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Research article

Five classes of binomial/harmonic series of convergence rate $-1/4$

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Abstract: By applying the "coefficient extraction method" to the symmetric transformation of hypergeometric series due to Chu and Zhang (2014), we examine systematically five classes of infinite series of convergence rate "$-1/4$" containing binomial coefficients and harmonic numbers. Numerous closed formulae in terms of universal constants (such as π, $\ln 2$, and the Riemann zeta values) are established.

Keywords: hypergeometric series; Gamma function; harmonic number; Riemann zeta function
Mathematics Subject Classification: Primary 11B65; Secondary 33C20, 65B10

1. Introduction and outline

Harmonic numbers have a very long history that can be traced back to Euler, who initiated his investigation in the course of a correspondence with Goldbach in 1742. Nowadays, these numbers and their variants appear frequently in the literature, for example, of mathematics (particularly, number theory [9, 17, 26] and combinatorial analysis [3, 4]), physics (standing waves in strings [6]), and computer sciences (analysis of algorithms [18]). They are defined, for $x \in \mathbb{R}$ and $n, \lambda \in \mathbb{N}_0$, as follows:

$$\mathbf{H}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{1}{(x+k)^\lambda}, \qquad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(x+k)^\lambda};$$
$$\mathbf{O}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{1}{(x+2k)^\lambda}, \qquad \bar{\mathbf{O}}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(x+2k)^\lambda}.$$

When $\lambda = 1$ and/or $x = 1$, they will be suppressed from these notations. We record the following simple, but useful relations:

$$\mathbf{H}_{2n}^{\langle\lambda\rangle} = \mathbf{O}_n^{\langle\lambda\rangle} + 2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle}, \qquad \mathbf{H}_n^{\langle\lambda\rangle}(\tfrac{1}{2}) = 2^\lambda \mathbf{O}_n^{\langle\lambda\rangle},$$

$$\bar{\mathbf{H}}_{2n}^{\langle\lambda\rangle}=\mathbf{O}_n^{\langle\lambda\rangle}-2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle},\qquad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(\tfrac{1}{2})=2^{\lambda}\bar{\mathbf{O}}_n^{\langle\lambda\rangle}.$$

There are numerous infinite series identities about binomial coefficients (cf. [16, 19, 30]). For instance, it is well-known that

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{1+2n}\binom{2n}{n}=\frac{\arcsin(2x)}{2x}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{1+2n}\binom{2n}{n}=\frac{\operatorname{arcsinh}(2x)}{2x}.$$

Letting $x=\frac{1}{4}$ yields two summation formulae for numerical series

$$\sum_{n=0}^{\infty}\Big(\frac{1}{16}\Big)^n\frac{\binom{2n}{n}}{1+2n}=\frac{\pi}{3}\quad\text{and}\quad\sum_{n=0}^{\infty}\Big(\frac{-1}{16}\Big)^n\frac{\binom{2n}{n}}{1+2n}=2\ln\Big(\frac{1+\sqrt{5}}{2}\Big).$$

More binomial series containing harmonic numbers will be evaluated, through the "coefficient extraction" method, in closed form in Section 2. In order to smoothly conduct our investigation, some basic facts about "coefficient extraction", "Γ-function expansion", and "hypergeometric series" are briefly summarized below as preliminaries.

1.1. Coefficient extraction

For $n\in\mathbb{N}_0$ and an indeterminate x, the shifted factorials are usually defined by

$$(x)_0=1\quad\text{and}\quad(x)_n=x(x+1)\cdots(x+n-1)\quad\text{for}\quad n\in\mathbb{N}.$$

Their product and quotient forms will be abbreviated, respectively, by

$$[a,b,\cdots,c]_n=(a)_n(b)_n\cdots(c)_n\quad\text{and}\quad\begin{bmatrix}a,b,\cdots,c\\ \alpha,\beta,\cdots,\gamma\end{bmatrix}_n=\frac{(a)_n(b)_n\cdots(c)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

Denote by $[x^m]\phi(x)$ the coefficient of x^m in the formal power series $\phi(x)$. Then harmonic numbers can be obtained by extracting coefficients:

$$\begin{aligned}
&[x]\frac{(1+x)_n}{n!}=\mathbf{H}_n, && [x^2]\frac{(1+x)_n}{n!}=\frac{\mathbf{H}_n^2-\mathbf{H}_n^{\langle 2\rangle}}{2},\\
&[x]\frac{n!}{(1-x)_n}=\mathbf{H}_n, && [x^2]\frac{n!}{(1-x)_n}=\frac{\mathbf{H}_n^2+\mathbf{H}_n^{\langle 2\rangle}}{2},\\
&[y]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n}=2\mathbf{O}_n, && [y^2]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n}=2(\mathbf{O}_n^2-\mathbf{O}_n^{\langle 2\rangle}),\\
&[y]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n}=2\mathbf{O}_n, && [y^2]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n}=2(\mathbf{O}_n^2+\mathbf{O}_n^{\langle 2\rangle}).
\end{aligned}$$

By means of the generating function method, it is not difficult to show that (cf. [21]) in general there hold the relations:

$$[x^m]\frac{(\lambda-x)_n}{(\lambda)_n}=\mathbf{Y}_m(-\mathrm{hh})\quad\text{and}\quad[x^m]\frac{(\lambda)_n}{(\lambda-x)_n}=\mathbf{Y}_m(\mathrm{hh}).\tag{1.1}$$

Here "hh_k" stands for the higher harmonic number $\mathrm{hh}_k := \mathbf{H}_n^{\langle k\rangle}(\lambda)$, and the Bell polynomials ([14, §3.3]) are expressed explicitly as

$$\mathbf{Y}_m(\pm\mathrm{hh}) = \sum_{\omega(m)} \prod_{k=1}^{m} \frac{\left\{\pm \mathbf{H}_n^{\langle k\rangle}(\lambda)\right\}^{\ell_k}}{\ell_k!\, k^{\ell_k}}, \tag{1.2}$$

where the multiple sum runs over $\omega(m)$, the set of m-partitions represented by m-tuples of $(\ell_1, \ell_2, \cdots, \ell_m) \in \mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^{m} k\ell_k = m$.

In general for $n \in \mathbb{Z}$, we can express the shifted factorial as the Γ-function ratio

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad \text{where} \quad \Gamma(x) = \int_0^\infty y^{x-1} e^{-y} \mathrm{d}y \quad \text{for} \quad \Re(x) > 0.$$

For the sake of brevity, the Γ-function quotient will be abbreviated to

$$\Gamma\begin{bmatrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{bmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

1.2. Γ-function expansion

The Γ-function is one of the important special functions (cf. Rainville [24, §8]). The following two power series expansions (cf. [20]) will be repeatedly utilized in this paper

$$\Gamma(1-x) = \exp\Big\{\sum_{k\geq 1} \frac{\sigma_k}{k} x^k\Big\}, \tag{1.3}$$

$$\Gamma(\tfrac{1}{2}-x) = \sqrt{\pi} \exp\Big\{\sum_{k\geq 1} \frac{\tau_k}{k} x^k\Big\}; \tag{1.4}$$

where σ_k and τ_k are defined respectively by

$$\begin{array}{llll} \sigma_1 = \gamma & \text{and} \quad \sigma_m = \zeta(m) & \text{for} & m \geq 2; \\ \tau_1 = \gamma + 2\ln 2 & \text{and} \quad \tau_m = (2^m - 1)\zeta(m) & \text{for} & m \geq 2; \end{array}$$

with the Euler–Mascheroni constant and the Riemann zeta function being given by

$$\gamma = \lim_{n\to\infty} (\mathbf{H}_n - \ln n) \quad \text{and} \quad \zeta(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} \quad \text{for} \quad \Re(\lambda) > 1.$$

During the past two decades, evaluating infinite series with harmonic numbers attracted much attention (cf. [1,3,5,12,17,22]). We limit ourselves to recording the following series as representatives for the squared binomial ones weighted by harmonic numbers:

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{\mathbf{H}_n}{16^n(2n-1)^2} = \frac{12}{\pi} - \frac{16\ln 2}{\pi}, \qquad \text{(cf. [8])}$$

$$\sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{\mathbf{O}_n}{16^n(2n-1)^2} = \frac{4G}{\pi} - \frac{4\ln 2}{\pi}, \qquad \text{(cf. [28])}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{H}_n^{\langle 2\rangle}}{16^n(2n-1)}=4-\frac{\pi}{3}-\frac{8}{\pi}, \qquad \text{(cf. [10])}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{O}_n^{\langle 2\rangle}}{16^n(2n-1)}=\frac{4G}{\pi}-\frac{\pi}{4}, \qquad \text{(cf. [23])}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{H}_n^{2}}{16^n(2n-1)}=\frac{5\pi}{3}-4-\frac{8(1-2\ln 2)^2}{\pi}, \qquad \text{(cf. [27])}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{O}_n^{2}}{16^n(2n-1)}=\frac{4G}{\pi}-\frac{\pi}{12}-\frac{2\ln^2 2}{\pi}, \qquad \text{(cf. [7]).}$$

Their alternating counterparts are more involved and can be found in [10]. In addition, there is an isolated alternating finite sum conjectured by Weideman [29]

$$0=\sum_{k=0}^{n}(-1)^k\binom{2k}{k}^3\Big\{3(\mathbf{H}_k-\mathbf{H}_{n-k})^2+(\mathbf{H}_k^{\langle 2\rangle}+\mathbf{H}_{n-k}^{\langle 2\rangle})\Big\},$$

which was confirmed by the second author [11] and Driver et al. [15], respectively, via the hypergeometric series approach and the computer algebra package `Sigma`.

