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Research article

Nonparametric bias reduction of diffusion function in stochastic volatility models

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Abstract: This paper proposed a novel bias correction method based on nonparametric kernel estimator of the diffusion function in stochastic volatility models. In the case of fixed time span, the asymptotic bias of kernel estimation and the proposed nonparametric estimation of diffusion function have been developed. The results showed that conventional kernel-based estimators of the diffusion function suffer from nonvanishing discretization bias, primarily due to the local constant approximation in Nadaraya-Watson regression. We addressed this limitation by developing a dual-stage estimation method that incorporated nonparametric drift estimation into the diffusion function estimation procedure, thereby eliminating the dominant first-order bias term caused by discretization errors compared to traditional kernel estimation. Through rigorous theoretical analysis, the weak consistency and asymptotic normality of the new proposed estimator under mild regularity conditions were established. Furthermore, simulation studies and empirical analysis were provided to evaluate the finite sample performance of the proposed method. The proposed method offered a more robust tool for modeling volatility in financial time series, with significant applications in derivative pricing and risk management.

Keywords: stochastic volatility models; diffusion function; bias reduction; nonparametric estimator; consistency; asymptotic normality
Mathematics Subject Classification: 62G05, 62G20

1. Introduction

As an important mathematical tool, the stochastic volatility model has been widely applied in various fields, including natural sciences and social sciences. For example, Huang et al. [1] applied the stochastic volatility modeling approach to the environmental field, effectively addressing the challenges of the continuity of environmental datasets in both time and space. Wang et al. [2] applied this model to the field of hydrological research, which better described the runoff series with time-varying variance and improved the accuracy of capturing the occurrence timing of runoff peaks. In the financial field, the application of the stochastic volatility model is particularly extensive and exhaustive. It not only significantly enhances the precision of option product pricing but also plays an irreplaceable role in core areas such as risk management and asset allocation, providing a scientific basis for the rationality of asset planning and the formulation of risk aversion strategies. Bekierman and Gribisch [3] proposed a mixed-frequency stochastic volatility model for intraday returns. He and Lin [4] studied the pricing problem of European options in the presence of liquidity risk with the aid of the stochastic volatility model. Abi Jaber [5] investigated the stochastic volatility model based on Gaussian processes, opening the door to the rapid approximation of the joint density of stocks and their realized variances and the pricing of derivatives through Fourier inversion techniques.

In the stochastic volatility model, the diffusion coefficient (the volatility function) plays a crucial role. It describes the dynamic changes of volatility, and the estimation of the diffusion function is a prerequisite for applying this model in practice. Therefore, scholars have carried out a great deal of exploration regarding the estimation of the diffusion coefficient. In terms of parametric estimation, Hosszejni and Kastner [6] derived new algorithms for the central and noncentral parameterizations of the stochastic volatility model with a leverage effect. Sadok and Masmoudi [7] introduced a new parametric specification and extended form, in which the diffusion of stock returns is generated by a translated compound Poisson distribution. However, the accuracy of parametric estimation methods highly depends on the preset model structure and distribution assumptions, leading to the risk of model misspecification and insufficient ability to capture complex features in financial time series, such as nonlinear dynamics, structural breaks, and fat tails. This, in turn, gives rise to estimation biases and limitations in adaptability, significantly affecting the accuracy of volatility forecasting and derivative pricing. Therefore, nonparametric estimation methods, with their greater flexibility, have become an indispensable complement.

In terms of nonparametric statistical inference, Bandi and Phillips [8] introduced a fully nonparametric diffusion model estimation method. Their fully nonparametric estimation method provided more accurate model support for estimating the volatility function. Through this approach, they addressed the bias issues inherent in traditional nonparametric estimation, particularly when dealing with high-dimensional diffusion models. Renò [9, 10] introduced a nonparametric estimator for the stochastic volatility model by utilizing the technique of estimating integrated volatility with high-frequency data, and also proposed a new type of fully nonparametric estimator for the diffusion coefficient of the continuous-time model based on Fourier analysis and realized volatility. Comte, Genon-Catalot, and Rozenholc [11] selected and proposed a nonparametric least squares estimator for the stochastic volatility model from a collection of functions in a finite-dimensional space whose dimension was determined by a data-driven method. Kanaya and Kristensen [12] employed adaptive nonparametric filtering techniques during the estimation process, reducing bias caused by model assumptions and therefore improves the accuracy and robustness of volatility estimation. Bandi and Renò [13] proposed a new nonparametric estimation method to reduce bias, which can handle different types of volatility models. This research not only enriched the nonparametric estimation theory but also provided a practical application framework, further advancing the development of nonparametric volatility estimation.

Although nonparametric estimation methods have many advantages, the existence of estimator bias is a problem that cannot be ignored. Reducing the estimation bias of the diffusion coefficient is crucial for improving the accuracy of the model. From the perspective of the model's description of asset price volatility, a smaller bias means a more realistic reflection of the volatility characteristics, which in turn helps investors and financial institutions to more accurately evaluate the prices of financial derivatives and manage risks. At the level of risk assessment and management, the reduction of bias can significantly improve the accuracy of the estimation of the range of asset price fluctuations. Therefore, in order to reduce the estimation bias, scholars have made many attempts. Related research includes improvements to nonparametric estimation methods themselves as well as effective ways to address biases in the volatility function.

Nicolau [14] proposed an effective bias correction method specifically aimed at the nonparametric estimation of the diffusion coefficient. This work provided a theoretical foundation for subsequent volatility function estimation, particularly in advancing methods to reduce systematic biases in nonparametric estimations. In addition, Iscoe and Lakhany [15] proposed an alternative simulation strategy. Takaishi [16] introduced a bias correction parameter. Sahalia [17] proposed an implicit stochastic volatility model and, by combining it with nonparametric regression techniques, effectively reduced the biases caused by the subjectivity and indirectness of the model specification.

In this paper, we conduct bias correction based on the nonparametric Nadaraya-Watson estimator of the diffusion coefficient in the stochastic volatility model. By incorporating the nonparametric estimator of the drift coefficient into the Nadaraya-Watson estimator of the diffusion coefficient, we construct a new nonparametric estimator. Through rigorous analysis and proof, we verify that under mild regularity conditions, this estimator not only satisfies consistency and asymptotic normality but also exhibits a smaller bias when compared with the traditional kernel-type estimator.

