)\right]^2 \lambda_k^{2\iota}|(v,e_k)|^2 \\
&\le t^{2\beta}\sum_{k=1}^{\infty}\frac{\lambda_k^{2(\gamma-\iota)}}{(1+\lambda_k t^\alpha)^2}\lambda_k^{2\iota}|(v,e_k)|^2 \\
&= t^{2(\beta-\alpha\gamma+\alpha\iota)}\sum_{k=1}^{\infty}\left[\frac{(\lambda_k t^\alpha)^{\gamma-\iota}}{1+\lambda_k t^\alpha}\right]^2\lambda_k^{2\iota}|(v,e_k)|^2 \\
&\le t^{2(\beta-\alpha\gamma+\alpha\iota)}\|v\|^2_{D(A^\iota)}.
\end{aligned}$$

The $D(A^\iota)$ is defined in a manner similar to $D(A^\gamma)$. Similar estimates for $\mathcal{P}_\alpha(t)v$ and $\mathcal{S}_\alpha(t)v$ follow through identical methodology. The proof is complete. □

4. Main results

4.1. Local existence and regularity of mild solutions

This subsection addresses the existence and regularity of mild solutions to the linear problem (1.1).

Theorem 4.1 (Existence). *Let $\gamma \in (0,1)$ and $u_0 \in D(A^\gamma)$. Suppose $f \in L^p(0,T;D(A^\gamma))$ with $p > \frac{1}{1-\alpha}$. Then problem* (1.1) *admits a unique mild solution satisfying*

$$\|u(t)\| \lesssim \|u_0\|_\gamma + \|f\|_{L^p(0,T;D(A^\gamma))}, \quad t \in [0,T]. \tag{4.1}$$

Proof. Define the solution operator on $C([0,T];L^2(\Omega))$ as

$$\mathcal{H}u(t) = \mathcal{S}_\alpha(t)u_0 + \mathcal{Q}_\alpha(t)u_0 + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds - \int_0^t \mathcal{P}_\alpha(t-s)u(s)ds. \tag{4.2}$$

Clearly, the existence of a mild solution is equivalent to the operator $\mathcal{H}$ having a fixed point in $C([0,T];L^2(\Omega))$. The proof is divided into three steps.

Step 1. Verify that $\mathcal{H}$ is a self-mapping.

For $0 \le t_1 < t_2 \le T$, consider

$$\begin{aligned}(\mathcal{H}u)(t_2) - (\mathcal{H}u)(t_1) = {} & \mathcal{S}_\alpha(t_2)u_0 - \mathcal{S}_\alpha(t_1)u_0 + \mathcal{Q}_\alpha(t_2)u_0 - \mathcal{Q}_\alpha(t_1)u_0 \\ & + \int_0^{t_2} \mathcal{S}_\alpha(t_2-s)f(s)ds - \int_0^{t_1} \mathcal{S}_\alpha(t_1-s)f(s)ds \\ & - \int_0^{t_2} \mathcal{P}_\alpha(t_2-s)u(s)ds + \int_0^{t_1} \mathcal{P}_\alpha(t_1-s)u(s)ds.\end{aligned} \tag{4.3}$$

By Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}\|\mathcal{S}_\alpha(t_2)u_0 - \mathcal{S}_\alpha(t_1)u_0\| &= \left(\sum_{k=1}^\infty \left|E_{\alpha,1}(-\lambda_k t_2^\alpha) - E_{\alpha,1}(-\lambda_k t_1^\alpha)\right|^2 |(u_0,e_k)|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k=1}^\infty \left|\int_{t_1}^{t_2} -\lambda_k s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k s^\alpha)ds\right|^2 |u_{0k}|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k=1}^\infty \left(\int_{t_1}^{t_2} \lambda_k^{1-\gamma} s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k s^\alpha)ds\right)^2 \lambda_k^{2\gamma}|u_{0k}|^2\right)^{\frac{1}{2}} \\ &\lesssim \left(t_2^{\alpha\gamma} - t_1^{\alpha\gamma}\right)\|u_0\|_\gamma.\end{aligned}$$

According to the definition of the fractional power space $D(A^\gamma)$ with $\gamma > 0$, and using the Sobolev embedding $D(A^\gamma) \hookrightarrow L^2(\Omega)$, we obtain the inequality $\|u_0\| \lesssim \|u_0\|_\gamma$. Furthermore, we derive that

$$\begin{aligned}\|\mathcal{Q}_\alpha(t_2)u_0 - \mathcal{Q}_\alpha(t_1)u_0\| &= \left(\sum_{k=1}^\infty \left|t_2^\beta E_{\alpha,\beta+1}(-\lambda_k t_2^\alpha) - t_1^\beta E_{\alpha,\beta+1}(-\lambda_k t_1^\alpha)\right|^2 |(u_0,e_k)|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k=1}^\infty \left|\int_{t_1}^{t_2} s^{\beta-1}E_{\alpha,\beta}(-\lambda_k s^\alpha)ds\right|^2 |u_{0k}|^2\right)^{\frac{1}{2}}\end{aligned}$$

$$\lesssim \left(t_2^\beta - t_1^\beta\right) \|u_0\|_\gamma.$$

On one hand, since

$$\left|E_{\alpha,1}(-\lambda_k t^\alpha)\right| \leq \frac{M}{1+\lambda_k t^\alpha} \leq \frac{M}{\lambda_1 t^\alpha},$$

we employ Hölder's inequality along with the Sobolev embedding $L^p(0,T;D(A^\gamma)) \hookrightarrow L(0,T;D(A^\gamma))$ $(p \geq 1)$ to obtain

$$\begin{aligned}\int_{t_1}^{t_2} \|\mathcal{S}_\alpha(t_2-s)f(s)\|\, ds &\lesssim \int_{t_1}^{t_2} (t_2-s)^{-\alpha}\|f(s)\|_\gamma ds \\ &\lesssim \left(\int_{t_1}^{t_2} ((t_2-s)^{-\alpha})^{\frac{p}{p-1}}\, ds\right)^{\frac{p-1}{p}} \|f\|_{L^p(0,T;D(A^\gamma))}.\end{aligned}$$

