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Research article

On the structure and applications of neutrosophic prime ideals

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Abstract: This study explores key characteristics associated with neutrosophic prime ideals. In particular, it is shown that the image corresponding to these ideals contains exactly two different elements. The work also extends the concept of classical prime ideals by incorporating neutrosophic logic. A new approach to identifying prime ideals using neutrosophic points is introduced. Furthermore, the paper compares various definitions of neutrosophic prime ideals and examines the connections between them. Finally, an application is used to illustrate the role of neutrosophic prime ideals in decision-making processes.

Keywords: neutrosophic set; neutrosophic prime ideal; ideal
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1. Introduction

Ambiguity is an inherent element of all human experience. To overcome the rigid limitations of traditional set approaches in modelling such unpredictability, Zadeh introduced the notion of fuzzy sets [15]. While this innovation advanced the representation of partial truths, later analyses revealed the limitations in scenarios that require simultaneous consideration of membership and non-membership parameters. Subsequently, Atanassov [1] extended the framework by proposing intuitionistic fuzzy sets containing dual degrees to capture acceptance and rejection better. Although these sets are widely used in various contexts they encounter difficulties in practise due to inherent limitations. To resolve inconsistencies arising from ambiguous or contradictory information, Smarandache [10] later introduced neutrosophic sets (NSs), which have a three-part structure for modelling truth, falsity, and indeterminacy.

Investigations into algebraic structures within neutrosophic theory have increasingly captured the attention of scholars in recent studies. For example, researchers have investigated neutrosophic rings [4,6], ideals [3,7], modules [8,9], and groups [2,5], aiming to extend classical algebraic concepts into the neutrosophic framework. Elrawy et al. [6] introduced a new approach for the neutrosophic sub-ring. Accordingly, Hummdi et al. [7] presented the concepts of neutrosophic ideal and prime ideal. This work is a continuation of the further study of the concept of neutrosophic prime ideal. This research is driven by two main objectives. The first is to investigate the extent to which the concepts from classical ideal and prime ideal theory can be extended to the framework of neutrosophic ideals and neutrosophic prime ideals. Notably, some characteristics valid in classical prime ideals do not necessarily apply to neutrosophic prime ideals, as highlighted in Remark 3.12 in [7]. The second goal is the usefulness of neutrosophic ideals and prime ideals in dealing with ambiguous, incomplete, or contradictory information, an essential feature for practical applications in fields such as artificial intelligence, economics, social, sciences, and decision-making scenarios where uncertainty is a key factor. Classical ideals and prime ideals require exact inclusion, while the neutrosophic counterparts allow for varying degrees of membership. This flexibility leads to more meaningful and adaptable algebraic systems that better capture the complexity of the real world.

The decision-making model introduced through neutrosophic ideals represents a notable progression beyond conventional, especially in environments marked by ambiguity, conflicting inputs, or missing data. Additionally, the use of algebraic structures based on ideals brings internal consistency to the evaluation process, ensuring coherent outcomes even when dealing with inconsistent or partial datasets. Such a framework proves particularly useful in practical applications like evaluating alternatives in complex selection processes or scenarios involving multiple criteria where data imperfections are common. By formally incorporating both vagueness and contradiction into the analysis, the neutrosophic ideal-based system strengthens decision reliability. It facilitates a more comprehensive understanding of each option's trustworthiness, ultimately leading to more consistent and defensible choices.

The next part of this paper is organized as follows: The second section presents key concepts and preliminary results that form the basis for the core contributions presented later. The third section establishes several properties of neutrosophic prime ideals over a ring and studies different definitions of neutrosophic prime ideals. The fourth section presents an application of the neutrosophic prime ideal in decision-making. Finally, the results and summaries of the main findings discussed in this paper are in the last section.

2. Some basic concepts

Here, we give important concepts and outcomes as follows.

Definition 2.1. [11, 14] An NS $\mathcal{L}$ over a domain $\mathcal{U}$ is characterized by $\mathcal{L} = \{\langle a, \alpha(a), \beta(a), \theta(a)\rangle \mid a \in \mathcal{U}\}$, where the component functions α, β, and θ satisfy $\alpha, \beta, \theta \mid \mathcal{U} \to [0,1]$.

Definition 2.2. [10,12,13] Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote distinct NSs on $\mathcal{U}$. Then

(1) $\mathcal{L}_1 \cap_1 \mathcal{L}_2 = \{\langle \ell, \alpha_1(\ell) \vee \alpha_2(\ell), \beta_1(\ell) \wedge \beta_2(\ell), \theta_1(\ell) \wedge \theta_2(\ell)\rangle \mid \ell \in \mathcal{U}\}$.

(2) $\mathcal{L}_1 \cup_1 \mathcal{L}_2 = \{\langle \ell, \alpha_1(\ell) \wedge \alpha_2(\ell), \beta_1(\ell) \vee \beta_2(\ell), \theta_1(\ell) \vee \theta_2(\ell)\rangle \mid \ell \in \mathcal{U}\}$.

(3) $\mathcal{L}_1 \subseteq_1 \mathcal{L}_2 = \{\langle \ell, \alpha_1(\ell) \leq \alpha_2(\ell), \beta_1(\ell) \geq \beta_2(\ell), \theta_1(\ell) \geq \theta_2(\ell)\rangle \mid \ell \in \mathcal{U}\}$.

Proposition 2.3. *[6] $\mathcal{J}$ is said to be a neutrosophic ideal over $\mathcal{S}$ if the following for all $n, u \in \mathcal{S}$ are satisfied:*
(i) $\alpha(n-u) \geqslant \min(\alpha(n), \alpha(u))$,
(ii) $\alpha(nu) \geqslant \max(\alpha(n), \alpha(u))$,
(iii) $\beta(n-u) \leqslant \max(\beta(n), \beta(u))$,
(iv) $\beta(nu) \leqslant \min(\beta(n), \beta(u))$,
(v) $\theta(n-u) \leqslant \max(\theta(n), \theta(u))$,
(vi) $\theta(nu) \leqslant \min(\theta(n), \theta(u))$.

