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Research article

Oscillation for neutral differential equations of canonical form of even-order

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Abstract: In this article, I aimed to investigate the oscillatory properties of a novel class of neutral differential equations in their canonical form. I established new relationships between the solutions of the studied equation and their higher-order derivatives to reinforce the monotonic properties of these solutions. Using an iterative process, I present novel and sophisticated oscillatory criteria that broaden the breadth of previous findings in the literature. Several examples are provided to support and elucidate our findings, thereby emphasizing the applicability and significance of the proposed criteria.

Keywords: oscillatory; even order; neutral differential equations
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

The oscillatory conditions of DDEs have an important role in modeling phenomena, principally in relating dynamics of population areas [1, 2]. There are many important uses for studying Laplace differential equations in mechanical systems, chemical process control, and electrical circuits. They are also helpful for modeling ecological systems, epidemiology, and population dynamics [3, 4].

In this manuscript, I discuss even-order equations and the oscillating conditions for their solutions in the following form

$$\left(m(t)\left|y^{(j-1)}(t)\right|^{p-2} y^{(j-1)}(t)\right)' + b(t)\, z^{p-2}(\delta(t))\, z(\delta(t)) = 0, t \geq t_0, \tag{1.1}$$

where $y(t) := z(t) + a(t) z(\varphi(t))$, and $m \in C[t_0, \infty)$, $m(t) > 0$, $m'(t) \geq 0$, $a, b \in C[t_0, \infty)$, $b(t) > 0$, $0 \leq a(t) < a_0 < \infty, \varphi \in C^1[t_0, \infty), \delta \in C[t_0, \infty)$, $\varphi'(t) > 0$, $\varphi(t) \leq t$, $\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \delta(t) = \infty$, $\int_{t_0}^{\infty} m^{-1/(p-1)}(s)\, ds = \infty$, $j \geq 4$ is an even natural number, $(1 < p < \infty)$, p-Laplace type operator.

Definition 1. *The solution to (1.1) is said to be oscillatory if it is neither positive nor negative. Otherwise, this solution is considered nonoscillatory.*

Definition 2. *The Eq (1.1) is oscillatory if it has oscillatory solutions.*

Definition 3. *Delay Differential Equations (DDEs) are a class of differential equations in which the derivative of the unknown function at a given time depends on the current state of the system and its state at one or more previous times.*

Definition 4. *Neutral Differential Equations (NDEs) are a class of functional differential equations where the highest-order derivative of the unknown function depends on both its current state and its past states, including delays in the derivatives.*

The study of even-order differential equations has become increasingly vital in modern technology, where complex dynamic systems demand precise modeling and control. These equations, which describe phenomena involving rates of change up to the even derivative, are pivotal in capturing intricate behaviors such as jerk dynamics, nonlinear oscillations, and transient stability in mechanical, electrical, and electromechanical systems. From advanced robotics and aerospace engineering to micro electromechanical systems (MEMS) and nonlinear optical communications, even-order models underpin the design and optimization of technologies that rely on rapid response, high precision, and resilience to perturbations [5]. Understanding the qualitative behavior of their solutions-such as stability, bifurcations, and chaos-enables engineers to predict system performance, mitigate instabilities, and innovate adaptive control strategies. As emerging technologies push the boundaries of miniaturization, speed, and autonomy, the ability to analytically and numerically resolve even-order dynamics will remain indispensable for translating theoretical insights into robust, real-world applications. Here, I explore the interplay between solution behavior and technological advancements, highlighting how deeper mathematical scrutiny of even-order systems can unlock new frontiers in engineering and applied sciences [6, 7].

