AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 16355–16370.
DOI: 10.3934/math.2025731
Received: 28 March 2025
Revised: 30 June 2025
Accepted: 09 July 2025
Published: 21 July 2025

Research article

Relative attack strength and its application in argumentation frameworks

Jiachao Wu*

School of Mathematics and Statistics, Shandong Normal University, Jinan 250358, China

* **Correspondence:** Email: wujiachao@sdnu.edu.cn; Tel: +8653186182501.

Abstract: Adjusting the values of arguments is a widely applied method to establish the semantics of argumentation frameworks (AFs). In this process, the influence of a single attacker's value is mentioned or implicitly used many times. However, this effect has not been studied specifically. As a result, its role in semantic exploration cannot be brought into full play. Thus, my objective was to study this effect, so as to provide a new tool for semantics research. In this paper, the strength of this influence was called the relative attack strength (RAS). It was formally defined and computed by a three-variable function *ras*. Then, nine basic properties of RAS were studied. As an application, I established a new semantics for fuzzy AFs based on RAS, which covered the Gödel semantics. The main contribution lies in the proposal of RAS and the discussion of its properties. This provides a new tool for the further study of argumentation semantics. The other contribution is the establishment of new semantics. On one hand, it fills a gap in theory. On the other hand, it shows the effectiveness of RAS in semantic research of AFs.

Keywords: argumentation framework; weighted argumentation framework; weighted arguments; fuzzy semantics
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

As a useful tool to deal with conflict argumentation, Dung's theory of argumentation frameworks (AFs) [1–3] has been applied in many fields, such as law [4], voting, multi-agent systems [5], reasoning [3], decision-making [6–8]. Dung's original AFs have been extended by adding different information, such as supports between arguments [9–11], joint attacks [12], high-order attacks [13–15], and preferences over arguments [16, 17]. In particular, values are assigned to arguments and/or attacks to indicate their ranks in various meanings. For instance, these values are interpreted as probabilities in [18, 19], as fuzzy degrees in [20–22], as weights in [23, 24], and as other explanations in [25–27].

Generally speaking, there are three ways to form these value. First, in crisp AFs, in order to divide the arguments into different levels, a value is captured for each argument according to the number of its attackers [28,29]. Second, the values can be captured through statistical methods. In most uncertain AFs, the initial values of the arguments and/or attacks are obtained in this way. Last, in the works about ASPIC, an argument is not atom. Its value is constructed according to its internal structure. In this paper, I focus on the modification of the arguments' values. As a result, it is assumed that the initial values already exist.

In the literature, several methods have been introduced to build semantics of AFs with values. For example, kinds of extensions [30, 31] are put forward in the form of fuzzy sets. Epistemic approaches [18, 32] adjust the probability of each argument according to the probabilities of its attackers. Ranking and gradual methods [28, 29] recursively calculate the weights of arguments according to their attackers' weights. In these works, the value of an argument is adjusted according to its attackers. The adjusted value is usually determined by four factors: The value of the argument itself, the number of its attackers, the values of its attacker arguments, and the values of the attacks. Particularly, in [33], 21 principles are summarized for adjusting values.

In the semantics research of AFs with values, two problems seem to be imperfect in theory:

(i) When adjusting the value of an argument, its attackers always work collectively [33–35]. The influence of a single attacker has never been considered. How much the value of an argument is influenced by the value of each single attacker has been skipped or missed. In this paper, I refer to this influence strength of a single attacker as relative attack strength (RAS) because it is related to both arguments and their values.

The idea of applying the RAS was first put forward in [36]. In the process of building semantics, RAS is usually used implicitly. For example, in [30], attack strength between two fuzzy sets of arguments is computed according to the attack strength between crisp arguments. In [21], a value $\min\{a, b, \rho_{BA}\}$ is applied to build semantics, where a, b are values of arguments, and ρ_{BA} is the attack strength of the attack from the argument B to A.

However, although the importance of RAS has been noticed, there is no formal study or explicit use of this term. The research on this strength is a gap in theory, and this blank brings inconvenience to the application of RAS. Thus, filling this gap will benefit the semantic research of valued AFs.

(ii) Dung's widely applied semantics puts forward a series of extensions, from the least grounded extensions to the maximal preferred extensions, etc. In the field of fuzzy AFs, extensions are built in the form of fuzzy sets. For example, [21,30] establish fuzzy extensions with additional values x, y, z, such as x-conflict-free sets, y-admissible extensions, z-stable extensions. [31] builds fuzzy extensions avoiding the use of these extra numbers. In the former, many aggregation operators have been used to compute the RAS. In the latter, however, only the Gödel t-norm appears. Other classical aggregation operators, each with its own special practical significance, have not been applied. Therefore, it is an interesting and meaningful job to explore extension-based fuzzy semantics using other operators.

In this paper, my goal is to study RAS and explore new semantics of fuzzy AFs. First, the concept of RAS is formally defined, and nine basic properties of RAS are proposed. Then, as an application of RAS, an extension-based semantics of AFs is established. It is proven that the Gödel semantics in [31] corresponds to a special case of the new semantics.