Definition 2.4. [6] Consider $\mathcal{L}_1$ and $\mathcal{L}_2$ as neutrosophic ideals over $\mathcal{S}$. Then the product of $\mathcal{I}_1$ and $\mathcal{I}_2$ is subsequently formulated in the manner described below for all $\ell, s_i, t_i \in \mathcal{S}$.

$$(\alpha_1 \circledast \alpha_2)(\ell) = \sup_{\ell=\sum_i s_i t_i} (min_i(min(\alpha_1(s_i), \alpha_2(t_i)))),$$
$$(\beta_1 \circledast \beta_2)(\ell) = \inf_{\ell=\sum_i s_i t_i} (max_i(max(\beta_1(s_i), \beta_2(t_i)))),$$
$$(\theta_1 \circledast \theta_2)(\ell) = \inf_{\ell=\sum_i s_i t_i} (max_i(max(\theta_1(s_i), \theta_2(t_i)))).$$

Figure 1 illustration presents the concept of a neutrosophic prime ideal as originally proposed in [7].

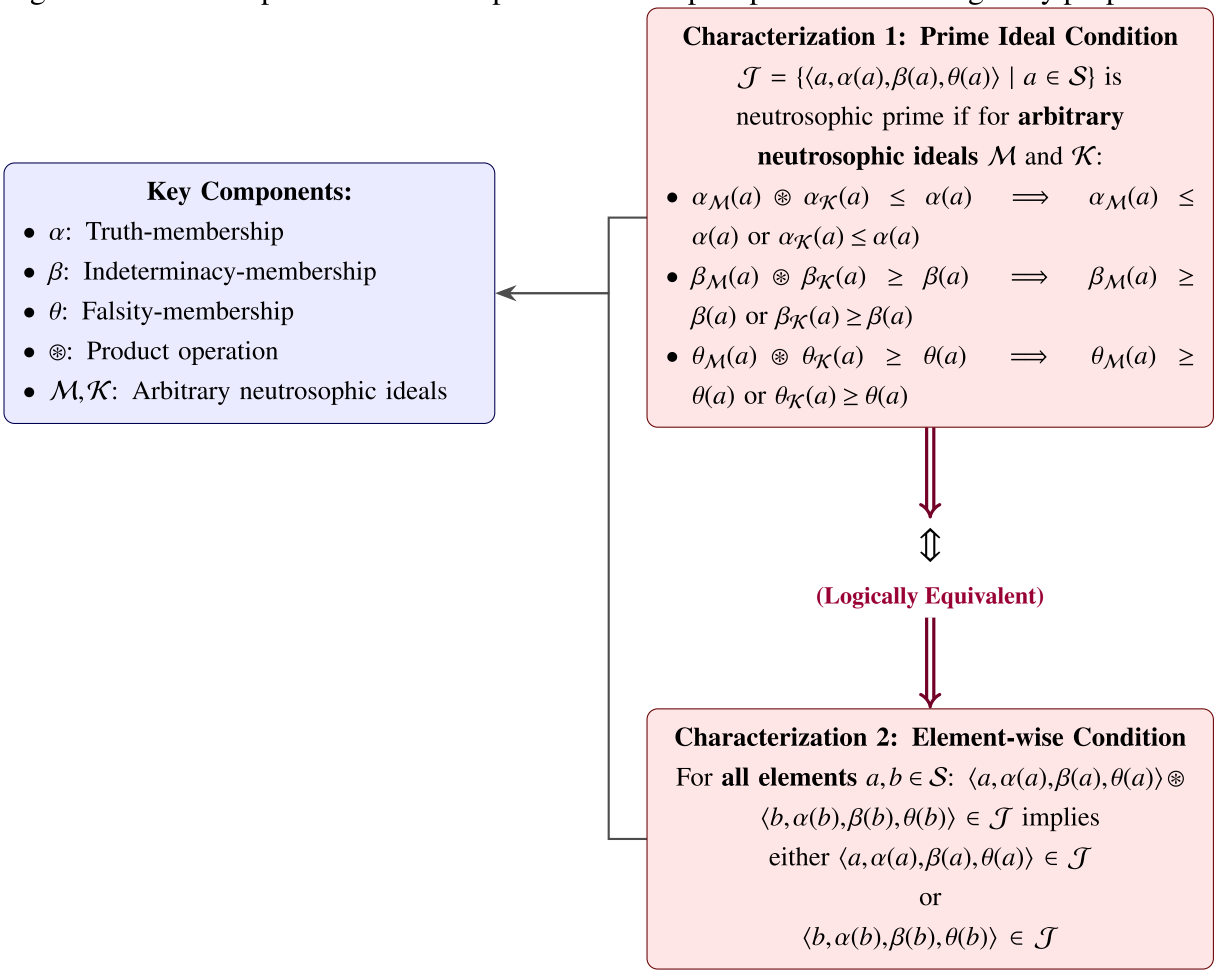

Figure 1. Neutrosophic prime ideal definition.

