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Research article

On Padovan numbers that are perfect powers of Lucas numbers

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Abstract: Let (P_n) and (L_n) be the Padovan and Lucas sequences. In this paper, we found all Padovan numbers that were perfect powers of Lucas numbers using advanced mathematical tools such as linear forms in logarithms and the reduction method. The results for this equation were

$$9 = P_9 = L_2^2 = 3^2, 16 = P_{11} = L_3^2 = 4^2, 49 = P_{15} = L_4^2 = 7^2.$$

Moreover, the intersection between the Padovan and Lucas sequences revealed that

$$1 = P_2 = L_1, 2 = P_3 = L_0, 2 = P_4 = L_0, 3 = P_5 = L_2, 4 = P_6 = L_3, 7 = P_8 = L_4.$$

This work not only solved a specific open problem in exponential Diophantine equations but also highlighted the wider applicability of the methods used. The results paved the way for further research on perfect powers in other linear iteration sequences and their intersections, emphasizing the importance of such research in modern mathematics.

Keywords: Padovan numbers; Lucas numbers; exponential Diophantine equations; Baker's method
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1. Introduction

Let (L_n) and (P_n) be the sequences of Lucas and Padovan numbers defined by

$$L_0 = 2,\ L_1 = 1;\ L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2$$

and

$$P_0 = P_1 = P_2 = 1; P_n = P_{n-2} + P_{n-3} \text{ for } n \geq 3,$$

respectively. There has been some studies on the case where the terms of linear recurrence sequences are perfect powers. In 1963, Moser and Carlitz [1] and Rollet [2] proposed the problem of finding all the perfect square Fibonacci numbers. This problem was solved independently in [3, 4], showing that the only perfect squares in the Fibonacci sequence are 0, 1, and 144. In later years, some authors continued to deal with the terms of Fibonacci or Lucas sequences that could be written as a square or a cube of an integer (for more details, see [5–7]). In [8], as a generalization of these studies, Bugeaud, Mignotte, and Siksek showed that the only perfect powers in the Fibonacci and Lucas sequences are $0, 1, 8, 144$, and $1, 4$, respectively. Applications and theoretical explorations involving Fibonacci and Lucas numbers can be seen in various studies. For instance, [9–11] focus on algebraic and number-theoretic properties of these sequences, including matrix representations and Diophantine equations. References [12–14] investigate generalized forms and algebraic structures such as bi-periodic sequences and Fibonacci-related spinors and quaternions. In contrast, [15–17]. examine unique numerical patterns and factorizations involving Fibonacci and Lucas numbers, including their expression as repdigits or concatenated values. In [18], authors proved that there are only finitely many perfect powers in any linear recurrence sequence of integers of order at least two and whose characteristic polynomial is irreducible and has a dominant root. Furthermore, we can say from Theorem 6 in [19], since the characteristic polynomial of the Padovan sequence is cubic and irreducible and has a dominating root, the perfect powers among its members are finite. From the proof of Pethő (and Bugeaud-Kaneko), it is clear that the exponents of perfect powers are effectively bounded. The problem is that we do not have effective methods to treat the equation

$$P_n = y^a \text{ for } a \geq 2.$$

Thus, first we solved this problem by taking k-generalized Fibonacci numbers instead of the Padovan number and Lucas numbers instead of y in [20]. For more information about k-generalized Fibonacci numbers, see [21].

In this paper, we dealt with the equations

$$P_n = L_m^a \text{ for } m \neq 1 \text{ and } a \geq 2 \tag{1.1}$$

and

$$P_n = L_m \text{ for } n \geq 2 \tag{1.2}$$

in nonnegative integers n, m, a. In these cases, we have the following results:

Theorem 1.1. *Padovan numbers which can be written as perfect powers of a Lucas number are given by*

$$\begin{aligned} 4 = P_6 = L_0^2 = 2^2,\ 16 = P_{11} = L_0^2 = 2^2, \\ 9 = P_9 = L_2^2 = 3^2, 16 = P_{11} = L_3^2 = 4^2, \\ 49 = P_{15} = L_4^2 = 7^2. \end{aligned}$$

If $a = 1$ in Eq (1.1), then we get the intersection of the Padovan sequence with the Lucas sequence. For technical reasons, this case will be tackled as a separate theorem.

