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Research article

Non-associative and non-commutative Poisson structures on Filiform Lie algebras

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Abstract: In this paper, we study the non-associative and non-commutative Poisson structures on filiform Lie algebras L_n and Q_{2m} and then characterize the commutative and associative Poisson structures on L_n and Q_{2m}. Besides, we discuss the relationships between some non-associative algebras.

Keywords: Poisson algebra; filiform Lie algebra; commutative algebra; associative algebra
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1. Introduction

Poisson algebras came from the study of symplectic manifolds [1, 2]. They are widely used in the study of mathematics and physics, such as algebraic geometry, Poisson quantization theory, manifolds, noncommutative geometry, quantum groups, and classical and quantum mechanics, and so on [3,4]. M. Van den Bergh introduced the notion of double Poisson brackets, which can induce Poisson brackets on representation spaces [5]. S. Arthamonov extended these brackets to the modified double Poisson brackets [6]. Furthermore, M. Fairon proposed the mixed modified double brackets [7]. Bai, Bai, Guo, and Wu exchanged the roles of the two bilinear operations in the Leibniz rule defining the Poisson algebra and proposed the definition of transposed Poisson algebras, which share common properties of the Poisson algebras [4]. Sartayev proved that every transposed Poisson algebra is an F-manifold, and proposed a conjecture stating that every transposed Poisson algebra can be embedded into a differential Poisson algebra [8]. Conventional Poisson algebra (in [4]) is a Lie algebra endowed with a commutative and associative multiplication, which, together with the Lie product, satisfies the Leibniz identity. But there are also some literatures in which the multiplication in Poisson algebra is not necessarily commutative or associative. For example, in [9–11], the multiplication in Poisson algebra is not necessarily associative. In [12–15], the multiplication in Poisson algebra is not necessarily commutative. Motivated by these papers, we want to investigate a type of Poisson algebra in which the multiplication is not necessarily commutative and not necessarily associative. In this process, we introduce the definition of anti-semi-commutative algebras, discuss the relationships between some non-associative algebras, and provide some examples of special algebras, such as non-associative and non-commutative Poisson algebras, Novikov algebras, and transposed Poisson algebras. It is important to describe the Poisson structure on a given Lie algebra. Considering the complexity of computation brought by using matrix methods, we choose filiform Lie algebras L_n and Q_{2m} with relatively simple Lie products. Filiform Lie algebras were defined by Vergne [16] in 1970 as nilpotent Lie algebras with good properties. It is shown in [16] that there exist only the following two isomorphism classes possessing rank two, and they are the only two isomorphic classes that are naturally graded nilpotent Lie algebras.

(1) $L_n(n \geqslant 3)$:

$$[x_0, x_i] = x_{i+1},\ 1 \leqslant i \leqslant n-1,$$

over the basis $\{x_0, x_1, \cdots, x_n\}$.

(2) $Q_{2m}(m \geqslant 3)$:

$$[x_1, x_i] = x_{i+1},\ 2 \leqslant i \leqslant 2m-1,$$

$$[x_j, x_{2m+1-j}] = (-1)^{j+1} x_{2m},\ 2 \leqslant j \leqslant 2m-1,$$

over the basis $\{x_1, x_2, \cdots, x_{2m}\}$.

The structure of this paper is as follows: In Section 2, we first give some notions and examples, discuss the relationships between some non-associative algebras, and then provide some lemmas that will be used later; In Section 3, we first investigate the Poisson structures on L_n, and then we characterize the commutative Poisson structures and associative Poisson structures on L_n, respectively; In Section 4, we perform the same research as the third section on Q_{2m}.

2. Basics

Throughout the paper, we assume that $3 \leqslant n, m$. $\mathbb{F}$ stands for a field whose characteristic is not 2, $\mathbb{F}^*$ is the set of all nonzero elements of $\mathbb{F}$, and $\mathbb{F}^k$ is the k-dimensional column vector space over $\mathbb{F}$. All vector spaces and all algebras mentioned in this paper are over $\mathbb{F}$. In an algebra $(G, \cdot)$, we sometimes simplify $x \cdot y$ as xy. All transformations mentioned in this paper are linear. The matrices of transformations of L_n are all with respect to the basis $\{x_0, x_1, \cdots, x_n\}$, and the matrices of transformations of Q_{2m} are all with respect to the basis $\{x_1, x_2, \cdots, x_{2m}\}$. I, E_{ij} and e_i represent the unit matrix, matrix unit, and unit vector, respectively. Their sizes are known from the context. For any Lie algebra L and $x \in L$, denote inner derivation $\mathrm{ad}_x(y) = [x, y], \forall y \in L$. For any positive integer k and $\alpha = (a_1, a_2, \cdots, a_k)^t \in \mathbb{F}^k$, we denote $k \times k$ matrices

$$J = \sum_{2 \leqslant i \leqslant k} E_{i,i-1},\ A_\alpha = a_1 I + a_2 J + a_3 J^2 + \cdots + a_k J^{k-1}.$$

Next, we give the definitions firstly.

Definition 2.1. *Let L be a vector space equipped with two bilinear operations "$\cdot$" and "$[\cdot,\cdot]$", the triple $(L, \cdot, [\cdot,\cdot])$ is called a Poisson algebra if $(L, [\cdot,\cdot])$ is a Lie algebra that satisfies the following identity:*

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z], \quad x, y, z \in L. \tag{2.1}$$

Definition 2.2. *Let $(L, \cdot, [\cdot, \cdot])$ be a Poisson algebra, where $(L, [\cdot, \cdot])$ is a Lie algebra. If $(L, \cdot)$ is an associative algebra, then we call $(L, \cdot, [\cdot, \cdot])$ is associative.*

Definition 2.3. *Let $(L, \cdot, [\cdot, \cdot])$ be a Poisson algebra, where $(L, [\cdot, \cdot])$ is a Lie algebra. If $(L, \cdot)$ is a commutative algebra, then we call $(L, \cdot, [\cdot, \cdot])$ is commutative.*

Definition 2.4. *Let $(L, [\cdot, \cdot])$ be a Lie algebra. A (resp. commutative or associative) Poisson structure on $(L, [\cdot, \cdot])$ is a bilinear operation " $\cdot$ " on L which makes $(L, \cdot, [\cdot, \cdot])$ a (commutative or associative) Poisson algebra.*

Obviously, if $(L, [\cdot, \cdot])$ is an abelian Lie algebra, then for any bilinear operation "$\cdot$" on L, $(L, \cdot, [\cdot, \cdot])$ a Poisson algebra. If an bilinear operation "$\cdot$" on a vector space V is zero, then for any Lie bracket "$[\cdot, \cdot]$" on V, $(V, \cdot, [\cdot, \cdot])$ is a Poisson algebra.