Example 2.5. Assume that $(Z_6, \oplus_6, \otimes_6)$ is a ring. Then a neutrosophic subset $\mathcal{S} = \{\langle a, \alpha(a), \beta(a), \theta(a)\rangle : a \in Z_6\}$ defined as follows:

$$\alpha(a) = \begin{cases} 0.7 & \text{if } a = 0. \\ 0.4 & \text{if } a \in \{2, 4\}. \\ 0.2 & \text{otherwise.} \end{cases}$$

$$\beta(a) = \begin{cases} 0.2 & \text{if } a = 0. \\ 0.7 & \text{if } a \in \{2, 4\}. \\ 0.8 & \text{otherwise.} \end{cases}$$

$$\theta(a) = \begin{cases} 0.2 & \text{if } u = 0. \\ 0.6 & \text{if } a \in \{2, 4\}. \\ 0.6 & \text{otherwise.} \end{cases}$$

Therefore, $\mathcal{S}$ is a neutrosophic ideal. In addition, we find $\mathcal{J} = \{< 0, 0.7, 0.2, 0.2 >, < 2, 0.4, 0.7, 0.6 >, < 4, 0.4, 0.7, 0.6 >\}$ is a neutrosophic prime ideal.

Lemma 2.6. *[7] If $\mathcal{J}_1$ represents a neutrosophic left ideal and $\mathcal{J}_2$ denotes a neutrosophic right ideal, then $\mathcal{J}_1 \circledast \mathcal{J}_2 \subseteq_1 \mathcal{J}_1 \cap_1 \mathcal{J}_2$.*

Before we show the next result, it is useful to define $\mathcal{J}_0 = \{\ell \in \mathcal{S} \mid \alpha(\ell) = \alpha(0), \beta(\ell) = \beta(1), \theta(\ell) = \theta(1)\}$.

Theorem 2.7. *[7] $\mathcal{J}_0$ is a prime ideal of $\mathcal{S}$.*

Definition 2.8. [6] Let $\mathcal{R}$ be a neutrosophic sub-ring (or ideal) over $\mathcal{S}$, where $0 \leqslant \Upsilon \leqslant \alpha(0)$ and $0 \leqslant \beta(0), \theta(0) \leqslant \Upsilon$. The sub-ring (or ideal) $\mathcal{R}_\Upsilon$ is referred to as a level sub-ring (or level ideal) of $\mathcal{R}$.

3. Main results

3.1. Characterization theorems

Here, we establish fundamental characteristics of a neutrosophic prime ideal in the context of neutrosophic algebra, each of which stems directly from the definition of a neutrosophic prime ideal [7].

Now, we define $\mathcal{J} = \{\langle x, \alpha(x), \beta(x), \theta(x)\rangle \mid x \in \mathcal{S}\}$, where $\mathcal{S}$ is a ring and the mappings $\alpha, \beta, \theta : \mathcal{S} \to [0, 1]$.

The next theorem asserts that any non-trivial neutrosophic prime ideal within a ring imposes particular constraints on the structure of its corresponding membership functions.

Theorem 3.1. *Let $\mathcal{J}$ be a non-constant neutrosophic prime ideal of $\mathcal{S}$; then $Im(\alpha) = \{\rho, 1\}$, $Im(\beta) = \{0, \rho'\}$, $Im(\theta) = \{0, \rho'\}$ where $0 \leqslant \rho < 1$, $0 < \rho' \leqslant 1$, $\alpha(0) = 1, \beta(1) = 0$, and $\theta(1) = 0$.*

Proof. Suppose that $c, v \in \mathcal{S}$ with $0 \leqslant \alpha(c), \alpha(v) < 1, 0 < \beta(c), \beta(v) \leqslant 1$, and $0 < \beta(c), \beta(v) \leqslant 1$. Within the framework of this examination, parameters c and v remain unaltered throughout all subsequent evaluations. Now, we define $\alpha_1, \beta_1, \theta_1, \alpha, \beta, \theta \mid \mathcal{S} \to [0,1]$ by

$$\alpha_1(n) = \begin{cases} 1 & \text{if } n \in < c > \\ 0 & \text{otherwise} \end{cases}, \qquad \alpha_2(n) = \alpha(c) \forall n \in \mathcal{S}$$

$$\beta_1(n) = \begin{cases} 0 & \text{if } n \in < c > \\ 1 & \text{otherwise} \end{cases}, \qquad \beta_2(n) = \beta(c) \forall n \in \mathcal{S}$$

$$\theta_1(n) = \begin{cases} 0 & \text{if } n \in < c > \\ 1 & \text{otherwise} \end{cases}. \qquad \theta_2(n) = \theta(c) \forall n \in \mathcal{S}$$

Since $\mathcal{J}_1 = \{\langle \ell, \alpha_1(\ell), \beta_1(\ell), \theta_1(\ell) \rangle \mid \ell \in \mathcal{S}\}$ and $\mathcal{J}_2 = \{\langle \ell, \alpha_2(\ell), \beta_2(\ell), \theta_2(\ell) \rangle \mid \ell \in \mathcal{S}\}$ are ideals over $\mathcal{S}$ and also

$$\alpha_1(n) \circledast \alpha_2(u) \le \alpha_2(nu),$$
$$\beta_1(n) \circledast \beta_2(u) \ge \beta_2(nu),$$
$$\theta_1(n) \circledast \theta_2(u) \ge \theta_2(nu),$$

for all $n, u \in \mathcal{S}$. When $\alpha_1(n) = 0, \beta_1(n) = 1,$ and $\theta_1(n) = 1$, then

$$\alpha_1(n) \circledast \alpha_2(u) = 0 \le \alpha_2(nu),$$
$$\beta_1(n) \circledast \beta_2(u) = 1 \ge \beta_2(nu),$$
$$\theta_1(n) \circledast \theta_2(u) = 1 \ge \theta_2(nu).$$

When $\alpha_1(n) = 1, \beta_1(n) = 0,$ and $\theta_1(n) = 0$, then $n = n_1 c$ for some $n_1 \in \mathcal{S}$ and

$$\alpha_1(n) \circledast \alpha_2(u) = \alpha(c) \le \alpha_2(nu),$$
$$\beta_1(n) \circledast \beta_2(u) = \beta(c) \ge \beta_2(nu),$$
$$\theta_1(n) \circledast \theta_2(u) = \theta(c) \ge \theta_2(nu).$$