Theorem 1.2. *The intersection between Padovan and Lucas sequences are given by*

$$1 = P_2 = L_1,\ 2 = P_3 = L_0,\ 2 = P_4 = L_0,$$
$$3 = P_5 = L_2,\ 4 = P_6 = L_3,\ 7 = P_8 = L_4.$$

The following figure is given to show how the values of each "Padovan numbers" and "perfect powers of Lucas numbers" are located in the 3D coordinate system and the relationship between them. In the Figure 1, the values that satisfy this theorem are shown with the symbol "o". When Figure 1 is examined, it can be seen that there are a finite number of common solutions and the values of the numbers gradually increase.

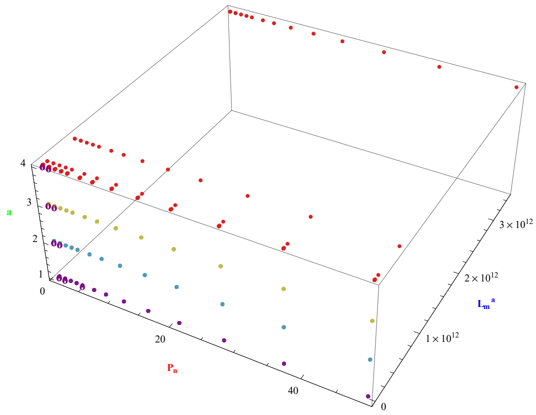

Figure 1. Plot of the relationship between Padovan numbers and perfect power of Lucas numbers.

Figure 2 presents the graphical representation of the intersection between the Padovan and Lucas numbers. These two sequences, although defined by different recurrence relations, occasionally produce the same values at certain indices, which is the focus of this plot. Clearly, the plot highlights the points where the two sequences intersect, meaning the values where a term from the Padovan sequence is equal to a term from the Lucas sequence. These intersection points are rare, and their occurrence is particularly interesting as it reveals a potential link between these two distinct sequences.

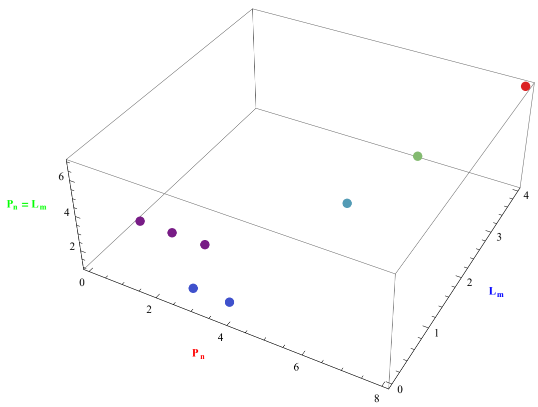

Figure 2. Plot of the intersection between Padovan numbers and Lucas numbers.

By examining the graphs, we can observe that these intersections occur at specific, distinct values of n, indicating that although both series have common properties, their paths often diverge. The intersections are not regularly spaced and do not follow a clear pattern, indicating that these series typically do not overlap except in these unique cases. As a result, these graphs visually show that two separate recursive sequences occasionally equate, providing a fascinating look at the properties and interactions of sequences of numbers.

2. The tools

In this section, we give the tools needed to prove our results.

2.1. Some properties of the Lucas and Padovan sequences

Here, we give some important information for the Lucas and Padovan sequences. Binet formula for Lucas numbers is

$$L_n = \varphi^n + \left(-\frac{1}{\varphi}\right)^n,$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$. Moreover, we have

$$\varphi^{n-1} \le L_n \le 2\varphi^n \text{ for } n \ge 0.$$

The characteristic equation $x^3 - x - 1 = 0$ has roots α, β, and γ. Here,

$$\alpha = \frac{m_1 + m_2}{6},\ \overline{\gamma} = \beta = \frac{-m_1 - m_2 + i\sqrt{3}(m_1 - m_2)}{12}$$

with

$$m_1 = \sqrt[3]{108 + 12\sqrt{69}},\ m_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Binet formula is

$$P_n = t \cdot \alpha^n + s \cdot \beta^n + r \cdot \gamma^n,$$

for the Padovan numbers, where

$$t = \frac{\alpha + 1}{-\alpha^2 + 3\alpha + 1},\ s = \frac{\beta+1}{-\beta^2 + 3\beta+1},\ r = \frac{\gamma+1}{-\gamma^2 + 3\gamma+1}.$$