The introduction of RAS fills a gap in AFs theory. This is the first time a special study of RAS is conducted, which lays a foundation for its application. The establishment of the new semantics, on one hand, shows that the tool of RAS is effective in the study of AFs. On the other hand, it is a development of the extension-based semantics in theory, where many other operators can be adopted in the new model.

The contents are arranged as follows. In Section 2, I recall some necessary notions. In Section 3, RAS is formally defined and some basic properties are studied. As an application, a new semantics of fuzzy AFs is established using RAS in Section 4. In Section 5, I discuss some related works. Finally, I conclude the study in Section 6.

2. Preliminaries

In this section, I briefly provide some necessary basic terminology.

A Dung's AF is a pair $(Args,\ Atts)$, where $Args$ is a set of arguments and $Atts \subseteq Args \times Args$ represents the attack relation between arguments.

For a set $S \subseteq Args$, if $\forall A, B \in S$, $(A, B) \notin Atts$, then S is called conflict-free. Given an argument $A \in Args$, if $\forall B$ with $(B, A) \in Atts$, there is some $C \in S$ such that $(C, B) \in Atts$, then A is acceptable w.r.t. (or defended by) S. Based on these two notions, a series of extensions are established. An argument set $S \subseteq Args$ is called

An admissible extension if S defends every argument in it;

A complete extension if it consists of all the arguments defended by itself;

A preferred extension if it is a maximal (w.r.t. set inclusion) admissible extension;

A grounded extension if it is the least complete extension (w.r.t. set inclusion);

A stable extension if it attacks every argument not in it.

The notion of fuzzy sets is a useful mathematical tool for dealing with uncertainty [37–39]. A fuzzy set S is a function from the universe U to [0, 1]. For a fuzzy set S, if there is a unique element $x \in U$ such that $S(x) \neq 0$, then S is called a fuzzy point.

In an AF, when every argument is assigned a value in [0, 1], this assignment is a function from the set $Args$ to [0, 1]. Formally speaking, it is the same as a fuzzy set. Borrowing the terminology of fuzzy sets, we call such an assignment a fuzzy set of arguments or a fuzzy arguments set. Additionally, for an argument with a value, we call it a fuzzy argument. In addition, AFs with initial fuzzy degrees are called fuzzy AFs.

Definition 2.1. *Given a Dung's AF* $(Args, Atts)$*, a fuzzy argumentation framework (FAF) is a tuple* $(Args, Atts, \mathcal{A}_0, \rho)$*, where* $\mathcal{A}_0\colon Args \to [0,1]$ *and* $\rho\colon Atts \to [0,1]$ *assign a fuzzy degree to every argument in* $Args$ *and every attack* $(B, A) \in Atts$*, respectively.*

The fuzzy degree of an argument/attack shows the membership degree at which the argument/attack belongs to the fuzzy sets. In this paper, a higher fuzzy degree represents a higher membership. In particular, 0 means not in the set at all, and 1 means completely in the set.

3. Relative attack strength

3.1. Motivation and definition of relative attack strength

When adjusting an argument's degree, it is reasonable to consider the influence of its attackers' degrees. For every attacker, this influence is not only related to itself as an argument, but also to its degree. Let us see a simple example, which is a widely existing fragment in argumentation.

Example 3.1. *Consider the FAF in Figure 1, where* $Args = \{A, B\}$, $Atts = \{(B, A)\}$, $\mathcal{A}_0(A) = a, \mathcal{A}_0(B) = b$ *and* $\rho(B, A) = \rho_{BA}$.

In the literature, though different researchers use different algorithms, the degree of A is influenced by three values: a, b, and ρ_{BA}*, where* ρ_{BA} *is related to argument B.*

$$\text{(B,b)} \xrightarrow[\rho_{BA}]{} \text{(A,a)}$$

Figure 1. An FAF example.

When considering the modification of the value of A, the attack strength between two numbers b and a plays a significant role. Strictly speaking, the attack strength is from the fuzzy argument (B, b) to the fuzzy argument (A, a). Compared with the attack strength ρ_{BA}, which is between arguments B and A, this attack strength is also relative to the degrees b and a. Hence, we call them relative attack strength (RAS).

Definition 3.1. *In an FAF* $(Args, Atts, \mathcal{A}_0, \rho)$*, suppose* (B, b) *and* (A, a) *are two fuzzy arguments. The level of an attack from* (B, b) *to* (A, a) *is called the relative attack strength from* (B, b) *to* (A, a)*, denoted by* $RAS((B, b), (A, a))$.

Similar to the arguments and attacks, in this paper, the RAS is represented by a number in [0, 1], where a higher number stands for a higher strength.