For any Lie algebra $(L, [\cdot, \cdot])$ and $a \in \mathbb{F}$, define $x * y = a[x, y]$ for all $x, y \in L$, then $(L, *, [\cdot, \cdot])$ is a Poisson algebra. If $(L, \cdot, [\cdot, \cdot])$ is a Poisson algebra, define $x * y = yx$ for all $x, y \in L$, then $(L, *, [\cdot, \cdot])$ is a Poisson algebra. If both $(L, \cdot, [\cdot, \cdot])$ and $(L, \circ, [\cdot, \cdot])$ are Poisson algebras, for any $a, b, c \in F$, define $\{x, y\} = a[x, y], \ x * y = bx \cdot y + cx \circ y$ for all $x, y \in L$, then $(L, *, \{\cdot, \cdot\})$ is a Poisson algebra.

For any associative algebra $(G, \cdot)$, define $[x, y] = xy - yx$ for all $x, y \in G$, then $(G, \cdot, [\cdot, \cdot])$ is an associative Poisson algebra. For any weakly associative algebra, we also have a similar conclusion.

Let $(G, \cdot)$ be an algebra. Denote the associator $(x, y, z) = (xy)z - x(yz)$ and the commutator $[x, y] = xy - yx$. Recall that $(G, \cdot)$ is called weakly associative if $(x, y, z) + (y, z, x) = (y, x, z)$; $(G, \cdot)$ is called a symmetric Leibniz algebra if $(xy)z = (xz)y + x(yz), \ x(yz) = (xy)z + y(xz)$; $(G, \cdot)$ is called shift associative (or nearly associative) if $(xy)z = y(zx)$; $(G, \cdot)$ is called semi-commutative if $(xy)z = (yx)z, \ z(yx) = z(xy)$; $(G, \cdot)$ is called a bicommutative algebra (or LR algebra) if $(xy)z = (xz)y, \ x(yz) = y(xz)$ ([17,18]).

Obviously, both commutative algebras and associative algebras are weakly associative.

Example 2.1. *Let $(G, \cdot)$ be an algebra. Define $[x, y] = xy - yx$. If $(G, \cdot)$ is weakly associative or semi-commutative, then $(G, [\cdot, \cdot])$ is a Lie algebra.*

In fact, if $(G, \cdot)$ is weakly associative, then,

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
=&x(yz) - x(zy) - (yz)x + (zy)x + y(zx) - y(xz) - (zx)y + (xz)y + z(xy) - z(yx) - (xy)z + (yx)z \\
=&[-(xy)z + x(yz) - (yz)x + y(zx) + (yx)z - y(xz)] + [(xz)y - x(zy) + (zy)x - z(yx) - (zx)y + z(xy)] \\
=&-(x, y, z) - (y, z, x) + (y, x, z) + (x, z, y) + (z, y, x) - (z, x, y) = 0.
\end{aligned}$$

Otherwise, if $(G, \cdot)$ is semi-commutative, then,

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
=&x(yz) - x(zy) - (yz)x + (zy)x + y(zx) - y(xz) - (zx)y + (xz)y + z(xy) - z(yx) - (xy)z + (yx)z \\
=&0.
\end{aligned}$$

Example 2.2. *Let $(G, \cdot)$ be an algebra. Define $[x, y] = xy - yx$. Since*

$$[x, yz] - [x, y]z + y[x, z] = x(yz) - (yz)x - (xy)z + (yx)z - y(xz) + y(zx) = -(x, y, z) - (y, z, x) + (y, x, z),$$

it can be inferred from Example 2.1 that $(G, \cdot, [\cdot, \cdot])$ is a Poisson algebra if and only if $(G, \cdot)$ is weakly associative.

Example 2.3. *Let $(G,\cdot)$ be a symmetric Leibniz algebra. Denote $[x,y] = x\cdot y - y\cdot x$. Then direct verification shows that $(G,\cdot,[\cdot,\cdot])$ is a Poisson algebra.*

Example 2.4. *Let $(G,\cdot)$ be a shift associative and bicommutative algebra. Then for all $x,y,z\in G$, we have*

$$\begin{aligned}&(x,y,z)+(y,z,x)-(y,x,z)\\=&(xy)z-x(yz)+(yz)x-y(zx)-(yx)z+y(xz)\\=&y(zx)-y(xz)+(yx)z-y(zx)-(yx)z+y(xz)\\=&0.\end{aligned}$$

Thus, $(G,\cdot)$ is weakly associative. Hence, $(G,\cdot,[\cdot,\cdot])$ is a Poisson algebra.

Similar to anti-commutative algebras, we give the definition of anti-semi-commutative algebras.

Definition 2.5. *Let $(G,\cdot)$ be an algebra. If $(x\cdot y)\cdot z = -(y\cdot x)\cdot z$ and $z\cdot(y\cdot x) = -z\cdot(x\cdot y)$ for all $x,y,z\in G$, then $(G,\cdot)$ is called anti-semi-commutative.*

Example 2.5. *Let $(G,\cdot)$ be a shift associative algebra. Denote $[x,y] = xy - yx$. If $(G,\cdot)$ is semi-commutative or anti-semi-commutative, then*

$$(xy)z = \delta(yx)z = \delta x(zy) = \delta^2 x(yz) = x(yz),\ \forall x,y,z\in G,$$

where $\delta = \pm1$. Thus, $(G,\cdot,[\cdot,\cdot])$ is an associative Poisson algebra.

Recall that an algebra $(G,\cdot)$ is called Novikov if $(x,y,z) = (y,x,z)$ and $(x\cdot y)\cdot z = (x\cdot z)\cdot y$ for all $x,y,z\in G$; The triple $(L,\cdot,[\cdot,\cdot])$ is called a $(L,*,[\cdot,\cdot])$ if $(L,*)$ is a commutative associative algebra $(L,[\cdot,\cdot])$ is a Lie algebra that satisfies $2x\cdot[y,z] = [x\cdot y,z]+[y,x\cdot z]$ for all $x,y,z\in L$ [4].

Example 2.6. *Let $(G,\cdot)$ be an anti-semi-commutative algebra. Define $x\circ y = xy+yx$. Then we have the following statements:*

(1) $(G,\circ)$ is a commutative associative algebra;

(2) $(G,\circ)$ is a Nivikov algebra.

Proof. Obviously

$$x(y\circ z) = x(yz+zy) = x(yz)+x(zy) = 0.$$

Similarly, $(x\circ y)z = 0$. Thus,

$$x\circ(y\circ z) = x(y\circ z)+(y\circ z)x = 0,$$

$$(x\circ y)\circ z = (x\circ y)z + z(x\circ y) = 0,$$

and therefore

$$(x,y,z) = (x\circ y)\circ z - x\circ(y\circ z) = 0.$$

Hence, both (1) and (2) hold. □

Example 2.7. *Let $(L,\cdot,[\cdot,\cdot])$ be a Poisson algebra. Define $x\circ y = xy+yx$ and $x*y = xy-yx$. Then we have the following statements.*

(1) If $x[y,z] = [x,y]z$, then $2x*[y,z] = [x*y,z]+[y,x*z],\ x,y,z\in L$;

(2) If $x[y,z] = -[x,y]z$, then $2x\circ[y,z] = [x\circ y,z]+[y,x\circ z],\ x,y,z\in L$.