The minimal polynomial of t over $\mathbb{Z}$ is given by $23x^3 - 23x^2 + 6x - 1$. Zeros of this equation are t, s, r. Moreover, with simple calculation, it can be shown that the following estimates hold:

$$1.32 < \alpha < 1.33 \text{ and } 0.86 < |\beta| = |\gamma| < \alpha^{-1/2} < 0.87, \tag{2.1}$$

$$0.72 < |t| < 0.73 \text{ and } 0.24 < |s| = |r| < 0.25. \tag{2.2}$$

Let $e(n) := P_n - t{\cdot}\alpha^n = s \cdot \beta^n + r \cdot \gamma^n$. From (2.1) and (2.2), we conclude that $|e(n)| < \frac{0.5}{\alpha^{n/2}}$ for $n \ge 1$. The relation between P_n and α is given by

$$\alpha^{n-3} \le P_n \le \alpha^{n-1} \text{ for } n \ge 1.$$

2.2. Linear forms in logarithms

The logarithmic height of an algebraic number η is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right),$$

where $a_0 > 0$ is the leading coefficient of the minimal polynomial of η, d is the degree of η over $\mathbb{Q}$, and $\left(\eta^{(i)}\right)_{1\le i\le d}$ are conjugates of η over $\mathbb{Q}$.

The following properties of the logarithmic height can be found in [22].

$$h(\eta \pm \gamma) \le h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta\gamma^{\pm 1}) \le h(\eta) + h(\gamma),$$
$$h(\eta^m) = |m|h(\eta).$$

Now, we give a lemma which is deduced from Corollary 2.3 of Matveev [23], and it can be found in [8].

Lemma 2.1. *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then,

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2 \cdots A_t\right),$$

where

$$B \ge \max\{|b_1|, |b_2|, ..., |b_t|\},$$

and $A_i \ge \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$ for all $i = 1, 2, ..., t$.

2.3. Reduction method

To reduce the upper bounds on a, m and n in Eqs (1.1) and (1.2), we will use the following lemma which is given in [24]. This lemma is an immediate variation of the lemma of Dujella and Pethő in [25].

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\epsilon := ||\mu q|| - M||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \le M \text{ and } w \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

2.4. Useful lemmas

The following lemma can be found in [26].

Lemma 2.3. *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$\left|\log(1+x)\right| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

To determine the relationships among the variables in Eq (1.1), we will use the following lemma.

Lemma 2.4. *Let $a, m \geq 2$. Suppose that the Eq (1.1) holds. Then, we have the inequalities*

(a) $n < 5ma$,
(b) $a < n$.

Proof. a) We can say

$$\alpha^{n-3} \leq P_n = L_m^a \leq (2\varphi^m)^a < \varphi^{(m+2)a} < \alpha^{2(m+2)a}.$$

So, it follows that $n < 2(m+2)a + 3 < 5ma$ for $m, a \geq 2$.

b) We have

$$\varphi^{(m-1)a} \leq L_m^a = P_n < \alpha^{n-1} < \varphi^{n-1}.$$

Thus, this implies that $a \leq (m-1)a < n-1 < n$ for $m \geq 2$. □

3. The proof of Theorem 1.1

Let's begin our proof by examining the cases of m. If $m = 0$ in Eq (1.1), then we obtain $P_n = 2^a$. In [27], authors found Padovan numbers which are sum of two Jacobsthal numbers. That is, they solved the equation

$$P_n = \frac{2^b - (-1)^b}{3} + \frac{2^a - (-1)^a}{3}.$$

Here, considering the equality $b = a + 1$, we get $P_n = 2^a$. Therefore, the solutions for the case $m = 0$, can be deduced from Theorem 1.1 in [27] as $(n, m, a) = (6, 0, 2), (11, 0, 4)$. From now on, we will take $n \geq 9$, since $a, m \geq 2$. To begin, we arrange Eq (1.1) as

$$t \cdot \alpha^n - L_m^a = -s \cdot \beta^n - r \cdot \gamma^n.$$

With the necessary arrangement, we obtain

$$\left|t \cdot \alpha^n - L_m^a\right| \leq |s \cdot \beta^n + r \cdot \gamma^n| < \frac{0.5}{\alpha^{n/2}} < 0.1,$$

i.e.,

$$\left|t \cdot \alpha^n \cdot L_m^{-a} - 1\right| < \frac{0.1}{\varphi^{(m-1)a}}, \tag{3.1}$$

