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Research article

A bounded Lindley exponential model with applications to industrial engineering and epidemiological data

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Abstract: The log-Lindley exponential (LLE), a bounded variant of the Lindley exponential distribution, is introduced using the inverse exponential transformation method. The LLE demonstrates significant flexibility, generating novel and well-known distributions with various parameters and providing support, including log-Lindley, Lindley, and two-parameter Lindley distributions, and exhibiting diverse density shapes, including right-skewed, approximately symmetric, left-skewed, decreasing, U-shaped, and increasing, while the hazard rate can be increasing, J-shaped, and bathtub-shaped. We explore several important statistical properties of the LLE model, including moments, entropy, quantile function, actuarial measures, mean residual life function, and stochastic orderings, which enhance its applicability in practical settings. Our study presents nine distinct frequentist strategies for estimating the model's parameters. In addition, the performance of the proposed estimation method is studied and evaluated through extensive numerical simulations. Finally, applications to real-world datasets from industrial engineering and epidemiology reveal the LLE model's practical utility and superiority, with it outperforming other existing models in terms of goodness-of-fit and flexibility.

Keywords: log-Lindley; entropy; quantile function; simulation; estimation methods; applications
Mathematics Subject Classification: 60E05, 60E99, 62E15

1. Introduction

The progression of unit interval distributions is gaining traction in the literature due to their utility in practical applications. Most of the time, new ideas are distributed with unbounded support. Nevertheless, in many real-world situations, such as percentages and proportions, the data takes values in a bounded interval. Additionally, unit interval models are preferred when reliability is quantified as the ratio of the number of successful trials out of all the trials there were. A common issue in many practical situations is dealing with uncertain phenomena noticed in the bounded interval $(0, 1)$. The well-known statistical distribution for modeling data sets on the interval $(0, 1)$ is the classical beta-type I distribution. However, its ability to model data may be insufficient to model leptokurtic and strongly left-skewed data sets. In the literature, several distributions have been proposed with the aim of improving the modeling accuracy of the data sets on $(0, 1)$. The readers may refer to log-Lindley distribution [1], then unit Lindley distribution [2], the cosine-sine distribution [3], the power logarithmic distribution [4], the unit-Chen distribution [5], the log exponential-power distribution [6], the unit log-log distribution [7], the unit folded normal distribution [8], the unit generalized log Burr XII distribution [9], the unit Weibull distribution [10], the unit generalized exponential distribution [11], and the unit generalized half-normal distribution [12], among others. Several transformation methods have been developed for converting unbounded probability distributions into the unit interval $(0, 1)$. Among the most commonly used methods is the inverse exponential ($X = e^{-Y}$), where Y is a random variable defined on $(0, \infty)$, which generate a unit distributions like unit gamma distribution [13], unit inverse power Lomax distribution [14], unit new X-Lindley distribution [15], unit Gompertz [16], unit Birnbaum-Saunders [17], unit inverse Gaussian [18], and a new 3-parameter bounded beta [19]. Motivated by the preceding discussion, it is essential to introduce other unit interval distributions in addition to those already present in the literature; hence, we introduce a new two-parameter model with bounded support that can be viewed as an alternative to the classical beta-type I distribution.

This new model is called the log-Lindley exponential (LLE) distribution, referred to simply as the LLE distribution. The LLE distribution is developed using the exponential transformation $X = e^{-Y}$, where Y follows the Lindley exponential (LE) distribution proposed by [20]. Although the LLE distribution is defined on the unit interval $(0, 1)$, its bounded support does not limit its practical application. On the contrary, many significant forms of real-world data are already constrained to this range. Probabilities, proportions, normalized scores, and bounded performance indexes are some examples. These data types are commonly used in fields including reliability engineering, survival analysis, economics, and quality control. As a result, creating flexible models defined on $(0, 1)$, such as the LLE distribution, addresses a common and significant class of modeling problems. Furthermore, the LLE distribution can be extended to model data defined over any bounded interval (a, b) by a simple transformation:

$$X_{(a,b)} = a + (b - a)X, \quad \text{where } X \sim \text{LLE distribution.}$$

This transformation preserves the statistical structure while extending its applicability to other bound domains.