The paper is structured as follows. In the next section, we will briefly introduce the stochastic volatility model and outline some general conditions and assumptions used in this paper. Section 3 presents a new nonparametric estimator for the diffusion coefficient in the stochastic volatility model, which reduces the bias of the diffusion coefficient. It also discusses the consistency and asymptotic normality of the nonparametric estimator. The proof of the main results can be found in Section 4.

2. Stochastic volatility models and conditions

In this paper, we consider a stochastic volatility model defined by the following equations:

$$\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t, \\ d\sigma_t^2 = \alpha(\sigma_t^2)dt + \beta(\sigma_t^2)dZ_t, \end{cases} \tag{2.1}$$

where $\{W_t\}$ and $\{Z_t\}$ are two standard Brownian motions which may be correlated, and $\{\mu_t\}$ and $\{\sigma_t\}$ are stochastic processes that are right-continuous everywhere and have left limits everywhere. The process $\{\sigma_t^2\}$ is usually referred to as the (spot) volatility process of $\{X_t\}$. The processes $\{\alpha(\sigma_t^2)\}$ and $\{\beta(\sigma_t^2)\}$ are the drift functions and diffusion functions, respectively.

Our objective is to consider nonparametric estimation of the diffusion function $\beta(\cdot)$ in the stochastic differential equation (2.1). Through the paper, assume the process $\{X_t\}$ is observed over the time span $[0, T]$ and a discrete sample $\{X_{i\Delta}, i = 1, \cdots, N\}$ at N equally spaced time points, where $\Delta = \frac{T}{N}$ is the step size between observations. By the method of Bandi and Phillips [8], we can get the following kernel estimators for the drift function and diffusion function:

$$\tilde{\alpha}(x) = \frac{\sum_{j=1}^{N-1} K(\frac{\sigma^2_{j\Delta}-x}{h_N})(\sigma^2_{(j+1)\Delta} - \sigma^2_{j\Delta})}{\Delta \sum_{j=1}^{N} K(\frac{\sigma^2_{j\Delta}-x}{h_N})},$$

$$\hat{\beta}^2_0(x) = \frac{\sum_{j=1}^{N-1} K(\frac{\sigma^2_{j\Delta}-x}{h_N})(\sigma^2_{(j+1)\Delta} - \sigma^2_{j\Delta})^2}{\Delta \sum_{j=1}^{N} K(\frac{\sigma^2_{j\Delta}-x}{h_N})},$$

where $K(\cdot)$ is a standard kernel function and h_N is defined as bandwidth.

However, in the stochastic volatility model (2.1), the point volatility σ_t is unobservable, which means that the estimation of these two functions involves a latent stochastic process, and we must consider the estimator of the point volatility first. In this paper, the unobservable spot point volatility $\sigma_{j\Delta}$ is replaced by suitable estimators $\tilde{\sigma}_{j\Delta}$ for all $j = 1, \cdots, N$. For a detailed discussion of the estimation of the unobservable volatility process σ_t, readers can refer to Renò [10] or Kristensen [18]. Throughout the paper, we assume that

$$\max_{1\leqslant j\leqslant N} |\tilde{\sigma}^2_{j\Delta} - \sigma^2_{j\Delta}| = O_p(\vartheta_N), \quad N \to \infty, \tag{2.2}$$

for some error bound $\vartheta_N \to 0$.

The results of our lemmas and theorems are mainly based on the following assumptions.

Condition 1. (1) The kernel function $K(\cdot)$ is symmetric and continuously differentiable and satisfies

$$\int_R K(x)dx = \int_R x^2 K(x)dx = 1, \int_R xK(x)dx = 0, \int_R K^2(x)dx < +\infty.$$

(2) There exist a constant $\bar{K} \in (0, \infty)$ such that $\sup_{x\in R}|K(x)| \leq \bar{K}$, $\sup_{x\in R}|K'(x)| \leq \bar{K}$.

(3) There exist a constant $c > 0$ such that $|K'(x)|$ is monotonically nondecreasing on $(-\infty, -c]$ and monotonically nonincreasing on $[c, +\infty)$.

Condition 2. The process $\{\sigma^2_t\}$ has range $I = (0, \bar{\sigma})$, where $\bar{\sigma} \leq \infty$. Let

$$s(y) = \exp\{-2 \int_c^y \alpha(u)\beta^{-2}(u)du\}$$

be the scale density function (c is an arbitrary point inside I), $S(x) = \int_c^x s(y)dy$, and assume that

(1) $\lim_{x\to 0} S(x) = -\infty, \ \lim_{x\to\bar{\sigma}} S(x) = \infty$;

(2) $\int_0^{\bar{\sigma}} m(x)dx < \infty$, where $m(x) = (\beta^2(x)s(x))^{-1}$ is a speed density function;

(3) $\sigma^2_0 = x$ has distribution P^0, where P^0 is the invariant distribution of the ergodic process $\{\sigma^2_t\}$.

Remark 2.1. *The (1) and (2) in Condition 2 assure that $\{\sigma^2_t\}$ is ergodic and has invariant distribution (Kanaya and Kristensen [12]). We will henceforth denote $\pi(x)$ as the invariant density of $\{\sigma^2_t\}$. Condition 2 assures that $\{\sigma^2_t\}$ is stationary.*

Condition 3. $\alpha(x)$ and $\beta^2(x)$ are twice continuously differentiable and satisfy

(1) $\beta^2(\cdot) > 0$ on I;

(2) $E[\sigma^4_t] < \infty$, $E[|\alpha(\sigma^2_t)|^q] < \infty$, and $E[|\beta(\sigma^2_t)|^q] < \infty$ for some $q > 2$;

(3) There exist constants $\lambda > 0, \rho > 0$, and $C > 0$, such that $E\left(\left|\sigma_t^2 - \sigma_s^2\right|^\lambda\right) \leqslant C|t-s|^{1+\rho}$.