Remark 3.1. The notions $RAS((B, b), (A, a))$ and ρ_{BA} are distinct, though each is a number in [0,1] and represents the level of an attack. The number ρ_{BA} is the attack strength from B to A, which issues from the process of building the FAF and has nothing to do with the fuzzy degrees a, b. Moreover, $RAS((B, b), (A, a))$ represents the attack strength between fuzzy arguments (B, b) and (A, a). For example, when $\rho_{BA} = 1$, the attack strength from B to A is very strong. However, the attack strength from $(B, 0.1)$ to $(A, 0.1)$ through $((B, A), 1)$, i.e., $RAS((B, 0.1), (A, 0.1))$, is generally recognized very small.

In general, if $(B, A) \in Atts$, then $RAS((B, b), (A, a))$ is determined by three values: A's fuzzy degree a, B's fuzzy degree b, and the fuzzy degree ρ_{BA} of (B, A). This relationship can be expressed by a three-variable function.

Definition 3.2. *In an FAF, suppose* (A, a) *and* (B, b) *are two fuzzy arguments, and* $\rho_{BA} = \rho(B, A)$ *is the attack strength from B to A. The RAS from* (B, b) *to* (A, a) *is a number in [0, 1] calculated by a ternary function* $ras : [0, 1]^3 \to [0, 1]$*, i.e.,*

$$RAS((B, b), (A, a)) = ras(b, \rho_{BA}, a). \tag{3.1}$$

Next, I discuss the properties of RAS, i.e., the properties of the function ras.

3.2. Properties of relative attack strength

In this part, I talk about some properties of RAS. Most of these properties are inspired by the principles in [33].

First, the most simple case is that one of the three variables is 0. If $b = 0$, then B cannot output any strength to A. If $a = 0$, then A cannot bear any attack strength and its degree cannot be effectively influenced. If $\rho_{BA} = 0$, then no force can go through this path. In all these three cases, the RAS should be 0. In this way, we have the next property.

Property 3.1. (Absorption) *For any* $x, y \in [0, 1]$, $ras(0, x, y) = ras(x, 0, y) = ras(x, y, 0) = 0$.

A particular case is that $(B, A) \notin Atts$. In this case, the attack strength $RAS((B, b), (A, a))$ does not exist. In general, when necessary, the strength of this "attack" can also be represented by 0, i.e., if $(B, A) \notin Atts$, then $RAS((B, b), (A, a)) = 0$.

The property of absorption comes from Principle 12 (Neutrality) in [33].

Second, when a and ρ_{BA} are given, a rational agent always accepts that $RAS((B, 0.9), (A, a))$ is no less than $RAS((B, 0.1), (A, a))$, i.e., $ras(0.9, \rho_{BA}, a) \geq ras(0.1, \rho_{BA}, a)$. Similarly, ras does not decrease when a (or ρ_{BA} correspondingly) does not decrease. Hence, ras is monotone w.r.t. its three variables.

Property 3.2. (Monotonicity) *Suppose* x, y, z *and* x', y', z' *are numbers in [0,1]. If* $x \leq x'$, $y \leq y'$ *and* $z \leq z'$, *then* $ras(x, y, z) \leq ras(x', y', z')$.

Property 3.3. (Strict Monotonicity) *If one of* $x \leq x'$, $y \leq y'$ *and* $z \leq z'$ *is strict, such as* $x < x'$, *then* $ras(x, y, z) < ras(x', y', z')$.

These two properties of monotonicity come from Principles 9 (Proportionality) and 13 (Reinforcement) in [33]. But in some works, the strict monotonicity is not valid, e.g., [20, 31].

The following two comments illustrate two cases, which might confuse some readers.

Remark 3.2. $RAS((B, b), (A, a))$ is monotone w.r.t. a. Some people may question it. In my opinion, it can be understood in two aspects. On one hand, $RAS((B, b), (A, a))$ is not only related to the strength output from (B, b), but also depends on the strength that (A, a) can accept or input. When the degree a is higher, A can bear more and (A, a) can accept more strength. Hence, the RAS is higher. On the other hand, $RAS((B, b), (A, a))$ is positively related to how much (A, a) can be changed. In this sense, a higher a means that (A, a) can be changed more. Therefore, the higher a is, the higher $RAS((B, b), (A, a))$ is.

Remark 3.3. In an FAF, the function ρ is given and fixed. Why is ras monotone w.r.t. ρ? In a given FAF, the function ρ is, of course, fixed. However, there are different attacks. For instance, when we consider $RAS((B, b), (A, a))$ and $RAS((D, d), (C, c))$, if $a = c$ and $b = d$, what is the relationship between the two RASs? Which is bigger? In this case, the RAS is monotone w.r.t. ρ, i.e., if $\rho_{DC} \leq \rho_{BA}$, then $ras(d, \rho_{DC}, c) \leq ras(b, \rho_{BA}, a)$.

The fourth property is about the boundary of the function ras. In general, $RAS((B, b), (A, a))$ cannot be strictly higher than b, because (B, b) outputs the attack. Furthermore, $RAS((B, b), (A, a))$ cannot be strictly higher than ρ_{BA}, which is the highest value the attack (B, A) can sustain. Also, $RAS((B, b), (A, a))$ cannot be strictly higher than a, because a represents the highest strength that A can bear. Hence, $RAS((B, b), (A, a))$ cannot be strictly higher than anyone of the three values a, ρ_{BA} and b.