Then, applying Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}&\int_0^{t_1} \|(\mathcal{S}_\alpha(t_2-s) - \mathcal{S}_\alpha(t_1-s))\, f(s)\|\, ds \\ &= \int_0^{t_1} \left(\sum_{k=1}^\infty \left|\int_{t_1-s}^{t_2-s} \lambda_k \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k\tau^\alpha) d\tau\right|^2 |(f(s), e_k)|^2\right)^{1/2} ds \\ &\lesssim \int_0^{t_1} \left(\sum_{k=1}^\infty \left|\int_{t_1-s}^{t_2-s} \tau^{\alpha\gamma-1} d\tau\right|^2 (\lambda_k^\gamma f_k(s))^2\right)^{1/2} ds \\ &\lesssim \int_0^{t_1} ((t_2-s)^{\alpha\gamma} - (t_1-s)^{\alpha\gamma})\, \|f(s)\|_\gamma ds \\ &\lesssim (t_2-t_1)^{\alpha\gamma}\|f\|_{L(0,T;D(A^\gamma))}.\end{aligned}$$

Noting that for any $v \in L^2(\Omega)$, we have

$$\|\mathcal{S}_\alpha(t)v\| \lesssim \|v\|, \quad \|\mathcal{Q}_\alpha(t)v\| \lesssim t^\beta\|v\|, \quad \|\mathcal{P}_\alpha(t)v\| \lesssim t^{\beta-1}\|v\|, \tag{4.4}$$

it follows that

$$\begin{aligned}\int_{t_1}^{t_2} \|\mathcal{P}_\alpha(t_2-s)u(s)\| ds &\lesssim \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\|u(s)\| ds \\ &\lesssim (t_2-t_1)^\beta \|u\|_C.\end{aligned}$$

Further applying Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}\int_0^{t_1} \|(\mathcal{P}_\alpha(t_2-s) - \mathcal{P}_\alpha(t_1-s))\, u(s)\|\, ds &\lesssim \int_0^{t_1} \left|\int_{t_1-s}^{t_2-s} \tau^{\beta-2} d\tau\right| ds \cdot \|u\|_C \\ &\lesssim \left((t_2-t_1)^\beta + t_1^\beta - t_2^\beta\right) \|u\|_C.\end{aligned}$$

Combining the above inequalities and triangle inequality, we conclude that

$$\|(\mathcal{H}u)(t_2) - (\mathcal{H}u)(t_1)\| \to 0, \quad (t_2 \to t_1).$$

Therefore, $\mathcal{H}u \in C([0,T];L^2(\Omega))$ for any $u \in C([0,T];L^2(\Omega))$.

Step 2. The operator $\mathcal{H}$ has a unique fixed point.

In fact, for any $u_1, u_2 \in C\left([0,T];L^2(\Omega)\right)$, from Eq (4.3), we obtain the following estimate:

$$\begin{aligned}\|(\mathcal{H}u_1)(t) - (\mathcal{H}u_2)(t)\| &\lesssim \int_0^t \|\mathcal{P}_\alpha(t-s)(u_1(s)-u_2(s))\|\, ds \\ &\lesssim \int_0^t (t-s)^{\beta-1}\|u_1(s)-u_2(s)\|\, ds \\ &\lesssim \frac{\Gamma(\beta)}{\Gamma(\beta+1)} t^\beta \|u_1-u_2\|_C .\end{aligned}$$

Next, we prove the following estimate by mathematical induction:

$$\left\|(\mathcal{H}^j u_1)(t) - (\mathcal{H}^j u_2)(t)\right\| \lesssim \frac{(\Gamma(\beta))^j}{\Gamma(j\beta+1)} t^{j\beta}\|u_1-u_2\|_C. \tag{4.5}$$

Assume that (4.5) holds for j. For $j+1$, applying (4.3), we obtain

$$\begin{aligned}\left\|(\mathcal{H}^{j+1}u_1)(t) - (\mathcal{H}^{j+1}u_2)(t)\right\| &\lesssim \int_0^t (t-\tau)^{\beta-1}\left\|(\mathcal{H}^j u_1)(\tau) - (\mathcal{H}^j u_2)(\tau)\right\| d\tau \\ &\lesssim \frac{(\Gamma(\beta))^j}{\Gamma(j\beta+1)}\|u_1-u_2\|_C \int_0^t (t-\tau)^{\beta-1}\tau^{j\beta} d\tau \\ &= \frac{(\Gamma(\beta))^{j+1}}{\Gamma((j+1)\beta+1)} t^{(j+1)\beta}\|u_1-u_2\|_C.\end{aligned}$$