for $n \geq 9$. Now, let us apply Lemma 2.1 with $(\gamma_1, b_1) := (t, 1), (\gamma_2, b_2) := (\alpha, n), (\gamma_3, b_3) := (L_m, -a)$. Furthermore, $D = 3$. Now, we show that $\Lambda_1 := t \cdot \alpha^n \cdot L_m^{-a} - 1$ is nonzero. If $\Lambda_1 = 0$, then we get $L_m^a/t = \alpha^n$. Applying an automorphism σ to both sides of this equation and taking absolute values, we get

$$\left| \frac{L_m^a}{s} \right| = |\sigma(\alpha^n)| = |\beta|^n < 1,$$

which is not possible. Since $|s| = |r| < |t| = t < 1$ by (2.2), we get $h(t) \leq \frac{1}{3} \log 23$. Moreover, since $h(\gamma_2) = h(\alpha) = \dfrac{\log \alpha}{3}$ and $h(\gamma_3) = h(L_m) < \log 2 + m \log \varphi$, we can take $A_1 := \log 23, A_2 := \log \alpha$ and $A_3 := 6m \log \varphi$. Considering Lemma 2.4(b), we can take $B := n$. Let $T = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3)$. Thus, by using Lemma 2.1 and (3.1), we obtain

$$0.1 \cdot \varphi^{-(m-1)a} > |\Lambda_1| > \exp\left(T \cdot (1 + \log n) \cdot \log 23 \cdot \log \alpha \cdot 6m \log \varphi\right),$$

i.e.,

$$(m-1)a \log \varphi - \log 0.1 < 6.89 \cdot 10^{12} \cdot m \cdot (1 + \log n).$$

So, we get

$$a < 5.73 \cdot 10^{13} \cdot \log n. \tag{3.2}$$

Now, we find an upper bound for m, a, and n.

Assume that $m \geq 80$. From Eq (3.2) and Lemma 2.4(a), we write $a < 5.73 \cdot 10^{13} \cdot \log(5ma)$. Thus, we can say that $a < 1.27 \cdot 10^{14} \cdot \log m \cdot \log a$. Using the fact that

$$\text{if } A \geq 3 \text{ and } \frac{x}{\log x} < A, \text{ then } x < 2A \log A,$$

we get

$$a < 2.54 \cdot 10^{14} \cdot \log m \cdot \log(1.27 \cdot 10^{14} \cdot \log m);$$

i.e.,

$$a < 2.54 \cdot 10^{14} \cdot \log m \cdot (8 \log m + \log m)$$

and so

$$a < 2.29 \cdot 10^{15} \cdot (\log m)^2. \tag{3.3}$$

Let $y := a/\varphi^{2m}$. For $m \geq 80$, we obtain

$$y = \frac{a}{\varphi^{2m}} < \frac{2.29 \cdot 10^{15} \cdot (\log m)^2}{\varphi^{2m}} < \frac{1}{\varphi^m}$$

by the inequality (3.3). Moreover, we can write

$$L_m^a = \varphi^{ma} \left(1 + \frac{(-1)^m}{\varphi^{2m}}\right)^a. \tag{3.4}$$

Taking into account $y < \frac{1}{\varphi^m} < \varphi^{-80} < 10^{-16}$ and $\log(1-\rho) \geq -2\rho$ for $0 < \rho < 0.79$, we can say

$$1 > \left(1 - \frac{1}{\varphi^{2m}}\right)^a = e^{(a \log(1-\varphi^{-2m}))} \geq e^{-2y} > 1 - 2y$$

and

$$1 < \left(1 + \frac{1}{\varphi^{2m}}\right)^a = 1 + \frac{a}{\varphi^{2m}} + \frac{a(a-1)}{2!\varphi^{4m}} + ...$$
$$< e^y < 1 + 2y.$$

The above inequalities and (3.4) give us

$$\begin{aligned} \left|L_m^a - \varphi^{ma}\right| &= \left|\varphi^{ma}\left(1 + \frac{(-1)^m}{\varphi^{2m}}\right)^a - \varphi^{ma}\right| \\ &= \varphi^{ma}\left|\left(1 + \frac{(-1)^m}{\varphi^{2m}}\right)^a - 1\right| \\ &< \frac{2 \cdot \varphi^{ma}}{\varphi^m}. \end{aligned} \tag{3.5}$$