Proof. (1)

$$\begin{aligned}&[x*y,z]+[y,x*z]\\=&[xy-yx,z]+[y,xz-zx]\\=&x[y,z]+[x,z]y-[y,z]x-y[x,z]+[y,x]z+x[y,z]-[y,z]x-z[y,x]\\=&2(x[y,z]-[y,z]x)+(-[z,x]y+z[x,y])+([y,x]z-y[x,z])\\=&2x*[y,z].\end{aligned}$$

(2)

$$\begin{aligned}&[x\circ y,z]+[y,x\circ z]\\=&[xy+yx,z]+[y,xz+zx]\\=&x[y,z]+[x,z]y+[y,z]x+y[x,z]+[y,x]z+x[y,z]+[y,z]x+z[y,x]\\=&2(x[y,z]+[y,z]x)+(-[z,x]y-z[x,y])+([y,x]z+y[x,z])\\=&2x\circ[y,z].\end{aligned}$$

□

From the above two examples, we have the following example.

Example 2.8. *Let $(L,\cdot,[\cdot,\cdot])$ be a Poisson algebra. Define $x\circ y=xy+yx$. Assume $(L,\cdot)$ is anti-semi-commutative, then we have the following statements.*
(1) If $x[y,z]=[x,y]z$, then $(L,\circ,[\cdot,\cdot])$ is a commutative associative Poisson algebra;
(2) If $x[y,z]=-[x,y]z$, then $(L,\circ,[\cdot,\cdot])$ is a transposed Poisson algebra.

The ideas of the following two examples come from references [4] and [19], respectively.

Example 2.9. *If 2-dimensional Lie algebra $(L,[\cdot,\cdot])$ is non-abelian, then there exists a basis $\{e_1,e_2\}$ such that $[e_1,e_2]=e_2$. A straightforward computation shows that a bilinear operation "$\cdot$" on L satisfies (2.1) if and only if there exist $a,b\in\mathbb{F}$ such that*

$$e_1\cdot e_1=(a+b)e_1,\ e_1\cdot e_2=ae_2,\ e_2\cdot e_1=be_2,\ e_2\cdot e_2=0.$$

Furthermore, these Poisson algebras are mutually non-isomorphic. It is easy to know that the Poisson algebra $(L,\cdot,[\cdot,\cdot])$ is associative if and only if $ab=0$, and the Poisson algebra $(L,\cdot,[\cdot,\cdot])$ is commutative if and only if $a=b$. Therefore, any commutative and associative Poisson structure on 2-dimensional non-abelian Lie algebra is trivial.

Let $(A,\cdot)$ be an algebra. Recall that the centroid of A is the linear space $\Gamma(A)$ of linear maps $\varphi:A\to A$ such that

$$x\cdot\varphi(y)=\varphi(x\cdot y)=\varphi(x)\cdot y,\ x,y\in A.$$

Example 2.10. *(1) Let $(L,[\cdot,\cdot])$ be a Lie algebra. Then $\Gamma(L)$ is the linear space of linear maps $\varphi:L\to L$ such that*

$$[x,\varphi(y)]=\varphi([x,y])=[\varphi(x),y],\ x,y\in L.$$

For any $\varphi\in\Gamma(L)$, define

$$x\cdot y=[x,\varphi(y)],\ x,y\in L.$$

Then, for all $x, y, z \in L$ *we have*

$$[x, y\cdot z]=[x,[y,\varphi(z)]]=[[x,y],\varphi(z)]+[y,[x,\varphi(z)]]=[[x,y],\varphi(z)]+[y,\varphi([x,z])]=[x,y]\cdot z+y\cdot[x,z].$$

Thus, $(L, \cdot, [\cdot,\cdot])$ *is a Poisson algebra.*

(2) Suppose $(L, \cdot)$ *is an associative algebra. For any* $\varphi \in \Gamma(L)$*, define*

$$x * y = x\cdot\varphi(y)\ , [x, y] = x\cdot y - y\cdot x,\ \ x, y \in L.$$

Then "$*$" is a Poisson structure on Lie algebra $(L, [\cdot,\cdot])$*. Moreover, by Lemma 2.3 in [19],* $(L, *, [\cdot,\cdot])$ *is an associative Poisson algebra.*

The following three lemmas, which will be used later, are easy to obtain.

Lemma 2.1. *The* $(k+1)\times(k+1)$ *matrix that is commutative with* $\sum\limits_{3\leqslant i\leqslant k+1} E_{i,i-1}$ *is of the form*

$$\begin{pmatrix} a & be_1^t \\ de_k & A_\alpha \end{pmatrix}, \tag{2.2}$$

where $a, b, d \in \mathbb{F}$ *and* $\alpha \in \mathbb{F}^k$*.*

Lemma 2.2. *Suppose* $(L, [\cdot,\cdot])$ *is a Lie algebra equipped with a bilinear operation "$\cdot$",* $L_z : y \mapsto z\cdot y, \forall y, z \in L$*. Then* $(L, \cdot, [\cdot,\cdot])$ *is a Poisson algebra if and only if*

$$\mathrm{ad}_x L_y - L_y \mathrm{ad}_x = L_{[x,y]},\quad x, y \in L. \tag{2.3}$$

Lemma 2.3. *Suppose "$\cdot$" is a Poisson structure on Lie algebra* $(L, [\cdot,\cdot])$*,* $L_z : y \mapsto z\cdot y$ *is the left multiplication. Then*

(1) $(L, \cdot, [\cdot,\cdot])$ *is commutative if and only if* $L_x(y) = L_y(x),\ \forall x, y \in L$*;*

(2) $(L, \cdot, [\cdot,\cdot])$ *is associative if and only if* $L_{x\cdot y} = L_x L_y,\ \forall x, y \in L$*.*

3. Poisson structures on L_n

In this section, we will discuss the (commutative or associative) Poisson structures on the filiform Lie algebra L_n. First, we give a result about the inner derivations of L_n.

Lemma 3.1. *The matrix of the inner derivation* ad_{x_i} *of the Lie algebra* L_n *is*

$$B^{(i)} = \begin{cases} \sum\limits_{2\leqslant k\leqslant n} E_{k+1,k}, & i = 0, \\ -E_{i+2,1}, & 0 < i < n, \\ O, & i = n. \end{cases} \tag{3.1}$$

Next, we use the result to investigate the Poisson structures on L_n.