The objective of establishing the LLE distribution is as follows:

- This new model, which is based on two parameters, can be viewed as an alternative to the classical beta-type I distribution.
- For the new model, the cumulative distribution function can be easily expressed in a closed form.
- Our model is useful to define new and well-known distributions with a diversity of parameters and supports, a few of which are mentioned as remarks in Section 2.

- Additionally, our model's hazard rate takes on a variety of shapes, including the bathtub shape. As a consequence, it might prove beneficial to analyze the biological and mortality data as well.

- As shown in Figure 1, the density function has a shape that can be seen as right-skewed, approximately symmetric, left-skewed, decreasing, U-shaped, and increasing. It is a good point for our models because they provide a wide range of shapes without additional parameters in their formulation. Also, this provides a new model as adaptable as the beta type 1 model for modelling data constrained to (0, 1) intervals.

- This model, when compared to beta, log-Lindley, power log-Lindley, cosine-sine, power logarithmic, reduced Kies, transformed gamma, log-gamma, log-weighted power, and transmuted power distributions, is more flexible in responding to actual data, as will be shown in Section 6.

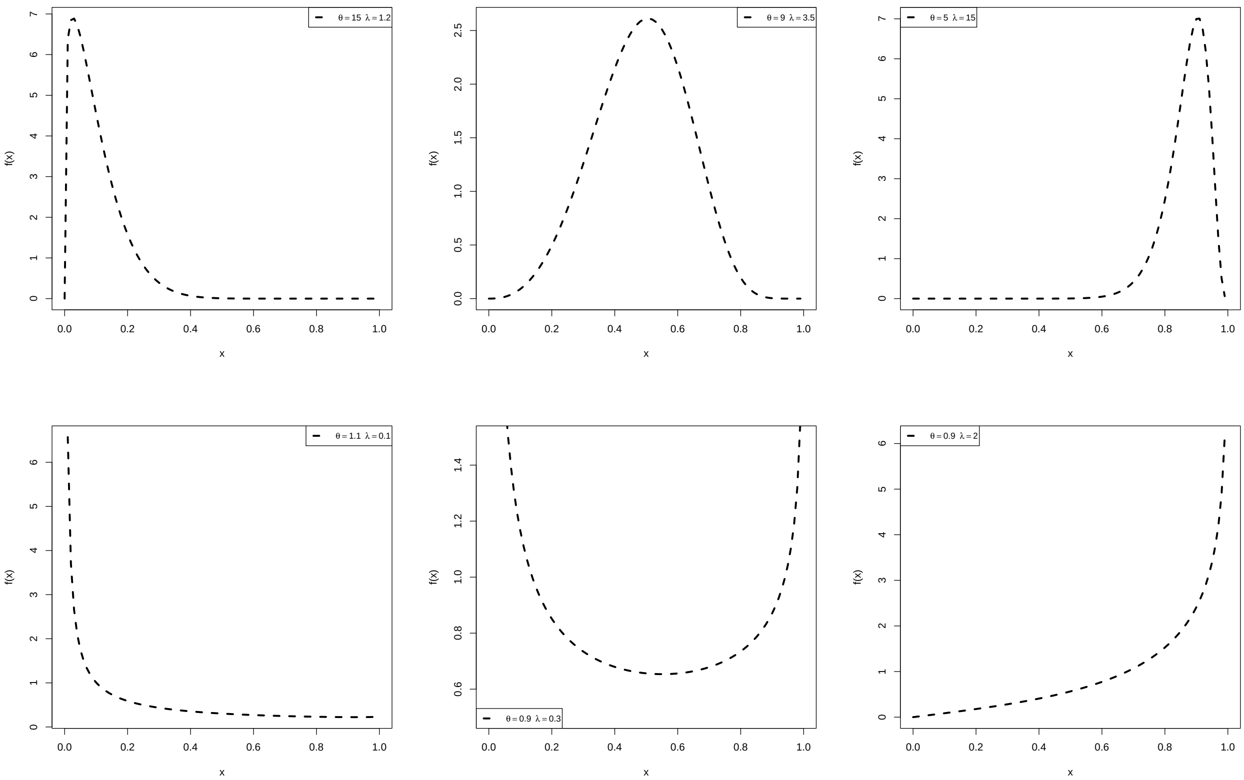

Figure 1. Plots of PDF of LLE distribution.