Oscillation theory has advanced significantly since Sturm's seminal work in 1836, especially for functional differential equations. In order to improve oscillation conditions, researchers have broadened the scope of NDEs using a variety of approaches, including the Riccati transformation, integral averaging, and comparison techniques (see [8]). Some models incorporate various factors that affect the population and interactions between these factors, enabling researchers to identify key drivers of population change, assess the impact of environmental changes, and develop effective conservation strategies. Furthermore, population models can be used to predict the long-term viability of species and inform decisions about habitat preservation, species reintroduction, and invasive species management. The complexity of biological systems often necessitates the use of nonlinear differential equations and oscillatory properties to accurately represent these interactions [9]. This mathematical field has received considerable attention because of its capability to model processes that exhibit anomalous behavior and memory effects, which are not adequately captured by integer-order models. Neutral and fractional differential equations (FDEs) are particularly useful in describing systems with long-range temporal or spatial dependencies, making them applicable in various disciplines such as physics, engineering, and biology [10–12] even natural phenomenon [13]. The versatility of FDEs lies in their ability to incorporate history-dependent effects, providing a more accurate and comprehensive framework for modeling complex dynamical systems [14, 15].

A fundamental component of mathematical modeling is the study of oscillatory processes in differential equation solutions, which connects theoretical understanding with practical applications [16, 17]. Oscillations, whether periodic, quasi-periodic, or chaotic, emerge ubiquitously in systems governed by differential equations, from classical mechanical vibrations to contemporary technologies such as microelectromechanical systems (MEMS), neural networks, and energy-harvesting devices. In modern technological sciences, understanding these dynamic behaviors is not merely an academic pursuit but a critical necessity. The stability, synchronization, and resonance of oscillatory solutions directly influence the design, efficiency, and reliability of innovations in fields as diverse as aerospace engineering, telecommunications, biomedical devices, and renewable energy systems [18, 19].

Several studies [20, 21] have yielded techniques and approaches to enhance the oscillatory properties of these equations. Furthermore, the researchers in [22, 23] expanded this investigation to include differential equations of the neutral variety. Studies like [24–26] demonstrate the substantial investigation of oscillation behaviors in higher-order DDEs using the p-Laplace type operator in recent years. Zafer [27] presented the following two conditions

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t} B(s)\,\mathrm{d}s > \frac{(j-1)\,2^{(j-1)(j-2)}}{\mathrm{e}}, \tag{1.2}$$

or

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t} B(s)\,\mathrm{d}s > (j-1)\,2^{(j-1)(j-2)},\ \ \delta'(t)\geq 0,$$

where $B(t) := \delta^{j-1}(t)\,(1-a(\delta(t)))\,b(t)$ to prove that the equation

$$y^{(j)}(t) + b(t)\,z(\delta(t)) = 0, \tag{1.3}$$

is oscillatory. While Zhang and Yan [28] gave the conditions

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t} B(s)\,\mathrm{d}s > \frac{(j-1)!}{\mathrm{e}}, \tag{1.4}$$

or

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t} B(s)\,\mathrm{d}s > (j-1)!,\ \ \delta(t)\geq 0,$$

to prove that (1.3) is oscillatory notice that $(j-1)\,2^{(j-1)(j-2)} > (j-1)!$ for $3 < j$, which generalizes the result of Zafer [27].

In [29], the authors proved that the conditions

$$\left(\delta^{-1}(t)\right)' \geq \delta_0 > 0,\ \varphi'(t)\geq \varphi_0 > 0,\ \varphi^{-1}(\delta(t))\,t,$$

and

$$\liminf_{t\to\infty}\int_{\varphi^{-1}(\delta(t))}^{t} \frac{\widehat{b}(s)}{m(s)}\left(s^{j-1}\right)^{(p-1)}\mathrm{d}s > \left(\frac{1}{\delta_0}+\frac{a_0^{(p-1)}}{\delta_0\varphi_0}\right)\frac{((j-1)!)^{(p-1)}}{\mathrm{e}}, \tag{1.5}$$

where $\widehat{b}(t) := \min\left\{b\left(\delta^{-1}(t)\right), b\left(\delta^{-1}(\varphi(t))\right)\right\}$ make the Eq (1.1) oscillatory.