Condition 4. Let interval $I = (0, \bar{\sigma})$ be the state space of $\{\sigma_t^2\}$, and we assume that

$$\lim_{x\to\bar{\sigma}} \sup([\alpha(x)/\beta(x)] - [\beta'(x)/2]) < 0,$$

$$\lim_{x\to 0} \sup([\alpha(x)/\beta(x)] - [\beta'(x)/2]) > 0.$$

Remark 2.2. *Condition 4 assures that the process is mixing.*

Condition 5. For a given error bound ϑ_N in Eq (2.2) and ρ and λ given in Condition 3, for any $\gamma \in (0, \rho/\lambda)$, assume that

(1) $\vartheta_N/\Delta^{1-\gamma} \to 0$, $\Delta^\gamma/h_N \to 0$, and $Nh_N \to \infty$;

(2) $\vartheta_N N({h_N}^{-1} + h_N\Delta^{-2+2\gamma}) \to 0$, $Nh_N^3 \to 0$, and $Nh_N\Delta^{2\gamma} \to 0$.

Remark 2.3. *Condition 5 gives the regularity conditions required to determine the asymptotic properties of the diffusion term estimator. An additional condition involving ϑ_N is introduced to ensure that the preliminary estimation error of $\{\sigma_t^2\}$ does not affect its asymptotic properties. For comprehensive technical discussions of these regularity conditions, readers may consult Bandi and Phillips [8] and Kanaya and Kristensen [12].*

3. Nonparametric estimator and asymptotic results

Substituting the unobservable spot volatility $\sigma_{j\Delta}$ by its suitable estimators $\tilde{\sigma}_{j\Delta}$ for all $j = 1, 2, \cdots, n$, the resulting Nadaraya-Watson estimators of $\alpha(\cdot)$ and $\beta(\cdot)$ take the form

$$\hat{\alpha}(x) = \frac{\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}_{j\Delta}^2 - x}{h_N})(\tilde{\sigma}_{(j+1)\Delta}^2 - \tilde{\sigma}_{j\Delta}^2)}{\Delta\sum_{j=1}^{N} K(\frac{\tilde{\sigma}_{j\Delta}^2 - x}{h_N})},$$

$$\hat{\beta}_1^2(x) = \frac{\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}_{j\Delta}^2 - x}{h_N})(\tilde{\sigma}_{(j+1)\Delta}^2 - \tilde{\sigma}_{j\Delta}^2)^2}{\Delta\sum_{j=1}^{N} K(\frac{\tilde{\sigma}_{j\Delta}^2 - x}{h_N})}.$$

According to Kanaya and Kristensen [12], under mild regularity conditions, the above nonparametric kernel type estimators $\hat{\alpha}(x)$ and $\hat{\beta}_1^2(x)$ for drift and diffusion functions achieve consistency and asymptotic normality in high-frequency settings.

To reduce the bias of the Nadaraya-Watson kernel estimator $\hat{\beta}_1^2(x)$ of the diffusion function, we first establish a decomposition of its asymptotic bias for a fixed time span Δ.

Theorem 3.1. *If Conditions 1–4 and Condition 5(1) hold, assume that $h_N \to 0, Nh_N \to +\infty$ as $N \to +\infty$, and $E[(\sigma_\Delta^2 - \sigma_0^2)^4] < +\infty$, then*

$$\hat{\beta}_1^2(x) \xrightarrow{P} \beta^2(x) + \alpha^2(x)\Delta + f(x)\Delta + O(\Delta^2),$$

where $f(x) = \beta^2(x)\alpha'(x) + \alpha(x)\beta(x)\beta'(x) + \frac{1}{2}\beta^2(x)(\beta'(x))^2 + \frac{1}{2}\beta^3(x)\beta''(x)$.

Remark 3.1. *As rigorously established in Nicolau [14] (Theorem 1, Eq (3)), when the discretization step Δ is fixed, the conventional kernel estimator converges to*

$$\beta^2(x) + \alpha^2(x)\Delta + f(x)\Delta + O(\Delta^2),$$

where the explicit $\alpha^2(x)\Delta$ term arises from the Euler discretization error inherent in the local constant approximation. This contrasts with the $\Delta \to 0$ literature (e.g., Bandi and Phillips [8]) where $\alpha^2(x)\Delta$ vanishes asymptotically, leaving only derivatives of $\beta(x)$ in the bias. Our Theorem 3.1 mirrors Nicolau's conclusion, confirming that the drift function's impact is an inherent property of discrete-time estimation.

Theorem 3.1 reveals that the discretization bias contains two distinct components, $\alpha^2(x)\Delta$ and $f(x)\Delta$, where $f(x)$ involves derivatives of both drift and diffusion coefficients. Therefore, in order to reduce the bias, we consider the following bias reduction estimator of $\beta^2(x)$,

$$\hat{\beta}^2(x) = \frac{\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})[\tilde{\sigma}^2_{(j+1)\Delta} - \tilde{\sigma}^2_{j\Delta} - \hat{\alpha}(x)\Delta]^2}{\Delta \sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})}. \tag{3.1}$$

Similar to Theorem 3.1, the asymptotic bias of the bias-adjusted estimator $\hat{\beta}^2(x)$ for a fixed time span Δ can be presented as follows:

Theorem 3.2. *If Conditions 1–4 and Condition 5(1) hold, assume that $h_N \to 0, Nh_N \to +\infty$, and $N\Delta \to +\infty$ as $N \to +\infty$ and $E[(\sigma^2_\Delta - \sigma^2_0 - \alpha(\sigma^2_0)\Delta)^4] < +\infty$, then*

$$\hat{\beta}^2(x) \xrightarrow{P} \beta^2(x) + f(x)\Delta + O(\Delta^2),$$

where $f(x) = \beta^2(x)\alpha'(x) + \alpha(x)\beta(x)\beta'(x) + \frac{1}{2}\beta^2(x)(\beta'(x))^2 + \frac{1}{2}\beta^3(x)\beta''(x)$.

Remark 3.2. *From the results of Theorems 3.1 and 3.2, it is not difficult to find that, compared with the traditional kernel estimation, the term $\alpha^2(x)\Delta$ is reduced in the bias term of the new estimator.*

In fact, the total error of the modified nonparametric diffusion estimator $\hat{\beta}^2(x)$ can be written as:

$$\hat{\beta}^2(x) - \beta^2(x) = [\hat{\beta}^2(x) - \tilde{\beta}^2(x)] + [\tilde{\beta}^2(x) - \beta^2(x)], \tag{3.2}$$

where

$$\tilde{\beta}^2(x) = \frac{\sum_{j=1}^{N-1} K(\frac{\sigma^2_{j\Delta}-x}{h_N})[\sigma^2_{(j+1)\Delta} - \sigma^2_{j\Delta} - \tilde{\alpha}(x)\Delta]^2}{\Delta \sum_{j=1}^{N} K(\frac{\sigma^2_{j\Delta}-x}{h_N})},$$

and $\tilde{\alpha}(x)$, the infeasible estimator of drift function based on latent volatility $\{\sigma^2_t, t \geq 0\}$, is defined in Section 2.