Theorem 3.1. *Suppose the filiform Lie algebra* $(L_n, [\cdot,\cdot])$ *is equipped with a bilinear operation "$\cdot$". Then* $(L_n, \cdot, [\cdot,\cdot])$ *is a Poisson algebra if and only if*

$$\begin{cases} x_0 \cdot x_0 = ax_0 + dx_n, \\ x_0 \cdot x_1 = bx_0 + \sum\limits_{1\leqslant k\leqslant n} b_k x_k, \\ x_0 \cdot x_i = \sum\limits_{i\leqslant k\leqslant n} b_{k-i+1} x_k,\ 2 \leqslant i \leqslant n, \\ x_1 \cdot x_0 = cx_0 + ax_1 + px_n - \sum\limits_{1\leqslant k\leqslant n-1} b_k x_k, \\ x_1 \cdot x_1 = (c+b)x_1 + qx_n, \\ x_1 \cdot x_i = cx_i,\ 2 \leqslant i \leqslant n, \\ x_i \cdot x_0 = ax_i - \sum\limits_{i\leqslant k\leqslant n} b_{k-i+1} x_k,\ 2 \leqslant i \leqslant n, \\ x_i \cdot x_1 = bx_i,\ 2 \leqslant i \leqslant n, \end{cases} \tag{3.2}$$

where $a, b, c, d, p, q, b_1, \cdots, b_n \in \mathbb{F}$ and all vanished products $x_i \cdot x_j$ are zero.

Proof. For any $z \in L_n$, let $L_z : y \mapsto z \cdot y, \forall y \in L_n$. Denote the matrix of L_{x_i} by $A^{(i)} = (A_0^{(i)}, A_1^{(i)}, \cdots, A_n^{(i)})$, $0 \leqslant i \leqslant n$. Then, by Lemmas 2.2 and 3.1, $(L_n, \cdot, [\cdot,\cdot])$ is a Poisson algebra if and only if the following identities hold:

$$A^{(0)}B^{(i)} - B^{(i)}A^{(0)} = A^{(i+1)},\quad 1 \leqslant i \leqslant n-1, \tag{3.3}$$

$$B^{(0)}A^{(i)} - A^{(i)}B^{(0)} = A^{(i+1)},\quad 1 \leqslant i \leqslant n-1, \tag{3.4}$$

$$B^{(i)}A^{(j)} = A^{(j)}B^{(i)},\quad 1 \leqslant i, j \leqslant n, \tag{3.5}$$

$$B^{(0)}A^{(n)} = A^{(n)}B^{(0)}, \tag{3.6}$$

$$B^{(0)}A^{(n)} = A^{(n)}B^{(0)}. \tag{3.7}$$

If $(L_n, \cdot, [\cdot,\cdot])$ is a Poisson algebra, by (3.6) and Lemma 2.1 we may assume

$$A^{(0)} = \begin{pmatrix} a & be_1^t \\ de_n & A_\alpha \end{pmatrix}, \tag{3.8}$$

where $a, b, d, \in \mathbb{F},\ \alpha = (b_1, \cdots, b_n) \in \mathbb{F}^n$. Then by (3.3) we have

$$A^{(i)} = aE_{i+1,1} + bE_{i+1,2} - \sum_{i\leqslant k\leqslant n} b_{k-1}E_{k+1,1},\quad 2 \leqslant i \leqslant n. \tag{3.9}$$

Suppose

$$A^{(1)} = \begin{pmatrix} a_{00} & a_{10} & a_{20} & \cdots & a_{n0} \\ a_{01} & a_{11} & a_{21} & \cdots & a_{n1} \\ a_{02} & a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{0n} & a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}.$$

Then taking $i = 1$ in (3.4), by $A^{(2)}$ in (3.9) we obtain

$$A^{(1)} = \begin{pmatrix} a_{00} & a_{10}e_1^t \\ ae_1 + (a_{0n} + b_n)e_n - \alpha & A_\beta - D \end{pmatrix}, \tag{3.10}$$

where

$$\beta = (a_{11}, a_{12}, \cdots, a_{1n})^t,\ D = \mathrm{diag}(0, b, b, \cdots, b).$$

Taking $i = j = 1$ in (3.5), by (3.1) and (3.10) we obtain

$$a_{10} = 0,\ a_{11} = a_{00} + b,\ a_{12} = a_{13} = \cdots = a_{1,n-1} = 0. \tag{3.11}$$

Denote $a_{00} = c, a_{0n} = p, a_{1n} = q$. Then by (3.10) and (3.11) we have

$$A^{(1)} = \begin{pmatrix} c & \\ ae_1 + (p + b_n)e_n - \alpha & cI + bE_{11} + qE_{n1} \end{pmatrix}. \tag{3.12}$$

Conversely, if we have (3.8), (3.9), and (3.12), then it is easy to verify that (3.3)–(3.7) hold. Thus, $(L_n, \cdot, [\cdot, \cdot])$ is a Poisson algebra. □

Now let us discuss the case that the Poisson algebra structures on $(L_n, [\cdot, \cdot])$ is commutative (resp. associative).

Corollary 3.1. *Suppose the filiform Lie algebra $(L_n, [\cdot, \cdot])$ is equipped with a bilinear operation "$\cdot$". Then $(L_n, \cdot, [\cdot, \cdot])$ is a commutative Poisson algebra if and only if*

$$\begin{cases} x_0 \cdot x_0 = 2ax_0 + dx_n, \\ x_0 \cdot x_1 = x_1 \cdot x_0 = bx_0 + ax_1 + px_n, \\ x_0 \cdot x_i = x_i \cdot x_0 = ax_i,\ 2 \leqslant i \leqslant n, \\ x_1 \cdot x_1 = 2bx_1 + qx_n, \\ x_1 \cdot x_i = x_i \cdot x_1 = bx_i,\ 2 \leqslant i \leqslant n, \end{cases} \tag{3.13}$$

where $a, b, d, p, q \in \mathbb{F}$ and all vanished products $x_i \circ x_j$ are zero.

Proof. We still use the symbols in the proof of Theorem 3.1.