Recently, the "coefficient extraction" method has been successfully applied to hypergeometric series in evaluating, in closed form, infinite series involving both harmonic numbers and binomial coefficients (see, e.g., [10, 27, 28]). In this paper, the same approach will be employed further to examine a useful symmetric transformation of hypergeometric series from [13]. In order to facilitate subsequent applications, it is reproduced below and will shortly be referred to "$\Omega(a;b,c,d)$".

1.3. Hypergeometric series

For five complex parameters $\{a,b,c,d,e\}$ subject to $\Re(1+2a-b-c-d-e)>0$, define the well-poised sum by

$$\sum_{k=0}^{\infty}(a+2k)\left[\begin{matrix} b,\ c,\ d,\ e \\ 1+a-b, 1+a-c, 1+a-d, 1+a-e\end{matrix}\right]_k.$$

When, in particular, $a=e$, this series can be evaluated by Dougall's theorem (cf. Bailey [2, §4.4]):

$$\Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d\end{matrix}\right].$$

In this case, the afore-mentioned symmetric transformation [13, Theorem 10] becomes as below:

$$\begin{aligned}\Omega(a;b,c,d)&=\Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d\end{matrix}\right]\\ &=\sum_{n=0}^{\infty}(-1)^n\frac{(1-b)_n(1-c)_n(1-d)_n}{(1+a-b-c-d)_{2n+2}}\Delta_n(a,b,c,d)\\ &\times\frac{(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{n!(1+a-b)_n(1+a-c)_n(1+a-d)_n},\end{aligned} \tag{1.5}$$

where the cubic polynomial $\Delta_n(a,b,c,d)$ is given by

$$\Delta_n(a,b,c,d) = (1+a-b-c+n)(1+a-b-d+n)(1+a-c-d+n) \\ + n(1+2a-b-c-d+2n)(2+a-b-c-d+2n).$$

According to (1.1)–(1.4), we shall examine, in the rest of the paper, five classes of infinite series involving both binomial coefficients and harmonic numbers by employing the "coefficient extraction" method. This will be fulfilled by manipulating the $\Omega(a;b,c,d)$ series under specific parameter settings. Numerous identities will be established for infinite series of convergence rate "$\frac{-1}{4}$". Most of them have not appeared in the literature except for a few known ones derived by the authors [20] via resolving linear systems.

2. Series with central binomial coefficient $\binom{2n}{n} = 4^n \times \frac{(\frac{1}{2})_n}{n!}$

This section is devoted to the series with central binomial coefficients in numerators. A few of them were evaluated in [20, §2.1 and §3.1]. However, the treatment presented here is more transparent.

2.1. *Series with* $\binom{2n}{n}(1+5n)$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a \to 1+ax,\ b \to \frac{1}{2}+bx,\ c \to 1+cx,\ d \to \frac{1}{2}+dx;$$

we obtain the following transformation formula.

Theorem 1. *For a variable x and four complex parameters $\{a,b,c,d\}$, there holds*

$$\Big(\frac{-1}{cx}\Big)\Gamma\left[\begin{matrix} 1+ax-bx-cx-dx, 1+ax-cx, \frac{1}{2}+ax-bx, \frac{1}{2}+ax-dx \\ 1+ax, 1+ax-bx-dx, \frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-cx-dx \end{matrix}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\left[\frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-cx-dx, \frac{1}{2}-bx, \frac{1}{2}-dx\right]_n}{(n-cx)(1+ax-cx)_n\left[\frac{1}{2}+ax-bx, \frac{1}{2}+ax-dx\right]_n}$$
$$\times \frac{[1+ax-bx-dx, 1-cx]_n\,\Delta_n(1+ax, \frac{1}{2}+bx, 1+cx, \frac{1}{2}+dx)}{(n+ax-bx+\frac{1}{2})(n+ax-dx+\frac{1}{2})(1+ax-bx-cx-dx)_{2n+1}}.$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,b,c,d\}$, we can derive from Theorem 1 several infinite series identities. As a showcase, we illustrate how to obtain from Ω_1 the three formulae involving the harmonic numbers of the first order. Recalling (1.1)–(1.4), it is not hard to deduce the following equality by extracting the coefficient of "x" across Theorem 1:

$$\boxed{\Omega_1}\quad 7a-3b+c-3d-\frac{\pi^2}{3}(2a-b-c-d)-8c\ln^2 2$$
$$= \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(-16)^n}\Big\{\frac{c+4cn-n^2(7a-3b-3c-3d)}{n^2(2n+1)^2}$$

$$-\frac{(a+b-c+d)(5n+1)\mathbf{H}_n}{2n(2n+1)}-\frac{(a+b+3c+d)(5n+1)\mathbf{O}_n}{n(2n+1)}\Big\}.$$

Then the three formulae below after the first one follow by specializing $\{a, b, c, d\}$ as indicated.

- Ω_0: Constant term identity.

$$4\ln 2-3=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{1+5n}{n(1+2n)}\Big\}.$$

- $\Omega_1:\ a=-b=1, c=d=0.$

$$\frac{\pi^2}{10}=\sum_{n=0}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{1}{(1+2n)^2}\Big\}.$$

- $\Omega_1:\ a=1, b=3, c=5, d=1$

$$\frac{7\pi^2}{15}-8\ln^2 2=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{1}{n^2}-\frac{4(1+5n)}{n(1+2n)}\mathbf{O}_n\Big\}.$$

- $\Omega_1:\ a=b=-c=d=1.$

$$\frac{\pi^2}{3}-8\ln^2 2=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{1}{n^2}+\frac{2(1+5n)}{n(1+2n)}\mathbf{H}_n\Big\}.$$

- $\Omega_2:\ a=0, b=1, c=0, d=-1.$

$$\frac{7\zeta(3)}{2}-4=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{4}{(1+2n)^3}-\frac{(1+5n)}{n(1+2n)}\mathbf{O}_n^{\langle 2\rangle}\Big\}.$$

- $\Omega_2:\ a=1, b=0, c=0, d=-1.$

$$30\zeta(3)-36=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{36}{(1+2n)^3}-\frac{(1+5n)}{n(1+2n)}\mathbf{H}_n^{\langle 2\rangle}\Big\}.$$

- $\Omega_2:\ a=0, b=1+\mathbf{i}\sqrt{7}, c=2, d=1-\mathbf{i}\sqrt{7}.$

$$7-3\zeta(3)-\frac{2\pi^2\ln 2}{3}+\frac{8\ln^3 2}{3}=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\left\{\begin{array}{l}\frac{1}{4n^3}-\frac{1+4n+6n^2}{n^2(1+2n)^2}\mathbf{O}_n\\ -\frac{7}{(1+2n)^3}+\frac{2(1+5n)}{n(1+2n)}\mathbf{O}_n^2\end{array}\right\}.$$

- $\Omega_2:\ a=-6-2\sqrt{5}, b=\sqrt{5}-\mathbf{i}\sqrt{15+4\sqrt{5}}, c=2, d=\sqrt{5}+\mathbf{i}\sqrt{15+4\sqrt{5}}.$
  This gives a combined equation of $\sqrt{5}$ and "1" that splits into two series:

$$28\zeta(3)-\pi^2\ln 2=\sum_{n=0}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{27}{(1+2n)^3}+\frac{5\mathbf{H}_n}{(1+2n)^2}\Big\},$$

$$114\zeta(3)-108+\frac{16\ln^3 2}{3}-\frac{14\pi^2\ln 2}{3}=\sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\left\{\begin{array}{l}\frac{1}{2n^3}+\Big(\frac{1}{n^2}+\frac{20}{(1+2n)^2}\Big)\mathbf{H}_n\\ +\frac{108}{(1+2n)^3}+\frac{(1+5n)}{n(1+2n)}\mathbf{H}_n^2\end{array}\right\}.$$

- Coefficient of "ab^2" in Ω_3 : $c \to 0, d \to -a-b$.

$$\frac{3\pi^4}{16} - 18 = \sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{18}{(1+2n)^4} - \frac{5\mathbf{O}_n^{\langle 2\rangle}}{(1+2n)^2} - \frac{(1+5n)}{n(1+2n)}\mathbf{O}_n^{\langle 3\rangle}\Big\}.$$

- Ω_3 : $a = 1, b = 0, c = 0, d = -1$.

$$\frac{7\pi^4}{10} - 68 = \sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{68}{(1+2n)^4} - \frac{5\mathbf{H}_n^{\langle 2\rangle}}{(1+2n)^2} - \frac{(1+5n)}{n(1+2n)}\mathbf{H}_n^{\langle 3\rangle}\Big\}.$$

- Ω_4 : $a = 0, b = 1, c = 0, d = -1$.

$$\frac{31\zeta(5)}{4} - 8 = \sum_{n=1}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{\frac{8}{(1+2n)^5} - \frac{8\mathbf{O}_n^{\langle 2\rangle}}{(1+2n)^3} + \frac{(1+5n)}{n(1+2n)}\Big((\mathbf{O}_n^{\langle 2\rangle})^2 - \mathbf{O}_n^{\langle 4\rangle}\Big)\Big\}.$$

2.2. *Series with* $\binom{2n}{n}(1+10n)$

Making the parameter replacements in $\Omega(a; b, c, d)$

$$a \to ax,\ b \to \frac{1}{2} + bx,\ c \to cx,\ d \to \frac{1}{2} + dx;$$

we obtain the following transformation formula.