The asymptotic properties of the second term $\tilde{\beta}^2(x) - \beta^2(x)$ can be derived from arguments similar to those presented in Nicolau [14], given the regularity conditions outlined above. Subsequently, the convergence of the first term $\hat{\beta}^2(x) - \tilde{\beta}^2(x)$ will be the primary focus of our discussion. The following theorem will give consistency and asymptotic normality of new estimator $\hat{\beta}^2(x)$ defined in Eq (3.1):

Theorem 3.3. *If Conditions 1–5(1) hold, $\Delta \to 0, h_N \to 0, Nh_N \to +\infty, Nh_N\Delta \to +\infty$ as $N \to +\infty$, and $E[(\sigma^2_\Delta - \sigma^2_0 - \alpha(\sigma^2_0)\Delta)^4] < +\infty$, then*

$$\hat{\beta}^2(x) \xrightarrow{P} \beta^2(x).$$

Furthermore, assume that Condition 5(2), and $Nh_N\Delta^2 \to 0$ as $N \to +\infty$, then

$$\sqrt{Nh_N}(\hat{\beta}^2(x) - \beta^2(x)) \xrightarrow{D} N\left(0, \frac{2K_2\beta^4(x)}{\pi(x)}\right),$$

where $K_2 = \int_R K^2(u)du$.

4. Lemmas and proofs

In this section, we establish foundational lemmas and rigorously develop the proofs for the above-mentioned theorems.

Lemma 4.1 (Kanaya and Kristensen [12]). *If Condition 3(3) holds, for any $\gamma \in (0, \rho/\lambda)$, there exists some constant $D > 0$ such that*

$$\Pr\left(\omega \in \Omega \mid \exists \bar{\Delta}(\omega)\ s.t. \sup_{|t-s|\in(0,\bar{\Delta}(\omega));s,t\in[0,\infty)} \frac{|\sigma^2_t(\omega) - \sigma^2_s(\omega)|}{|t-s|^\gamma} \leqslant D\right) = 1.$$

Lemma 4.2. *If Conditions 1–4 hold, and there exists some $\bar{q} > 0$ such that $(\vartheta_N/h_N)^{\bar{q}} = O(h_N)$, assume that $N\Delta \to +\infty$, $h_N \to 0$, $\Delta \to 0$, and $\vartheta_N/\Delta^{1-\gamma} \to 0$ as $N \to +\infty$, then*

$$\hat{\beta}^2(x) - \tilde{\beta}^2(x) = O_p(\vartheta_N/h_N) + O_p(\vartheta_N/\Delta^{1-\gamma}),$$

where ϑ_N is a given error bound in Eq (2.2).

The Proof of Lemma 4.2. Consider the structure of the new proposed bias reduction estimator (3.1) of diffusion function $\beta^2(x)$, and let

$$A_1 = \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\sigma^2_{j\Delta}-x}{h_N})\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})} - \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\sigma^2_{j\Delta}-x}{h_N})\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\sigma^2_{j\Delta}-x}{h_N})};$$

$$A_2 = \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})} - \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\sigma^2_{j\Delta}-x}{h_N})\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})};$$

$$A_3 = \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})\frac{[\tilde{\sigma}^2_{(j+1)\Delta}-\tilde{\sigma}^2_{j\Delta}-\hat{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})} - \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})},$$

then we have

$$\hat{\beta}^2(x) - \tilde{\beta}^2(x) = A_1 + A_2 + A_3.$$

For the first part A_1, we have

$$A_1=\frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1}K(\frac{\sigma^2_{j\Delta}-x}{h_N})\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N}K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})\times\frac{1}{Nh_N}\sum_{j=1}^{N}K(\frac{\sigma^2_{j\Delta}-x}{h_N})}\times\frac{1}{Nh_N}\sum_{j=1}^{N}\left[K\left(\frac{\sigma^2_{j\Delta}-x}{h_N}\right)-K\left(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N}\right)\right].$$

By Bandi and Phillips [8], we have

$$\frac{1}{Nh_N}\sum_{j=1}^{N}K\left(\frac{\sigma^2_{j\Delta}-x}{h_N}\right)=\pi(x)+o_p(1). \tag{4.1}$$

By Nicolau [14], we have

$$\frac{1}{Nh_N}\sum_{j=1}^{N-1}K\left(\frac{\sigma^2_{j\Delta}-x}{h_N}\right)\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}=\beta^2(x)\pi(x)+o_p(1). \tag{4.2}$$

By Kanaya and Kristensen [12], we have

$$\frac{1}{Nh_N}\sum_{j=1}^{N}\left[K\left(\frac{\sigma^2_{j\Delta}-x}{h_N}\right)-K\left(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N}\right)\right]=O_p\left(\frac{\vartheta_N}{h_N}\right). \tag{4.3}$$

By (4.1), (4.2), and (4.3), we can get $A_1=O_P(\vartheta_N/h_N)$.

For the second part A_2, we have

$$A_2=\frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1}\left[K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})-K(\frac{\sigma^2_{j\Delta}-x}{h_N})\right]\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N}K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})}.$$

By mean-value theorem, the same arguments as those for Eq (B.13) in Kanaya and Kristensen [12], and the fact that

$$\frac{1}{N}\sum_{j=1}^{N-1}\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}=O_p(1),$$

we can get

$$\frac{1}{Nh_N}\sum_{j=1}^{N-1}\left[K\left(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N}\right)-K\left(\frac{\sigma^2_{j\Delta}-x}{h_N}\right)\right]\frac{[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}=O_p\left(\frac{\vartheta_N}{h_N}\right).$$

On the other hand, by (4.1) and (4.3), we have

$$\frac{1}{Nh_N}\sum_{j=1}^{N-1}K\left(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N}\right)=\pi(x)+o_p(1). \tag{4.4}$$

Thus, $A_2=O_P(\vartheta_N/h_N)$.