If $(L_n, \cdot, [\cdot, \cdot])$ is a commutative Poisson algebra, then by Theorem 3.1 we have (3.8), (3.9), and (3.12). Using (1) of Lemma 2.3, we have $A_0^{(1)} = A_1^{(0)}$. Thus, by (3.2) we obtain

$$c = b,\ a = 2b_1,\ b_2 = \cdots = b_{n-1} = 0,\ b_n = p.$$

Hence, by (3.8), (3.9), and (3.12), we have

$$A^{(0)} = \begin{pmatrix} 2b_1 & b & & & & & \\ 0 & b_1 & & & & & \\ 0 & 0 & b_1 & & & & \\ 0 & 0 & 0 & b_1 & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & \vdots & \cdots & \ddots & b_1 & \\ d & p & 0 & \cdots & \cdots & 0 & b_1 \end{pmatrix},\ A^{(1)} = \begin{pmatrix} b & & & & & & \\ b_1 & 2b & & & & & \\ 0 & 0 & b & & & & \\ 0 & 0 & 0 & b & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & \vdots & \cdots & \ddots & b & \\ p & q & 0 & \cdots & \cdots & 0 & b \end{pmatrix}, \tag{3.14}$$

$$A^{(2)}=\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ b_1 & b & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & \cdots & \end{pmatrix}, \ \cdots, \ A^{(n)}=\begin{pmatrix} 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \\ b_1 & b & \cdots & \cdots & \end{pmatrix}, \tag{3.15}$$

where the last $n-1$ columns of $A^{(i)}$ are all 0, $2 \leqslant i \leqslant n$.

Conversely, if all $A^{(i)}$ are of the form (3.14) and (3.15), then it is easy to see that $A_i^{(j)} = A_j^{(i)}$, $0 \leqslant i, j \leqslant n$. By the bilinearity of " $\cdot$ ", " $\cdot$ " satisfies the commutative law. Note that Theorem 3.1, $(L_n, \cdot, [\cdot,\cdot])$ is a commutative Poisson algebra. □

Corollary 3.2. *Suppose the filiform Lie algebra $(L_n, [\cdot,\cdot])$ is equipped with a bilinear operation* " $\cdot$ ". *Then $(L_n, \cdot, [\cdot,\cdot])$ is an associative Poisson algebra if and only if* " $\cdot$ " *is in one of the following five cases:*

(1) $x_0 \cdot x_i = ax_i,\ x_1 \cdot x_i = bx_i,\ 0 \leqslant i \leqslant n,\ a \in \mathbb{F}, b \in \mathbb{F}^*$;

(2) $x_i \cdot x_0 = ax_i,\ x_i \cdot x_1 = bx_i,\ 0 \leqslant i \leqslant n,\ a \in \mathbb{F}, b \in \mathbb{F}^*$;

(3) $x_0 \cdot x_i = ax_i,\ x_1 \cdot x_1 = bx_n,\ 0 \leqslant i \leqslant n,\ a \in \mathbb{F}^*, b \in \mathbb{F}$;

(4) $x_i \cdot x_0 = ax_i,\ x_1 \cdot x_1 = bx_n,\ 0 \leqslant i \leqslant n,\ a \in \mathbb{F}^*, b \in \mathbb{F}$;

(5) $x_0 \cdot x_0 = ax_n,\ x_0 \cdot x_i = \sum\limits_{i+1 \leqslant k \leqslant n} b_{k-i+1} x_k,\ 1 \leqslant i \leqslant n-1,$

$x_1 \cdot x_1 = bx_n,\ x_1 \cdot x_0 = cx_n - \sum\limits_{2 \leqslant k \leqslant n-1} b_i x_i,;$

$x_i \cdot x_0 = -\sum\limits_{i+1 \leqslant k \leqslant n} b_{k-i+1} x_k,\ a, b, c, b_2, b_3, \cdots, b_n \in \mathbb{F},\ 2 \leqslant i \leqslant n-1,$

where all vanished products $x_i \cdot x_j$ in the above five cases are zero.

Proof. We still use the symbols in the proof of Theorem 3.1.

If $(L_n, \cdot, [\cdot,\cdot])$ is an associative Poisson algebra, then by Theorem 3.1 we have (3.8), (3.9), and (3.12). Using (2) of Lemma 2.3 and the bilinearity of " $\cdot$ ", $(L_n, \cdot, [\cdot,\cdot])$ is associative if and only if

$$L_{x_i \cdot x_j} = L_{x_i} L_{x_j},\ 0 \leqslant i, j \leqslant n. \tag{3.16}$$

Taking $i = j = 1$ in (3.16), we have $(c+b)A^{(1)} + qA^{(n)} = A^{(1)}A^{(1)}$. Using (3.8), (3.9), and (3.12), we obtain

$$cb = 0, \tag{3.17}$$

$$ca = cb_1, \tag{3.18}$$

$$cb_i = bb_i = 0,\ 2 \leqslant i \leqslant n-1, \tag{3.19}$$

$$cp = cq = bp = bq = 0. \tag{3.20}$$

Similarly, taking $i = j = 0$ in (3.16), we have $aA^{(0)} + dA^{(n)} = A^{(0)}A^{(0)}$, and therefore we obtain

$$bb_1 = 0, \tag{3.21}$$

$$ab_1 = b_1^2, \tag{3.22}$$

$$(a - 2b_1)d = (a - 2b_1)b_n = 0. \tag{3.23}$$

Taking $i = 0$ and $j = 1$ in (3.16), we have

$$bd = cd = cb_n = bb_n = 0. \tag{3.24}$$

Case 1. $c \neq 0$.

By (3.17)–(3.20) and (3.24) we have

$$a = b_1,\ b = d = p = q = b_i = 0,\ 2 \leqslant i \leqslant n.$$

Thus,

$$A^{(0)} = aI,\ A^{(1)} = cI,\ A^{(i)} = O,\ 2 \leqslant i \leqslant n, \tag{3.25}$$

where $a \in \mathbb{F}$ and $c \in \mathbb{F}^*$.

Case 2. $b \neq 0$.

By (3.17), (3.19)–(3.21) and (3.24) we have

$$c = d = p = q = b_i = 0,\ 1 \leqslant i \leqslant n.$$

Thus,

$$A^{(i)} = aE_{i+1,1} + bE_{i+1,2},\ 0 \leqslant i \leqslant n, \tag{3.26}$$

where $a \in \mathbb{F}$ and $b \in \mathbb{F}^*$.

Case 3. $b = c = 0$.

Now, by (3.8), (3.9), and (3.12), we have

$$A^{(0)} = \begin{pmatrix} a & \\ de_n & A_\alpha \end{pmatrix},\ A^{(1)} = \begin{pmatrix} 0 & \\ ae_1 + (p + b_n)e_n - \alpha & qE_{n1} \end{pmatrix}, \tag{3.27}$$

$$A^{(i)} = aE_{i+1,1} - \sum_{i \leqslant k \leqslant n} b_{k-1}E_{k+1,1},\ \ 2 \leqslant i \leqslant n, \tag{3.28}$$

where $a, d, p, q \in \mathbb{F},\ \alpha = (b_1, b_2, \cdots, b_n)^t \in \mathbb{F}^n$ which satisfy (3.22) and (3.23). Taking $i = 1$ and $j = 0$ in (3.16), we have

$$(a - b_1)A^{(1)} - b_2A^{(2)} - b_3A^{(3)} - \cdots - b_{n-1}A^{(n-1)} + pA^{(n)} = A^{(1)}A^{(0)}.$$

Then using (3.27) and (3.28), we obtain

$$b_i = (a - 2b_1)b_j = (a - 2b_1)p = 0,\ 2 \leqslant i \leqslant s,\ \ s + 1 \leqslant j \leqslant n - 1, s = [\frac{n+1}{2}]. \tag{3.29}$$

Subcase 3.1. $b_1 \neq 0$.