Therefore, the expression in (4.5) holds for any $j \in \mathbb{N}$. There exists a constant $C > 0$ such that

$$\left\|(\mathcal{H}^{j+1}u_1)(t) - (\mathcal{H}^{j+1}u_2)(t)\right\| \le \frac{C(\Gamma(\beta))^{j+1}}{\Gamma((j+1)\beta+1)} t^{(j+1)\beta}\|u_1-u_2\|_C.$$

Choosing $\hat{j} \in \mathbb{N}$ such that

$$\varsigma = \frac{C(\Gamma(\beta))^{\hat{j}}}{\Gamma(\hat{j}\beta+1)} T^{\hat{j}\beta} < 1,$$

holds, then we obtain the contraction property

$$\left\|\mathcal{H}^{\hat{j}}u_1 - \mathcal{H}^{\hat{j}}u_2\right\|_C \le \varsigma\|u_1-u_2\|_C.$$

According to the Banach contraction mapping theorem, $\mathcal{H}^{\hat{j}}$ admits a unique fixed point $u^* \in C([0,T];L^2(\Omega))$. Moreover, the commutation relation $\mathcal{H}\mathcal{H}^{\hat{j}} = \mathcal{H}^{\hat{j}}\mathcal{H}$ guarantees that u^* is also the unique fixed point of $\mathcal{H}$.

Step3. Mild solution estimates.

Using Lemma 2.2, we have the following estimates:

$$\|\mathcal{S}_\alpha(t)u_0\| \lesssim \|u_0\|_\gamma, \quad \|\mathcal{Q}_\alpha(t)u_0\| \lesssim t^\beta\|u_0\|_\gamma .$$

Therefore, we obtain

$$u(t) \lesssim \|u_0\|_\gamma + t^\beta \|u_0\|_\gamma + \int_0^t \|f(s)\|_\gamma \, ds + \int_0^t (t-s)^{\beta-1} \|u(s)\| \, ds.$$

Using the generalized Gronwall inequality and Lemma 2.3, there exists a constant C such that

$$\|u(t)\| \leq \varrho(t) \exp\left((C\Gamma(\beta))^{\frac{1}{\beta}} t \right),$$

where $\varrho(t) \lesssim \|u_0\|_\gamma + t^\beta \|u_0\|_\gamma + \|f\|_{L^p(0,T;D(A^\gamma))}$. The proof is complete. □

Next, we investigate the regularity of the solution in weighted Hölder continuous function space. To begin, we introduce the space.

Definition 4.1. *(Weighted Hölder continuous function space).*

$$X_T = \left\{ u \mid u \in C\left([0,T], L^2(\Omega)\right), \quad u' \in L^1\left(0,T; L^2(\Omega)\right) \right\},$$

where the norm is defined as

$$\|u\|_{X_T} = \|u\|_C + \|u'\|_{L^1(0,T;L^2(\Omega))}.$$

It is straightforward to verify that X_T, equipped with the norm $\|\cdot\|_{X_T}$, forms a Banach space.

Theorem 4.2. *Let $\gamma \in (0,1)$ and $u_0 \in D(A^\gamma)$ such that $\alpha\gamma < \beta$. Suppose that $f \in W^{1,1}(0,T; D(A^\gamma))$. Then problem (1.1) admits a unique mild solution in X_T. Furthermore, the following estimate holds:*

$$\|\partial_t u(t)\| \lesssim t^{\alpha\gamma-1} \|u_0\|_\gamma + \|f_0\|_\gamma + \|f'\|_{L^1(0,T;D(A^\gamma))}. \tag{4.6}$$

Proof. Consider the solution operator $\mathcal{H} : X_T \to X_T$ defined in (4.2). By leveraging the continuous embedding $W^{1,1}(0,T; D(A^\gamma)) \hookrightarrow L^p(0,T; D(A^\gamma))$ for $p \geq 1$ and applying Theorem 4.1, we establish that $\mathcal{H}u \in C([0,T]; L^2(\Omega))$ for any $u \in X_T$.

The key remaining task is to verify the temporal differentiability property $\mathcal{H}u \in C([0,T]; L^2(\Omega))$. The proof is structured in two steps.

Step 1. Prove that $(\mathcal{H}u)' \in L^1\left(0,T; L^2(\Omega)\right)$.

First, differentiate Eq (4.2) with respect to t. Using Eq (3.3) and Lemma 2.1, we obtain

$$(\mathcal{H}u)'(t) = S'_\alpha(t)u_0 + P_\alpha(t)u_0 + \partial_t \left(\int_0^t S_\alpha(t-s) f(s) \, ds \right) - \partial_t \left(\int_0^t P_\alpha(t-s) u(s) \, ds \right).$$

For any $v \in L^2(\Omega)$, we obtain

$$S'_\alpha(t)v = \sum_{k=1}^{\infty} -\lambda_k t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k t^\alpha)(v, e_k) e_k.$$

On the other hand, we have

$$\partial_t \left(\int_0^t S_\alpha(t-s) f(s) \, ds \right) = \partial_t \left(\int_0^t S_\alpha(s) f(t-s) \, ds \right)$$

$$= S_\alpha(t)f(0) + \int_0^t S_\alpha(s)f'(t-s)\,ds$$
$$= S_\alpha(t)f(0) + \int_0^t S_\alpha(t-s)f'(s)\,ds.$$

Similarly, we get

$$\partial_t\left(\int_0^t P_\alpha(t-s)u(s)\,ds\right) = P_\alpha(t)u_0 + \int_0^t P_\alpha(s)\partial_t u(t-s)\,ds$$
$$= P_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)u'(s)\,ds.$$