Therefore, $\mathcal{J}$ is a neutrosophic prime ideal, but $\alpha_1(c) = 1 > \alpha(c), \beta_1(c) = 0 < \beta(c)$, and $\theta_1(c) = 0 < \theta(c)$; this leads to $\alpha_2(v) \le \alpha(v)$, $\beta_2(v) \ge \beta(v)$, and $\theta_2(v) \ge \theta(v)$, so $\alpha(c) = \alpha(v)$, $\beta(c) = \beta(v)$, and $\zeta(c) = \zeta(v)$; hence, $|Im(\alpha)| = |Im(\beta)| = |Im(\theta)| = 2$. Again, consider a neutrosophic ideals $\alpha_1, \beta_1, \theta_1, \alpha, \beta, \theta \mid \mathcal{S} \to [0,1]$ as follows:

$$\alpha_1(n) = \begin{cases} 1 & \text{if } n \in \mathcal{J}_0 \\ 0 & \text{otherwise} \end{cases}, \qquad \alpha_2(n) = \alpha(0) \forall n \in \mathcal{S}$$

$$\beta_1(n) = \begin{cases} 0 & \text{if } n \in \mathcal{J}_0 \\ 1 & \text{otherwise} \end{cases}, \qquad \beta_2(n) = \beta(0) \forall n \in \mathcal{S}$$

$$\theta_1(n) = \begin{cases} 0 & \text{if } n \in \mathcal{J}_0 \\ 1 & \text{otherwise} \end{cases}. \qquad \theta_2(n) = \theta(0) \forall n \in \mathcal{S}$$

For all $n, u \in \mathcal{S}$, we obtain

$$\alpha_1(n) \circledast \alpha_2(u) \le \alpha_2(nu),$$
$$\beta_1(n) \circledast \beta_2(u) \ge \beta_2(nu),$$
$$\theta_1(n) \circledast \theta_2(u) \ge \theta_2(nu),$$

but $\alpha_1(0) = 1 > \alpha(0)$, $\beta_1(1) = 0 < \beta(1)$, $\theta_1(1) = 0 < \theta(1)$, and $\alpha_2(k) = \alpha(0) > \alpha(k)$, $\beta_2(k) = \beta(0) < \beta(k)$, $\theta_2(k) = \theta(0) < \theta(k)$; this lead to $\alpha_1(n) \nleq \alpha(nu)$, $\beta_1(n) \ngeq \beta(nu)$, $\theta(n)_1 \ngeq \theta(nu)$ and $\alpha_2(u) \nleq \alpha(nu)$, $\beta_2(u) \ngeq \beta(nu)$, $\theta(u)_2 \ngeq \theta(nu)$ which gives a contradiction. Therefore, $\alpha(0) = 1$, $\beta(1) = 0$, $\theta(1) = 0$, and $Im(\alpha) = \{\rho, 1\}$, $Im(\beta) = \{0, \rho'\}$, $Im(\theta) = \{0, \rho'\}$ with $0 \leqslant \rho < 1$, $0 < \rho' \leqslant 1$. □

Theorem 3.2. *Let $\mathcal{J}$ be a neutrosophic ideal over $\mathcal{S}$ with $Im(\alpha) = \{\rho, 1\}$, $Im(\beta) = \{0, \rho'\}$, $Im(\theta) = \{0, \rho'\}$ where $0 \leqslant \rho < 1$, and $0 < \rho' \leqslant 1$ and $\mathcal{J}_0$ is a neutrosophic prime ideal over $\mathcal{S}$. Then $\mathcal{J}$ is a neutrosophic prime ideal.*

Proof. We can prove this by contradiction. Assume that $\mathcal{J}_1 = \{\langle \ell, \alpha_1(\ell), \beta_1(\ell), \theta_1(\ell)\rangle \mid \ell \in \mathcal{S}\}$ and $\mathcal{J}_2 = \{\langle \ell, \alpha_2(\ell), \beta_2(\ell), \theta_2(\ell)\rangle \mid \ell \in \mathcal{S}\}$ are ideals over $\mathcal{S}$ and

$$\begin{aligned} \alpha_1(n) \circledast \alpha_2(u) &\leq \alpha_2(nu), \\ \beta_1(n) \circledast \beta_2(u) &\geq \beta_2(nu), \\ \theta_1(n) \circledast \theta_2(u) &\geq \theta_2(nu), \end{aligned}$$

where $n, u \in \mathcal{S}$. Suppose that $\alpha_1(n) \nleq \alpha(nu)$, $\beta_1(n) \ngeq \beta(nu)$, $\theta(n)_1 \ngeq \theta(nu)$ and $\alpha_2(u) \nleq \alpha(nu)$, $\beta_2(u) \ngeq \beta(nu)$, $\theta(u)_2 \ngeq \theta(nu)$ this leads to $\alpha_1(n) > \alpha(n)$, or $\beta_1(n) < \beta(n)$, or $\theta_1(n) < \theta(n)$ also $\alpha_2(u) > \alpha(u)$, or $\beta_2(u) < \beta(u)$, or $\theta_2(u) < \theta(u)$. So $n, u \notin \mathcal{J}_0$. Since $\mathcal{J}_0$ is the prime ideal of $\mathcal{S}$ (see [7]), there exist $r \in \mathcal{S}$ with $nru \notin \mathcal{S}$. So

$$\begin{aligned} \alpha(n) = \alpha(u) = \alpha(nru) &= \rho, \\ \beta(n) = \beta(u) = \beta(nru) &= \rho', \\ \theta(n) = \theta(u) = \theta(nru) &= \rho'. \end{aligned}$$