The following is how the rest of the article is structured: We introduce the LLE distribution in Section 2. Section 3 discusses the LLE distribution's properties, such as moments, entropy, quantile function, actuarial measures, and mean residual life function. Different frequentist estimation approaches are described in Section 4. Section 5 assesses the performance of these approaches through a Monte Carlo simulation study. Two real data sets are used in Section 6 to show how the LLE model can fit better than well-known distributions with bounded support. Finally, in Section 7, the study's conclusion is offered.

2. The LLE distribution

In this section, we introduce a new bounded model derived from the LE distribution [20] by the exponential transformation. Using the LE distribution proposed by Bhati [20], whose probability density function (PDF) is defined by

$$f(y) = \frac{\theta^2 \lambda e^{-\lambda y}\left(1-e^{-\lambda y}\right)^{\theta-1}\left(1-\log\left(1-e^{-\lambda y}\right)\right)}{\theta+1},\ y, \theta, \lambda > 0.$$

Then, our distribution can be viewed as a result of LE by employing the following transformation $X = e^{-Y}$. Now, we define the cumulative distribution function (CDF) and PDF of our proposed model, which we refer to as the LLE distribution which are, respectively, defined as

$$F(x) = 1-\left(1-x^{\lambda}\right)^{\theta} + \frac{\theta\left(1-x^{\lambda}\right)^{\theta}\log\left(1-x^{\lambda}\right)}{\theta+1},\ 0 < x < 1,\ \theta, \lambda > 0, \tag{2.1}$$

$$f(x) = \frac{\theta^2 \lambda x^{\lambda-1}\left(1-x^{\lambda}\right)^{\theta-1}\left(1-\log\left(1-x^{\lambda}\right)\right)}{\theta+1}. \tag{2.2}$$

The corresponding survival function (SF) and hazard rate function (HRF) are, respectively, given by

$$S(x) = \left(1-x^{\lambda}\right)^{\theta} - \frac{\theta\left(1-x^{\lambda}\right)^{\theta}\log\left(1-x^{\lambda}\right)}{\theta+1}, \tag{2.3}$$

$$h(x) = \frac{\theta^2 \lambda x^{\lambda-1}\left(\log\left(1-x^{\lambda}\right)-1\right)}{(1-x^{\lambda})(-\theta+\theta\log(1-x^{\lambda})-1)}. \tag{2.4}$$

Different plots of the PDF of the LLE model are displayed in Figure 1 for several parameter values, which show that our proposed model PDF can be right-skewed, approximately symmetric, left-skewed, decreasing, U-shaped, and increasing.

Our proposed model HRF can be a bathtub, J-shaped, or increasing function by different parameter values as shown in Figure 2.

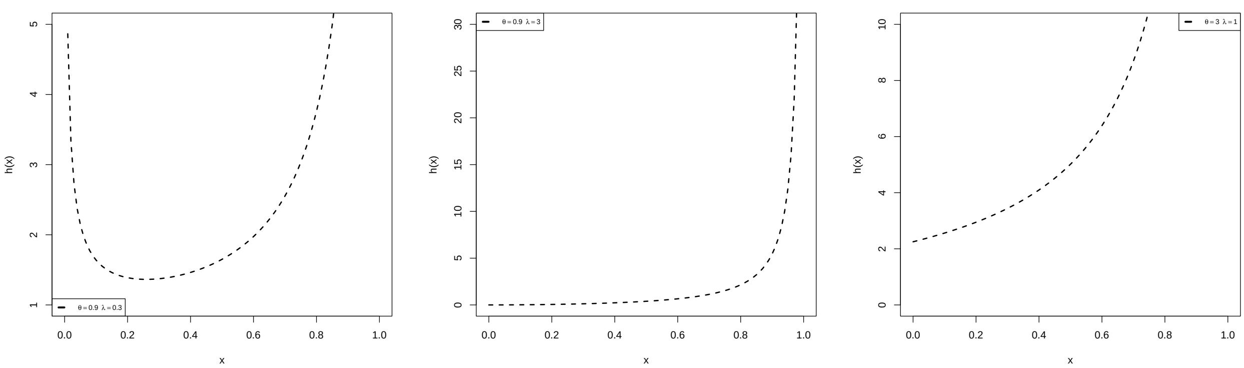

Figure 2. Plots of HRF of LLE distribution.