Let $f(0,x) = f_0$, and then, it follows that

$$(\mathcal{H}u)'(t) = S'_\alpha(t)u_0 + S_\alpha(t)f_0 + \int_0^t S_\alpha(t-s)f'(s)\,ds - \int_0^t P_\alpha(t-s)u'(s)\,ds.$$

Using Lemmas 2.2 and 3.1, we obtain

$$\begin{aligned}\|(\mathcal{H}u)'(t)\| &\leq \left\|S'_\alpha(t)u_0\right\| + \|S_\alpha(t)f_0\| + \left\|\int_0^t S_\alpha(t-s)f'(s)\,ds\right\| \\ &\quad + \left\|\int_0^t P_\alpha(t-s)u'(s)\,ds\right\| \\ &\lesssim t^{\alpha\gamma-1}\|u_0\|_\gamma + \|f_0\|_\gamma + \|f'\|_{L^1(0,T;D(A^\gamma))} + \int_0^t (t-s)^{\beta-1}\|u'(s)\|\,ds.\end{aligned}$$

By applying Young's inequality, we deduce that $u' \in L^1(0,T;L^2(\Omega))$.

Subsequently, following **Step 2** in Theorem 4.1 and utilizing the Banach contraction mapping principle, we establish the existence and uniqueness of the solution to Eq (1.1) in X_T.

Step 2. Using Lemma 2.3, we have

$$\left\|S'_\alpha(t)u_0\right\|^2 = \sum_{k=1}^{\infty}\left(\lambda_k^{1-\gamma}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)\right)^2\left(\lambda_k^\gamma u_{0k}\right)^2 \lesssim t^{\alpha\gamma-1}\|u_0\|_\gamma. \tag{4.7}$$

Therefore, using Eqs (4.4) and (4.7), we obtain

$$\begin{aligned}\|\partial_t u(t)\| &= \left\|S'_\alpha(t)u_0\right\| + \|S_\alpha(t)f_0\| + \left\|\int_0^t S_\alpha(t-s)f'(s)\,ds\right\| \\ &\quad + \left\|\int_0^t P_\alpha(t-s)u'(s)\,ds\right\| \\ &\lesssim t^{\alpha\gamma-1}\|u_0\|_\gamma + \|f_0\|_\gamma + \|f'\|_{L^1(0,T;D(A^\gamma))} + \int_0^t (t-s)^{\beta-1}\|u'(s)\|\,ds.\end{aligned}$$

By applying the generalized Gronwall inequality and Lemma 2.3, there exists a constant K such that

$$\|\partial_t u(t)\| \leq \varrho(t)\exp\left((K\Gamma(\beta))^{\frac{1}{\beta}}t\right),$$

where $\varrho(t) \lesssim \|u_0\|_\gamma + t^\beta\|u_0\|_\gamma + \|f\|_{L^p(0,T;D(A^\gamma))}$. The proof is complete. □

Theorem 4.3. *Let $\gamma \in (0,1)$, $u_0 \in D(A^\gamma)$, and assume that $\alpha\gamma < \beta$. Suppose $f \in W^{1,1}(0,T;D(A^\gamma))$, and then the solution $u \in C((0,T];H^{2\gamma}(\Omega))$ and satisfies the following inequality:*

$$\|u(t)\|_{H^{2\gamma}(\Omega)} \lesssim t^{\beta-1}\|u_0\|_\gamma + \|f\|_{W^{1,1}(0,T;D(A^\gamma))} + \|f_0\|_\gamma. \tag{4.8}$$

Proof. By Theorem 4.2, for any $t \in [0,T]$, there exists a unique mild solution u satisfying $u(t), \partial_t u(t) \in L^2(\Omega)$. Consequently, applying Eq (3.3), the condition $\mathcal{Q}_\alpha(0)v = 0$, and applying integration by parts, we have

$$\int_0^t \mathcal{P}_\alpha(t-s)u(s)\,ds = \mathcal{Q}_\alpha(t)u_0 + \int_0^t \mathcal{Q}_\alpha(t-s)\partial_s u(s)\,ds.$$

It follows that

$$u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds - \int_0^t \mathcal{Q}_\alpha(t-s)\partial_s u(s)\,ds.$$

Therefore, it remains to establish that $u \in C((0,T],H^{2\gamma}(\Omega))$ and that u satisfies the estimate in Eq (4.8).

First, from the Sobolev embedding relation $D(A^\gamma) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$, we conclude that if $u \in D(A^\gamma)$, then u must also belong to $H^{2\gamma}(\Omega)$. Therefore, we now prove that $u \in D(A^\gamma)$.