For the third part A_3, we have

$$A_3 = \frac{\frac{1}{Nh_N}\sum_{j=1}^{N-1} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})\frac{[\tilde{\sigma}^2_{(j+1)\Delta}-\tilde{\sigma}^2_{j\Delta}-\hat{\alpha}(x)\Delta]^2-[\sigma^2_{(j+1)\Delta}-\sigma^2_{j\Delta}-\tilde{\alpha}(x)\Delta]^2}{\Delta}}{\frac{1}{Nh_N}\sum_{j=1}^{N} K(\frac{\tilde{\sigma}^2_{j\Delta}-x}{h_N})}.$$

According to Kanaya and Kristensen [12],we can note that $\hat{\alpha}(x) = \tilde{\alpha}(x) + o_p(1)$, $\hat{\alpha}(x) - \tilde{\alpha}(x) = o_p(1)$, and by Lemma 4.1 and Eq (2.2), we have

$$\begin{aligned}
&\sup_{1\leqslant j\leqslant N-1}\left|[\tilde{\sigma}^2_{(j+1)\Delta} - \tilde{\sigma}^2_{j\Delta} - \hat{\alpha}(x)\Delta]^2 - [\sigma^2_{(j+1)\Delta} - \sigma^2_{j\Delta} - \tilde{\alpha}(x)\Delta]^2\right| \\
=& \sup_{1\leqslant j\leqslant N-1}\left|[\tilde{\sigma}^2_{(j+1)\Delta} - \tilde{\sigma}^2_{j\Delta} - \hat{\alpha}(x)\Delta + \sigma^2_{(j+1)\Delta} - \sigma^2_{j\Delta} - \tilde{\alpha}(x)\Delta]\cdot\right. \\
&\left.[\tilde{\sigma}^2_{(j+1)\Delta} - \tilde{\sigma}^2_{j\Delta} - \hat{\alpha}(x)\Delta - \sigma^2_{(j+1)\Delta} + \sigma^2_{j\Delta} + \tilde{\alpha}(x)\Delta]\right| \\
\leqslant& \left(2\sup_{1\leqslant j\leqslant N}\left|\tilde{\sigma}^2_{j\Delta} - \sigma^2_{j\Delta}\right| + 2\sup_{1\leqslant j\leqslant N-1}\left|\sigma^2_{(j+1)\Delta} - \sigma^2_{j\Delta}\right| + |\hat{\alpha}(x) + \tilde{\alpha}(x)|\,\Delta\right)\cdot \\
&\left(2\sup_{1\leqslant j\leqslant N}\left|\tilde{\sigma}^2_{j\Delta} - \sigma^2_{j\Delta}\right| + |\hat{\alpha}(x) - \tilde{\alpha}(x)|\,\Delta\right) \\
=& [O_p(\vartheta_N) + O_{a.s.}(\Delta^\gamma) + o_p(\Delta)][O_p(\vartheta_N) + o_p(\Delta)] \\
=& O_p(\vartheta_N^2) + O_p(\vartheta_N)O_{a.s.}(\Delta^\gamma).
\end{aligned}$$

Observe that $\vartheta_N/\Delta^{1-\gamma} \to 0$ implies $\vartheta_N/\Delta^\gamma = \vartheta_N/\Delta^{1-\gamma} \times \Delta^{1-2\gamma} \to 0$, so

$$\vartheta_N^2\Delta^{-1} = \vartheta_N/\Delta^{1-\gamma} \times \vartheta_N/\Delta^\gamma \leqslant \vartheta_N/\Delta^{1-\gamma},$$

and considering that Eq (4.4), we have $A_3 = O_P(\vartheta_N/\Delta^{1-\gamma})$. This completes the proof. □

Lemma 4.3 (Nicolau [14])**.** *If Conditions 1–4 hold,* $\Delta \to 0, h_N \to 0, Nh_N \to +\infty, Nh_N\Delta \to +\infty$ *as* $N \to +\infty$ *, and* $E[(\sigma^2_\Delta - \sigma^2_0 - \alpha(\sigma^2_0)\Delta)^4] < +\infty$*, then*

$$\tilde{\beta}^2(x) \xrightarrow{P} \beta^2(x).$$

Furthermore, assume that $Nh_N^3 \to 0$ *and* $Nh_N\Delta^2 \to 0$ *as* $N \to +\infty$*, then*

$$\sqrt{\frac{Nh_N}{2K_2}C_N(x)}\left(\frac{\tilde{\beta}^2(x)}{\beta^2(x)} - 1\right) \xrightarrow{D} N(0,1),$$

where $K_2 = \int_R K^2(u)du$ *and* $C_N(x) = (Nh_N)^{-1}\sum_{j=1}^N K((\sigma^2_{j\Delta} - x)/h_N)$.

The Proof of Theorem 3.1. By Theorem 1 in Nicolau [14], we have

$$\hat{\beta}_0^2(x) \xrightarrow{P} \beta^2(x) + \alpha^2(x)\Delta + f(x)\Delta + O(\Delta^2),$$

and by Kanaya and Kristensen [12], we have

$$\hat{\beta}_1^2(x) - \hat{\beta}_0^2(x) = O_p(\vartheta_N/h_N) + O_p(\vartheta_N/\Delta^{1-\gamma}).$$

This completes the proof. □

The Proof of Theorem 3.2. By Lemma 2 and Theorem 4 in Nicolau [14], the proof of Theorem 2 is similar to that of Theorem 1 and is omitted here. □

The Proof of Theorem 3.3. Consider that decomposition (3.2) of $\hat{\beta}^2(x)$, where we consider the two terms of the right side separately. For the first term of the right side of (3.2), by Condition 5(1), we can note that

$$\frac{\vartheta_N}{\Delta^{1-\gamma}} \to 0, \quad \frac{\Delta^\gamma}{h_N} \to 0,$$

therefore,

$$\vartheta_N < \Delta^{1-\gamma}, \quad \Delta^{1-\gamma} < h_N^{\frac{1-\gamma}{\gamma}},$$

and this implies that

$$\frac{\vartheta_N}{h_N} < \frac{\Delta^{1-\gamma}}{h_N} < \frac{h_N^{\frac{1-\gamma}{\gamma}}}{h_N} = h_N^{\frac{1}{\gamma}-2}.$$

If we choose $0 < \gamma < \frac{1}{2}$, then $h_N^{\frac{1}{\gamma}-2} \to 0$, and it follows that there exists some $\bar{q} = \frac{1}{\gamma} - 2 > 0$ such that $(\vartheta_N/h_N)^{\bar{q}} = O(h_N)$; thus, by Lemma 4.2, we have

$$\hat{\beta}^2(x) - \tilde{\beta}^2(x) = O_p(\vartheta_N/h_N) + O_p(\vartheta_N/\Delta^{1-\gamma}).$$

For the second term of the right side of (3.2), by Lemma 4.3, we have $\tilde{\beta}^2(x) - \beta^2(x) \xrightarrow{P} 0$. This completes the consistency of $\hat{\beta}^2(x)$.