Theorem 2. *For a variable x and four complex parameters $\{a, b, c, d\}$, there holds*

$$\Gamma\begin{bmatrix} 1+ax-bx-cx-dx, 1+ax-cx, \frac{1}{2}+ax-bx, \frac{1}{2}+ax-dx \\ ax, 1+ax-bx-dx, \frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-cx-dx \end{bmatrix}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{[1+ax-bx-dx, 1-cx]_n\,\Delta_n(ax, \frac{1}{2}+bx, cx, \frac{1}{2}+dx)}{(n+ax-bx-dx)(1+ax-bx-cx-dx)_{2n+1}}$$
$$\times\frac{\Big[\frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-cx-dx, \frac{1}{2}-bx, \frac{1}{2}-dx\Big]_n}{(1+ax-cx)_n\Big[\frac{1}{2}+ax-bx, \frac{1}{2}+ax-dx\Big]_n}.$$

By specifying particular values for parameters $\{a, b, c, d\}$, we derive from Theorem 2 the following infinite series identities.

- Ω_0: Constant term identity.

$$0 = \sum_{n=0}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}(1+10n).$$

- Ω_1 : $a = b = 0, c = d = 1$.

$$0 = \sum_{n=0}^{\infty}\Big(\frac{-1}{16}\Big)^n\binom{2n}{n}\Big\{1 + (1+10n)\mathbf{O}_n\Big\}.$$

- $\Omega_1: \ a = b = 0, c = 1, d = -3.$

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ 8 - (1 + 10n)\mathbf{H}_n \Big\}.$$

- $\Omega_2: \ a = 0, b = 1, c = 0, d = -1.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{1}{1+2n} + (1 + 10n)\mathbf{O}_n^{\langle 2\rangle} \Big\}.$$

- $\Omega_2: \ a = 1, b = 0, c = 0, d = -1.$

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{8}{n} - (1 + 10n)\mathbf{H}_n^{\langle 2\rangle} \Big\}.$$

- $\Omega_2: \ a = 0, b = 1 + \mathbf{i}\sqrt{7}, c = 2, d = 1 - \mathbf{i}\sqrt{7}.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{1}{1+2n} + 2\mathbf{O}_n + (1 + 10n)\mathbf{O}_n^2 \Big\}.$$

- $\Omega_2: \ a = -6 - 2\sqrt{5}, b = \sqrt{5} - \mathbf{i}\sqrt{15 + 4\sqrt{5}}, c = 2, d = \sqrt{5} + \mathbf{i}\sqrt{15 + 4\sqrt{5}}$. This gives a combined equation of $\sqrt{5}$ and "1" that splits into two series, where the first one recovers "Constant term identity" given in Section 2.1:

$$4\ln 2 - 3 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{1+5n}{n(1+2n)} \Big\},$$
$$24\ln 2 - 18 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{2(7+23n)}{n(1+2n)} - 16\mathbf{H}_n + (1 + 10n)\mathbf{H}_n^2 \Big\}.$$

- Coefficient of "a^2b" in $\Omega_3: \ c \to 0, d \to -a - b.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{1}{(1+2n)^2} + (1 + 10n)\mathbf{O}_n^{\langle 3\rangle} \Big\}.$$

- $\Omega_3: \ a = 1, b = 0, c = 0, d = -1.$

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{8}{n^2} - (1 + 10n)\mathbf{H}_n^{\langle 3\rangle} \Big\}.$$

- Coefficient of "b^2c" in $\Omega_3: \ a \to 0, d \to c - b.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{1}{(1+2n)^2} - \frac{2\mathbf{O}_n}{1+2n} - 2\mathbf{O}_n^{\langle 2\rangle} + (1 + 10n)(3\mathbf{O}_n^{\langle 3\rangle} - 2\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle}) \Big\}.$$

- Ω_3 : $a = 0, b = 1 + \mathbf{i}\sqrt{7}, c = 2, d = 1 - \mathbf{i}\sqrt{7}$.

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{3}{(1+2n)^2} - \frac{6\mathbf{O}_n}{1+2n} - 6\mathbf{O}_n^2 + (1+10n)(5\mathbf{O}_n^{\langle 3\rangle} - 2\mathbf{O}_n^3) \Big\}.$$

- Ω_4 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{2\mathbf{O}_n^{\langle 2\rangle}}{1+2n} + (1+10n)\Big((\mathbf{O}_n^{\langle 2\rangle})^2 - \mathbf{O}_n^{\langle 4\rangle}\Big) \Big\}.$$

- Ω_4 : $a = 1, b = 0, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{ \frac{64}{n^3} - \frac{16\mathbf{H}_n^{\langle 2\rangle}}{n} + (1+10n)\Big((\mathbf{H}_n^{\langle 2\rangle})^2 - 7\mathbf{H}_n^{\langle 4\rangle}\Big) \Big\}.$$

2.3. Series with $\binom{2n}{n}(1-4n^2)$

Making the parameter replacements in $\Omega(a; b, c, d)$

$$a \to ax - 1,\ b \to \frac{1}{2} + bx,\ c \to cx,\ d \to dx - \frac{1}{2};$$

we obtain the following transformation formula.

Theorem 3. *For a variable x and four complex parameters $\{a, b, c, d\}$, there holds*

$$\begin{aligned}
&\Big(\frac{1}{2} - dx\Big)\Gamma\left[\begin{matrix} 1+ax-bx-cx-dx, 1+ax-cx, \frac{1}{2}+ax-bx, \frac{1}{2}+ax-dx \\ ax-1, 1+ax-bx-dx, \frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-cx-dx \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\begin{matrix} 1+ax-bx-dx, 1-cx, \frac{1}{2}+ax-bx-cx, \frac{1}{2}+ax-cx-dx, \frac{1}{2}-bx, \frac{1}{2}-dx \\ 1+ax-cx,\ \frac{1}{2}+ax-bx,\ \frac{1}{2}+ax-dx \end{matrix}\right]_n \\
&\quad \times \frac{(n+ax-cx)(n-dx+\frac{1}{2})(n+ax-bx-\frac{1}{2})\Delta_n(ax-1, \frac{1}{2}+bx, cx, dx-\frac{1}{2})}{n!\,(n+ax-bx-dx)(n+ax-bx-cx-\frac{1}{2})(1+ax-bx-cx-dx)_{2n+1}}.
\end{aligned}$$

By specifying particular values for parameters $\{a, b, c, d\}$, we derive from Theorem 3 the following infinite series identities.

- Ω_0: Constant term identity.

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} n(1-2n)(1+2n).$$

- Ω_1 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} n(7-10n).$$

- Ω_1 : $a = 1, b = 7, c = 8, d = 0$.

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} (1-2n)(1+2n)\Big\{11 + 160n\mathbf{O}_n\Big\}.$$

- Ω_1 : $a = b = 4, c = -3, d = 1$.

$$3 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} (1-2n)(1+2n)\Big\{13 - 10n\mathbf{H}_n\Big\}.$$

- Ω_2 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{n(3+2n)}{5(1+2n)} - n(1-2n)(1+2n)\mathbf{O}_n^{\langle 2\rangle}\Big\}.$$

- Ω_2 : $a = 1, b = 0, c = 0, d = -1$.

$$\frac{2}{5} = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{2(4+3n-18n^2)}{5n} - n(1-2n)(1+2n)\mathbf{H}_n^{\langle 2\rangle}\Big\}.$$

- Ω_2 : $a = 0, b = 1 + \mathbf{i}\sqrt{7}, c = 2, d = 1 - \mathbf{i}\sqrt{7}$ (Real part).

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{1 - 22n + 60n^2 + 88n^3}{4(1+2n)(2n-1)} - (1 + 5n - 18n^2)\mathbf{O}_n - 10n(1-4n^2)\mathbf{O}_n^2\Big\}.$$

- Ω_2 : $a = 0, b = 1 + \mathbf{i}\sqrt{7}, c = 2, d = 1 - \mathbf{i}\sqrt{7}$ (Imaginary part).

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{(3+2n)(1-6n)}{4(1-2n)} + n(7-10n)\mathbf{O}_n\Big\}.$$

- Ω_2 : $a = -6 - 2\sqrt{5}, b = \sqrt{5} - \mathbf{i}\sqrt{15 + 4\sqrt{5}}, c = 2, d = \sqrt{5} + \mathbf{i}\sqrt{15 + 4\sqrt{5}}$.
  The real part gives a combined equation of "1" and $\sqrt{5}$ that splits into two series:

$$23 - 48\ln 2 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{(28 + 7n - 406n^2 + 148n^3 + 952n^4)}{n(1+2n)(2n-1)} + 2(8 - 22n - 92n^2)\mathbf{H}_n + 10n(2n-1)(1+2n)\mathbf{H}_n^2\Big\}.$$

$$8\ln 2 - 3 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{(2 + 7n - 60n^2 - 108n^3)}{n(1+2n)} + 3(1 + 3n + 10n^2)\mathbf{H}_n\Big\}.$$

- Ω_2 : $a = -6 - 2\sqrt{5}, b = \sqrt{5} - \mathbf{i}\sqrt{15 + 4\sqrt{5}}, c = 2, d = \sqrt{5} + \mathbf{i}\sqrt{15 + 4\sqrt{5}}$.
  The imaginary part gives a combined equation of "1" and $\sqrt{5}$ that splits into two series:

$$0 = \sum_{n=1}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{16 - 84n + 72n^2}{2n-1} + 2n(7-10n)\mathbf{H}_n\Big\},$$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{16}\Big)^n \binom{2n}{n} \Big\{\frac{1 + 8n - 20n^2}{2n-1}\Big\}.$$

- Ω_3 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{n}{1+2n} - n(7-10n)\mathbf{O}_n^{\langle 2 \rangle} \right\}.$$

- Coefficient of "ab^2" in Ω_3 : $c \to 0, d \to -a-b$.

$$0 = \sum_{n=0}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{3+6n+8n^2}{2(1+2n)^2} + \frac{3}{2}(1-4n+20n^2)\mathbf{O}_n^{\langle 2 \rangle} + 5n(1-2n)(1+2n)\mathbf{O}_n^{\langle 3 \rangle} \right\}.$$

- Ω_3 : $a = 1, b = 0, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{8(2+n)}{n^2} - (3+2n+40n^2)\mathbf{H}_n^{\langle 2 \rangle} - 10n(1-2n)(1+2n)\mathbf{H}_n^{\langle 3 \rangle} \right\}.$$

- Ω_3 : $a = 0, b = 1 + \sqrt{7}\mathbf{i}, c = 2, d = 1 - \sqrt{7}\mathbf{i}$ (Imaginary part).