Now, we give some remarks on the LLE distribution.

Remark 1. The particular case of density (2.2) for $Y = X^{\lambda}$ is the log-Lindley distribution $LL(\theta, 1)$ proposed by Bhati et al. [20] with PDF:

$$f(y) = \frac{\theta^2 (1-y)^{\theta-1} (1 - \log(1-y))}{\theta+1}, \; 0 < y < 1.$$

Remark 2. Under X having LLE distribution, $\lambda = 1$ and the transformation $Y = -\log(1-X)$, Eq (2.2) reduces to the Lindley distribution introduced by Lindley [21] with PDF:

$$f(y) = \frac{\theta^2}{\theta+1}(1+y)e^{-\theta y}, \; y > 0, \theta > 0.$$

Remark 3. Further, if $Y = -\frac{\log(1-x^{\lambda})}{\beta}$, $\beta > 0$, Eq (2.2) gives the new form of Lindley with two parameters, which was not proposed before, and its PDF is given by:

$$f(y) = \frac{\theta^2\beta}{\theta+1}(1+\beta y)e^{-\theta\beta y}, \; y, \theta, \beta > 0.$$

The definitions and results listed below will be used throughout the rest of the paper.

i. The polygamma function of order n, $\psi^{(n)}(z)$, which is given by the $(n+1)\,th$ derivative of the logarithm of the gamma function $\Gamma(z)$:

$$\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \ln[\Gamma(z)] = \frac{d^n}{dz^n}\frac{\Gamma^{/}(z)}{\Gamma(z)} = \frac{d^n}{dz^n}\psi^{(0)}(z) = \frac{d^n}{dz^n}\psi(z), \tag{2.5}$$

where $\psi(z)$ is the digamma function, and for $n > 0$ the polygamma function, it can be written as

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}.$$

ii. The $n-th$ harmonic number defined by

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

iii. Euler gamma γ is the Euler-Mascheroni constant with a numerical value $\simeq 0.577216$.

iv. The Lambert W function is the function that satisfies $W(z)\exp(W(z)) = z$, with z a complex number.

v. The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as

$${}_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots.$$

vi. The exponential integral function for positive values of the real part of z defined by

$$E_1(z) = \int_1^{\infty} e^{-tz} t^{-1} dt.$$

vii. The beta function $B(a, b)$ is defined as

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt. \tag{2.6}$$

3. Mathematical properties of the LLE

Several mathematical features of the LLE distribution, including moments, entropy, quantile function, and mean residual life function, are presented in this section.

3.1. Moments

Theorem 1. Let X be a random variable with LLE density (2.2). Then the r^{th} moment and logarithmic moments are given, respectively, for $r = 1, 2, ...$

i.

$$E[X^r] = \frac{\theta^2}{(\theta+1)} B\left(\theta, \frac{r}{\lambda}+1\right)\left(1 - \psi^{(0)}(\theta) + \psi^{(0)}\left(1+\theta+\frac{r}{\lambda}\right)\right).$$

ii.

$$E[\log(X^r)] = \frac{r}{\theta(\theta+1)\lambda}\left(\theta^2\psi^{(1)}(\theta+1) - \theta\psi^{(0)}(\theta) - \theta^2 H_\theta - \theta\gamma - 1\right).$$

Proof. Using the definition of the r^{th} moment of X around the origin, we have

$$E[X^r] = \frac{\theta^2\lambda}{\theta+1}\int_0^1 x^{r+\lambda-1}\left(1-x^\lambda\right)^{\theta-1}\left(1-\log\left(1-x^\lambda\right)\right)dx.$$