Then by (3.22) we have $a = b_1$, and therefore $a - 2b_1 = -b_1 \neq 0$. Thus, it follows from (3.23) and (3.29) that $p = d = b_j = 0,\ \ s + 1 \leqslant j \leqslant n$. Hence,

$$A^{(0)} = aI,\ A^{(1)} = qE_{n+1,2},\ A^{(i)} = O,\ 1 \leqslant i \leqslant n, \tag{3.30}$$

where $q \in \mathbb{F}$ and $a \in \mathbb{F}^*$.

Subcase 3.2. $b_1 = 0$ and $a \neq 0$.

It follows from (3.23) and (3.29) that $p = d = b_j = 0,\ s+1 \leqslant j \leqslant n$. Hence,

$$A^{(0)} = aE_{11},\ A^{(1)} = aE_{21} + qE_{n+1,2},\ A^{(i)} = aE_{i+1,1},\ 2 \leqslant i \leqslant n, \tag{3.31}$$

where $q \in \mathbb{F}$ and $a \in \mathbb{F}^*$.

Subcase 3.3. $b_1 = a = 0$.

By (3.8), (3.9), and (3.12), we obtain

$$A^{(0)} = \begin{pmatrix} 0 & \\ de_n & A_\gamma \end{pmatrix},\ A^{(1)} = \begin{pmatrix} 0 & \\ pe_n - \gamma & qE_{n1} \end{pmatrix}, \tag{3.32}$$

$$A^{(i)} = \begin{pmatrix} 0 & \\ -J^{i-1}\gamma & O \end{pmatrix},\ 2 \leqslant i \leqslant n, \tag{3.33}$$

where $d, p, q \in \mathbb{F}$ and $r = (0, \cdots, 0, b_{s+1}, \cdots, b_n) \in \mathbb{F}^n$.

Conversely, if " · " is in one of five cases in Corollary 3.2, then it is easy to verify that (3.16) holds, and therefore $(L_n, \cdot, [\cdot,\cdot])$ is an associative Poisson algebra. □

Using Corollaries 3.1 and 3.2, the next result is obvious.

Corollary 3.3. *Suppose* " · " *is a bilinear multiplication in Lie algebra* $(L_n, [\cdot,\cdot])$. *Then* $(L_n, \cdot, [\cdot,\cdot])$ *is a commutative and associative Poisson algebra if and only if there exist* $a, b, c \in \mathbb{F}$ *such that*

$$\begin{cases} x_0 \cdot x_0 = ax_n, \\ x_0 \cdot x_1 = x_1 \cdot x_0 = bx_n, \\ x_1 \cdot x_1 = cx_n, \\ x_i \cdot x_j = 0,\ i > 1 \ or\ j > 1. \end{cases}$$

4. Poisson structures on Q_{2m}

Similarly, we also first give a result about the inner derivations of the filiform Lie algebra Q_{2m}. Next, we use the result to investigate the Poisson structures on Q_{2m}.

Lemma 4.1. *The matrix of the inner derivation* ad_{x_i} *of the Lie algebra* Q_{2m} *is*

$$C^{(i)} = \begin{cases} \sum\limits_{2\leqslant k\leqslant 2m-1} E_{k+1,k}, & i = 1, \\ -E_{i+1,1} + (-1)^{i+1}E_{2m,2m+1-i}, & 1 < i < 2m, \\ O, & i = 2m. \end{cases} \tag{4.1}$$

Theorem 4.1. *Suppose the filiform Lie algebra* $(Q_{2m}, [\cdot,\cdot])$ *is equipped with a bilinear operator* " · ". *Then* $(Q_{2m}, \cdot, [\cdot,\cdot])$ *is a Poisson algebra if and only if*

$$\begin{cases} x_1 \cdot x_1 = ax_1 + dx_{2m}, \\ x_1 \cdot x_2 = bx_1 + b_2x_2 + b_{2m-1}x_{2m-1} + b_{2m}x_{2m}, \\ x_1 \cdot x_3 = b_2x_3 + b_{2m-1}x_{2m}, \\ x_1 \cdot x_i = b_2x_i, \ 4 \leqslant i \leqslant 2m, \\ x_2 \cdot x_1 = cx_1 + (a - b_2)x_2 - b_{2m-1}x_{2m-1} + px_{2m}, \\ x_2 \cdot x_2 = (c + b)x_2 + qx_{2m}, \\ x_2 \cdot x_k = cx_k, \ 3 \leqslant k \leqslant 2m, \\ x_3 \cdot x_1 = (a - b_2)x_3 - b_{2m-1}x_{2m}, \\ x_3 \cdot x_2 = bx_3 + b_{2m-1}x_{2m}, \\ x_i \cdot x_1 = (a - b_2)x_i, \ 4 \leqslant i \leqslant 2m, \\ x_i \cdot x_2 = bx_i, \ 4 \leqslant i \leqslant 2m, \end{cases} \tag{4.2}$$

where $a, b, c, d, p, q, b_2, b_{2m-1}, b_{2m} \in \mathbb{F}$ and all vanished products $x_i \cdot x_j$ are zero.

Proof. For any $z \in Q_{2m}$, let $L_z : y \mapsto z \cdot y, \forall y \in Q_{2m}$. Denote the matrix of L_{x_i} by $A^{(i)} = (A_1^{(i)}, A_2^{(i)}, \cdots, A_{2m}^{(i)})$, $1 \leqslant i \leqslant 2m$. Then, by Lemmas 2.2 and 4.1, $(Q_{2m}, \cdot, [\cdot, \cdot])$ is a Poisson algebra if and only if the following identities hold

$$A^{(1)}C^{(i)} - C^{(i)}A^{(1)} = A^{(i+1)}, \ \ 2 \leqslant i \leqslant 2m - 1, \tag{4.3}$$

$$C^{(1)}A^{(i)} - A^{(i)}C^{(1)} = A^{(i+1)}, \ \ 2 \leqslant i \leqslant 2m - 1, \tag{4.4}$$

$$C^{(j)}A^{(2m+1-j)} - A^{(2m+1-j)}C^{(j)} = (-1)^{j+1}A^{(2m)}, \ \ 2 \leqslant j \leqslant 2m - 1, \tag{4.5}$$

$$C^{(i)}A^{(j)} = A^{(j)}C^{(i)}, \ \ 2 \leqslant i, j \leqslant 2m \ \ and \ \ i + j \neq 2m + 1, \tag{4.6}$$