By repeating the proof process of **Step 1** in Theorem 4.1, for any $0 < t_1 \le t_2 \le T$,

$$\begin{aligned}
&\|u(t_2) - u(t_1)\|_\gamma \\
&\le \|(\mathcal{S}_\alpha(t_2) - \mathcal{S}_\alpha(t_1))u_0\|_\gamma + \int_0^{t_1} \|(\mathcal{S}_\alpha(t_2-s) - \mathcal{S}_\alpha(t_1-s))f(s)\|_\gamma\,ds \\
&\quad + \int_0^{t_1} \|(\mathcal{P}_\alpha(t_2-s) - \mathcal{P}_\alpha(t_1-s))\partial_s u(s)\|_\gamma\,ds \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{S}_\alpha(t_2-s)f(s)\|_\gamma\,ds + \int_{t_1}^{t_2} \|\mathcal{P}_\alpha(t_2-s)\partial_s u(s)\|_\gamma\,ds \\
&\lesssim \left(\frac{t_2-t_1}{t_2t_1}\right)^{\alpha\gamma}\|u_0\|_\gamma + \left(\frac{t_2-t_1}{t_2t_1}\right)^{\alpha\gamma}\|f\|_{L(0,T;D(A^\gamma))} + \left(\frac{t_2-t_1}{t_2t_1}\right)^{1+\alpha\gamma-\beta}\|\partial_s u(s)\|_{L(0,T;L^2(\Omega))}.
\end{aligned}$$

From the preceding analysis, it follows that $u \in C((0,T],D(A^\gamma))$. Thus, $u \in C\big((0,T],H^{2\gamma}(\Omega)\big)$.

Next, it remains to establish the validity of Eq (4.8).

By the definition of fractional powers of operators, for any $u \in L^2(\Omega)$ and $\gamma \in (0,1)$, we obtain

$$A^\gamma u(t) = A^\gamma\mathcal{S}_\alpha(t)u_0 + \int_0^t A^\gamma\mathcal{S}_\alpha(t-s)f(s)\,ds + \int_0^t A^\gamma\mathcal{Q}_\alpha(t-s)\partial_s u(s)\,ds.$$

On the other hand, from the Sobolev embedding theorem, we know that $\|u(t)\|_{H^{2\gamma}(\Omega)} \lesssim \|u(t)\|_\gamma$. Therefore, it is sufficient to estimate $A^\gamma u(t)$ as follows:

$$\|u(t)\|_\gamma \le \|\mathcal{S}_\alpha(t)u_0\|_\gamma + \int_0^t \|A^\gamma\mathcal{S}_\alpha(t-s)f(s)\|\,ds + \int_0^t \|A^\gamma\mathcal{Q}_\alpha(t-s)\partial_s u(s)\|\,ds.$$

According to Lemma 3.1, we have

$$\|\mathcal{S}_\alpha(t)u_0\|_\gamma = \left(\sum_{k=1}^\infty \left|E_{\alpha,1}(-\lambda_k t^\alpha)\right|^2 \lambda_k^{2\gamma}|u_{0k}|^2\right)^{1/2} \lesssim \|u_0\|_\gamma. \tag{4.9}$$

Using Lemma 3.1 again, we obtain

$$\int_0^t \|A^\gamma \mathcal{S}_\alpha(t-s)f(s)\|\, ds \lesssim \|f\|_{L(0,T;D(A^\gamma))}. \tag{4.10}$$

By Lemma 3.1, we have

$$\int_0^t \|A^\gamma \mathcal{Q}_\alpha(t-s)\partial_s u(s)\|\, ds \lesssim t^{\beta-\alpha\gamma} \|\partial_s u(s)\|_{L\left(0,T;L^2(\Omega)\right)}. \tag{4.11}$$

Combining Eqs (4.9), (4.10), (4.11), and (4.6), we obtain

$$\begin{aligned}\|u(t)\|_\gamma &\lesssim \|u_0\|_\gamma + t^{\beta-1}\|u_0\|_\gamma + \|f\|_{L(0,T;D(A^\gamma))} + \|f_0\|_\gamma + \|f'\|_{L^1(0,T,D(A^\gamma))}\\ &\lesssim t^{\beta-1}\|u_0\|_\gamma + \|f\|_{L(0,T;D(A^\gamma))} + \|f_0\|_\gamma + \|f'\|_{L^1(0,T,D(A^\gamma))}.\end{aligned}$$

The proof is complete. □

4.2. Existence of mild solutions in the weighted Hölder continuous function space

In this section, we investigate the existence and uniqueness of mild solutions to Eq (1.1) in a weighted Hölder continuous function space.

First, we introduce the weighted Hölder continuous function space.

Definition 4.2. *We introduce a weighted Hölder continuous function space, defined as*

$$\bar{X}_T = \left\{u \in C\left([0,T], L^2(\Omega)\right) \partial^{1-\beta} u \in \mathcal{F}^{\eta,\theta}\left((0,T], L^2(\Omega)\right)\right\},$$

where $0 < \theta < \eta < \beta$. *The norm on this space is defined as*

$$\|u\|_{\bar{X}_T} = \|u\|_C + \left\|\partial^{1-\beta} u\right\|_{\mathcal{F}_\eta,\theta}.$$

It is clear that $\bar{X}_T$ *is a Banach space under the norm* $\|\cdot\|_{\bar{X}_T}$.

Theorem 4.4. *Let* $u_0 \in D(A^\gamma)$ *and* $f \in \mathcal{F}^{\eta,\theta}((0,T], D(A^\gamma))$. *Then, Eq* (1.1) *admits a unique solution* $u \in \bar{X}_T$.

Proof. **Step 1.** We define an operator on $\bar{X}_T$ as follows:

$$\mathcal{H}u(t) = \mathcal{S}_\alpha(t)u_0 + \mathcal{Q}_\alpha(t)u_0 + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds - \int_0^t \mathcal{P}_\alpha(t-s)u(s)\,ds.$$

Since $\mathcal{F}^{\eta,\theta}((0,T], D(A^\gamma)) \hookrightarrow L^p(0,T; D(A^\gamma))$ for all $p \geq 1$, it follows from Theorem 4.1 that for any $u \in C([0,T], L^2(\Omega))$, we have $\mathcal{H}u \in C([0,T], L^2(\Omega))$. Consequently, the critical step is to demonstrate that $(\partial_t^{1-\beta}\mathcal{H})u \in \mathcal{F}^{\eta,\theta}((0,T], L^2(\Omega))$.