However,

$$\begin{aligned} (\alpha_1 \circledast \alpha_2)(nru) &= \sup_{nru=\sum_i s_i t_i} (\min_i(\min(\alpha_1(s_i), \alpha_2(t_i)))) \\ &\geqslant \alpha_1(n) \wedge \alpha_2(ru) \\ &\geqslant \alpha_1(n) \wedge \alpha_2(u) \\ &> \alpha(n) \wedge \alpha(u) \\ &= \rho = \alpha(nru), \\ (\beta_1 \circledast \beta_2)(nru) &= \inf_{nru=\sum_i s_i t_i} (\max_i(\max(\beta_1(s_i), \beta_2(t_i)))) \\ &\leqslant \beta_1(n) \vee \beta_2(ru) \\ &\leqslant \beta_1(n) \vee \beta_2(u) \\ &< \beta(n) \vee \beta(u) \\ &= \rho' = \beta(nru), \\ (\theta_1 \circledast \theta_2)(nru) &= \inf_{nru=\sum_i s_i t_i} (\max_i(\max(\theta_1(s_i), \theta_2(t_i)))) \\ &\leqslant \theta_1(n) \vee \theta_2(ru) \\ &\leqslant \beta_1(n) \vee \theta_2(u) \\ &< \theta(n) \vee \theta(u) \\ &= \rho' = \theta(nru), \end{aligned}$$

thus gives contradiction. Therefore, either $\alpha_1(n) \leq \alpha(nu), \beta_1(n) \geq \beta(nu)$, $\theta(n)_1 \geq \theta(nu)$ or $\alpha_2(u) \leq \alpha(nu)$, $\beta_2(u) \geq \beta(nu)$, $\theta(u)_2 \geq \theta(nu)$. □

Remark 3.3. In the context of neutrosophic algebraic structures, verifying that every level subset of a neutrosophic prime ideal $\mathcal{J}$ in a ring $\mathcal{S}$ satisfies prime ideal properties is straightforward. However, the reverse implication fails to hold in general.

Example 3.4. Consider the neutrosophic ideal $\mathcal{J}$ over $\mathbb{Z}$ defined as follows:

$$\alpha(n) = \begin{cases} 0.7 & \text{if } n \in < 3 >. \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta(n) = \begin{cases} 0 & \text{if } n \in < 3 >. \\ 0.9 & \text{otherwise.} \end{cases}$$

$$\theta(n) = \begin{cases} 0 & \text{if } n \in < 3 >. \\ 0.6 & \text{otherwise.} \end{cases}$$

Therefore, every level subset of it constitutes a prime ideal of $\mathbb{Z}$; however, $\mathcal{J}$ fails to be a neutrosophic prime ideal as $\alpha(0) \neq 1, \beta(1) \neq 0$ and $\theta(1) \neq 0$.

The following result establishes an exact condition under which a ring can be characterized as a field, using the structural properties of its neutrosophic ideals.

Corollary 3.5. *A ring $\mathcal{S}$ is a field if and only if each neutrosophic ideal $\mathcal{J}$ over $\mathcal{S}$ that is not constant and satisfies $\alpha(0) = 1, \beta(1) = 0$, and $\theta(1) = 0$ is prime.*

Proof. Suppose that $n \in \mathcal{S} - \{0\}$, then

$$\alpha(n) = \alpha(1n) \geqslant \alpha(1) = \alpha(nn^{-1}) \geqslant \alpha(n), \text{ i.e., } \alpha(n) = \alpha(1),$$
$$\beta(n) = \beta(1n) \leqslant \beta(1) = \beta(nn^{-1}) \leqslant \beta(n), \text{ i.e., } \beta(n) = \beta(1),$$
$$\theta(n) = \theta(1n) \leqslant \theta(1) = \theta(nn^{-1}) \leqslant \theta(n), \text{ i.e., } \theta(n) = \theta(1),$$

which leads to $Im(\alpha) = \{\alpha(1), 1\}$, $Im(\beta) = \{0, \beta(1)\}$, $Im(\theta) = \{0, \theta(1)\}$, and $\mathcal{J}_0 = \{\langle 0, 1, 1 \rangle\}$ is a prime ideal of $\mathcal{S}$. Thus, $\mathcal{J}$ is a neutrosophic prime ideal over $\mathcal{S}$. The converse is clear. □

3.2. Behavior under intersection

In most cases, the intersection of two neutrosophic prime ideals does not necessarily remain a neutrosophic prime ideal, which is different in the case of the classical set.

Example 3.6. Consider two neutrosophic ideals, $\mathcal{J}_1$ and $\mathcal{J}_2$, over $\mathbb{Z}$ defined as follows:

$$\alpha_1(n) = \begin{cases} 0.7 & \text{if } n \in < 2 > \\ 0 & \text{otherwise} \end{cases}, \qquad \alpha_2(n) = \begin{cases} 0.8 & \text{if } n \in < 3 > \\ 0 & \text{otherwise} \end{cases},$$

$$\beta_1(n) = \begin{cases} 0 & \text{if } n \in < 2 > \\ 0.9 & \text{otherwise} \end{cases}, \qquad \beta_2(n) = \begin{cases} 0 & \text{if } n \in < 3 > \\ 0.4 & \text{otherwise} \end{cases},$$

$$\theta_1(n) = \begin{cases} 0 & \text{if } n \in < 2 > \\ 0.6 & \text{otherwise} \end{cases}, \qquad \theta_2(n) = \begin{cases} 0 & \text{if } n \in < 3 > \\ 0.5 & \text{otherwise} \end{cases}.$$

Therefore, it is clear that $\mathcal{J}_1 \cap_1 \mathcal{J}_2$ is not a neutrosophic prime ideal.