$$C^{(1)}A^{(1)} = A^{(1)}C^{(1)}, \tag{4.7}$$

$$C^{(1)}A^{(2m)} = A^{(2m)}C^{(1)}. \tag{4.8}$$

By Lemma 2.1 and (4.7) we may assume

$$A^{(1)} = \begin{pmatrix} a & be_1^t \\ de_{2m-1} & A_\alpha \end{pmatrix}, \tag{4.9}$$

where $a, b, d, \in \mathbb{F}$, $\alpha = (b_2, \cdots, b_{2m}) \in \mathbb{F}^{2m-1}$. Then, using (4.3) and by Lemma 4.1, we get

$$A^{(i)} = aE_{i1} + bE_{i,2} - \sum_{i \leqslant k \leqslant 2m} b_{k-i+2}E_{k1} + (-1)^{i+1} \sum_{3 \leqslant k \leqslant 2m+2-i} b_k E_{2m,2m+4-i-k}, \ \ 3 \leqslant i \leqslant 2m - 1, \tag{4.10}$$

$$A^{(2m)} = (a - b_2)E_{2m,1} + bE_{2m,2}. \tag{4.11}$$

Suppose $A^{(2)} = (a_{ji})_{2m \times 2m}$. Then taking $i = 2$ in (4.4), by $A^{(3)}$ in (4.10) we get

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{21}e_1^t \\ ae_1 + (a_{1,2m} + b_{2m})e_{2m-1} - \alpha & A_\beta - D - \sum\limits_{i \leqslant k \leqslant 2m} b_{2m+2-k}E_{2m,k} \end{pmatrix}, \tag{4.12}$$

where

$$\beta = (a_{22}, a_{23}, \cdots, a_{2,2m})^t, \ D = \mathrm{diag}(0, b, b, \cdots, b).$$

Taking $i = j = 2$ in (4.6), by (4.12) and Lemma 4.1 we get

$$a_{21} = 0, \ a_{22} = a_{11} + b, \ a_{23} = a_{24} = \cdots = a_{2,2m-1} = 0, \tag{4.13}$$

Similarly, for any $3 \leqslant k \leqslant 2m-2$, taking $i = k$ and $j = 2$ in (4.6), we have

$$b_{2m+1-k} = 0. \tag{4.14}$$

Denote $a_{11} = c, a_{1,2m} = p, a_{2,2m} = q$. Then by (4.9)–(4.14) we have

$$A^{(1)} = \begin{pmatrix} a & be_1^t \\ de_{2m-1} & A_\gamma \end{pmatrix}, \tag{4.15}$$

$$A^{(2)} = \begin{pmatrix} c & O \\ (a-b_2)e_1 - b_{2m-1}e_{2m-2} + pe_{2m-1} & cI + bE_{11} + qE_{2m-1,1} \end{pmatrix}, \tag{4.16}$$

$$A^{(3)} = (a-b_2)E_{31} + bE_{32} + b_{2m-1}(E_{2m,2} - E_{2m,1}), \tag{4.17}$$

$$A^{(k)} = (a-b_2)E_{k1} + bE_{k2}, \ 4 \leqslant k \leqslant 2m, \tag{4.18}$$

where $a, b, c, d, p, q \in \mathbb{F}$ and $\gamma = (b_2, 0, \cdots 0, b_{2m-1}, b_{2m}) \in \mathbb{F}^{2m-1}$. Thus we have (4.2).

Conversely, if we have (4.2), then it is easy to verify that (4.3)–(4.8) hold, and therefore $(Q_{2m}, \cdot, [\cdot,\cdot])$ is a Poisson algebra. □

Now, using Theorem 4.1, we characterize the commutative (resp. associative) Poisson structures on Q_{2m}.

Corollary 4.1. *Suppose the filiform Lie algebra $(Q_{2m}, [\cdot,\cdot])$ is equipped with a bilinear operator "$\cdot$". Then $(Q_{2m}, \cdot, [\cdot,\cdot])$ is a commutative Poisson algebra if and only if*

$$\begin{cases} x_1 \cdot x_1 = 2ax_1 + dx_n, \\ x_1 \cdot x_2 = x_2 \cdot x_1 = bx_1 + ax_2 + px_{2m}, \\ x_1 \cdot x_i = x_i \cdot x_1 = ax_i, \ 3 \leqslant i \leqslant 2m, \\ x_2 \cdot x_2 = 2bx_2 + qx_{2m}, \\ x_2 \cdot x_i = x_i \cdot x_1 = bx_i, \ 2 \leqslant i \leqslant 2m, \end{cases} \tag{4.19}$$

where $a, b, p, q \in \mathbb{F}$ and all vanished products $x_i \cdot x_j$ are zero.

Proof. We still use the symbols in the proof of Theorem 4.1.

If $(Q_{2m}, \cdot, [\cdot,\cdot])$ is a Poisson algebra, then by Theorem 4.1 we have (4.15)–(4.18). In this case, by (1) of Lemma 2.3, $(Q_{2m}, \cdot, [\cdot,\cdot])$ is commutative if and only if

$$A_i^{(j)} = A_j^{(i)}, \ 1 \leqslant i, j \leqslant 2m. \tag{4.20}$$

Taking $i = 1$ and $j = 2$ in (4.20) and by (4.15)–(4.18) we get

$$c = b, \ a = 2b_2, \ b_{2m-1} = 0, b_{2m} = p.$$

Hence,

$$A^{(1)} = \begin{pmatrix} 2b_2 & be_1^t \\ de_{2m-1} & b_2 I + pE_{2m-1,1} \end{pmatrix}, \tag{4.21}$$

$$A^{(2)} = \begin{pmatrix} b & O \\ b_2 e_1 + pe_{2m-1} & bI + bE_{11} + qE_{2m-1,1} \end{pmatrix}, \tag{4.22}$$

$$A^{(3)} = b_2 E_{31} + bE_{32} + b_{2m-1}(E_{2m,2} - E_{2m,1}), \tag{4.23}$$

$$A^{(k)} = b_2 E_{k1} + bE_{k2},\ 4 \leqslant k \leqslant 2m. \tag{4.24}$$

Conversely, if all $A^{(i)}$ are of the form (4.21)–(4.24), then it is easy to see that $A_i^{(j)} = A_j^{(i)}$, $1 \leqslant i$, $j \leqslant 2m$. By the bilinearity of " $\cdot$ ", " $\cdot$ " satisfies the commutative law. Note that Theorem 4.1, $(Q_{2m}, \cdot, [\cdot,\cdot])$ is a commutative Poisson algebra. □

Corollary 4.2. *Suppose the filiform Lie algebra $(Q_{2m}, [\cdot,\cdot])$ is equipped with a bilinear operator " $\cdot$ ". Then $(Q_{2m}, \cdot, [\cdot,\cdot])$ is an associative Poisson algebra if and only if " $\cdot$ " is in one of the following five cases:*