Applying this, I obtain the following equation:

$$\left(z(t)+\frac{7}{8}z\left(\frac{1}{\mathrm{e}}t\right)\right)^{(4)} + \frac{b_0}{t^4}z\left(\frac{1}{\mathrm{e}^2}t\right) = 0,\ t\geq 1, \tag{1.6}$$

so, we find that (1.6) is oscillatory if

The condition	(1.2)	(1.4)	(1.5)
The criterion	$b_0 > 113981.3$	$b_0 > 3561.9$	$b_0 > 3008.5$

Thus, [29] improved the results in [27, 28]. The authors in [25, 26] establish the oscillation conditions for equation

$$\left(m(t)\left|y^{(j-1)}(t)\right|^{p-2} y^{(j-1)}(t)\right)' + b(t)\left|z(\delta(t))\right|^{p-2} z(\delta(t)) = 0, t \geq t_0,$$

where $\int_{s_0}^{\infty-} m^{1/(p-1)}(\upsilon)\,\mathrm{d}\upsilon = \infty$. Liu et al. [24] extended some oscillatory conditions for equation

$$\left(m(t)\left|y^{(n)}(t)\right|^{p-2} y^{(n)}(t)\right)' + r(t)\left|y^{(n)}(s)\right|^{p-2} y^{(n)}(t) + b(t)\left|z(\delta(t))\right|^{p-2} z(\delta(t)) = 0,$$

where

$$y(t) := z(t) + a(t)\,z(\varphi(t)).$$

In order to progress the oscillation theory for even-order neutral equations, the main objective of the endeavor is to determine sufficient conditions that guarantee oscillatory solutions.

I aim to broaden the scope of research and improve the results in [27–29] by applying the comparison approach with first-order equations to produce a new theorem of (1.1) under condition $\int_{t_0}^{\infty} m^{-1/(p-1)}(s)\,\mathrm{d}s = \infty$. Within this perspective, the paper offers unique criteria for assessing oscillatory solutions of (1.1). I apply two examples to see the validity and efficiency of the obtained criteria.

Now, I introduce some notations:

$$A_\ell(t) = \frac{1}{a(\varphi^{-1}(t))}\left(1 - \frac{\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^{\ell-1}}{\left(\varphi^{-1}(t)\right)^{\ell-1} a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}\right), \quad \text{for } \ell = 2, j,$$

$$D_0(t) = \left(\frac{1}{m(t)}\int_t^\infty b(s)\,A_2^{(p-1)}(\delta(s))\,\mathrm{d}s\right)^{1/(p-1)},$$

and

$$D_\nu(t) = \int_t^\infty D_{\nu-1}(s)\,\mathrm{d}s, \quad \nu = 1, 2, ..., j-3.$$

I first introduce some important lemmas.

Lemma 1. *[30, Lemmas 1 and 2] If $w, g \geq 0$. Then*

$$(w+g)^k \leq w^k + g^k, \textit{ for } k \leq 1,$$

and

$$(w+g)^k \leq 2^{k-1}\left(w^k + g^k\right), \textit{ for } k \geq 1.$$

Lemma 2. *[31] Let $z \in C^j([t_0,\infty),(0,\infty))$, $z^{(j-1)}(t)\,z^{(j)}(t) \leq 0$, where $z^{(j)}(t)$ is fixed sign and not identically zero on $[t_0,\infty)$. If $\lim_{t\to\infty} z(t) \neq 0$, then*

$$z(t) \geq \frac{\varepsilon}{(j-1)!} t^{j-1}\left|z^{(j-1)}(t)\right|,$$

where $t \geq t_\varepsilon, \varepsilon \in (0,1)$.

Lemma 3. *[25] Let $z^{(j+1)}(t) < 0$, then*

$$\frac{z(t)}{t^j/j!} \geq \frac{z'(t)}{t^{j-1}/(j-1)!},$$

where $z^{(i)}(t) > 0$, $i = 0, 1, ..., j$.