Proposition 3.7. *Presume $\mathcal{J}_1$ and $\mathcal{J}_2$ are two neutrosophic ideals; then $\mathcal{J}_1 \cap_1 \mathcal{J}_2$ is a neutrosophic prime ideal if and only if either $\mathcal{J}_1 \subseteq_1 \mathcal{J}_2$ or $\mathcal{J}_2 \subseteq_1 \mathcal{J}_1$.*

Proof. The proof is concluded by using Lemma 2.6. □

The next result explains the characterization of a neutrosophic prime ideal.

Proposition 3.8. *Let $\{\mathcal{J}_i\}_{i\in\Gamma}$ be a chain of neutrosophic prime ideals over $\mathcal{S}$, then $\cap_{1,i\in\Gamma}\mathcal{J}_i$ and $\cup_{1,i\in\Gamma}\mathcal{J}_i$ are neutrosophic prime ideals over $\mathcal{S}$.*

Proof. We only prove $\cup_{1,i\in I}\mathcal{J}_i$ is a neutrosophic prime ideal over $\mathcal{S}$ and $\cap_{1,i\in I}\mathcal{J}_i$ is easy to show as a neutrosophic prime ideal over $\mathcal{S}$. Now, since $\alpha_i(0) = 1$, $\beta_i(1) = 0$, and $\theta_(1) = 0$; this implies $\bigwedge_{i\in\Gamma}\alpha_i(0) = 1$, $\bigvee_{i\in\Gamma}\beta_i(0) = 1$, and $\bigvee_{i\in\Gamma}\theta_i(0) = 1$ for some $i \in \Gamma$ by Theorem 3.1. Again, since $\{(\mathcal{J}_i)_0\}$ is a chain, $\cup_{1,i\in\Gamma}(\mathcal{J}_i)_0$ is a neutrosophic prime ideal over $\mathcal{S}$. Assume that $\mathcal{P}_1 = \{\langle \ell, \alpha_1(\ell), \beta_1(\ell), \theta_1(\ell)\rangle \mid \ell \in \mathcal{S}\}$, $\mathcal{P}_2 = \{\langle \ell, \alpha_2(\ell), \beta_2(\ell), \theta_2(\ell)\rangle \mid \ell \in \mathcal{S}\}$ are neutrosophic ideals such that

$$\alpha_1 \circledast \alpha_2 \le \bigwedge_{i\in\Gamma}\alpha_i,$$
$$\beta_1 \circledast \beta_2 \ge \bigvee_{i\in\Gamma}\beta_i,$$
$$\theta_1 \circledast \theta_2 \ge \bigvee_{i\in\Gamma}\theta_i.$$

Let

$$\alpha_1 \nleq \bigwedge_{i\in\Gamma}\alpha_i \quad \Rightarrow \alpha_1(n) > (\bigwedge_{i\in\Gamma}\alpha_i)(n), \text{ and } \alpha_2 \nleq \bigwedge_{i\in\Gamma}\alpha_i \quad \Rightarrow \alpha_2(u) > (\bigwedge_{i\in\Gamma}\alpha_i)(u),$$
$$\beta_1 \ngeq \bigvee_{i\in\Gamma}\beta_i \quad \Rightarrow \beta_1(n) < (\bigvee_{i\in\Gamma}\beta_i)(n), \text{ and } \beta_2 \ngeq \bigvee_{i\in\Gamma}\beta_i \quad \Rightarrow \beta_2(u) < (\bigvee_{i\in\Gamma}\beta_i)(u),$$
$$\theta_1 \ngeq \bigvee_{i\in\Gamma}\theta_i \quad \Rightarrow \theta_1(n) < (\bigvee_{i\in\Gamma}\theta_i)(n), \text{ and } \theta_2 \ngeq \bigvee_{i\in\Gamma}\theta_i \quad \Rightarrow \theta_2(u) < (\bigvee_{i\in\Gamma}\theta_i)(u),$$

where $n, u \in \mathcal{S}$, which lead to $n, u \notin \cup_{1,i\in\Gamma}(\mathcal{J}_i)_0$. Since $\cup_{1,i\in\Gamma}(\mathcal{J}_i)_0$ is prime ideal, thus

$$\alpha_i(n) = \alpha_i(u) = \alpha_i(un) = \rho_i,$$
$$\beta_i(n) = \beta_i(u) = \beta_i(un) = \rho_i',$$
$$\theta_i(n) = \theta_i(u) = \theta_i(un) = \rho_i',$$

for all $i \in \Gamma$. So

$$(\bigwedge_{i\in\Gamma}\alpha_i)(n) = (\bigwedge_{i\in\Gamma}\alpha_i)(u) = (\bigwedge_{i\in\Gamma}\alpha_i)(un) = \inf_{i\in\Gamma}\rho_i,$$
$$(\bigvee_{i\in\Gamma}\beta_i)(n) = (\bigvee_{i\in\Gamma}\beta_i)(u) = (\bigvee_{i\in\Gamma}\beta_i)(un) = \sup_{i\in\Gamma}\rho_i',$$
$$(\bigvee_{i\in\Gamma}\theta_i)(n) = (\bigvee_{i\in\Gamma}\theta_i)(u) = (\bigvee_{i\in\Gamma}\theta_i)(un) = \sup_{i\in\Gamma}\rho_i',$$

implying

$$(\alpha_1 \circledast \alpha_2)(nu) \geq \alpha_1(n) \wedge \alpha_2(u) > (\bigwedge_{i\in\Gamma} \alpha_i)(nu),$$
$$(\beta_1 \circledast \beta_2)(nu) \leq \beta_1(n) \vee \beta_2(u) < (\bigvee_{i\in\Gamma} \beta_i)(nu),$$
$$(\theta_1 \circledast \theta_2)(nu) \leq \theta_1(n) \vee \theta_2(u) < (\bigvee_{i\in\Gamma} \theta_i)(nu),$$

which gives contradiction. Therefore, the result is true. □

Let us now examine how the various definitions of a neutrosophic prime ideal relate to each other in terms of their equivalence.