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{2n(5+12n+20n^2)}{(1-2n)^2(1+2n)} - \frac{(3+2n)(1-6n)}{1-2n}\mathbf{O}_n - 2n(7-10n)\mathbf{O}_n^2 \right\}.$$

- Ω_4 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{2n(3+2n)}{1+2n}\mathbf{O}_n^{\langle 2 \rangle} - 5n(1-2n)(1+2n)\Big((\mathbf{O}_n^{\langle 2 \rangle})^2 - \mathbf{O}_n^{\langle 4 \rangle}\Big) \right\}.$$

- Ω_5 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{2n}{1+2n}\mathbf{O}_n^{\langle 2 \rangle} - n(7-10n)\Big((\mathbf{O}_n^{\langle 2 \rangle})^2 - \mathbf{O}_n^{\langle 4 \rangle}\Big) \right\}.$$

- Ω_6 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{3n(3+2n)}{5(1+2n)}\Big(((\mathbf{O}_n^{\langle 2 \rangle})^2 - \mathbf{O}_n^{\langle 4 \rangle}) - n(1-2n)(1+2n)((\mathbf{O}_n^{\langle 2 \rangle})^3 - 3\mathbf{O}_n^{\langle 2 \rangle}\mathbf{O}_n^{\langle 4 \rangle} + 2\mathbf{O}_n^{\langle 6 \rangle})\Big) \right\}.$$

- Ω_7 : $a = 0, b = 1, c = 0, d = -1$.

$$0 = \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \binom{2n}{n} \left\{ \frac{3n}{1+2n}\Big((\mathbf{O}_n^{\langle 2 \rangle})^2 - \mathbf{O}_n^{\langle 4 \rangle}\Big) - n(7-10n)((\mathbf{O}_n^{\langle 2 \rangle})^3 - 3\mathbf{O}_n^{\langle 2 \rangle}\mathbf{O}_n^{\langle 4 \rangle} + 2\mathbf{O}_n^{\langle 6 \rangle})\Big) \right\}.$$

3. Series with central binomial coefficient $\binom{4n}{2n} = 4^{2n} \times \frac{(\frac{1}{2})_{2n}}{(2n)!}$

The series with the same binomial coefficient has been investigated in [20, §2.4], where the harmonic numbers in the summands have different polynomial factors.

3.1. Series with $\frac{(\frac{1}{2})_{2n+1}}{(2n+1)!}$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a \to ax-1,\ b \to \frac{1}{4}+cx,\ c \to cx-\frac{1}{4},\ d \to ex;$$

and then applying Dougall's summation theorem for the ${}_5F_4$-series, we obtain the following transformation formula.

Theorem 4. *For a variable x and three complex parameters $\{a,c,e\}$, there holds*

$$\begin{aligned}&(ax-1)\Big(\frac{1}{4}-cx\Big)\Gamma\begin{bmatrix}1+ax-ex, 1+ax-2cx-ex, \frac{1}{4}+ax-cx, \frac{3}{4}+ax-cx\\ ax, 1+ax-2cx, \frac{1}{4}+ax-cx-ex, \frac{3}{4}+ax-cx-ex\end{bmatrix}\\ &=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(n-cx+\frac{1}{4})(n+ax-cx-\frac{1}{4})\Delta_n(ax-1,\frac{1}{4}+cx,cx-\frac{1}{4},ex)}{n!(n+ax-2cx)(n+ax-cx-ex-\frac{1}{4})}\\ &\quad\times\frac{(n+ax-ex)\,[1+ax-2cx,1-ex]_n\left[\frac{1}{2}+2ax-2cx-2ex,\frac{1}{2}-2cx\right]_{2n}}{(1+ax-ex)_n(\frac{1}{2}+2ax-2cx)_{2n}(1+ax-2cx-ex)_{2n+1}}.\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,c,e\}$, we derive from Theorem 4 the following infinite series identities.

- Ω_0: Constant term identity.

$$0=\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(\frac{1}{2})_{2n+1}}{(2n+1)!}\Big\{n(17-80n^2)\Big\}.$$

- $\Omega_1:\ a=e=1, c=0.$

$$2=\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(\frac{1}{2})_{2n+1}}{(2n+1)!}\Big\{\frac{4+5n-24n^2-48n^3}{4n-1}-n(17-80n^2)\mathbf{O}_{2n}\Big\}.$$

- $\Omega_1:\ a=c=-e=1.$

$$\frac{7}{2}=\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(\frac{1}{2})_{2n+1}}{(2n+1)!}\Big\{\frac{25-8n-272n^2+128n^3+1024n^4}{(1-4n)(1+4n)}-n(17-80n^2)\mathbf{H}_n\Big\}.$$

- $\Omega_1:\ a=1, c=e=0.$

$$\frac{15}{2}=\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(\frac{1}{2})_{2n+1}}{(2n+1)!}\Big\{(1-16n-144n^2)-n(17-80n^2)\mathbf{H}_{2n+1}\Big\}.$$

- $\Omega_2:\ a=1, c=e=0.$

$$15=\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(\frac{1}{2})_{2n+1}}{(2n+1)!}\begin{Bmatrix}160n+2(1-16n-144n^2)\mathbf{H}_{2n+1}\\ -n(17-80n^2)(\mathbf{H}^2_{2n+1}+\mathbf{H}^{\langle 2\rangle}_{2n+1})\end{Bmatrix}.$$

3.2. Series with $\frac{(3+4n)(\frac{1}{2})_{2n}}{(2n+1)!}$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a \to ax-1,\ b \to \frac{3}{4}+cx,\ c \to cx-\frac{3}{4},\ d \to ex;$$

we obtain the following transformation formula.

Theorem 5. *For a variable x and three complex parameters $\{a,c,e\}$, there holds*

$$(ax-1)\Big(\frac{3}{4}-cx\Big)\Gamma\left[\begin{matrix}1+ax-ex, 1+ax-2cx-ex, \frac{1}{4}+ax-cx, \frac{3}{4}+ax-cx\\ ax, 1+ax-2cx, \frac{1}{4}+ax-cx-ex, \frac{3}{4}+ax-cx-ex\end{matrix}\right]$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{[1+ax-2cx, 1-ex]_n\left[\frac{1}{2}+2ax-2cx-2ex, \frac{1}{2}-2cx\right]_{2n}}{n!(1+ax-ex)_n(\frac{1}{2}+2ax-2cx)_{2n}(1+ax-2cx-ex)_{2n+1}}$$
$$\times\frac{(n+ax-ex)(n-cx+\frac{3}{4})(n+ax-cx-\frac{3}{4})\Delta_n(ax-1, \frac{3}{4}+cx, cx-\frac{3}{4}, ex)}{(n+ax-2cx)(n+ax-cx-ex-\frac{3}{4})}.$$

By appropriately specifying values for parameters $\{a,c,e\}$, we deduce from Theorem 5 the following infinite series identities.

- Ω_0: Constant term identity.

$$0=\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(3+4n)(\frac{1}{2})_{2n}}{(2n+1)!}\Big\{n(5-16n^2)\Big\}.$$

- $\Omega_1:\ a=e=1, c=0.$

$$0=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(3+4n)(\frac{1}{2})_{2n}}{(2n+1)!}\Big\{\frac{12+21n-40n^2-48n^3}{5(3-4n)}+n(5-16n^2)\mathbf{O}_{2n}\Big\}.$$

- $\Omega_1:\ a=c=-e=1.$

$$\frac{96}{5}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(3+4n)(\frac{1}{2})_{2n}}{(2n+1)!}\Big\{\frac{297-72n-1424n^2+128n^3+1024n^4}{5(3+4n)(3-4n)}-n(5-16n^2)\mathbf{H}_n\Big\}.$$

- $\Omega_1:\ a=1, c=e=0.$

$$48=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(3+4n)(\frac{1}{2})_{2n}}{(2n+1)!}\Big\{(9-16n-144n^2)-5n(5-16n^2)\mathbf{H}_{2n+1}\Big\}.$$

3.3. Series with $\frac{(\frac{1}{2})_{2n}}{(2n+1)!}$

Under the parameter setting

$$a \to ax,\ b \to \tfrac{1}{4}+cx,\ c \to \tfrac{3}{4}+cx,\ d \to ex;$$

the summation formula $\Omega(a;b,c,d)$ becomes the following one.