Considering (1.1) and (3.5), we rearrange the equality

$$t \cdot \alpha^n - \varphi^{ma} = L_m^a - \varphi^{ma} - s \cdot \beta^n - r \cdot \gamma^n$$

so that

$$\begin{aligned} \left|t \cdot \alpha^n \cdot \varphi^{-ma} - 1\right| &\leq \frac{\left|L_m^a - \varphi^{ma}\right|}{\varphi^{ma}} + \frac{|s \cdot \beta^n + r \cdot \gamma^n|}{\varphi^{ma}} \\ &< \frac{2}{\varphi^m} + \frac{0.5}{\alpha^{n/2} \cdot \varphi^{2m}} \\ &\leq \frac{2}{\varphi^m} + \frac{0.15}{\varphi^{2m}} < \frac{2.15}{\varphi^m}, \end{aligned} \tag{3.6}$$

where we have used the fact that $0.5/\alpha^{n/2} < 0,15$ for $n \geq 9$. Now, put $\gamma_1 := t, \gamma_2 := \alpha, \gamma_3 := \varphi$ and $b_1 := 1, b_2 := n, b_3 := -ma$ in order to apply Lemma 2.1. Here, $D = 6$. Let $\Lambda_2 := t \cdot \alpha^n \cdot \varphi^{-ma} - 1$. It can be shown that Λ_2 is nonzero. Moreover, since $h(\gamma_3) = \frac{\log \varphi}{2}$, we can choose $A_1 := 2 \log 23$, $A_2 := 2 \log \alpha$, and $A_3 := 3 \log \varphi$. From Lemma 2.4(a), it can be easily seen that $B := 5ma$. Thus, we are ready to apply Lemma 2.1. Using the inequality (3.6), we obtain

$$\frac{2.15}{\varphi^m} > |\Lambda_2| > \exp\left(K \cdot (1 + \log 5ma) \cdot 12 \cdot \log \alpha \cdot \log \varphi \cdot \log 23\right),$$

where $K = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 \cdot (1 + \log 6)$. By using the inequality $1 + \log 5ma < 2 \log ma$ for $m \geq 80$ and $a \geq 2$, the above inequality and (3.3) tell us

$$m < 2.31 \cdot 10^{14} \cdot \log(ma);$$

i.e.,

$$m < 2.31 \cdot 10^{14} \cdot \log(m \cdot 2.29 \cdot 10^{15} \cdot (\log m)^2),$$

which implies that

$$m < 2.31 \cdot 10^{14} \cdot (\log m + 8.1 \log m + 2 \log m),$$

and so, we obtain

$$m < 1.1 \cdot 10^{17}. \tag{3.7}$$

Combining the inequalities (3.3) and (3.7), we arrive at the conclusion that

$$a < 2.29 \cdot 10^{15} \cdot (\log(1.1 \cdot 10^{17}))^2 < 3.53 \cdot 10^{18}. \tag{3.8}$$

Furthermore, using the Lemma 2.4(a), (3.7), and (3.8), we get

$$n < 5ma < 1.95 \cdot 10^{36}.$$

Now, assume that $m \leq 79$. From Eq (3.2) and Lemma 2.4(a), we write

$$a < 5.73 \cdot 10^{13} \cdot \log(395a),$$

which leads to

$$a < 2.38 \cdot 10^{15} \text{ and so } n < 9.41 \cdot 10^{17}.$$

Consequently, considering both cases, we can choose

$$m < 1.1 \cdot 10^{17}, a < 3.53 \cdot 10^{18} \text{ and } n < 1.95 \cdot 10^{36}.$$

Put

$$z_2 := n \log \alpha - ma \log \varphi + \log t$$

and $\Lambda_2 := e^{z_2} - 1$. Combining the inequality (3.3) and (3.6), it can be seen that

$$|\Lambda_2| = |e^{z_2} - 1| < (2.15) \cdot \varphi^{-m} < 0.01,$$

for $m \geq 80$. According to Lemma 2.3, we obtain

$$0 < |z_2| = \left| n \log \alpha - ma \log \varphi + \log t \right| < \frac{\log(100/99)}{0.01} \cdot \frac{2.15}{\varphi^m} < (2.17) \cdot \varphi^{-m};$$

i.e.,

$$0 < \left| n \frac{\log \alpha}{\log \varphi} - ma + \frac{\log t}{\log \varphi} \right| < (4.51) \cdot \varphi^{-m}. \tag{3.9}$$