Property 3.4. (Boundedness) *For any* $x, y, z \in [0, 1]$, *then* $0 \leq ras(x, y, z) \leq min\{x, y, z\} \leq 1$.

This property develops Principle 4 (maximality) in [33].

From this property, we directly obtain the next property.

Property 3.5. (Imperfection) *If one of x, y, z is not 1, then $ras(x, y, z) < 1$.*

On the other hand, when two of the three factors a, b, and ρ_{BA} are 1, a rational agent commonly recognizes $RAS((B, b), (A, a))$ to be the third one, i.e., $ras(b, 1, 1) = b$, $ras(1, \rho_{BA}, 1) = \rho_{BA}$, and $ras(1, 1, a) = a$.

Property 3.6. (Identity) *For any $x \in [0, 1]$, $ras(x, 1, 1) = ras(1, x, 1) = ras(1, 1, x) = x$. Particularly, $ras(1, 1, 1) = 1$.*

The above properties, except the property of strict monotonicity, are always valid. The next two properties show two opposite views on whether or not to permit the RAS to be 0 when none of the three variables are 0.

Property 3.7. (Non-Archimedean) *For all $x, y, z \in (0, 1]$, $ras(x, y, z) > 0$. In other words, if $ras(x, y, z) = 0$, then at least one of the three is 0, i.e., $xyz = 0$.*

Property 3.8. (Archimedean) *There exist some $x, y, z \in (0, 1]$ such that $ras(x, y, z) = 0$.*

A particular case of Property Archimedean is that when $x + z \leq 1$ or $y + z \leq 1$, $ras(x, y, z) = 0$. This commonly exists in many works: If A is attacked by B and $a + b \leq 1$, then a is not modified. For example, in [20, 31, 35], if $a = 0.5 = b$ and $\rho_{BA} = 1$, then a remains unchanged. In this case, we can say $ras(b, \rho_{BA}, a) = 0$. There are also some algorithms using Property 3.7. For example, in [30, 34], when the operator in the examples are selected to be the Gödel t-norm or the product t-norm, then $ras(b, \rho_{BA}, a)$ is not 0, for all a, b and ρ_{BA}.

The property of commutativity also appears, e.g., [30].

Property 3.9. (Commutativity) *Suppose x, y, z is a permutation of $a, b, \rho_{BA} \in [0, 1]$. Then, we have $ras(b, \rho_{BA}, a) = ras(x, y, z)$.*

Example 3.2. *Suppose x, y, z are three numbers in [0,1]. Consider the next functions from $[0, 1]^3$ to $[0, 1]$, which are constructed by common t-norms.*

(1) $ras_1(x, y, z) = xyz$.

(2) $ras_2(x, y, z) = \min\{x, y, z\}$.

(3) $ras_3(x, y, z) = \max\{0, \max\{0, x + y - 1\} + z - 1\}$.

(4) $ras_4(x, y, z) = \max\{0, xy + z - 1\}$.

(5) $ras_5(x, y, z) = \max\{0, \min\{x, y\} + z - 1\}$.

The first three are constructed by product t-norms, Gödel t-norms, and Łukasiewicz t-norms, respectively. ras_4 is a combination of a product t-norm and a Łukasiewicz t-norm, and ras_5 is a combination of a Gödel t-norm and a Łukasiewicz t-norm. Properties of these functions are shown in Table 1, where ✓ means validity and ✕ stands for invalidity.

Table 1. Properties of the five *ras*.

Properties	ras_1	ras_2	ras_3	ras_4	ras_5
Absorbtion	✓	✓	✓	✓	✓
Monotonicity	✓	✓	✓	✓	✓
Strictly Monotonicity	✓	×	×	×	×
Boundedness	✓	✓	✓	✓	✓
Imperfection	✓	✓	✓	✓	✓
Identity	✓	✓	✓	✓	✓
Non-Archimedean	✓	✓	×	×	×
Archimedean	×	×	✓	✓	✓
Commutativity	✓	✓	✓	×	×

4. An extension-based semantics for FAFs based on RAS

As an application of RAS, in this section, I introduce a new extension-based semantics for FAFs. The new semantics is compared to the semantics in [31].

4.1. New semantics

In general, a fuzzy semantics arranges every argument in $Args$ a suitable degree. Formally, it is a fuzzy set S on $Args$, where for each $A \in Args$, $S(A)$ is the degree. In this subsection, I build an extension-based semantics for FAFs based on RAS. I recall and adapt a scenario from [20].

Example 4.1. *A judge is going to make a decision according to four testimonies, where A, B, C_1 are from three witnesses and C_2 is from a coroner.*

A**:** *I saw John killing Mary, so John is guilty.*

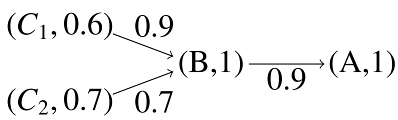
B**:** *John was at the theater with me when Mary was killed, thus John did not kill Mary.*

C_1**:** *That afternoon, I saw a man who looked like John near the murder site.*

C_2**:** *Mary was killed before 6 p.m.. Thus when Mary was killed, the show was still to begin.*

These four arguments and the attack relationship between them constitute an an AF $(Args, Atts)$, where $Args = \{A, B, C_1, C_2\}$, $Atts = \{(C_1, B), (C_2, B), (B, A)\}$.