Now, lets define a new variable $U = X^\lambda$, $dx = \frac{1}{\lambda}u^{\frac{1}{\lambda}-1}du$. Then, $E[X^r] = E[U^{\frac{r}{\lambda}}]$, and $E[\log(X^r)] = \frac{r}{\lambda}E[\log(U)]$. Now, we compute $E[U^m]$ for $m = \frac{r}{\lambda}$:

$$E[U^m] = \frac{\theta^2}{\theta+1}\left[\int_0^1 u^m(1-u)^{\theta-1}du - \int_0^1 u^m(1-u)^{\theta-1}\log(1-u)du\right].$$

Using the definition of beta function (2.6) and the polygamma function (2.5), we have the first integral:

$$\int_0^1 u^m(1-u)^{\theta-1}du = B(m+1,\theta).$$

Also, the second integral:

$$\int_0^1 u^m(1-u)^{\theta-1}\log(1-u)du = B(m+1,\theta)\left[\psi^{(0)}(\theta) - \psi^{(0)}(m+\theta+1)\right].$$

Putting it all together and substituting $m = \frac{r}{\lambda}$, we have

$$E[X^r] = \frac{\theta^2}{(\theta+1)} B\left(\theta, \frac{r}{\lambda}+1\right)\left(1 - \psi^{(0)}(\theta) + \psi^{(0)}\left(1+\theta+\frac{r}{\lambda}\right)\right).$$

Also, the proof for the logarithmic moments, $E[\log(X^r)]$, follows similarly. □

According to numerical values of moments, we can use any suitable software like R or Wolfram Mathematica to compute our moments values.

3.2. Entropy

Entropy is a measure of the variation in the degree of uncertainty in a system characterized by a probability distribution $f(x)$. Two major entropy measures of $f(x)$ are the Shannon entropy (Shannon [22]) and Rényi entropy (Rényi [23]), both of which are defined as

$$\Lambda = E[-\log(f(x))], \tag{3.1}$$

$$\Omega = \frac{1}{1-\delta}\log\left(\int_R f^{\delta}(x)dx\right),\ \delta > 0, \delta \neq 1, \tag{3.2}$$

respectively. The following two theorems investigate explicit expressions of these kinds of entropies for the LLE distribution.

Theorem 2. The Shannon entropy for the LLE model is given by

$$\Lambda = \log(\theta+1) - \log(\theta^2\lambda) - \frac{(\lambda-1)}{\theta(\theta+1)\lambda}\left(\theta^2\psi^{(1)}(\theta+1) - \theta\psi^{(0)}(\theta) - \theta^2 H_\theta - \theta\gamma - 1\right)$$
$$+ \frac{1}{\theta+1}\sum_{i=0}^{\infty}\frac{(-1)^{i+1}\Gamma(2+i)(2+i+\theta\log(e))}{(i+1)\theta^{i+1}(\log(e))^{i+3}}.$$

Proof. Using the definition of Shannon entropy (3.1), we have

$$\Lambda = \log(\theta+1) - \log(\theta^2\lambda) - (\lambda-1)E[\log(x)] - (\theta+1)E[\log(1-x^\lambda)] - E[\log(1-\log(1-x^\lambda))].$$

The expression $E[log(x)]$ is immediate from Theorem 1 by setting $r = 1$ in item (iii). Hence,

$$E[log(x)] = \frac{\theta^2\psi^{(1)}(\theta+1) - \theta\psi^{(0)}(\theta) - \theta^2 H_\theta - \theta\gamma - 1}{\theta(\theta+1)\lambda}.$$

Also, the quantity

$$E[log(1-x^\lambda)] = -\frac{(\theta+2)}{\theta(\theta+1)}.$$

Finally, the quantity $E[\log(1-\log(1-x^\lambda))]$ can be computed as

$$E[\log(1-\log(1-x^\lambda))] = \frac{\theta^2\lambda}{(\theta+1)}\int_0^1 x^{\lambda-1}(1-x^\lambda)^{\theta-1}(1-\log(1-x^\lambda))\log(1-\log(1-x^\lambda))dx.$$

Substituting $u = -\log(1-x^\lambda)$, then $x = (1-e^{-u})^{\frac{1}{\lambda}}$ and using the expansion of

$$log(1+z) = -\sum_{i=0}^{\infty}\frac{(-1)^{i+1}z^{i+1}}{i+1},\ |z| < 1,$$

the above expression reduces to

$$E[\log(1-\log(1-x^\lambda))] = -\frac{1}{\theta+1}\sum_{i=0}^{\infty}\frac{(-1)^{i+1}\Gamma(2+i)(2+i+\theta\log(e))}{(i+1)\theta^{i+1}(\log(e))^{i+3}}.$$

By putting all of the preceding results in a Λ expression, we have the demanded proof.