Taking into account the inequality (3.9), we can take

$$\gamma := \frac{\log \alpha}{\log \varphi}, \ \mu := \frac{\log t}{\log \varphi}, \ M := 1.95 \cdot 10^{36}, A := 4.51, \ B := \varphi,$$

and $w := m$ in Lemma 2.2. Here, γ is irrational. If it were not, then we could write $\frac{\log \alpha}{\log \varphi} = \frac{a}{b}$ for a, $b \in \mathbb{Z}^+$. Thus, it follows that $\alpha^b = \varphi^a$. As $a \geq 1$, it is seen that φ^a, and therefore, α^b are irrationals. This is impossible since the minimal polynomial of φ^a over $\mathbb{Q}$ has degree 2, but the minimal polynomial of α^b over $\mathbb{Q}$ has degree 3. PARI/GP, an open source computer algebra system, gives us that q_{69}, the denominator of the 69 rd convergent of γ exceeds $6M$, and

$$\epsilon := ||\mu q_{69}|| - M||\gamma q_{69}|| > 0.35.$$

All the conditions of Lemma 2.2 are fulfilled. Hence, we have

$$m < \frac{\log(Aq_{69}/\epsilon)}{\log B} < 184.1,$$

which yields $m \leq 184$. From (3.3) and Lemma 2.4(a), we obtain $a < 6.23 \cdot 10^{16}$ and $n < 5ma < 5.74 \cdot 10^{19}$ for $m \leq 184$. Taking $M := 5.74 \cdot 10^{19}$ and applying reduction again to inequality (3.9), we have that $q_{37} > 6M, \epsilon > 0.36$, and $m \leq 104$. Thus, we can say $m \leq 104, a < 4.94 \cdot 10^{16}$ and $n < 2.57 \cdot 10^{19}$ by (3.3) and Lemma 2.4(a). Now, let

$$z_1 := n \log \alpha - a \log L_m + \log t$$

and $\Lambda_1 := e^{z_1} - 1$. For $m, a \geq 2$, it is clear that

$$|\Lambda_1| = |e^{z_1} - 1| < \frac{0.1}{\varphi^{(m-1)a}} < 0.04$$

by (3.1). According to Lemma 2.3, we get

$$0 < |z_1| = \left|n \log \alpha - a \log L_m + \log t\right| < -\frac{\log(0.96)}{0.04} \cdot \frac{0.1}{\varphi^a} < (0.11) \cdot \varphi^{-a},$$

and so

$$0 < \left|n \frac{\log \alpha}{\log L_m} - a + \frac{\log t}{\log L_m}\right| < (0.04) \cdot \varphi^{-a}. \tag{3.10}$$

Now, we show that $\frac{\log \alpha}{\log L_m}$ is irrational. If it were not, then we could write $\frac{\log \alpha}{\log L_m} = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$. Thus, it follows that $L_m^a = \alpha^b$. By applying an automorphism σ from both sides of this equation and taking absolute values, we get

$$\left|L_m^a\right| = \left|\sigma(\alpha^b)\right| = |\beta|^b < 1,$$

which is imposible for $m \geq 2$. Taking into account the inequality (3.10), we can take

$$\gamma(m) := \frac{\log \alpha}{\log L_m}, \mu(m) := \frac{\log t}{\log L_m}, M := 2.57 \cdot 10^{19}, A := 0.04, B := \varphi$$

and $w := a$ in Lemma 2.2. It can be seen that $q_{51}(m) > 6M$ and

$$\epsilon(m) := ||\mu q_{51}|| - M||\gamma q_{51}|| > 0.001$$

for $2 \leq m \leq 104$. Thus, we compute each value of $(\log(Aq_{51}/\epsilon)) / \log B$, and so we can say that all its values are at most 159.28, which yields $a \leq 159$. Considering Lemma 2.4(a), we obtain $n < 82680$ for $m \leq 104$ and $a \leq 159$. Applying Lemma 2.2 again with $M := 82680$, we have that $q_{17}(m) > 6M, \epsilon(m) > 0.0006, a \leq 61$, and $n < 5ma < 31720$. Finally, we use PARI/GP to find solutions of the Eq (1.1). Thus, these solutions can be seen as

$$\begin{aligned} 9 &= P_9 = L_2^2 = 3^2, \\ 16 &= P_{11} = L_3^2 = 4^2, \\ 49 &= P_{15} = L_4^2 = 7^2 \end{aligned}$$

for $2 \leq m \leq 104, 2 \leq a \leq 61$, and $9 \leq n < 31720$.