Theorem 6. *For a variable x and three complex parameters $\{a, c, e\}$, there holds*

$$\begin{aligned}
&\Gamma\left[\begin{matrix} 1+ax-ex, \frac{1}{4}+ax-cx, \frac{3}{4}+ax-cx, 1+ax-2cx-ex \\ ax, \frac{1}{4}+ax-cx-ex, \frac{3}{4}+ax-cx-ex, 1+ax-2cx \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2}+2ax-2cx-2ex)_{2n}(\frac{1}{2}-2cx)_{2n}}{(n+ax-2cx)(\frac{1}{2}+2ax-2cx)_{2n}} \\
&\quad \times \frac{(1+ax-2cx)_n(1-ex)_n\Delta_n(ax, \frac{1}{4}+cx, \frac{3}{4}+cx, ex)}{n!(1+ax-ex)_n(1+ax-2cx-ex)_{2n+1}}.
\end{aligned}$$

By specifying particular values for parameters $\{a, c, e\}$, we derive from Theorem 6 the following infinite series identities.

- Ω_0: Constant term identity.

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2})_{2n}}{(2n+1)!} \{3+48n+80n^2\}.$$

- $\Omega_1 :\ a = 3, c = -1, e = 1.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2})_{2n}}{(2n+1)!} \Big\{32n - (3+48n+80n^2)\mathbf{O}_{n+1}\Big\}.$$

- $\Omega_1 :\ a = c = -e = 1.$

$$32 = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2})_{2n}}{(2n+1)!} \Big\{32(1+2n) - (3+48n+80n^2)\mathbf{H}_n\Big\}.$$

- $\Omega_1 :\ a = e = 1, c = 0.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2})_{2n}}{(2n+1)!} \Big\{4(1+2n) + (3+48n+80n^2)\mathbf{O}_{2n}\Big\}.$$

- $\Omega_2 :\ a = 1, c = e = 0.$

$$0 = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2})_{2n}}{(2n+1)!} \Big\{32 - 32(1+4n)\mathbf{H}_{2n+1} + (3+48n+80n^2)(\mathbf{H}^{\langle 2\rangle}_{2n+1} + \mathbf{H}^2_{2n+1})\Big\}.$$

4. Series with multinomial coefficient $\binom{4n}{n,n,n,n} = 4^{4n} \times \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1 \end{matrix}\right]_n$

The series evaluated in this section are quite different from those examined in [20, §2.3] since the harmonic numbers in the summands are tied with different weight factors.

4.1. Series with $\left[\begin{matrix}\frac12,\frac34,\frac54\\1,1,1\end{matrix}\right]_n$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a\to ax-\frac{1}{2},\ b\to\frac{1}{4}+cx,\ c\to cx-\frac{1}{4},\ d\to\frac{1}{2}+ex;$$

we obtain the following transformation formula.

Theorem 7. *For a variable x and three complex parameters* $\{a,c,e\}$*, there holds*

$$\begin{aligned}&\Big(ax-\frac{1}{2}\Big)\Big(\frac{1}{4}-cx\Big)\Gamma\left[\begin{matrix}1+ax-ex,1+ax-2cx-ex,\frac14+ax-cx,\frac34+ax-cx\\\frac12+ax,\frac12+ax-2cx,\frac14+ax-cx-ex,\frac34+ax-cx-ex\end{matrix}\right]\\&=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(\frac12+ax-2cx)_n(\frac12-ex)_n(\frac12+2ax-2cx-2ex)_{2n}(\frac12-2cx)_{2n}}{(1+ax-2cx-ex)_{2n+1}(\frac12+2ax-2cx)_{2n}}\\&\quad\times\frac{(n+ax-ex)(n-cx+\frac14)\Delta_n(ax-\frac12,\frac14+cx,cx-\frac14,\frac12+ex)}{n!\,(1+ax-ex)_n(n+ax-cx-ex-\frac14)}.\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,c,e\}$, we derive from Theorem 7 the following infinite series identities.

- Ω_0: Constant term identity.

$$\frac{-4}{\pi}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac12,\frac34,\frac54\\1,\ 1,\ 1\end{matrix}\right]_n\Big\{n(1+20n)\Big\}.$$

- $\Omega_1:\ a=c=-e=1.$

$$\frac{12}{\pi}(\ln 2-1)=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac12,\frac34,\frac54\\1,\ 1,\ 1\end{matrix}\right]_n\Big\{\frac{1+21n+4n^2-176n^3-192n^4}{(1+2n)(1+4n)(4n-1)}+n(1+20n)\mathbf{H}_n\Big\}.$$

- $\Omega_1:\ a=1,c=-e=-3.$

$$\frac{4}{\pi}(\ln 2-5)=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac12,\frac34,\frac54\\1,\ 1,\ 1\end{matrix}\right]_n\Big\{\frac{1+9n-60n^2+336n^3+1088n^4}{(1+2n)(1+4n)(4n-1)}+2n(1+20n)\mathbf{O}_n\Big\}.$$

- $\Omega_1:\ a=e=1,c=0.$

$$\frac{4}{\pi}(1+\ln 2)=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac12,\frac34,\frac54\\1,\ 1,\ 1\end{matrix}\right]_n\Big\{\frac{n(-1+16n+48n^2)}{(1+2n)(4n-1)}-2n(1+20n)\mathbf{O}_{2n}\Big\}.$$

4.2. Series with $\left[\begin{matrix}\frac{1}{2}, \frac{-3}{4}, \frac{7}{4}\\ 1, 1, 1\end{matrix}\right]_n$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a \to ax - \frac{1}{2},\ b \to \frac{3}{4} + cx,\ c \to cx - \frac{3}{4},\ d \to \frac{1}{2} + ex;$$

we obtain the following transformation formula.

Theorem 8. *For a variable x and three complex parameters $\{a, c, e\}$, there holds*

$$\begin{aligned}(2ax-1)\Big(\frac{3}{4} - cx\Big)\Gamma&\left[\begin{matrix}1 + ax - ex, 1 + ax - 2cx - ex, \frac{1}{4} + ax - cx, \frac{3}{4} + ax - cx\\ \frac{1}{2} + ax, \frac{1}{2} + ax - 2cx, \frac{1}{4} + ax - cx - ex, \frac{3}{4} + ax - cx - ex\end{matrix}\right]\\ =\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n&\frac{(\frac{1}{2} + ax - 2cx)_n(\frac{1}{2} - ex)_n(\frac{1}{2} + 2ax - 2cx - 2ex)_{2n}(\frac{1}{2} - 2cx)_{2n}}{(1 + ax - ex)_n(1 + ax - 2cx - ex)_{2n+1}(\frac{1}{2} + 2ax - 2cx)_{2n-1}}\\ \times&\frac{(n + ax - ex)(n - cx + \frac{3}{4})\Delta_n(ax - \frac{1}{2}, \frac{3}{4} + cx, cx - \frac{3}{4}, \frac{1}{2} + ex)}{n!\,(n + ax - cx + \frac{1}{4})(n + ax - cx - ex - \frac{3}{4})}.\end{aligned}$$

By specifying particular values for parameters $\{a, c, e\}$, we derive from Theorem 8 the following infinite series identities.

- Ω_0: Constant term identity.

$$\frac{12}{\pi} = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{-3}{4}, \frac{7}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n\Big\{n(1-4n)(9-20n)\Big\}.$$

- $\Omega_1:\ a = c = -e = 1.$

$$\begin{aligned}\frac{4}{\pi}(5 - 9\ln 2) = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{-3}{4}, \frac{7}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n&(4n-1)\Big\{n(20n-9)\mathbf{H}_n\\ &- \frac{81 + 225n - 720n^2 - 1728n^3 + 512n^4 + 768n^5}{(1+2n)(1+4n)(3+4n)(4n-3)}\Big\}.\end{aligned}$$

- $\Omega_1:\ a = 1, c = -3, e = 3.$

$$\begin{aligned}\frac{12}{\pi}(1 - \ln 2) = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{-3}{4}, \frac{7}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n&(4n-1)\Big\{2n(20n-9)\mathbf{O}_n\\ &+ \frac{81 + 657n + 588n^2 - 3648n^3 - 8704n^4 - 3328n^5 + 17408n^6}{(1+2n)(1+4n)(4n-1)(3+4n)(4n-3)}\Big\}.\end{aligned}$$

- $\Omega_1:\ a = e = 1, c = 0.$

$$\begin{aligned}\frac{12}{\pi}(1 + \ln 2) = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n&\left[\begin{matrix}\frac{1}{2}, \frac{-3}{4}, \frac{7}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n(4n-1)\\ &\times\Big\{2n(20n-9)\mathbf{O}_{2n} + \frac{n(27 + 44n - 176n^2 - 192n^3)}{(1+2n)(1+4n)(4n-1))}\Big\}.\end{aligned}$$

4.3. Series with $\left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1, 1, 1\end{matrix}\right]_n$

Making the parameter replacements in $\Omega(a; b, c, d)$

$$a \to \frac{1}{2} + ax,\ b \to \frac{1}{4} + cx,\ c \to cx + \frac{3}{4},\ d \to \frac{1}{2} + ex;$$

we obtain the following transformation formula.