(1) $x_1 \cdot x_i = ax_i,\ x_2 \cdot x_i = cx_i,\ 1 \leqslant i \leqslant 2m,\ a \in \mathbb{F}, c \in \mathbb{F}^*$;

(2) $x_i \cdot x_1 = ax_i,\ x_i \cdot x_2 = bx_i,\ 1 \leqslant i \leqslant 1m,\ a \in \mathbb{F}, b \in \mathbb{F}^*$;

(3) $x_1 \cdot x_i = ax_i,\ 1 \leqslant i \leqslant 2m,\ a \in \mathbb{F}^*$;

(4) $x_i \cdot x_1 = ax_i,\ 1 \leqslant i \leqslant 2m,\ a \in \mathbb{F}^*$;

(5) $x_1 \cdot x_1 = dx_n,\ x_1 \cdot x_2 = b_{2m-1}x_{2m-1} + b_{2m}x_{2m}, x_1 \cdot x_2 = b_{2m-1}x_{2m},$
$x_2 \cdot x_1 = -b_{2m-1}x_{2m-1} + px_{2m}, x_2 \cdot x_2 = qx_{2m},$
$x_3 \cdot x_1 = -x_3 \cdot x_2 = -b_{2m-1}x_{2m},\ d, p, q, b_{2m-1}, b_{2m} \in \mathbb{F},$

where all vanished products $x_i \cdot x_j$ in the above five cases are zero.

Proof. We still use the symbols in the proof of Theorem 4.1.

If $(Q_{2m}, \cdot, [\cdot,\cdot])$ is a Poisson algebra, then by Theorem 4.1 we have (4.15)–(4.18). By (2) of Lemma 2.3, $(Q_{2m}, \cdot, [\cdot,\cdot])$ is associative if and only if

$$L_{x_i \cdot x_j} = L_{x_i} L_{x_j},\ 1 \leqslant i, j \leqslant 2m. \tag{4.25}$$

Taking $i = j = 1$ in (4.25), we have $aA^{(1)} + dA^{(2m)} = A^{(1)}A^{(1)}$. Using (4.15) and (4.18), we obtain

$$bb_2 = 0, \tag{4.26}$$

$$ab_2 = b_2^2, \tag{4.27}$$

$$ad = b_2 d = ab_i = b_2 b_i = 0,\ 2m - 1 \leqslant i \leqslant 2m. \tag{4.28}$$

Similarly, taking $i = j = 2$ in (4.25), we have $(a_1 + b)A^{(2)} + qA^{(2m)} = A^{(2)}A^{(2)}$, and therefore we get

$$cb = 0, \tag{4.29}$$

$$(a - b_2)c = 0, \tag{4.30}$$

$$cb_{2m-1} = bb_{2m-1} = cp = cq = bp = bq = 0. \tag{4.31}$$

Taking $i = 1$ and $j = 2$ in (4.25) and using (4.29), we have

$$cd = bd = cb_{2m} = bb_{2m} = 0. \tag{4.32}$$

Taking $i = 2$ and $j = 1$ in (4.25) and using (4.27), we have

$$ap = b_2 p = aq = b_2 q = 0. \tag{4.33}$$

Case 1. $c \neq 0$.

By (4.29)–(4.32) we have

$$a = b_2,\ b = d = p = q = b_i = 0,\ i = 2m-1, 2m.$$

Thus,

$$A^{(1)} = aI,\ A^{(2)} = cI,\ A^{(i)} = O,\ 3 \leqslant i \leqslant 2m,$$

where $a \in \mathbb{F}$ and $c \in \mathbb{F}^*$.

Case 2. $b \neq 0$.

By (4.26), (4.27), (4.29), (4.31), and (4.32), we have

$$b_2 = c = d = p = q = b_i = 0,\ i = 2, 2m-1, 2m.$$

Thus,

$$A^{(k)} = aE_{k1} + bE_{k2},\ 0 \leqslant i \leqslant n,$$

where $a \in \mathbb{F}$ and $b \in \mathbb{F}^*$.

Case 3. $b = c = 0$.

Subcase 3.1. $b_2 \neq 0$.

By (4.27), (4.28), and (4.33), we have

$$a = b_2,\ d = p = q = b_i = 0,\ i = 2m-1, 2m.$$

Thus,

$$A^{(1)} = aI,\ A^{(i)} = O,\ 2 \leqslant i \leqslant 2m,$$

where $a \in \mathbb{F}^*$.

Subcase 3.2. $b_2 = 0$ and $a \neq 0$.

It follows from (4.28) and (4.33) that $p = q = d = b_i = 0,\ i = 2m-1, 2m$. Thus,

$$A^{(i)} = aE_{i1},\ 1 \leqslant i \leqslant 2m,$$

where $q \in \mathbb{F}$ and $a \in \mathbb{F}^*$.

Subcase 3.3. $b_2 = a = 0$.

By (4.15)–(4.18) we obtain

$$A^{(1)} = dE_{2m,1} + b_{2m-1}(E_{2m-1,2} + E_{2m,3}) + b_{2m}E_{2m,2},$$

$$A^{(2)} = -b_{2m-1}E_{2m-1,1} + pE_{2m,1} + qE_{2m,2},$$

$$A^{(3)} = -b_{2m-1}E_{2m,1} + b_{2m-1}E_{2m,2},\ A^{(i)} = O,\ 4 \leqslant i \leqslant 2m,$$

where $d, b_{2m-1}, b_{2m}, p, q \in \mathbb{F}$.

To sum up, an associative Poisson algebra structure " $\cdot$ " on $(Q_{2m}, [\cdot,\cdot])$ is of the form in the five cases in Corollary.

Conversely, if " $\cdot$ " is in one of five cases in Corollary, then it is easy to verify that (4.25) holds. By the bilinearity of " $\cdot$ ", " $\cdot$ " satisfies the associative law. Note that Theorem 4.1, $(Q_{2m}, \cdot, [\cdot,\cdot])$ is an associative Poisson algebra. □

Naturally, we also have the following result.

Corollary 4.3. *Suppose " $\cdot$ " is a bilinear binary multiplication in Lie algebra $(Q_{2m}, [\cdot,\cdot])$. Then $(Q_{2m}, \cdot, [\cdot,\cdot])$ is a commutative and associative Poisson algebra if and only if there exist $a, b, c \in \mathbb{F}$ such that*

$$\begin{cases} x_1 \cdot x_1 = ax_{2m}, \\ x_1 \cdot x_2 = x_2 \cdot x_1 = bx_{2m}, \\ x_2 \cdot x_2 = cx_{2m}, \\ x_i \cdot x_j = 0, \ i > 1 \ or \ j > 1, \end{cases}$$

Author contributions

The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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