From Eq (3.2), we obtain

$$\mathcal{H}u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds - \int_0^t \mathcal{S}_\alpha(t-s)\partial_s^{1-\beta}u(s)\,ds.$$

Let $I_t^\beta u(t) = w(t)$. Using Eq (3.3), we derive

$$\left(I_t^\beta \mathcal{H}u\right)(t) = \mathcal{Q}_\alpha(t)u_0 + \int_0^t \mathcal{Q}_\alpha(t-s)f(s)\,ds - \int_0^t \mathcal{P}_\alpha(t-s)w(s)\,ds,$$

where

$$\begin{aligned} I_t^\beta\left(\int_0^t \mathcal{S}_\alpha(t-s)f(s)ds\right) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\int_0^s \mathcal{S}_\alpha(s-\tau)f(\tau)d\tau ds \\ &= \frac{1}{\Gamma(\beta)}\int_0^t\int_\tau^t (t-s)^{\beta-1}\mathcal{S}_\alpha(s-\tau)f(\tau)dsd\tau \\ &= \int_0^t \mathcal{Q}_\alpha(t-s)f(s)ds. \end{aligned}$$

Since $w(0) = 0$, integration by parts yields, we have

$$\begin{aligned} I_t^\beta\left(\int_0^t \mathcal{S}_\alpha(t-s)\partial_s^{1-\beta}u(s)ds\right) &= \int_0^t \mathcal{Q}_\alpha(t-s)\partial_s^{1-\beta}u(s)ds \\ &= \int_0^t \mathcal{Q}_\alpha(t-s)dw(s) \\ &= \int_0^t \mathcal{P}_\alpha(t-s)w(s)ds. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \left(\partial_t^{1-\beta}\mathcal{H}\right)u(t) = \mathcal{P}_\alpha(t)u_0 &+ \partial_t\left(\int_0^t \mathcal{Q}_\alpha(t-s)f(s)ds\right) \\ &- \partial_t\left(\int_0^t \mathcal{P}_\alpha(t-s)w(s)ds\right). \end{aligned}$$

Since $Q_\alpha(0) = 0$, it follows that

$$\partial_t\left(\int_0^t \mathcal{Q}_\alpha(t-s)f(s)ds\right) = \int_0^t \mathcal{P}_\alpha(t-s)f(s)ds.$$

On the other hand, we have

$$\begin{aligned} \partial_t\left(\int_0^t \mathcal{P}_\alpha(t-s)I_s^\beta u(s)ds\right) &= \mathcal{P}_\alpha(t)w(0) + \int_0^t \mathcal{P}_\alpha(s)\partial_t w(t-s)ds \\ &= \int_0^t \mathcal{P}_\alpha(t-s)w'(s)ds. \end{aligned}$$

Since $w'(t) = \partial_t^{1-\beta}u(t)$, we have

$$\begin{aligned} \left(\partial_t^{1-\beta}\mathcal{H}u\right)(t) &= \mathcal{P}_\alpha(t)u_0 + \int_0^t \mathcal{P}_\alpha(t-s)f(s)ds - \int_0^t \mathcal{P}_\alpha(t-s)\partial_s^{1-\beta}u(s)ds \\ &:= \mathcal{H}_1(t) + \mathcal{H}_2(t) + \mathcal{H}_3(t). \end{aligned}$$

Next, we will now prove separately that $\mathcal{H}_i \in \mathcal{F}^{\eta,\theta}\left((0,T], L^2(\Omega)\right)$ for $i = 1, 2, 3$.

Step 2. First, according to the definition of $P_\alpha(t)$, it follows that for $\beta > \eta$, the $\lim_{t\to 0} t^{1-\eta} P_\alpha(t) u_0$ exists. Moreover, for $0 \le t_1 < t_2 \le T$, applying Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}\|P_\alpha(t_2) u_0 - P_\alpha(t_1) u_0\| &\lesssim \|u_0\|_\gamma \left(t_1^{\beta-1} - t_2^{\beta-1}\right) \\ &\lesssim \|u_0\|_\gamma \frac{(t_2 - t_1)^{1-\beta}}{(t_1 t_2)^{1-\beta}}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\frac{t_1^{1-\eta+\theta} \|\mathcal{H}_1(t_2) - \mathcal{H}_1(t_1)\|}{(t_2 - t_1)^\theta} &\lesssim \|u_0\|_\gamma t_1^{\beta-\eta} \left(\frac{t_1}{t_2}\right)^\theta \left(\frac{t_2 - t_1}{t_2}\right)^{1-\beta-\theta} \\ &\lesssim \|u_0\|_\gamma T^{\beta-\eta},\end{aligned}$$

and it follows that

$$\lim_{t_2\to 0} \sup_{0\le t_1 < t_2} \frac{t_1^{1-\eta+\theta} \|\mathcal{H}_1(t_2) - \mathcal{H}_1(t_1)\|}{(t_2 - t_1)^\theta} = 0.$$

Therefore, it follows that $\mathcal{H}_1 \in \mathcal{F}^{\eta,\theta}\left((0,T], L^2(\Omega)\right)$.