Theorem 3. The Rényi entropy for the LLE distribution is

$$\Omega = \frac{1}{1-\delta}\log\left(\frac{\theta^{2\delta}\lambda^{\delta-1}}{(\theta+1)^{\delta}}\sum_{i=1}^{\infty}(-1)^i\binom{\frac{\delta\lambda-\delta+1}{\lambda}-1}{i}e^{\delta(\theta-1)+i+1}E_{-\delta}(\delta(\theta-1)+i+1)\right).$$

Proof. For our distribution

$$\int_0^1 f^{\delta}(x)dx = (\frac{\theta^2\lambda}{\theta+1})^{\delta}\int_0^1 x^{\delta(\lambda-1)}\left(1-x^{\lambda}\right)^{\delta(\theta-1)}(\left(1-\log\left(1-x^{\lambda}\right)\right))^{\delta}dx.$$

Substituting $u = -log(1-x^{\lambda})$, then $x = (1-e^{-u})^{\frac{1}{\lambda}}$ and using the expansion of power series

$$(1-z)^t = \sum_{i=0}^{\infty}(-1)^i\binom{t}{i}z^i,\ |z| < 1,$$

the above expression reduces to

$$\int_0^1 f^{\delta}(x)dx = \frac{\theta^{2\delta}\lambda^{\delta-1}}{(\theta+1)^{\delta}}\sum_{i=1}^{\infty}(-1)^i\binom{\frac{\delta\lambda-\delta+1}{\lambda}-1}{i}e^{\delta(\theta-1)+i+1}E_{-\delta}(\delta(\theta-1)+i+1).$$

Using the definition of Rényi entropy (3.2), it follows that

$$\Omega = \frac{1}{1-\delta}\log\left(\frac{\theta^{2\delta}\lambda^{\delta-1}}{(\theta+1)^{\delta}}\sum_{i=1}^{\infty}(-1)^i\binom{\frac{\delta\lambda-\delta+1}{\lambda}-1}{i}e^{\delta(\theta-1)+i+1}E_{-\delta}(\delta(\theta-1)+i+1)\right).$$

This completes the proof.

3.3. Quantile function

The quantile function of $f(x)$ is useful in statistical applications and Monte Carlo simulation, so we provide the quantile function for the LLE distribution in the next proposition.

Proposition 1. The quantile function $Q(u)$ for the LLE distribution is obtained as

$$x = Q(u) = \left(1-\exp\left(\frac{1}{\theta}W\left(-(1-u)(1+\theta)e^{-(1+\theta)}\right)+1+\theta\right)\right)^{\frac{1}{\lambda}},\ 0 < u < 1, \tag{3.3}$$

where $W(.)$ is the Lambert function.

Proof. For any $0 < u < 1$, we solve $F(x) = u$, with respect to x, that is

$$\frac{(\theta+1)\left(1-x^{\lambda}\right)^{\theta}-\theta\left(1-x^{\lambda}\right)^{\theta}\log\left(1-x^{\lambda}\right)}{\theta+1} = 1-u,$$

it can also be written as

$$\left(1-x^{\lambda}\right)^{-\theta}\exp\left(-(1-u)(\theta+1)\left(1-x^{\lambda}\right)^{-\theta}\right) = \exp\left(-(1+\theta)\right).$$

Multiplying both sides of the above equation by $-(1-u)(1+\theta)$, we obtain

$$-(1-u)(1+\theta)\left(1-x^{\lambda}\right)^{-\theta}\exp\left(-(1-u)(\theta+1)\left(1-x^{\lambda}\right)^{-\theta}\right)=-(1-u)(1+\theta)\exp\left(-(1+\theta)\right).$$