For the asymptotic normality, the bias of the original estimator can be decomposed into two components,

$$\sqrt{Nh_N}(\hat{\beta}^2(x) - \beta^2(x)) = \sqrt{Nh_N}[\hat{\beta}^2(x) - \tilde{\beta}^2(x) + \tilde{\beta}^2(x) - \beta^2(x)].$$

By Lemma 4.2 and Condition 5(2), we can get

$$\begin{aligned}
&\sqrt{Nh_N}[\hat{\beta}^2(x) - \tilde{\beta}^2(x)] \\
= &\sqrt{Nh_N}\left[O_p\left(\frac{\vartheta_N}{h_N}\right) + O_p\left(\frac{\vartheta_N}{\Delta^{1-\gamma}}\right)\right] \\
= &O_p\left(\vartheta_N \sqrt{Nh_N^{-1}}\right) + O_p\left(\vartheta_N \sqrt{Nh_N}\Delta^{-1+\gamma}\right) \\
= &o_p(1),
\end{aligned}$$

and by Lemma 4.3, we have

$$\sqrt{Nh_N}(\tilde{\beta}^2(x) - \beta^2(x)) \xrightarrow{D} N\left(0, \frac{2K_2\beta^4(x)}{\pi(x)}\right).$$

This completes the proof. □

5. Simulation results

In this section, we conduct Monte Carlo simulations to evaluate the finite-sample performance of the proposed bias-reduced estimator $\hat{\beta}^2(x)$ of the diffusion coefficient against the conventional Nadaraya-Watson estimator $\hat{\beta}_1^2(x)$. The data-generating process follows the following stochastic volatility model:

$$\begin{cases} dX_t = \sigma_t dW_t, \\ d\sigma_t^2 = \beta(\alpha - \sigma_t^2)dt + \kappa\sigma_t^2 dZ_t, \end{cases} \tag{5.1}$$

where $\{W_t\}$ and $\{Z_t\}$ are independent standard Brownian motions, X_t denotes the asset price at time t, σ_t^2 represents the volatility of the asset price, and $\beta(\alpha - \sigma_t^2)$ and $\kappa\sigma_t^2$ are the drift coefficient and diffusion coefficient of volatility, respectively. Throughout the parameters are calibrated to financial data: $\alpha = 0.476, \beta = 0.510, \kappa = \sqrt{0.0518}$, the simulation time span is fixed at $T = 250$, which roughly corresponds to 250 business days per year, and the sampling time interval is $\bar{\Delta} = \frac{1}{60\times 24}$, which means sampling every one minute.

We discretize the model via the Euler-Maruyama scheme:

$$\begin{cases} \Delta X_{i\bar{\Delta}} = \sigma_{(i-1)\bar{\Delta}} \sqrt{\bar{\Delta}}\varepsilon_{1,i}, \\ \Delta\sigma_{i\bar{\Delta}}^2 = \beta(\alpha - \sigma_{(i-1)\bar{\Delta}}^2)\bar{\Delta} + \kappa\sigma_{(i-1)\bar{\Delta}}^2 \sqrt{\bar{\Delta}}\varepsilon_{2,i}, \end{cases} \tag{5.2}$$

where $\{\varepsilon_{1,i}\}$ and $\{\varepsilon_{2,i}\}$ are two independent standard normal random variables.

In this simulation, we select the Gaussian kernel $K(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ and the common bandwidth $h_N = cSN^{-\frac{1}{5}}$, where c is an adjustable constant, S is the sample standard deviation, and N is the sample size. The pseudo-sampling intervals are selected as $\Delta = \frac{1}{2}, \Delta = \frac{1}{4}$, and $\Delta = \frac{1}{8}$, respectively.

Figure 1 depicts the sample paths of the price process X_t and its stochastic volatility σ_t^2. Figure 2 compares the performance of the conventional Nadaraya-Watson diffusion estimator $\hat{\beta}_1^2(x)$ with the proposed bias-reduced estimator $\hat{\beta}^2(x)$, demonstrating the superior accuracy of $\hat{\beta}^2(x)$ particularly pronounced at lower volatility levels ($\sigma^2(x) < 0.3$), where it yields greater improvement over the $\hat{\beta}_1^2(x)$. Both estimators benefit from higher pseudo-sampling frequencies ($\Delta \downarrow$), and $\hat{\beta}^2(x)$ exhibits steeper error decay.

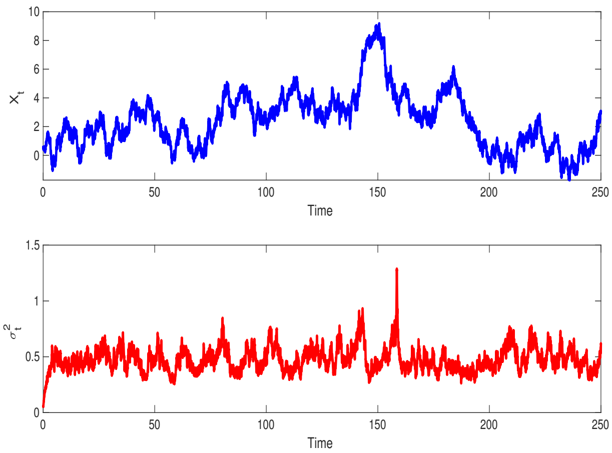

Figure 1. The sample paths of the process X_t and σ_t^2.