Proposition 3.9. *Consider $\mathcal{J} = \{\langle a, \alpha(a), \beta(a), \theta(a)\rangle \mid a \in \mathcal{S}\}$ is a neutrosophic ideal over $\mathcal{S}$, then*

(i) *When $\mathcal{J}$ is a neutrosophic prime ideal, then either $\alpha(nu) = \alpha(n), \beta(nu) = \beta(n)$, and $\theta(nu) = \theta(n)$ or $\alpha(nu) = \alpha(u), \beta(nu) = \beta(u)$, and $\theta(nu) = \theta(u)$, for all $n, u \in \mathcal{S}$.*

(ii) *When $\mathcal{J}$ is a neutrosophic prime ideal, then $\alpha(nu) = \alpha(0), \beta(nu) = \beta(1)$, and $\theta(nu) = \theta(1)$; this leads to either $\alpha(n) = \alpha(0), \beta(n) = \beta(1)$, and $\theta(n) = \theta(1)$ or $\alpha(u) = \alpha(1), \beta(u) = \beta(1)$ and $\theta(u) = \theta(1)$, for all $n, u \in \mathcal{S}$.*

Proof. (i) Assume that $\mathcal{J}$ is a neutrosophic prime ideal. Then Theorems 3.1 and 2.7 hold, according that when $nu \in \mathcal{J}_0$, then either $n \in \mathcal{J}_0$ or $u \in \mathcal{J}_0$ where $n, u \in \mathcal{S}$, this lead to either $\alpha(nu) = \alpha(n)$, $\beta(nu) = \beta(n)$ and $\theta(nu) = \theta(n)$ or $\alpha(nu) = \alpha(u), \beta(nu) = \beta(u)$ and $\theta(nu) = \theta(u)$. When $nu \notin \mathcal{J}_0$, then

$$\alpha(nu) = \rho \geq \alpha(n) \wedge \alpha(u),$$
$$\beta(nu) = \rho' \leq \beta(n) \vee \beta(u),$$
$$\theta(nu) = \rho' \leq \theta(n) \vee \theta(u).$$

If

$$\alpha(n) = 1 \text{ and } \alpha(u) = 1 \Rightarrow \rho \geqslant 1,$$
$$\beta(n) = 0 \text{ and } \beta(u) = 0 \Rightarrow \rho' \leqslant 0,$$
$$\theta(n) = 0 \text{ and } \theta(u) = 0 \Rightarrow \rho' \leqslant 0,$$

which gives a contradiction. Thus, either $\alpha(n) = \rho, \beta(n) = \rho', \theta(n) = \rho'$ or $\alpha(u) = \rho, \beta(u) = \rho', \theta(u) = \rho'$. Therefore, the proof is establishes.

(ii) This is a direct proof with no complications. □

Remark 3.10. The converse of Proposition 3.9 is not true in general. This can be illustrated by the following examples.

Example 3.11. Consider $\mathcal{J}$ is a neutrosophic ideal over $\mathbb{Z}$ defined as follows:

$$\alpha(n) = \begin{cases} 0.7 & \text{if } n = 0 \\ 0.6 & \text{if } n \in < 5 > -\{0\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta(n) = \begin{cases} 0.3 & \text{if } n = 1 \\ 0.6 & \text{if } n \in < 5 > -\{0\} \\ 1 & \text{otherwise,} \end{cases}$$

$$\theta(n) = \begin{cases} 0.4 & \text{if } n = 1 \\ 0.6 & \text{if } n \in < 5 > -\{0\} \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that $\alpha(nu) = \alpha(0), \beta(nu) = \beta(1)$, and $\theta(nu) = \theta(1)$ this leads to either $\alpha(n) = \alpha(0), \beta(n) = \beta(1)$, and $\theta(n) = \theta(1)$ or $\alpha(u) = \alpha(1)$, $\beta(u) = \beta(1)$ and $\theta(u) = \theta(1)$, for all $n, u \in \mathbb{Z}$, but $\mathcal{J}$ is not a neutrosophic prime ideal for $\alpha(0) \neq 1, \beta(1) \neq 0$ and $\theta(1) \neq 0$.

4. Application

In this section, we attempt to build a novel decision-making method by using the concept of neutrosophic ideal. Also, a neutrosophic prime ideal represents a kind of filter that helps preserve logical consistency.

4.1. Problem statement

Here, a basic description of a system has multiple sources. Each gives a reading with neutrosophic evaluation information.

In realistic machine overheating detection problems, there exist some kinds of sensors, say (A, B, C) monitoring a machine to discover (overheating, malfunctioning, not overheating).

Suppose that $S = \{A, B, C, ..., D\}$ is a set of n sensors and can represent this set by the classical ring Z_n. The A sensor has three readings α_A, β_A, and θ_A, which means overheating, malfunctioning, and not overheating, respectively. Now, we can represent the data by using, neutrosophic set as follows: $\mathcal{M} = \{\langle i, \alpha_i, \beta_i, \theta_i \rangle : i \in Z_n\}$.

4.2. Decision-making method

Based on the outlined problem, we propose an approach utilizing the framework of neutrosophic prime ideals. The importance of a neutrosophic prime ideal guarantees that if the combination of any two sensors leads to a decision, at least one must be in the ideal.

The suggested approach involves these fundamental steps. Now, we show an algorithm for neutrosophic data fusion in sensor systems

First step: After representing data by using a neutrosophic set, we check if this system represents a neutrosophic prime ideal.