By further combining with the assumption on f, we obtain

$$\begin{aligned}\|\mathcal{H}_2(t)\| &\lesssim \int_0^t (t-s)^{\beta-1} \|f(s)\|_\gamma ds \\ &\lesssim \frac{\Gamma(\beta)\Gamma(\eta)}{\Gamma(\beta+\alpha)} \|f\|_{\mathcal{F}^{\eta,\theta}} t^{\beta+\eta-1}.\end{aligned}$$

This implies that $\lim_{t\to 0} t^{1-\eta} \|\mathcal{H}_2(t)\|$ exists.

Then, we define

$$\begin{aligned}\mathcal{H}_{21}(t) &:= \int_0^t \mathcal{P}_\alpha(t-s)[f(s) - f(t)]ds, \\ \mathcal{H}_{22}(t) &:= \int_0^t \mathcal{P}_\alpha(t-s) f(t) ds = Q(t) f(t).\end{aligned}$$

Combing with Lemma 3.1 and Lemma 6.1 in [12], we obtain

$$\begin{aligned}&\|\mathcal{H}_{21}(t_2) - \mathcal{H}_{21}(t_1)\| \\ &\le \left\| \int_0^{t_1} [\mathcal{P}_\alpha(t_2 - s) - \mathcal{P}_\alpha(t_1 - s)] [f(s) - f(t_1)] ds \right\| + \left\| \int_0^{t_1} \mathcal{P}_\alpha(t_2 - s) [f(t_1) - f(t_2)] ds \right\| \\ &\quad + \left\| \int_{t_1}^{t_2} \mathcal{P}_\alpha(t_2 - s) [f(s) - f(t_2)] ds \right\| \\ &\lesssim \|f\|_{\mathcal{F}^{\eta,\theta}} (t_2 - t_1)^{1-\beta} \int_0^{t_1} (t_1 - s)^{\beta+\theta-1} (t_2 - s)^{\beta-1} s^{-(1-\eta+\theta)} ds \\ &\quad + \|f\|_{\mathcal{F}^{\eta,\theta}} (t_2 - t_1)^\theta \int_0^{t_1} (t_2 - s)^{\beta-1} s^{-(1-\eta+\theta)} ds + \|f\|_{\mathcal{F}^{\eta,\theta}} \int_{t_1}^{t_2} (t_2 - s)^{\beta+\theta-1} s^{-(1-\eta+\theta)} ds \\ &\lesssim \|f\|_{\mathcal{F}^{\eta,\theta}} \left(t_1^{\eta-\theta-1} t_2^\beta (t_2 - t_1)^\theta + \frac{\Gamma(\beta)\Gamma(\eta-\theta)}{\Gamma(\beta+\eta-\theta)} t_1^{\beta+\eta-\theta-1} (t_2 - t_1)^\theta + t_1^{\eta-\theta-1} (t_2 - t_1)^{\beta+\theta} \right).\end{aligned}$$

Thus, we derive

$$\begin{aligned}\frac{t_1^{1-\eta+\theta}\left\|\mathcal{H}_{21}(t_2)-\mathcal{H}_{21}(t_1)\right\|}{(t_2-t_1)^{\theta}} &\lesssim \|f\|_{\mathcal{F}^{\eta,\theta}}\left[t_2^{\beta}+B(\beta,\beta-\theta)t_1^{\beta}+(t_2-t_1)^{\beta}\right]\\ &\lesssim \|f\|_{\mathcal{F}^{\eta,\theta}}T^{\beta},\end{aligned}$$

and it follows that

$$\lim_{t_2\to 0}\sup_{0\le t_1<t_2}\frac{t_1^{1-\eta+\theta}\left\|\mathcal{H}_{21}(t_2)-\mathcal{H}_{21}(t_1)\right\|}{(t_2-t_1)^{\theta}}=0.$$

On the other hand, it is evident that

$$\lim_{t\to 0}t^{1-\eta}Q(t)f(t)=0.$$

Applying a similar approach, we obtain

$$\begin{aligned}&\left\|\mathcal{H}_{22}(t_2)-\mathcal{H}_{22}(t_1)\right\|\\ \le&\left\|\int_0^{t_1}\left[\mathcal{P}_\alpha(t_2-s)-\mathcal{P}_\alpha(t_1-s)\right]f(t_2)\,ds\right\|+\left\|\int_0^{t_1}\mathcal{P}_\alpha(t_1-s)\left[f(t_2)-f(t_1)\right]ds\right\|\\ &+\left\|\int_{t_1}^{t_2}\mathcal{P}_\alpha(t_2-s)f(t_2)\,ds\right\|\\ \lesssim&\|f\|_{\mathcal{F}^{\eta,\theta}}t_2^{\eta-1}\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\right]ds+\|f\|_{\mathcal{F}^{\eta,\theta}}t_1^{\eta-\theta-1}(t_2-t_1)^{\theta}\int_0^{t_1}(t_1-s)^{\beta-1}\,ds\\ &+\|f\|_{\mathcal{F}^{\eta,\theta}}t_2^{\eta-1}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\\ \lesssim&\|f\|_{\mathcal{F}^{\eta,\theta}}t_1^{\eta-\theta-1}(t_2-t_1)^{\theta}\left(t_1^{\theta}(t_2-t_1)^{\beta-\theta}+t_1^{\beta}\right),\end{aligned}$$