Theorem 9. *For a variable x and three complex parameters $\{a, c, e\}$, there holds*

$$\begin{aligned}
&\frac{1}{2}\Gamma\left[\begin{matrix}1 + ax - ex, 1 + ax - 2cx - ex, \frac{1}{4} + ax - cx, \frac{3}{4} + ax - cx\\ \frac{1}{2} + ax, \frac{1}{2} + ax - 2cx, \frac{1}{4} + ax - cx - ex, \frac{3}{4} + ax - cx - ex\end{matrix}\right]\\
&= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \frac{(\frac{1}{2} + ax - 2cx)_n(\frac{1}{2} - ex)_n(\frac{1}{2} + 2ax - 2cx - 2ex)_{2n}}{n!\ (1 + ax - ex)_n(\frac{1}{2} + 2ax - 2cx)_{2n+1}}\\
&\quad \times \frac{(\frac{1}{2} - 2cx)_{2n}\Delta_n(\frac{1}{2} + ax, \frac{1}{4} + cx, \frac{3}{4} + cx, \frac{1}{2} + ex)}{(1 + ax - 2cx - ex)_{2n+1}}.
\end{aligned}$$

By specifying particular values for parameters $\{a, c, e\}$, we derive from Theorem 9 the following four infinite series identities, where the last two have previously appeared in [20].

- Ω_0: Constant term identity (Ramanujan [25]).

$$\frac{8}{\pi} = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n\Big\{(3 + 20n)\Big\}.$$

- $\Omega_1:\ a = e = 1, c = 0.$

$$\frac{8\ln 2}{\pi} = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n\Big\{\frac{3 + 8n}{1 + 2n} + 2(3 + 20n)\mathbf{O}_{2n}\Big\}.$$

- $\Omega_1:\ a = c = -e = 1.$

$$\frac{24\ln 2}{\pi} = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n\Big\{\frac{5 + 12n}{1 + 2n} - (3 + 20n)\mathbf{H}_n\Big\}.$$

- $\Omega_1:\ a = 1, c = -e = -3.$

$$\frac{8\ln 2}{\pi} = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n\Big\{\frac{1 + 12n}{1 + 2n} - 2(3 + 20n)\mathbf{O}_n\Big\}.$$

5. Series with factorial structure $\binom{6n}{2n}\Big/\binom{3n}{n} = \Big(\frac{27}{4}\Big)^n \times \left[\begin{matrix}\frac{1}{6}, \frac{5}{6}\\ \frac{1}{4}, \frac{3}{4}\end{matrix}\right]_n$

In this section, we are going to evaluate the infinite series whose summands contain the factorial quotients highlighted in the title and weighted by harmonic numbers.

5.1. Series with $\frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}}$

Under the parameter setting

$$a \to ax,\ b \to \tfrac{1}{2} + cx,\ c \to \tfrac{1}{3} + ex,\ d \to \tfrac{2}{3} + ex;$$

the summation formula $\Omega(a; b, c, d)$ becomes the following one.

Theorem 10. *For a variable x and three complex parameters $\{a, c, e\}$, there holds*

$$\begin{aligned}
&\Gamma\begin{bmatrix} \frac{1}{2} + ax - cx, \frac{1}{3} + ax - ex, \frac{2}{3} + ax - ex, \frac{1}{2} + ax - cx - 2ex \\ ax, \frac{1}{6} + ax - cx - ex, \frac{5}{6} + ax - cx - ex, 1 + ax - 2ex \end{bmatrix}\\
&= \sum_{n=0}^{\infty} \Big(\frac{-1}{27}\Big)^n \frac{(\frac{1}{2} - cx)_n(1 + ax - 2ex)_n(1 + ax - ex)_n(1 - 3ex)_{3n}}{n!(ax - 2ex + n)(\frac{1}{2} + ax - cx)_n(1 - ex)_n(1 + 3ax - 3ex)_{3n}}\\
&\quad\times \frac{(\frac{1}{2} + 3ax - 3cx - 3ex)_{3n}\Delta_n(ax, \frac{1}{2} + cx, \frac{1}{3} + ex, \frac{2}{3} + ex)}{(n - \frac{1}{6} + ax - cx - ex)(\frac{1}{2} + ax - cx - ex)_n(\frac{1}{2} + ax - cx - 2ex)_{2n+1}}.
\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, c, e\}$, we derive from Theorem 10 the following infinite series identities.

- Ω_0: Constant term identity.

$$\frac{1}{5} = \sum_{n=1}^{\infty} (-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}} \{18n^2 - 1\}.$$

- $\Omega_1 :\ a = 0, c = -e = 1.$

$$\frac{1}{5} = \sum_{n=1}^{\infty} (-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}} \Big\{(18n^2 - 1)(\mathbf{O}_{2n+1} - \mathbf{H}_n) + \frac{9(8n^2 - 1)}{10n}\Big\}.$$

- $\Omega_1 :\ a = c = e = 1.$

$$\frac{2 + \sqrt{3}\pi}{10} = \sum_{n=1}^{\infty} (-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}} \Big\{(18n^2 - 1)(3\mathbf{H}_{6n} - 2\mathbf{O}_{2n+1}) - \frac{3(3 - 44n + 84n^2 + 72n^3)}{20n(6n - 1)}\Big\}.$$

- $\Omega_1 :\ a = -e = 1, c = 3.$

$$\frac{6 + \sqrt{3}\pi}{10} = \sum_{n=1}^{\infty} (-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}} \Big\{(18n^2 - 1)(3\mathbf{H}_{6n} - 2\mathbf{H}_n) + \frac{3(9 - 28n - 180n^2 + 504n^3)}{20n(6n - 1)}\Big\}.$$

- $\Omega_1 :\ a = c = 1, e = 0.$

$$\frac{\sqrt{3}\pi}{5} = \sum_{n=1}^{\infty} (-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}} \Big\{(18n^2 - 1)(3\mathbf{H}_{3n} + 2\mathbf{O}_n - 2\mathbf{H}_n) + \frac{9(2n - 1)(4n + 1)}{10n}\Big\}.$$

- $\Omega_1: \ a = e = 1, c = 0.$

$$\frac{\sqrt{3}\pi - 2}{5} = \sum_{n=1}^{\infty}(-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}}\Big\{(18n^2-1)(3\mathbf{H}_{3n}+2\mathbf{O}_n-2\mathbf{O}_{2n+1}) - \frac{9(2n+1)(4n-1)}{10n}\Big\}.$$

- $\Omega_1: \ a = -c = e = 1.$

$$\frac{\sqrt{3}\pi - 6}{10} = \sum_{n=1}^{\infty}(-1)^n \frac{(\frac{1}{6})_n(-\frac{1}{6})_n}{(\frac{1}{2})_{2n+1}}\Big\{(18n^2-1)(3\mathbf{H}_{3n}-3\mathbf{H}_{6n}+2\mathbf{O}_n) + \frac{9(2n-1)(4n+1)}{10n}\Big\}.$$

5.2. *Series with* $\frac{(\frac{5}{6})_n(-\frac{5}{6})_n}{(\frac{1}{2})_{2n+1}}$

Making the parameter replacements in $\Omega(a; b, c, d)$

$$a \to ax,\ b \to \tfrac{1}{2} + cx,\ c \to ex - \tfrac{1}{3},\ d \to \tfrac{4}{3} + ex;$$

we obtain the following transformation formula.

Theorem 11. *For a variable x and three complex parameters $\{a, c, e\}$, there holds*

$$\Big(\frac{ex-\frac{1}{3}}{ex+\frac{1}{3}}\Big)\Gamma\begin{bmatrix}\frac{1}{2}+ax-cx,\frac{1}{3}+ax-ex,\frac{2}{3}+ax-ex,\frac{1}{2}+ax-cx-2ex\\ ax,\frac{1}{6}+ax-cx-ex,\frac{5}{6}+ax-cx-ex,1+ax-2ex\end{bmatrix}$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{27}\Big)^n\frac{(1+ax-2ex)_n(1+ax-ex)_n(3n-1+3ax-3ex)(1-3ex)_{3n+1}}{n!(3n-1-3ex)(\frac{1}{2}+ax-cx-ex)_n(1-ex)_n(1+3ax-3ex)_{3n+1}}$$
$$\times\frac{(\frac{1}{2}-cx)_n(\frac{1}{2}+3ax-3cx-3ex)_{3n}\Delta_n(ax,\frac{1}{2}+cx,-\frac{1}{3}+ex,\frac{4}{3}+ex)}{(\frac{1}{2}+ax-cx)_n(ax-2ex+n)(n-\frac{5}{6}+ax-cx-ex)(\frac{1}{2}+ax-cx-2ex)_{2n+1}}.$$

By specifying particular values for parameters $\{a, c, e\}$, we derive from Theorem 11 the following infinite series identities.

- Ω_0: Constant term identity.

$$25 = \sum_{n=1}^{\infty}(-1)^n \frac{(\frac{5}{6})_n(-\frac{5}{6})_n}{(\frac{1}{2})_{2n+1}}\{90n^2-17\}.$$

- $\Omega_1: \ a = 0, c = -e = 1.$

$$25 = \sum_{n=1}^{\infty}(-1)^n \frac{(\frac{5}{6})_n(-\frac{5}{6})_n}{(\frac{1}{2})_{2n+1}}\Big\{(90n^2-17)(\mathbf{O}_{2n+1}-\mathbf{H}_n) + \frac{9(8n^2-1)}{2n}\Big\}.$$

- $\Omega_1: \ a = 0, c = 1, e = 0.$

$$10 = \sum_{n=1}^{\infty}(-1)^n \frac{(\frac{5}{6})_n(-\frac{5}{6})_n}{(\frac{1}{2})_{2n+1}}\Big\{(90n^2-17)(\mathbf{O}_n-3\mathbf{O}_{3n}+\mathbf{O}_{2n+1}) + \frac{3(17-90n+18n^2)}{5-6n}\Big\}.$$

- $\Omega_1: \ a = c = 1, e = 0.$

$$5(30-\sqrt{3}\pi) = \sum_{n=1}^{\infty}(-1)^n \frac{(\frac{5}{6})_n(-\frac{5}{6})_n}{(\frac{1}{2})_{2n+1}}\left\{\begin{matrix}(90n^2-17)(3\mathbf{H}_{3n+1}+2\mathbf{O}_n-2\mathbf{H}_n)\\ +\dfrac{3(3+31n-42n^2-108n^3)}{2n(3n-1)}\end{matrix}\right\}.$$

5.3. Series with $\frac{(\frac{7}{6})_n(-\frac{7}{6})_n}{(\frac{1}{2})_{2n+1}}$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a \to ax,\ b \to \tfrac{1}{2}+cx,\ c \to ex-\tfrac{2}{3},\ d \to \tfrac{5}{3}+ex;$$

we obtain the following transformation formula.