On the other hand, the coroner is drunk and the third witness has a grudge against John, so their testimonies are not convincing. Therefore, some initial values can be associated with the arguments and attacks to show their credibility. For instance, $\mathcal{A}_0(A) = \mathcal{A}_0(B) = 1$, $\mathcal{A}_0(C_1) = 0.6$, $\mathcal{A}_0(C_2) = 0.7$, $\rho_{C_1B} = 0.9$, $\rho_{C_2B} = 0.7$ and $\rho_{BA} = 0.9$. Then, we get an FAF $(Args, Atts, \mathcal{A}_0, \rho)$ as shown in the figure below.



In the literature, different ways have been put forward to identify conflict-free fuzzy sets [30, 31]. In these works, weak attacks between fuzzy arguments are ignored. For example, the influence from $(C_1, 0.6)$ to $(B, 0.3)$ is ignored [20,31]. When calculating the RAS with ras_5, $RAS((C_1, 0.6), (B, 0.3)) = 0$. Similarly, in this paper, if $RAS((B, b), (A, a)) = 0$, then the attack from (B, b) to (A, a) is considered to be very weak, so it is ignored. These weak attacks are called tolerable attacks. Others are called sufficient attacks. With this idea, conflict-free sets can be defined as follows.

Definition 4.1. (Conflict-freeness) *Let $(Args, Atts, \mathcal{A}_0, \rho)$ be an FAF. A fuzzy set S is conflict-free, if and only if for all $A, B \in Args$, $RAS((B, S(B)), (A, S(A))) = 0$.*

In other words, a fuzzy set S is conflict-free, if and only if there are no sufficient attacks between fuzzy arguments in S.

Example 4.2. *Consider the FAF in Example 4.1. Let us see the conflict-freeness of two fuzzy sets $S = \{(C_1, 0.6), (C_2, 0.7), (B, 0.3), (A, 0.7)\}$ and $S' = \{(C_1, 0.6), (C_2, 0.7), (B, 0.5), (A, 0.5)\}$.*

If the RAS is calculated by ras_1 or ras_2, then neither S nor S' is conflict-free.

If the RAS is calculated by ras_3, then both S and S' are conflict-free.

If the RAS is calculated by ras_4 or ras_5, then S is conflict-free. However, S' is not conflict-free, because

$$\begin{aligned} &RAS_4((C_1, 0.6), (B, 0.5)) = ras_4(0.6, 0.9, 0.5) \\ &= \max\{0, 0.6 \times 0.9 + 0.5 - 1\} = 0.04 > 0, \text{ and} \\ &RAS_5((C_2, 0.7), (B, 0.5)) = ras_5(0.7, 0.7, 0.5) \\ &= \max\{0, \min\{0.7, 0.7\} + 0.5 - 1\} = 0.2 > 0. \end{aligned} \tag{4.1}$$

This shows that the conflict-freeness of a fuzzy set of arguments depends on how one sees it. In other words, the degree to which the judge justifies John's crime depends on the algorithm chosen by the judge.

In fact, when the function ras is constructed by non-Archimedean t-norms, like ras_1, ras_2, then $ras(x, y, z) = 0$ iff one of the three variables is 0. In the view of a judge, this means that no conflicts are allowed. Otherwise, some weak conflicts are permitted.

Acceptability is another key concept for the extension-based semantics. With the notion of RAS, we have the intuitive idea that a stronger RAS can defeat a weaker one. For example, if one of the defenders' RASs is bigger than the attacker's RAS, then it is a successful defense. Then, the acceptability can be defined as follows.

Definition 4.2. (Acceptability) *Suppose S is a fuzzy set and (A, a) is a fuzzy argument in an FAF $(Args, Atts, \rho, \mathcal{A}_0)$. (A, a) is acceptable w.r.t. (or defended by) S, iff $\forall (B, b)$ in the FAF, there is some $C \in Args$ such that $RAS((C, S(C)), (B, b)) \geq RAS((B, b), (A, a))$.*

Example 4.3. (Continued to Example 4.2) *Consider the defence relation between the fuzzy set S and the fuzzy argument $(A, 0.7)$ under ras_4 and ras_5. Obviously, $(B, 0.9)$ sufficiently attacks $(A, 0.7)$.*

Assume RAS is calculated by ras_5. Then,

$$\begin{aligned} &RAS_5((C_2, 0.7), (B, 0.9)) = ras_5(0.7, 0.7, 0.9) \\ &= \max\{0, \min\{0.7, 0.7\} + 0.9 - 1\} = 0.6, \text{ and} \\ &RAS_5((B, 0.9), (A, 0.7)) = ras_5(0.9, 0.9, 0.7) \\ &= \max\{0, \min\{0.9, 0.9\} + 0.7 - 1\} = 0.6. \end{aligned} \tag{4.2}$$