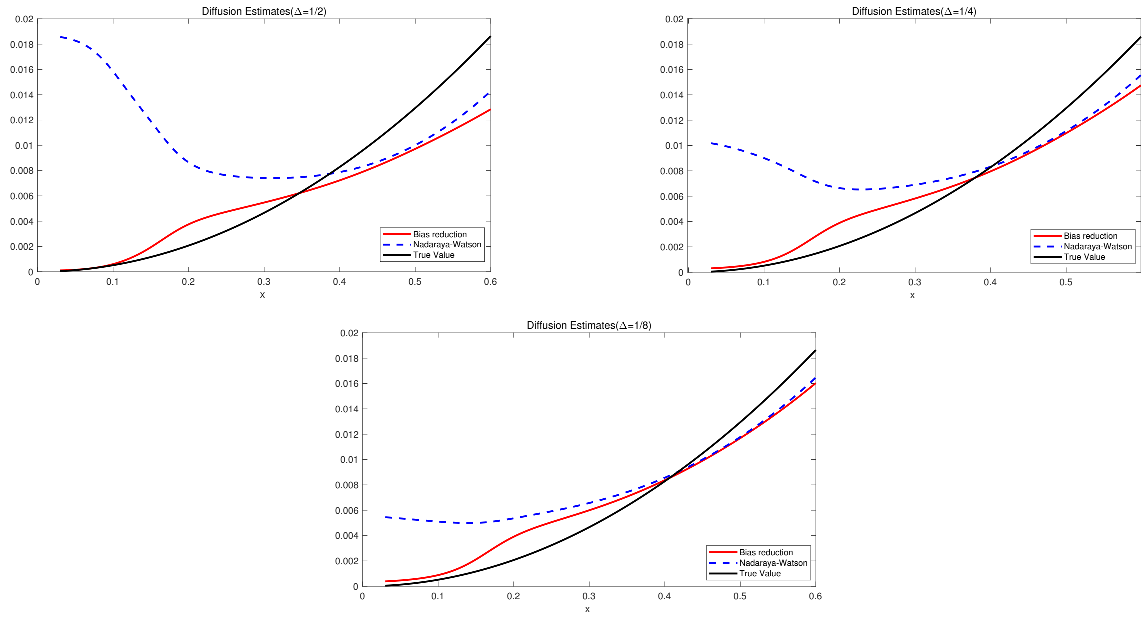

Figure 2. Behavior of the diffusion estimators $\hat{\beta}_1^2(x)$ and $\hat{\beta}^2(x)$.

Table 1 compares the mean squared error (MSE) performance of the bias reduction estimator $\hat{\beta}^2(x)$ and the Nadaraya-Watson estimator $\hat{\beta}_1^2(x)$, indicating superior accuracy of $\hat{\beta}^2(x)$ across varying sampling frequencies, while both estimators show improved performance with increasing sampling frequency.

Table 1. The MSE ($\times 10^{-4}$) of diffusion function estimators.

pseudo-sampling interval	$\hat{\beta}_1^2(x)$	$\hat{\beta}^2(x)$
$\Delta = \frac{1}{2}$	0.7711	0.0608
$\Delta = \frac{1}{4}$	0.2540	0.0318
$\Delta = \frac{1}{8}$	0.0884	0.0197

6. Empirical analysis

To validate the performance of the proposed bias reduction estimator in real financial markets, this section conducts an empirical analysis using high-frequency data of the Shanghai Composite Index. The sample data consists of 5-minute high-frequency transaction data of the Shanghai Composite Index from September 25, 2023 to September 24, 2024, sourced from the Wind Financial Data Platform. The dataset contains 241 valid trading days, with forty-eight 5-minute intervals per day (9:30-11:30 and 13:00-15:00), resulting in a total of 11,568 high-frequency observations. The data from September 25, 2023 to June 14, 2024 (8098 high-frequency observations) is used as the in-sample data for fitting the stochastic volatility model (2.1) and estimation, and the data spanning from June 14, 2024 to September 24, 2024 (3470 high-frequency observations) is used as the out-of-sample data to evaluate the strategy performance.

For kernel estimation, we employ a Gaussian kernel $K(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ with bandwidth $h_N =$

$cSN^{-\frac{1}{5}}$, where c is an adjustable constant, S is the sample standard deviation, and N is the sample size.

Figure 3 illustrates the logarithmic price series of the Shanghai Composite Index during the sample period, reflecting typical nonstationary characteristics of financial time series such as intermittent sharp fluctuations, while the first-differenced log returns (Figure 4) show stationarity, verified by the Augmented Dickey–Fuller (ADF) test with a p-value less than 0.05.

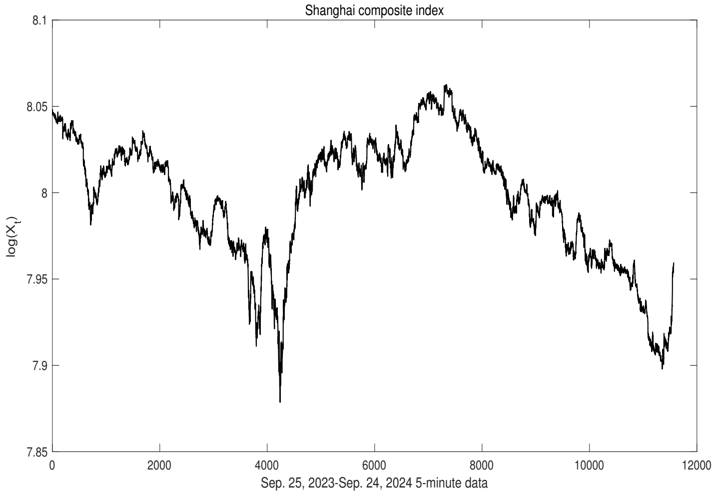

Figure 3. Logarithmic price series of the Shanghai Composite Index during the sample period.

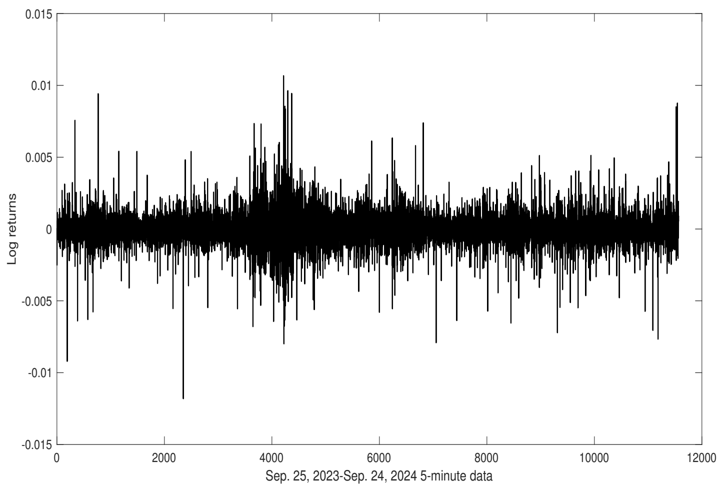

Figure 4. Log returns series of 5-minute high-frequency data of the Shanghai Composite Index.