Theorem 12. *For a variable x and three complex parameters $\{a,c,e\}$, there holds*

$$\begin{aligned}
&3\left(\frac{ex-\frac{2}{3}}{ex+\frac{2}{3}}\right)\Gamma\left[\begin{matrix}\frac{1}{2}+ax-cx,\frac{1}{3}+ax-ex,\frac{2}{3}+ax-ex,\frac{1}{2}+ax-cx-2ex\\ ax,\frac{1}{6}+ax-cx-ex,\frac{5}{6}+ax-cx-ex,1+ax-2ex\end{matrix}\right]\\
&=\sum_{n=0}^{\infty}\Big(\frac{-1}{27}\Big)^n\frac{(\frac{1}{2}-cx)_n(1+ax-2ex)_{n-1}(1-3ex)_{3n}(n-\frac{2}{3}+ax-ex)(n+\frac{2}{3}-ex)}{n!(\frac{1}{2}+ax-cx)_n(1+3ax-3ex)_{3n}(n+\frac{2}{3}+ax-ex)(n-\frac{2}{3}-ex)}\\
&\quad\times\frac{(1+ax-ex)_n(\frac{1}{2}+3ax-3cx-3ex)_{3n+1}\Delta_n(ax,\frac{1}{2}+cx,-\frac{2}{3}+ex,\frac{5}{3}+ex)}{(\frac{1}{6}-ax+cx+ex-n)_2(\frac{1}{2}+ax-cx-ex)_n(1-ex)_n(\frac{1}{2}+ax-cx-2ex)_{2n+1}}.
\end{aligned}$$

By specifying particular values for parameters $\{a,c,e\}$, we derive from Theorem 12 the following infinite series identities.

- Ω_0: Constant term identity.

$$49=\sum_{n=1}^{\infty}(-1)^n\frac{(\frac{7}{6})_n(-\frac{7}{6})_n}{(\frac{1}{2})_{2n+1}}\{90n^2-29\}.$$

- $\Omega_1:\ a=0, c=-e=1.$

$$49=\sum_{n=1}^{\infty}(-1)^n\frac{(\frac{7}{6})_n(-\frac{7}{6})_n}{(\frac{1}{2})_{2n+1}}\Big\{(90n^2-29)(\mathbf{O}_{2n+1}-\mathbf{H}_n)+\frac{9(8n^2-1)}{2n}\Big\}.$$

- $\Omega_1:\ a=0, c=1, e=0.$

$$266=\sum_{n=1}^{\infty}(-1)^n\frac{(\frac{7}{6})_n(-\frac{7}{6})_n}{(\frac{1}{2})_{2n+1}}\left\{\begin{matrix}(29-90n^2)(\mathbf{O}_n-3\mathbf{O}_{3n+1}+\mathbf{O}_{2n+1})\\ -\dfrac{6(116-237n+72n^2+216n^3)}{(6n-1)(6n-7)}\end{matrix}\right\}.$$

- $\Omega_1:\ a=c=1, e=0.$

$$7(21+\sqrt{3}\pi)=\sum_{n=1}^{\infty}(-1)^n\frac{(\frac{7}{6})_n(-\frac{7}{6})_n}{(\frac{1}{2})_{2n+1}}\left\{\begin{matrix}(90n^2-29)(3\mathbf{H}_{3n}+2\mathbf{O}_n-2\mathbf{H}_n)\\ +\dfrac{3(12+256n-123n^2-774n^3+216n^4)}{2n(3n-2)(3n+2)}\end{matrix}\right\}.$$

6. Series with factorial structure $\binom{6n}{n,2n,3n}\Big/\binom{2n}{n}=108^n\times\left[\begin{matrix}\frac{1}{6},\frac{5}{6}\\ 1,\frac{1}{2}\end{matrix}\right]_n$

Finally, we examine the series with the factorial structure displayed in the title.

6.1. Series with $\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}$

Performing the parameter replacements in $\Omega(a;b,c,d)$

$$a\to ax-1,\ b\to\frac{1}{6}+cx,\ c\to cx-\frac{1}{6},\ d\to ex;$$

we obtain the following transformation formula.

Theorem 13. *For a variable x and three complex parameters $\{a,c,e\}$, there holds*

$$\begin{aligned}&(ax-1)\Big(\frac{1}{6}-cx\Big)\Gamma\begin{bmatrix}1+ax-ex,1+ax-2cx-ex,\frac{1}{6}+ax-cx,\frac{5}{6}+ax-cx\\ ax,1+ax-2cx,\frac{1}{6}+ax-cx-ex,\frac{5}{6}+ax-cx-ex\end{bmatrix}\\ &=\sum_{n=0}^{\infty}\Big(\frac{-1}{27}\Big)^n\begin{bmatrix}1+ax-2cx,1-ex,\frac{1}{2}+ax-cx\\ 1+ax-ex,\frac{1}{2}+ax-cx-ex,\frac{1}{2}-cx\end{bmatrix}_n\frac{\Big[\frac{1}{2}+3ax-3cx-3ex,\frac{1}{2}-3cx\Big]_{3n}}{n!(\frac{1}{2}+3ax-3cx)_{3n}}\\ &\times\frac{(n+ax-ex)(n-cx+\frac{1}{6})(n+ax-cx-\frac{1}{6})\Delta_n(ax-1,\frac{1}{6}+cx,cx-\frac{1}{6},ex)}{(n+ax-2cx)(n+ax-cx-ex-\frac{1}{6})(1+ax-2cx-ex)_{2n+1}}.\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,c,e\}$, we derive from Theorem 13 the following infinite series identities.

- Ω_0: Constant term identity.

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}\Big\{n(37-180n^2)\Big\}.$$

- $\Omega_1:\ a=c=-e=1.$

$$72=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}\Big\{n(180n^2-37)\mathbf{H}_n+\frac{55-18n-972n^2+648n^3+5184n^4}{(1+6n)(1-6n)}\Big\}.$$

- $\Omega_1:\ a=3,c=-1,e=1.$

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}\Big\{2n(37-180n^2)\mathbf{O}_{n+1}+\frac{53+54n-252n^2-1944n^3-18144n^4}{(1+6n)(1-6n)}\Big\}.$$

- $\Omega_1:\ a=1,c=e=0.$

$$36=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}\Big\{n(180n^2-37)\mathbf{H}_{2n+1}+1-36n-324n^2\Big\}.$$

- $\Omega_1:\ a=-c=e=1.$

$$36=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}\Big\{n(37-180n^2)\bar{\mathbf{H}}_{2n+1}+\frac{18(3+n-34n^2-36n^3-360n^4)}{(1+6n)(1-6n)}\Big\}.$$

- $\Omega_1:\ a=5,c=-1,e=3.$

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{5}{6})_n(\frac{7}{6})_n}{(2n+1)!}\Big\{n(37-180n^2)\mathbf{O}_{3n+2}+\frac{89+90n+1044n^2-3240n^3-28512n^4}{6(1-6n)(1+6n)}\Big\}.$$

6.2. *Series with* $\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}$

Making the parameter replacements in $\Omega(a;b,c,d)$

$$a \to ax-1,\ b \to \frac{5}{6}+cx,\ c \to cx-\frac{5}{6},\ d \to ex;$$

we obtain the following transformation formula.

Theorem 14. *For a variable x and three complex parameters $\{a,c,e\}$, there holds*

$$(ax-1)\Big(\frac{5}{6}-cx\Big)\Gamma\left[\begin{matrix}1+ax-ex, 1+ax-2cx-ex, \frac{1}{6}+ax-cx, \frac{5}{6}+ax-cx\\ ax, 1+ax-2cx, \frac{1}{6}+ax-cx-ex, \frac{5}{6}+ax-cx-ex\end{matrix}\right]$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{27}\Big)^n\left[\begin{matrix}1+ax-2cx, 1-ex, \frac{1}{2}+ax-cx\\ 1+ax-ex, \frac{1}{2}+ax-cx-ex, \frac{1}{2}-cx\end{matrix}\right]_n\frac{\big[\frac{1}{2}+3ax-3cx-3ex, \frac{1}{2}-3cx\big]_{3n}}{n!(\frac{1}{2}+3ax-3cx)_{3n}}$$
$$\times\frac{(n+ax-ex)(n-cx+\frac{5}{6})(n+ax-cx-\frac{5}{6})\Delta_n(ax-1,\frac{5}{6}+cx,cx-\frac{5}{6},ex)}{(n+ax-2cx)(n+ax-cx-ex-\frac{5}{6})(1+ax-2cx-ex)_{2n+1}}.$$

By specifying particular values for parameters $\{a,c,e\}$, we derive from Theorem 14 the following infinite series identities.

- Ω_0: Constant term identity.