Because $0.6 \geq 0.6$*, we have* $(C_2, 0.7)$ *defends* $(A, 0.7)$*. Hence,* S *defends* $(A, 0.7)$*. Back to the scenario, if one of* $(C_1, 0.6)$ *and* $(C_2, 0.7)$ *has a greater influence on B than B has on* $(A, 0.7)$*, then the judge can trust A with a degree of 0.7.*

Assume RAS is calculated by ras_4*. Then,*

$$\begin{aligned} &RAS_4((C_1, 0.6), (B, 0.9)) = ras_4(0.1, 0.9, 0.9) \\ &= \max\{0, 0.6 \times 0.9 + 0.9 - 1\} = 0.44, \text{ and} \\ &RAS_4((C_2, 0.7), (B, 0.9)) = ras_4(0.7, 0.7, 0.9) \\ &= \max\{0, 0.7 \times 0.7 + 0.9 - 1\} = 0.39. \end{aligned} \tag{4.3}$$

However, $RAS_4((B, 0.9), (A, 0.7)) = \max\{0, 0.9 \times 0.9 + 0.7 - 1\} = 0.51$*. Because both* $0.44 < 0.51$ *and* $0.39 < 0.51$*, we have that* S *cannot defend* $(A, 0.7)$*.*

From Examples 4.2 and 4.3, both the conflict-freeness and the acceptability depend on the RAS. Therefore, different RASs lead to different kinds of conflict-freeness and acceptability, hence, different semantics. This is natural: Different RASs reflect different standards to measure the strength of an attack between fuzzy arguments, and different measurements bring different semantics.

Definition 4.3. (Characteristic function) *Let* $(Args, Atts, \mathcal{A}_0, \rho)$ *be an FAF. A function* $F\colon [0, 1]^{Args} \to [0, 1]^{Args}$ *is called the characteristic function of the FAF, if for any fuzzy set* $S \subseteq \mathcal{A}_0$*,*

$$F(S) = \cup\{(A, a)\colon (A, a) \in \mathcal{A}_0 \text{ is defended by } S\}. \tag{4.4}$$

The monotonicity of the characteristic function is obvious.

Proposition 4.1. *In an FAF, the characteristic function* F *is monotonic (w.r.t. set inclusion), i.e., if* $S \subseteq S'$*, then* $F(S) \subseteq F(S')$*.*

Then, kinds of extensions can be introduced.

Definition 4.4. (Extensions) *In an FAF* $(Args, Atts, \mathcal{A}_0, \rho)$*, suppose* $S \subseteq \mathcal{A}_0$ *is a conflict-free set. Then,* S *is*

An admissible extension *if* S *defends every fuzzy argument in it, i.e.,* $S \subseteq F(S)$*;*

A complete extension *if it consists of all the fuzzy arguments defended by itself, i.e.,* $S = F(S)$*;*

A preferred extension *if it is a maximal (w.r.t. set inclusion) admissible extension;*

A grounded extension *if it is the least complete extension (w.r.t. set inclusion);*

A stable extension *if it sufficiently attacks every fuzzy argument in* $\mathcal{A}_0 \setminus S$*.*

Example 4.4. (Continued to Example 4.3) *Consider the fuzzy set* S *based on the RAS calculated by* ras_5*.*

C_1 *and* C_2 *are not attacked. Then,* $(C_1, 0.6)$ *and* $(C_2, 0.7)$ *is defended by any fuzzy argument. Hence,* S *defends them.*

B is attacked by C_1 and C_2. However,

$$\begin{aligned}
&RAS_5((C_1, 0.6), (B, 0.3)) = ras_5(0.6, 0.9, 0.3) \\
&= \max\{0, \min\{0.6, 0.9\} + 0.3 - 1\} = 0, \text{ and} \\
&RAS_5((C_2, 0.7), (B, 0.3)) = ras_5(0.7, 0.7, 0.3) \\
&= \max\{0, \min\{0.7, 0.7\} + 0.3 - 1\} = 0.
\end{aligned} \tag{4.5}$$

Hence, $(B, 0.3)$ is defended by any fuzzy argument, i.e., S defends $(B, 0.3)$.

A is attacked by B. From Example 4.3, $(A, 0.7)$ is defended by S. Therefore, S is admissible.

Furthermore, for any $a > 0.7$, we have

$$\begin{aligned}
&RAS_5((B, 0.9), (A, a)) = ras_5(0.9, 0.9, a) \\
&= \max\{0, \min\{0.9, 0.9\} + a - 1\} = 0.9 + a - 1 \\
&> 0.9 + 0.7 - 1 = 0.6.
\end{aligned} \tag{4.6}$$

But

$$\begin{aligned}
&RAS_5((C_1, 0.6), (B, 0.9)) = ras_5(0.6, 0.9, 0.9) \\
&= \max\{0, \min\{0.6, 0.9\} + 0.9 - 1\} = 0.5, \text{ and} \\
&RAS_5((C_2, 0.7), (B, 0.9)) = ras_5(0.7, 0.7, 0.9) \\
&= \max\{0, \min\{0.7, 0.7\} + 0.9 - 1\} = 0.6.
\end{aligned} \tag{4.7}$$

Therefore, S cannot defend (A, a) for $a > 0.7$.