Consequently, we derive

$$\begin{aligned}\frac{t_1^{1-\eta+\theta}\left\|\mathcal{H}_{22}(t_2)-\mathcal{H}_{22}(t_1)\right\|}{(t_2-t_1)^{\theta}} &\le \|f\|_{\mathcal{F}^{\eta,\theta}}\left(t_1^{\theta}(t_2-t_1)^{\beta-\theta}+t_2^{\beta}\right)\\ &\le \|f\|_{\mathcal{F}^{\eta,\theta}}T^{\beta},\end{aligned}$$

and it follows that

$$\lim_{t_2\to 0}\sup_{0\le t_1<t_2}\frac{t_1^{1-\eta+\theta}\left\|\mathcal{H}_{21}(t_2)-\mathcal{H}_{21}(t_1)\right\|}{(t_2-t_1)^{\theta}}=0.$$

Therefore, we conclude that $\mathcal{H}_2\in\mathcal{F}^{\eta,\theta}\left((0,T],L^2(\Omega)\right)$. By employing the same approach, it follows that $\mathcal{H}_3\in\mathcal{F}^{\eta,\theta}\left((0,T],L^2(\Omega)\right)$.

In summary, we obtain $\left(\partial_t^{1-\beta}\mathcal{H}u\right)(t)\in\mathcal{F}^{\eta,\theta}\left((0,T],L^2(\Omega)\right)$. Applying the Banach contraction mapping principle and Theorem 4.1, we conclude that the existence and uniqueness of the solution to equation (1.1) in the space $\bar{X}_T$. □

Author contributions

Hujing Tan and Pu Wang: Writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no competing interests.

References

1. M. A. Abdelkawy, R. T. Alqahtani, Shifted Jacobi collocation method for solving multi-dimensional fractional Stokes' first problem for a heated generalized second grade fluid, *Adv. Differ. Equations*, **1** (2016), 114. https://doi.org/10.1186/s13662-016-0845-zz
2. A. H. Bhrawy, M. A. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear Cable equation, *Nonlinear Dyn.*, **80** (2015), 101–116. https://doi.org/10.1007/s11071-014-1854-7
3. M. Dehghan, M. Abbaszadeh, Analysis of the element free Galerkin (EFG) method for solving fractional Cable equation with Dirichlet boundary condition, *Appl. Numer. Math.*, **109** (2016), 208–234. https://doi.org/10.1016/j.apnum.2016.07.002
4. M. A. Khan, N. Alias, I. Khan, F. M. Salama, S. M. Eldin, A new implicit high-order iterative scheme for the numerical simulation of the two-dimensional time fractional Cable equation, *Sci. Rep.*, **13** (2023), 1549. https://doi.org/10.1038/s41598-023-28741-7
5. C. Li, W. H. Deng, Analytical solutions, moments, and their asymptotic behaviors for the time-space fractional Cable equation, *Commun. Theor. Phys.*, **62** (2014), 54–60. https://doi.org/10.1088/0253-6102/62/1/09
6. T. A. M. Langlands, B. I. Henry, S. L. Wearne, Fractional Cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions, *J. Math. Biol.*, **59** (2009), 761–808. https://doi.org/10.1007/s00285-009-0251-1
7. T. A. M. Langlands, B. I. Henry, S. L. Wearne, Fractional Cable equation models for anomalous electrodiffusion in nerve cells: finitedomain solutions, *SIAM J. Appl. Math.*, **71** (2011), 1168–1203. https://doi.org/10.1137/090775920
8. Y. Liu, Y.-W. Du, H. Li, J.-F. Wang, A two-grid finite element approximation for a nonlinear time-fractional Cable equation, *Nonlinear Dyn.*, **85** (2016), 2535–2548. https://doi.org/10.1007/s11071-016-2843-9
9. Y. Z. Lin, W. Jiang, Numerical method for Stokes' first problem for a heated generalized second grade fluid with fractional derivative, *Numer. Meth. Part. Differ. Equ.*, **27** (2011), 1599–1609. https://doi.org/10.1002/num.20598
10. Y. Ma, L. Chen, Error bounds of a finite difference/spectral method for the generalized time fractional Cable equation, *Fractal Fract.*, **6** (2022), 439. https://doi.org/10.3390/fractalfract6080439
11. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

12. L. Peng, Y. Zhou, The existence of mild and classical solutions for time fractional Fokker–Planck equations, *Monatsh. Math.*, **199** (2022), 377–410. https://doi.org/10.1007/s00605-022-01710-4

13. R. K. Saxena, Z. Tomovski, T. Sandev, Analytical solution of generalized space-time fractional Cable equation, *Mathematics*, **3** (2015), 153–170. https://doi.org/10.3390/math3020153

14. H. X. Zhang, X. H. Yang, X. L. Han, Discrete-time orthogonal spline collocation method with application to two-dimensional fractional Cable equation, *Comput. Math. Appl.*, **68** (2014), 1710–1722. https://doi.org/10.1016/j.camwa.2014.10.019

15. P. Zhuang, F. Liu, I. Turner, V. Anh, Galerkin finite element method and error analysis for the fractional Cable equation, *Numer. Algor.*, **72** (2016), 447–466. https://doi.org/10.1007/s11075-015-0055-x

16. Y. Zhou, J. W. He, B. Ahmad, N. H. Tuan, Existence and regularity results of a backward problem for fractional diffusion equations, *Math. Method. Appl. Sci.*, **42** (2019), 6775–6790. https://doi.org/10.1002/mma.5781

17. Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2014. https://doi.org/10.1142/9069