Lemma 4. *[32, Lemma 1.2] The ultimate positive solution to (1.1) is found to be the function z. Thus, I identify two instances:*

$$\begin{array}{ll} (\mathbf{I}_1) & y(t) > 0,\ y^{(j)}(t) < 0,\ y^{(j-1)}(t) > 0,\ y^{(c)}(t) > 0, c = 1, 2, \\ (\mathbf{I}_2) & y(t) > 0, y^{(r+1)}(t) < 0, y^{(r)}(t) > 0 \text{ for } r \in \{1, 3, ..., j-3\}, \\ & y^{(j-1)}(t) > 0, y^{(j)}(t) < 0, \end{array}$$

for $t \geq t_1$.

2. Oscillation criteria

Using the comparison technique and Riccati approach with a variety of substitutions, this section presents criteria that guarantee the oscillatory behavior of solutions to (1.1). These specifications are derived from a thorough analysis that eventually concentrates on positive solutions while accounting for the equation's particular structure. In order to simplify the proof without sacrificing generality, we assume that the functional inequalities hold greatly.

Lemma 5. *Assume that Case (I_1) holds, and the function z is a positive solution of (1.1). If*

$$\frac{\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^{j-1}}{\left(\varphi^{-1}(t)\right)^{j-1} a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)} \leq 1, \tag{2.1}$$

then

$$y(t) \geq \frac{\varepsilon}{(j-1)!} t^{j-1} y^{(j-1)}(t). \tag{2.2}$$

Proof. Let us assume that $z(t)$ is the final positive solution of (1.1) and Case (I_1) holds, using

$$y(t) := z(t) + a(t) z(\varphi(t)).$$

I find

$$z(t) = \frac{1}{a\left(\varphi^{-1}(t)\right)}\left(y\left(\varphi^{-1}(t)\right) - z\left(\varphi^{-1}(t)\right)\right).$$

Thus, I obtain

$$\begin{aligned} z(t) &= \frac{y\left(\varphi^{-1}(t)\right)}{a\left(\varphi^{-1}(t)\right)} - \frac{1}{a\left(\varphi^{-1}(t)\right)}\left(\frac{y\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}{a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)} - \frac{z\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}{a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}\right) \\ &\geq \frac{y\left(\varphi^{-1}(t)\right)}{a\left(\varphi^{-1}(t)\right)} - \frac{1}{a\left(\varphi^{-1}(t)\right)} \frac{y\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}{a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}. \end{aligned} \tag{2.3}$$

Applying Lemma 3, we see

$$y(t) \geq \frac{1}{(j-1)} t y'(t),$$

so $t^{1-j} y(t)$ is nonincreasing, and from $\varphi(t) \leq t$, I see

$$\left(\varphi^{-1}(t)\right)^{j-1} y\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right) \leq \left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^{j-1} y\left(\varphi^{-1}(t)\right). \tag{2.4}$$

Combining (2.3) and (2.4), I conclude that

$$\begin{aligned} z(t) &\geq \frac{1}{a\left(\varphi^{-1}(t)\right)}\left(1-\frac{\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^{j-1}}{\left(\varphi^{-1}(t)\right)^{j-1} a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}\right) y\left(\varphi^{-1}(t)\right) \\ &= A_j(t)\, y\left(\varphi^{-1}(t)\right). \end{aligned} \tag{2.5}$$

By (1.1) and (2.5), I have

$$\left(m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}\right)' + b(t) A_j^{(p-1)}(\delta(t))\, y^{(p-1)}\left(\varphi^{-1}(\delta(t))\right) \leq 0.$$

Since $w(t) \leq \delta(t)$ and $y'(t) > 0$, I get

$$\left(m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}\right)' \leq -b(t) A_j^{(p-1)}(\delta(t))\, y^{(p-1)}\left(\varphi^{-1}(w(t))\right). \tag{2.6}$$