Similarly, S cannot defend (B, b) for $b > 0.3$. As a result, when the RAS is calculated by ras_5, S is the unique complete extension. Hence, it is both grounded and preferred.

It is not difficult to check that S is also stable.

From this complete set S, the judge will trust with a degree of 0.7 that John is guilty, if he adopts the function ras_5. Thus, it is possible for the judge to make a guilty verdict.

4.2. Relation to Gödel fuzzy semantics

In this section, I compare my work to the Gödel fuzzy semantics in [31]. For convenience, the semantics in Subsection 4.1 is named relative fuzzy semantics (r-semantics, for short) since it is based on the relative attack strength. Moreover, the Gödel fuzzy semantics in [31] is abbreviated as g-semantics.

I recall the g-semantics first.

Definition 4.5. (Definitions in [31]) *In an FAF $(Args, Atts, \mathcal{A}_0, \rho)$, suppose $S \subseteq \mathcal{A}_0$ is a fuzzy argument set, and (A, a), (B, b) are two fuzzy arguments with $(B, A) \in Atts$.*

g-tolerable attacks: *If $\min\{b, \rho_{BA}\} + a \leq 1$, then we say (A, a) is tolerably attacked by (B, b). Otherwise, (A, a) is said to be sufficiently attacked by (B, b).*

g-conflict-free: *S is g-conflict-free iff every attack between the fuzzy arguments in S is tolerable.*

Weaken: *A fuzzy argument (A, a) weakens (B, b) to (B, b') through the attack $((A, B), \rho_{AB})$, where $b' = \min\{1 - \min\{a, \rho_{AB}\}, b\}$.*

g-acceptability: *(A, a) is g-defended by S iff for any (B, b) (sufficiently) attacks (A, a), there exists $(C, c) \in S$, such that (C, c) weakens (B, b) to (B, b'), which is no longer sufficiently attacks (A, a).*

Following, a consequent of semantics, e.g., g-admissible extensions, g-complete extensions, and g-preferred extensions, are established in Dung's extension-based way.

Theorem 4.1. *Given an FAF $(Args, Atts, \mathcal{A}_0, \rho)$, suppose $S \subseteq \mathcal{A}_0$ is a fuzzy set of arguments and (A, a) is a fuzzy argument. If the r-semantics is computed by ras_5, i.e., $RAS((B, b), (A, a)) = \max\{0, \min\{b, \rho_{BA}\} + a - 1\}$, then we have*

(1) S is r-conflict-free, if and only if S is g-conflict-free.

(2) S r-defends (A, a), if S g-defends (A, a).

Proof. (1) **Necessity:** Suppose S is r-conflict-free. From Definition 4.1, for all $A, B \in Args$, $RAS((B, S(B)), (A, S(A))) = 0$, i.e., $\max\{0, \min\{S(B), \rho_{BA}\} + S(A) - 1\} = 0$. Hence, $\min\{S(B), \rho_{BA}\} + S(A) - 1 \leq 0$. In other words, $\min\{S(B), \rho_{BA}\} + S(A) \leq 1$. Then, for all $(A, a), (B, b) \in S$, i.e., $a \leq S(A), b \leq S(B)$, the attack from (B, b) to (A, a) is tolerable. As a result, S is g-conflict-free.

Sufficiency: Just reverse the proof process of Necessity.

(2) Suppose S g-defends (A, a). For any $(B, b) \in \mathcal{A}_0$, if $RAS((B, b), (A, a)) = 0$, then for all $C \in Args$, $RAS((C, S(C)), (B, b)) \geq 0 = RAS((B, b), (A, a))$.

If $RAS((B, b), (A, a)) > 0$, i.e., (B, b) sufficiently attacks (A, a), then $a + \min\{b, \rho_{BA}\} > 1$, i.e., $a + \min\{b, \rho_{BA}\} - 1 > 0$. Hence,

$$
\begin{aligned}
&RAS((B, b), (A, a)) \\
=&\max\{0, \min\{b, \rho_{BA}\} + a - 1\} \\
=&a + \min\{b, \rho_{BA}\} - 1 > 0.
\end{aligned} \tag{4.8}
$$

On the other hand, from the g-acceptability, there is some $(C, c) \in S$ such that (C, c) weakens (B, b) to (B, b') and (B, b') does not sufficiently attack (A, a). Because $(C, c) \in S$, we have $c \leq S(C)$. Hence, $(C, S(C))$ weakens (B, b) to (B, b''), and (B, b'') does not sufficiently attack (A, a). Therefore, $a + \min\{b'', \rho_{BA}\} \leq 1$. Together with $a + \min\{b, \rho_{BA}\} > 1$, we have $b'' < \min\{b, \rho_{BA}\}$, and

$$
\begin{aligned}
1 \geq& a + \min\{b'', \rho_{BA}\} \\
=& a + b'' = a + \min\{1 - \min\{S(C), \rho_{CB}\}, b\} \\
=& a + 1 - \min\{S(C), \rho_{CB}\}.
\end{aligned} \tag{4.9}
$$