Figure 5 depicts the kernel estimate of volatility process $\sigma^2(x)$ from the 5-minute high-frequency data of the Shanghai Composite Index. The graph exhibits pronounced volatility clustering, with alternating periods of high volatility (peaks exceeding 6×10^{-3}) and relative stability (troughs near baseline levels).

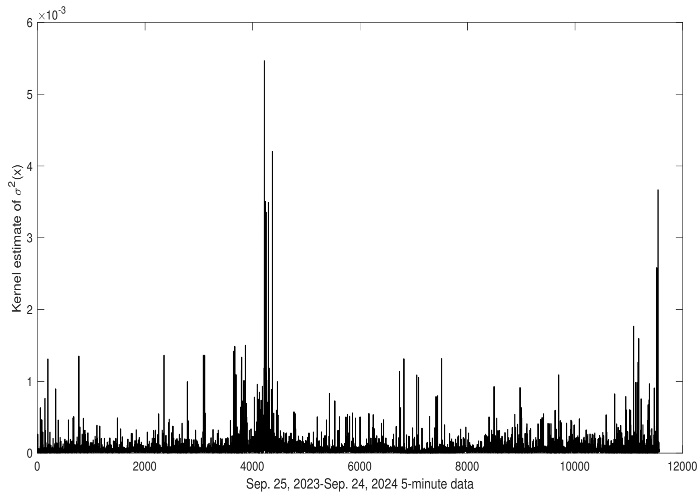

Figure 5. Kernel estimate of $\sigma^2(x)$ in 5-minute high-frequency data of the Shanghai Composite Index.

Figure 6 describes the behavior of the conventional Nadaraya-Watson estimator $\hat{\beta}_1^2(x)$ and the proposed bias-reduced estimator $\hat{\beta}^2(x)$ in 5-minute high-frequency data of the Shanghai Composite Index. Table 2 presents a comparative analysis of the MSE, root mean squared error (RMSE), and mean absolute error (MAE) for the two diffusion function estimators using Shanghai Composite Index data, demonstrating the significantly superior accuracy of the bias reduction estimator.

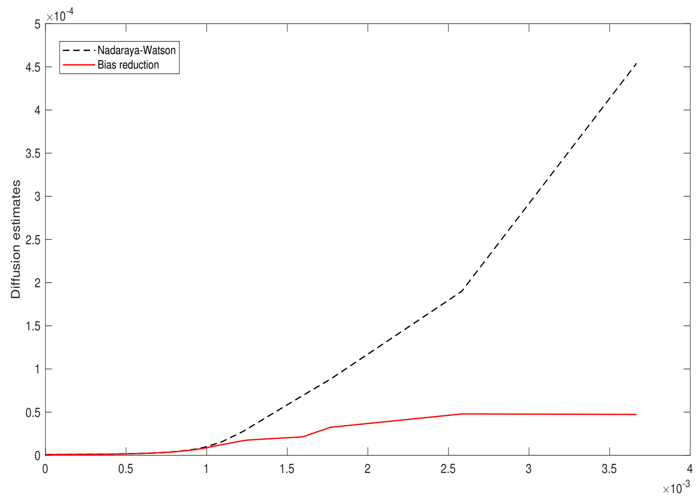

Figure 6. Behavior of the diffusion estimators $\hat{\beta}_1^2(x)$ and $\hat{\beta}^2(x)$ under the Shanghai Composite Index.

Table 2. The statistical properties of diffusion function estimators under the Shanghai Composite Index.

Diffusion estimates	MSE ($\times 10^{-11}$)	RMSE($\times 10^{-6}$)	MAE ($\times 10^{-6}$)
$\hat{\beta}_1^2(x)$	5.7698	7.5960	1.1878
$\hat{\beta}^2(x)$	0.1655	1.2864	0.9901

7. Conclusions and directions for future work

This study develops a novel nonparametric estimator that reduces smoothing bias of diffusion function in stochastic volatility models. By incorporating the nonparametric estimator of the drift coefficient into the estimation procedure for the diffusion coefficient, we achieve a reduction in the smoothing bias inherent in traditional kernel-based methods. Theoretical analysis proves that the proposed estimator retains desirable asymptotic properties, including consistency and asymptotic normality, while effectively eliminating the dominant bias term $\alpha^2(x)\Delta$ present in the Nadaraya-Watson estimator.

The results of this article demonstrate that bias correction can be achieved without affecting asymptotic efficiency. This theoretical framework is based on strict assumptions about the ergodicity, stationarity, and smoothness of the volatility process, ensuring robustness under real market conditions.

The proposed estimator exhibits limitations despite its bias reduction advantages. First, while effective in reducing discretization bias, it inherits the boundary bias inherent in kernel-based methods. Specifically, the reliance on Nadaraya-Watson regression with local constant approximation leads to asymmetric smoothing near the boundaries of the state space in the volatility process. This results in significantly higher residual bias in these regions compared to interior points, as confirmed by simulation results. Second, the new bias reduction estimation requires estimation of the drift function $\alpha(x)$, and its accuracy directly affects the correction effect of the diffusion function $\beta^2(x)$. If the estimation of $\alpha(x)$ is poor (such as sparse samples or boundary regions), errors will be transmitted to the diffusion estimation.

For future research, due to the large boundary deviation of Nadaraya-Watson kernel estimation, local polynomial regression can be considered to adaptively adjust the weight function of the boundary region and reduce boundary deviation when correcting the diffusion function deviation. On the other hand, although the boundary bias of the local linear estimator is well controlled, under finite sample conditions, the estimation of the diffusion function $\beta^2(x)$ may exhibit negative values, which is inconsistent with the nonnegativity of the diffusion function. Therefore, it is possible to consider constructing a nonparametric re-weighted estimator that can maintain the attractive bias characteristics of local linear estimators and ensure nonnegativity in finite samples.

Author contributions

Yunyan Wang: Conceptualization, resources, supervision, methodology, project administration, writing review & editing; Shiguang Peng: Software, data curation, formal analysis, validation, investigation, writing-original draft, writing-review & editing; Mingtian Tang: Conceptualization, software, formal analysis, supervision, funding acquisition, methodology, writing review & editing.

All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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