Now, using Lemma 2, I have

$$y(t) \geq \frac{\varepsilon}{(j-1)!} t^{j-1} y^{(j-1)}(t),$$

for some $\varepsilon \in (0,1)$. Therefore, the proof is complete. □

Lemma 6. *If (2.1) and Case* (I_2) *hold. Then,*

$$\left(m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}\right)' \leq -b(t) A_2^{(p-1)}(\delta(t))\, y^{(p-1)}\left(\varphi^{-1}(\varrho(t))\right). \tag{2.7}$$

Proof. Given that $z(t)$ constitutes the final positive solution of (1.1), if Case (I_2) holds, by Lemma 3, I have

$$y(t) \geq t y'(t), \tag{2.8}$$

so $t^{-1} y(t)$ is nonincreasing eventually.

Since

$$\varphi^{-1}(t) \leq \varphi^{-1}\left(\varphi^{-1}(t)\right),$$

thus, I find

$$\varphi^{-1}(t)\, y\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right) \leq \varphi^{-1}\left(\varphi^{-1}(t)\right) y\left(\varphi^{-1}(t)\right). \tag{2.9}$$

Combining (2.3) and (2.9), I find

$$z(t) \geq \frac{1}{a\left(\varphi^{-1}(t)\right)}\left(1-\frac{\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}{\left(\varphi^{-1}(t)\right) a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)}\right) y\left(\varphi^{-1}(t)\right)$$

$$= \quad A_2(t)\, y\left(\varphi^{-1}(t)\right),$$

which with (1.1) yields

$$\left(m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}\right)' + b(t)\, A_2^{(p-1)}(\delta(t))\, y^{(p-1)}\left(\varphi^{-1}(\delta(t))\right) \le 0.$$

Since $\varrho(t) \le \delta(t)$ and $y'(t) > 0$, I have that

$$\left(m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}\right)' \le -b(t)\, A_2^{(p-1)}(\delta(t))\, y^{(p-1)}\left(\varphi^{-1}(\varrho(t))\right).$$

The proof is complete. □

Theorem 1. *Let positive functions* $w,\ \varrho \in C^1([t_0,\infty),\mathbb{R})$, *satisfying*

$$w(t) \le \delta(t),\ w(t) < \varphi(t),\ \varrho(t) \le \delta(t),\ \varrho(t) < \varphi(t),\ \varrho'(t) \ge 0,\ \text{and}\ \lim_{t\to\infty} w(t) = \lim_{t\to\infty} \varrho(t) = \infty. \tag{2.10}$$

If the equations

$$y'(t) + \left(\frac{\varepsilon\left(\varphi^{-1}(w(t))\right)^{j-1}}{(j-1)!\,m^{1/(p-1)}\left(\varphi^{-1}(w(t))\right)}\right)^{(p-1)} b(t)\, A_j^{(p-1)}(\delta(t))\, y\left(\varphi^{-1}(w(t))\right) = 0, \tag{2.11}$$

where $\varepsilon \in (0,1)$, *and*

$$\phi'(t) + \varphi^{-1}(\varrho(t))\, D_{j-3}(t)\, \phi\left(\varphi^{-1}(\varrho(t))\right) = 0, \tag{2.12}$$

are oscillatory, then (1.1) is oscillatory.

Proof. Given that $z(t)$ constitutes the final positive solution of (1.1). Cases (I_1) and (I_2) are holds from Lemma 4. Using Lemma 5, see for (2.6) and (2.2) that, for all $\varepsilon \in (0,1)$,

$$\left(\frac{\varepsilon\left(\varphi^{-1}(w(t))\right)^{j-1}}{(j-1)!}\right)^{(p-1)} b(t)\, A_j^{(p-1)}(\delta(t)) \left(y^{(j-1)}\left(\varphi^{-1}(w(t))\right)\right)^{(p-1)} \le -\left(m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}\right)'.$$

Thus, if I choose $y(t) = m(t)\left(y^{(j-1)}(t)\right)^{(p-1)}$, have that y is a positive solution of

$$y'(t) + \left(\frac{\varepsilon\left(\varphi^{-1}(w(t))\right)^{j-1}}{(j-1)!\,m^{1/(p-1)}\left(\varphi^{-1}(w(t))\right)}\right)^{(p-1)} b(t)\, A_j^{(p-1)}(\delta(t))\, y\left(\varphi^{-1}(w(t))\right) \le 0.$$

From [33, Theorem 1], it is also clear that Eq (2.11) becomes a positive solution. Therefore, it is clear that this is a contradiction.