Hence, $a \leq \min\{S(C), \rho_{CB}\}$. Together with Eq (4.8), we have $\min\{S(C), \rho_{CB}\} + b - 1 \geq a + b - 1 > 0$. As a result, we have

$$
\begin{aligned}
&RAS((C, S(C)), (B, b)) \\
=&\max\{0, \min\{S(C), \rho_{CB}\} + b - 1\} \\
=&\min\{S(C), \rho_{CB}\} + b - 1 \\
\geq& a + \min\{b, \rho_{BA}\} - 1 \\
=&RAS((B, b), (A, a)).
\end{aligned} \tag{4.10}
$$

It means that S r-defends (A, a). □

Following from this theorem, the g-semantics, e.g., g-admissible extensions, g-complete extensions, and g-preferred extensions, are r-admissible sets based on ras_5.

When Dung's classical extensions are read as fuzzy sets, every Dung's extension belongs to the corresponding kind of g-extensions [31]. Therefore, my r-semantics is also compatible with Dung's semantics. The papers [21, 30] put forward extensions of FAFs in form of fuzzy sets, including x-conflict-free sets, y-admissible extensions, y-preferred extensions, and z-stable extensions. These extensions depend on extra numbers $x, y, z \in [0,\ 1]$. This is different from my r-semantics.

5. Discussion

In this paper, I have done two jobs. The first is the proposal and research of the RAS in Section 3. The other is the establishment of the r-semantics in Section 4.

In [30, 31], extension-based FAF semantics are also established in the form of fuzzy sets. In Section 4.2, it is proven that my r-semantics cover Dung's original semantics in [3] and the Gödel semantics in [31]. In [20], fuzzy reinstatement labelings are introduced and computed recursively by convergent sequences. In [31], each fuzzy reinstatement labeling in [20] is proven to be a preferred extension in [31]. As a result, my r-semantics also cover the fuzzy reinstatement labelings in [20].

In order to well apply the RAS, nine basic properties are proposed. These properties are similar to the principles in [33], which are summarized for the revision of the degrees. On one hand, when an RAS is a requisite for a concrete scenario, it can be built under the guidance of these properties. This is similar to the role of the principles in [33]. On the other hand, some of the nine properties in this paper can be traced back to the principles [33], such as Properties 3.1–3.5.

In addition, it is necessary to note that though RAS is a useful tool for argumentation semantics with values, there are AFs or semantics that do not suit to apply RAS. For example, in [23], a weighted AF is transferred to a Dung's AF, where attacks whose strength are less than a budget β are neglected and the values of the remaining attacks are also omitted. In this case, there are no values to be adjusted. Thus, the RAS cannot take effect. Similarly, in [32], the probabilistic AFs are transferred into a family of Dung's AFs. Then, extension semantics are computed without values in each Dung's AF. In this case, the RAS also cannot play a role. Therefore, the RAS is suitable only for the semantics, which is built by modifying the degrees according to the attackers' degrees.

There are some limitations of my work. For example, the cooperation of some RASs has not been studied, though I implicitly aggregate some RASs when building the new semantics. As a result, in the new semantics, the Gödel t-norm seems indispensable. In fact, if the cooperation is fully studied, there may be some other ways to define acceptability.

6. Conclusions

In the literature, the extent to which the value of an argument is adjusted is determined by its attackers. However, the effect of a single attacker has not been specially studied. This paper is an investigation of this aspect.

First, I formally study the attack strength between fuzzy arguments, named RAS. RAS is calculated by a three-variable function *ras*. Nine properties of RAS are proposed to promote its application. Moreover, as an application, a new semantics is established based on RAS. The Gödel semantics in [31] is proven to coincide with the new semantics with ras_5.

There are three contributions of this work. First, I introduce the notion of RAS and represent it by a three-variable function. It is the major contribution and provides a new tool for the study of argumentation semantics.

The second contribution is the research of the properties of RAS. People can build RAS by choosing suitable properties according to the practical requirement in application. Hence, these properties provide a foundation for building RAS.

The last contribution is the establishment of the new semantics. On one hand, it shows that the RAS can be applied to build fuzzy semantics. On the other hand, it enriches the argumentation theory by providing new kinds of semantics.

This paper is the beginning of the formal research on RAS. There are some limitations of this work. For example, the properties of RAS have not been fully studied, and the cooperation of RAS has not been discussed. In the future, I will attempt to overcome these shortcomings, and additional semantics and properties of RAS will be discussed.

Use of Generative-AI tools declaration

The author declares he has not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11601288) and the Natural Science Foundation of Shandong (ZR2016AQ21).

Conflict of interest

The author declares no conflicts of interest.

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