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}\Big\{n(61-180n^2)\Big\}.$$

- $\Omega_1:\ a=c=-e=1.$

$$72=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}\Big\{n(180n^2-61)\mathbf{H}_n+\frac{1975-450n-8748n^2+648n^3+5184n^4}{(5+6n)(5-6n)}\Big\}.$$

- $\Omega_1:\ a=-3c=3e=3.$

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}\Big\{2n(61-180n^2)\mathbf{O}_{n+1}+\frac{725+1350n+9252n^2-1944n^3-18144n^4}{(5-6n)(5+6n)}\Big\}.$$

- $\Omega_1:\ a=-c=e=1.$

$$36=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}\Big\{n(61-180n^2)\bar{\mathbf{H}}_{2n+1}+\frac{18(75+25n+14n^2-36n^3-360n^4)}{(5+6n)(5-6n)}\Big\}.$$

- $\Omega_1:\ a=1, c=e=0.$

$$36=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}\Big\{n(180n^2-61)\mathbf{H}_{2n+1}+25-36n-324n^2\Big\}.$$

- $\Omega_1:\ a=-5c=5, e=3.$

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{11}{6})_n}{(2n+1)!}\Big\{n(61-180n^2)\mathbf{O}_{3n}+\frac{1625+10380n+16512n^2-5328n^3-39312n^4-31104n^5}{6(1+2n)(5-6n)(5+6n)}\Big\}.$$

6.3. Series with $\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(2n+1)!}$

Performing the parameter replacements in $\Omega(a;b,c,d)$

$$a \to ax,\ b \to \tfrac{1}{6}+cx,\ c \to \tfrac{5}{6}+cx, d \to ex;$$

we obtain the following transformation formula.

Theorem 15. *For a variable x and three complex parameters $\{a,c,e\}$, there holds*

$$\begin{aligned}
&(ax)\Gamma\left[\begin{matrix}1+ax-ex, 1+ax-2cx-ex, \frac{1}{6}+ax-cx, \frac{5}{6}+ax-cx\\ 1+ax, 1+ax-2cx, \frac{1}{6}+ax-cx-ex, \frac{5}{6}+ax-cx-ex\end{matrix}\right]\\
=\ &\sum_{n=0}^{\infty}\Big(\frac{-1}{27}\Big)^n\left[\begin{matrix}1+ax-2cx, 1-ex, \frac{1}{2}+ax-cx\\ 1+ax-ex, \frac{1}{2}+ax-cx-ex, \frac{1}{2}-cx\end{matrix}\right]_n\\
&\times\frac{\left[\frac{1}{2}+3ax-3cx-3ex, \frac{1}{2}-3cx\right]_{3n}\Delta_n(ax, \frac{1}{6}+cx, \frac{5}{6}+cx, ex)}{n!(n+ax-2cx)(\frac{1}{2}+3ax-3cx)_{3n}(1+ax-2cx-ex)_{2n+1}}.
\end{aligned}$$

By specifying particular values for parameters $\{a,c,e\}$, we derive from Theorem 15 the following infinite series identities.

- Ω_0: Constant term identity.

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(1)_{2n+1}}\{5+108n+180n^2\}.$$

- Ω_1 : $a=c=-e=1$.

$$72=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(1)_{2n+1}}\Big\{72(1+2n)-(5+108n+180n^2)\mathbf{H}_n\Big\}.$$

- Ω_1 : $a=-3c=3e=3$.

$$0=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(2n+1)!}\Big\{72n-(5+108n+180n^2)\mathbf{O}_{n+1}\Big\}.$$

- Ω_1 : $a=1, c=e=0$.

$$36=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(1)_{2n+1}}\Big\{36(1+4n)-(5+108n+180n^2)\mathbf{H}_{2n+1}\Big\}.$$

- Ω_1 : $a=-c=e=1$.

$$36=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(2n+1)!}\Big\{36+(5+108n+180n^2)\bar{\mathbf{H}}_{2n+1}\Big\}.$$

- Ω_1 : $a = -5c = 5, e = 3.$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(2n+1)!}\Big\{(5+108n+180n^2)\mathbf{O}_{3n} + \frac{23+108n+108n^2}{3(1+2n)}\Big\}.$$

- Ω_2 : $a = 1, c = e = 0.$

$$0 = \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(1)_{2n+1}}\Big\{72 - 72(1+4n)\mathbf{H}_{2n+1} + (5+108n+180n^2)(\mathbf{H}^{\langle 2\rangle}_{2n+1} + \mathbf{H}^2_{2n+1})\Big\}.$$

7. Conclusions and further comments

By employing the "coefficient extraction method", we have exhibited "95" summation formulae for infinite series of convergence rate "-1/4". The major part of them concerns infinite series expressions for simple rational numbers. There are also series with their values involving important mathematical constants. For example, the four series recorded in Section 4.3 are remarkable, since the initial formula (also the simplest one) recovers one of the 17 significant series discovered by Ramanujan [25] one century ago. If we extract the coefficient of x^2 and x^3 in the transformation given in Theorem 9, we find, among the resulting series, the following four typical ones:

$$\begin{aligned}
&\frac{128\ln^2 2}{\pi} + \frac{32\pi}{3}\\
&= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \left[\begin{matrix}\frac{1}{2}, \frac{3}{4}, \frac{5}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n \Big\{\tfrac{3+20n}{1+4n}\Big(5\mathbf{H}^{\langle 2\rangle}_n - 76\mathbf{O}^{\langle 2\rangle}_n + 4\mathbf{H}^2_{2n}\Big) + \tfrac{24}{(1+2n)^2} - \tfrac{24\mathbf{H}_{2n}}{1+2n}\Big\},\\
&\frac{512\ln^2 2}{\pi} - \frac{64\pi}{3}\\
&= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \left[\begin{matrix}\frac{1}{2}, \frac{3}{4}, \frac{5}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n \left\{\begin{matrix}\frac{3+20n}{1+4n}\Big(5\mathbf{H}^{\langle 2\rangle}_n - 12\mathbf{O}^{\langle 2\rangle}_n + (3\mathbf{H}_n - 2\mathbf{O}_n)^2\Big)\\ +\frac{8}{(1+2n)^2} - \frac{4(7+12n)(3\mathbf{H}_n - 2\mathbf{O}_n)}{(1+2n)(1+4n)}\end{matrix}\right\};\\
&42\pi\ln 2 - \frac{216\ln^3 2}{\pi} - \frac{96\zeta(3)}{\pi}\\
&= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n \left\{\begin{matrix}\frac{24}{(1+2n)(1+4n)} - \frac{24\mathbf{H}_n}{1+4n} - \frac{3(5+12n)}{1+2n}\Big(\mathbf{H}^2_n + \mathbf{H}^{\langle 2\rangle}_n - 2\mathbf{O}^{\langle 2\rangle}_n - 8\mathbf{O}^{\langle 2\rangle}_{2n}\Big)\\ +(3+20n)\Big(\mathbf{H}^3_n + 2\mathbf{H}^{\langle 3\rangle}_n + 3\mathbf{H}_n(\mathbf{H}^{\langle 2\rangle}_n - 2\mathbf{O}^{\langle 2\rangle}_n - 8\mathbf{O}^{\langle 2\rangle}_{2n})\Big)\end{matrix}\right\},\\
&\frac{3}{2}\pi\ln 2 + \frac{2\ln^3 2}{\pi} + \frac{21\zeta(3)}{\pi}\\
&= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \left[\begin{matrix}\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\\ 1,\ 1,\ 1\end{matrix}\right]_n \left\{\begin{matrix}\frac{6(3+8n)}{(1+2n)(1+4n)^2} + \frac{6(3+8n)\mathbf{O}_{2n}}{(1+2n)(1+4n)} - \frac{3(3+8n)}{2(1+2n)}\Big(\mathbf{O}^{\langle 2\rangle}_n - 2\mathbf{O}^2_{2n} - 2\mathbf{O}^{\langle 2\rangle}_{2n}\Big)\\ +(3+20n)\Big(4\mathbf{O}^{\langle 3\rangle}_{2n} + \mathbf{O}_{2n}(2\mathbf{O}^2_{2n} - 3\mathbf{O}^{\langle 2\rangle}_n + 6\mathbf{O}^{\langle 2\rangle}_{2n})\Big)\end{matrix}\right\}.
\end{aligned}$$

However, it remains intriguing to determine, in closed form, the explicit value of similar series corresponding to any "singleton" of quadratic and cubic harmonic numbers

$$\Big\{\mathbf{H}^{\langle 2\rangle}_n,\ \mathbf{O}^{\langle 2\rangle}_n,\ \mathbf{H}^2_n,\ \mathbf{O}^2_n,\ \mathbf{H}_n\mathbf{O}_n;\ \mathbf{H}^{\langle 3\rangle}_n,\ \mathbf{O}^{\langle 3\rangle}_n,\ \mathbf{H}^3_n,\ \mathbf{O}^3_n,\ \mathbf{H}_n\mathbf{O}^{\langle 2\rangle}_n,\ \mathbf{H}_n\mathbf{O}^{\langle 2\rangle}_{2n},\ \mathbf{O}_{2n}\mathbf{O}^{\langle 2\rangle}_n,\ \mathbf{O}_{2n}\mathbf{O}^{\langle 2\rangle}_{2n}\Big\}.$$

The interested reader is enthusiastically encouraged to make further investigation.

Author contributions

C. Li: Writing, computation and review; W. Chu: Methodology and supervision. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

Prof. Wenchang Chu is the Guest Editor of special issue "Special Functions and q-Series" for AIMS Mathematics. Prof. Wenchang Chu was not involved in the editorial review and the decision to publish this article.

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