By Lemma 4, I integrate the (2.7) from t to ∞, I obtain

$$y^{(j-1)}(t) \ge D_0(t)\, y\left(\varphi^{-1}(\varrho(t))\right).$$

Integrating from t to ∞ a total of $j-3$ times, I obtain

$$y''(t) + D_{j-3}(t)\, y\left(\varphi^{-1}(\varrho(t))\right) \le 0. \tag{2.13}$$

Thus, if I choose $\phi(t) := y'(t)$ and apply (2.8), I find

$$\phi'(t) + \varphi^{-1}(\varrho(t)) D_{j-3}(t)\, \phi\left(\varphi^{-1}(\varrho(t))\right) \leq 0, \tag{2.14}$$

where ϕ is a positive solution. Based on [33, Theorem 1], it is also clear that Eq (2.12) becomes a positive solution. Therefore, it is clear that this is a contradiction, so the proof is complete. □

Corollary 1. *Let (2.1) and (2.10) hold. If*

$$\liminf_{t\to\infty} \int_{\varphi^{-1}(w(t))}^{t} \left(\frac{\left(\varphi^{-1}(w(s))\right)^{j-1}}{m^{1/(p-1)}\left(\varphi^{-1}(w(s))\right)} \right)^{(p-1)} b(s) A_j^{(p-1)}(\delta(s))\, \mathrm{d}s > \frac{((j-1)!)^{(p-1)}}{\mathrm{e}}, \tag{2.15}$$

and

$$\liminf_{t\to\infty} \int_{\varphi^{-1}(\varrho(t))}^{t} \varphi^{-1}(\varrho(s)) D_{j-3}(s)\, \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.16}$$

then (1.1) is oscillatory.

Proof. Based on ([34, Theorem 2]), I see that Conditions (2.15) and (2.16) imply oscillation of (2.11) and (2.12), respectively. □

3. Applications

In this section, I stress the importance of the conditions, I derived in Corollary 1 by discussing specific applications.

Example 1. *Consider the equation:*

$$\left[t\left(z(t) + \frac{1}{2} z\left(\frac{t}{3}\right) \right)''' \right]' + \frac{b_0}{t}\left(z^2 + z\right)\left(\frac{t}{2}\right) = 0, \iota \geq 1, \tag{3.1}$$

where $b_0 > 0$ is a constant. Let $p = 2,\ m(t) = t,\ a(t) = 1/2,\ \varphi(t) = t/3,\ b(t) = b_0/t,\ \delta(t) = t/2$.

Now, I find

$$\begin{aligned} &\int_{t_0}^{\infty} m^{-1/(p-1)}(s)\, \mathrm{d}s \\ = &\int_{t_0}^{\infty} 1/s \mathrm{d}s = \infty. \end{aligned}$$

Let $w(t) = \varrho(t) = \beta t$, I see that (2.1) and (2.10) are satisfied.

Also, I obtain

$$\begin{aligned} A_\ell(t) &= \frac{1}{a\left(\varphi^{-1}(t)\right)} \left(1 - \frac{\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^{\ell-1}}{\left(\varphi^{-1}(t)\right)^{\ell-1} a\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)} \right) \\ &= \frac{1}{2}\ \left(1 - 2t^{2(\ell-1)} 3^{(1-\ell)}\right)\ \textit{for } \ell = 2, j, \end{aligned}$$

and

$$\begin{aligned} D_0(t) &= \left(\frac{1}{m(t)}\int_t^{\infty} b(s)\,A_2^{(p-1)}(\delta(s))\,\mathrm{d}s\right)^{1/(p-1)} \\ &= \frac{1}{t}\int_t^{\infty}\frac{b_0}{2s}\left(1-\frac{s^2}{6}\right)\mathrm{d}s \\ &= \infty. \end{aligned}$$

By Corollary 1, see for (3.2) is oscillatory.