Second step: Define thresholds Υ_α, Υ_β, and Υ_θ with $\Upsilon_\alpha \leqslant \alpha_i$, which means high confidence in overheating; $\Upsilon_\beta \geqslant \beta_i$, which means low malfunctioning, and $\Upsilon_\theta \geqslant \theta_i$, which means low confidence in not overheating. Therefore, we can make decision according to this result.

In the next subsection, we will present an algorithm for the suggested decision-making approach.

4.3. Algorithm for the method

Inputs: Sensor readings represented as neutrosophic triplets say (sensor A: $(\alpha_A, \beta_A, \theta_A)$).

Output: Data representation by neutrosophic set.

Step 1. Check the set of neutrosophic is neutrosophic prime ideal.

Step 2. Apply neutrosophic prime ideal filtering (thresholds).

Step 3. Decision-making.

Figure 2 illustrates the schematic diagram of the developed algorithm applied to machine sensor data.

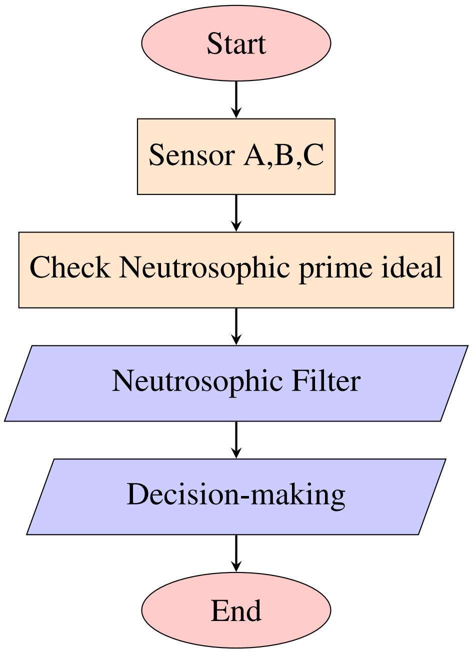

Figure 2. Flow chart of the proposed decision-making method.

Example 4.1. Consider the three sensors A, B, and C monitoring a machine. Each gives a reading with:

α: Says a machine is overheating.

β: Says a machine is malfunctioning.

θ: Says a machine is not overheating.

The readings are given in Table 1.

Table 1. Data of sensors monitoring a machine.

Sensors	α	β	θ
A	0.8	0.3	0.1
B	0.3	0.5	0.6
C	0.4	0.7	0.9

Form the above result, we can contact it by Z_2, where $A := 0$, $B := 1$, and $C := 2$. It can easily show that the neutrosophic set $\mathcal{M} = \{\langle 0, 0.8, 0.3, 0.1\rangle, \langle 1, 0.3, 0.5, 0.6\rangle, \langle 2, 0.4, 0.7, 0.9\rangle\}$ is a neutrosophic prime ideal. Now, we apply neutrosophic prime ideal filter by choosing thresholds $\langle \Upsilon_\alpha, \Upsilon_\beta, \Upsilon_\theta\rangle = \langle 0.6, 0.4, 0.3\rangle$. In Table 2, we apply the rule:

Table 2. Data of neutrosophic prime ideal filtering.

Sensors	$\alpha \geq 0.6$	$\beta \leq 0.4$	$\theta \leq 0.3$
A	✓	✓	✓
B	X	X	X
C	X	X	X

According to the result in Table 2, the decision-making is based only on A, which is partially reliable.

In the following example, we demonstrate a method based on the structure of prime ideals.

Example 4.2. Consider the three sensors A, B, and C monitoring a machine. Each gives a reading of a heating according to Table 3.

Table 3. Data of sensors monitoring a machine.

Sensors	A	B	C
Temperature Sensor (T)	92° C	85° C	95° C

Form the above result we can contact it by Z_2 which $A := 0$, $B := 1$, and $C := 2$. Now, we consider the ideal filter by choosing the thresholds $T = 90°$C. In Table 4 we apply the rule in Table 4.

Table 4. Data of sensors monitoring a machine.

Sensors	$T_A > 90°$C	$T_B > 90°$C	$T_C > 90°$C
Case of machine	Overload	No Alarm	Overload

Remark 4.3. Based on the preceding examples, we present a comparison between the method grounded in neutrosophic prime ideals and the classical ideal-based approach in Table 5.

Table 5. Comparison between the method grounded in neutrosophic prime ideals and the classical ideal-based approach.

Decision	Prime ideal	Neutrosophic prime ideal
Pros	Simple, fast decisions	Handles uncertainty, reduces false alarms, provides gradual warnings
Cons	Fails in ambiguous cases	More complex computations required.
Handles noisy data	Limited	Unlimited
Real-time graded decision	No	Yes

5. Conclusions

In this paper, we investigated the basic properties of neutrosophic prime ideals by highlighting their particular behavior and how they extend classical prime ideals within algebraic structures. An important result of this research is the identification of the structure of their image, which consistently consists of two specific elements. Moreover, we have introduced a new perspective by interpreting the prime ideals through neutrosophic points, thus extending the theoretical framework. Furthermore, we have demonstrated the usefulness of neutrosophic ideals by applying them in decision-making contexts.

Future work may aim to extend the scope of this research by investigating neutrosophic prime ideals in more complex algebraic systems, such as non-commutative rings or modules. Furthermore, establishing deeper connections between the different definitions of neutrosophic prime ideals could lead to a more unified theory. Potential applications in logic, fuzzy systems, and computer algebra can also be explored, offering new avenues for theoretical and practical progress in this field.

Author contributions

Ali Yahya Hummdi: Writing-review and editing; Amr Elrawy: Conceptualization, methodology, writing-original draft, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

This work has no conflict of interest.

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