Now, we determine intersection of the Padovan numbers with the Lucas numbers.

4. The proof of Theorem 1.2

To start, we can write Eq (1.2) as

$$t \cdot \alpha^n - \varphi^m = (-1)^m \cdot \varphi^{-m} - s \cdot \beta^n - r \cdot \gamma^n.$$

From this, we get

$$\left| t \cdot \alpha^n \cdot \varphi^{-m} - 1 \right| < \frac{1.07}{\varphi^m}, \tag{4.1}$$

for $n \geq 2$. Now, let us apply Lemma 2.1 with $(\gamma_1, b_1) := (t, 1), (\gamma_2, b_2) := (\alpha, n), (\gamma_3, b_3) := (\varphi, -m)$. Furthermore, $D = 6$. Put $\Lambda = t \cdot \alpha^n \cdot \varphi^{-m} - 1$. It can be easily shown that $\Lambda \neq 0$. We can choose $A_1 := 2 \log 23, A_2 := 2 \log \alpha$, and $A_3 := 3 \log \varphi$. Considering the inequality

$$\alpha^{n-3} \leq P_n = L_m \leq 2\varphi^m < \varphi^{m+2} < \alpha^{2(m+2)},$$

we obtain $n < 2m + 7$. From this, we can take $B := 2m + 7$. Put $T = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 \cdot (1 + \log 6)$. Lemma 2.1 with (4.1) give that

$$1.07 \cdot \varphi^{-m} > |\Lambda| > \exp\left(T \cdot (1 + \log(2m + 7)) \cdot 2 \log 23 \cdot 2 \log \alpha \cdot 3 \log \varphi\right)$$

and so we get

$$m < 5.8 \cdot 10^{15} \text{ and } n < 2m + 7 < 1.16 \cdot 10^{16}.$$

Now, we find an upper bound for m. Put

$$z := n \log \alpha - m \log \varphi + \log t$$

and $\Lambda := e^z - 1$. Considering the inequality (4.1), it can be seen that

$$|\Lambda| = |e^z - 1| < (1.07) \cdot \varphi^{-m} < 0.7$$

for $m \geq 1$. Thanks to Lemma 2.3, we have

$$0 < |z| = \left| n \log \alpha - m \log \varphi + \log t \right| < -\frac{\log(0.3)}{0.7} \cdot \frac{1.07}{\varphi^m} < (1.85) \cdot \varphi^{-m}.$$

This implies that

$$0 < \left| n \frac{\log \alpha}{\log \varphi} - m + \frac{\log t}{\log \varphi} \right| < (3.85) \cdot \varphi^{-m}. \tag{4.2}$$

To apply Lemma 2.2, we can take

$$\gamma := \frac{\log \alpha}{\log \varphi}, \mu := \frac{\log t}{\log \varphi}, M := 1.16 \cdot 10^{16}, A := 3.85, B := \varphi$$

and $w := m$ in the inequality (4.2). PARI/GP gives us the inequality

$$\epsilon := ||\mu q_{32}|| - M||\gamma q_{32}|| > 0.18$$

and $\log(Aq_{32}/\epsilon)/\log B < 92.31$. This leads to $m \leq 92$, and so $n < 2m + 7 \leq 191$. Consequently, by using PARI/GP, we have only the solutions given by

$$1 = P_2 = L_1, 2 = P_3 = L_0,$$
$$2 = P_4 = L_0, 3 = P_5 = L_2,$$
$$4 = P_6 = L_3, 7 = P_8 = L_4,$$

for $0 \leq m \leq 92$ and $2 \leq n \leq 191$.

5. Conclusion and future work

In this article, we have identified the Padovan numbers, which are perfect powers of the Lucas numbers. We have also found the common terms of Padovan sequences and Lucas sequences. With this work, we have obtained solutions to a problem connected with exponential Diophantine equations. The significance of these intersections lies in the rare alignment of the sequences. The Padovan numbers grow more slowly than the Lucas numbers, and their terms are typically not equal. However, at the identified intersection points, the values align, which could suggest underlying mathematical properties or relationships between the sequences that are worth further investigation. We have once again demonstrated the power of modern mathematical tools, such as reduction methods and linear forms in logarithms, in solving difficult problems involving linear iteration sequences. The next work could be to find perfect powers of Padovan numbers. However, it is clear that more material will be needed to solve this difficult problem. We leave this work to the readers.

Author contributions

All authors contributed equally to this work. They have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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