Example 2. *Consider the equation*

$$(z(t)+a_0 z(\pi t))^{(j)}+\frac{b_0}{t^j}z(\beta t)=0, t\geq 1,\ b_0>0, \tag{3.2}$$

where $\pi \in \left(a_0^{-1/(j-1)}, 1\right)$, *and* $\beta \in (0,\pi)$, $m(t) = 1, p = 2$, $a(t) = a_0$, $\varphi(t) = \pi t$, $\delta(t) = \beta t$ *and* $b(t) = b_0/t^j$.

If I set $w(t) = \varrho(t) = \beta t$, *then it is easy to obtain that (2.1) and (2.10) are satisfied.*

Moreover, I see

$$\begin{aligned} A_\ell(t) &= \frac{1}{a_0}\left(1-\frac{\pi^{1-\ell}}{a_0}\right), \text{ for } \ell = 2, j, \\ D_0(t) &= \frac{b_0}{a_0}\left(1-\frac{1}{\pi a_0}\right)\frac{t^{1-j}}{(j-1)}, \end{aligned}$$

and

$$D_{j-3}(t)=\frac{1}{(j-3)!}\frac{b_0}{a_0}\left(1-\frac{1}{\pi a_0}\right)\frac{1}{(j-2)(j-1)t^2}.$$

Thus, Conditions (2.15) and (2.16) become

$$b_0\frac{1}{a_0}\left(\frac{\beta}{\pi}\right)^{j-1}\left(1-\frac{\pi^{1-j}}{a_0}\right)\ln\frac{\pi}{\beta}>\frac{(j-1)!}{\mathrm{e}}, \tag{3.3}$$

and

$$b_0\frac{1}{a_0}\frac{\beta}{\pi}\left(1-\frac{1}{\pi a_0}\right)\ln\frac{\pi}{\beta}>\frac{(j-1)!}{\mathrm{e}}. \tag{3.4}$$

So, it is clear that (3.3) implies (3.4).

Thus, applying Corollary 1, I obtain (3.2) is oscillatory if (3.3) holds.

Remark 1. *Regarding (3.2) as a particular case if I set* $j = 4$, $a_0 = 16$, $\pi = 1/2$ *and* $\beta = 1/3$, *we see that Condition (3.3) yields* $b_0 > 587.93$. *Moreover, the condition obtained from the results of [29] is* $b_0 > 4850.4$. *Therefore, the results I obtained improve the results in [29].*

4. Conclusions

The topic of defining suitable criteria to guarantee oscillatory behavior for every solution of a class of nonlinear NDEs has been solved in this paper. I examined new monotonic properties of positive solutions and derived unique oscillation criteria for (1.1), with a special focus on the canonical situation. I have advanced a more comprehensive understanding of oscillatory processes in differential equations by implementing and refining their methodologies.

Using comparable approaches to investigate even-order nonlinear NDEs of the form $\left(m(t)\, y^{(j-1)}(t)\right)' + b(t)\, z(\delta(t)) = 0$ would be an attractive avenue for future research.

The methods used in this paper can be used to reveal new oscillation patterns and expand the scope of study to include more general models, for example:

$$\left(m_1(t)\left(m_2(t)\left(\left|y^{(j-1)}(t)\right|^{p-2} y^{(j-1)}(t)\right)\right)\right)' + b(t)\, z^{p-2}(\delta(t))\, z(\delta(t)) = 0.$$

Use of Generative-AI tools declaration

The authors declare that they have not use any artificial intelligence (AI) tools in the writing of this article.

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Conflict of interest

The authors declare no competing financial interest.

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