It is clear that $-(1-u)(1+\theta)\left(1-x^{\lambda}\right)^{-\theta}$ is the Lambert W function of a real argument $-(1-u)(1+\theta)\exp\left(-(1+\theta)\right)$, that is

$$W\left(-(1-u)(1+\theta)\exp\left(-(1+\theta)\right)\right)=-(1-u)(1+\theta)\left(1-x^{\lambda}\right)^{-\theta}.$$

Therefore, we have

$$\left(1-x^{\lambda}\right)^{\theta}=-\frac{(1-u)(1+\theta)}{W\left(-(1-u)(1+\theta)\exp\left(-(1+\theta)\right)\right)}.$$

Finally, solving for x, we have

$$x=\left(1-\exp\left(\frac{1}{\theta}W\left(-(1-u)(1+\theta)e^{-(1+\theta)}\right)+1+\theta\right)\right)^{\frac{1}{\lambda}}.$$

3.4. Actuarial measures

One of the most popular tasks in the field of actuarial science is estimating market risk in a portfolio of instruments. Risk estimation is required when purchasing and selling products. In this section, we calculated two important actuarial measures for the proposed distribution, known as value at risk (VaR) or *quantile function* $Q(u)$ (see Proposition 1) and tail value at risk ($TVaR$). The $TVaR$ is the conditional tail expectation, and it is stated as follows

$$TVaR=\frac{1}{1-p}\int_{Q(u)}^{1} yf(y)\,dy.$$

$$TVaR=\frac{\left(1-q^{\lambda}\right)^{\theta}{}_{3}F_{2}\left(\theta,\theta,-\frac{1}{\lambda};\theta+1,\theta+1;1-q^{\lambda}\right)-\theta^{2}\left(\log\left(1-q^{\lambda}\right)-1\right)B_{1-q^{\lambda}}\left(\theta,1+\frac{1}{\lambda}\right)}{(\theta+1)(1-p)}.$$

We presented some numerical values of VaR and TVar for the LLE model for various parametric values in Table 1, which were calculated by using Wolfram Mathematica software version 12.0. We conclude that our model risk measures increase as the significance level increases.

Table 1. Numerical computations of VaR and TVaR for the LLE distirbution.

				$\theta = 0.75,\ \lambda = 0.25$				
Significance Level	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
VaR	0.5822724	0.6612069	0.7355526	0.8036486	0.8640308	0.9153851	0.9564482	0.9857406
TVaR	0.8399455	0.8710738	0.8997804	0.9256994	0.9484895	0.9678152	0.9833082	0.9944620
				$\theta = 1.5,\ \lambda = 0.5$				
Significance Level	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
VaR	0.3513219	0.4077332	0.4683828	0.5335934	0.6039048	0.6802529	0.7644360	0.8608003
TVaR	0.6226111	0.6573864	0.6940018	0.7326841	0.7737821	0.8178692	0.8660162	0.9207690
				$\theta = 0.75,\ \lambda = 1.5$				
Significance Level	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
VaR	0.9138068	0.9333751	0.9500992	0.9642235	0.9759364	0.9853730	0.9926060	0.9976092
TVaR	0.9697290	0.9762846	0.9820048	0.9869319	0.9910963	0.9945149	0.9971863	0.9990736
				$\theta = 2,\ \lambda = 1.5$				
Significance Level	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
VaR	0.6043484	0.6415265	0.6790675	0.7173778	0.7570238	0.7988986	0.8446670	0.8984831
TVaR	0.7655140	0.7858843	0.8068230	0.8285600	0.8514354	0.8760057	0.9033335	0.9360980

3.5. *Residual life function*

Residual life is defined by the conditional random variable $R_{(t)} = X - t|X > t,\ t \geq 0$. It is widely used in risk theory and reliability analysis. As a result, we explore several of their associated statistical functions, including the survival function, hazard function, and mean residual life, in relevance to the LLE model.

Proposition 2. The survival function of the residual lifetime $R_{(t)}$ for the LLE model is given by

$$S_{R_{(t)}}(x) = \frac{\left(1-(t+x)^{\lambda}\right)^{\theta}\left(\theta+1-\theta\log\left(1-(t+x)^{\lambda}\right)\right)}{\left(1-t^{\lambda}\right)^{\theta}\left(\theta+1-\theta\log\left(1-t^{\lambda}\right)\right)},\quad x\in(0,1). \tag{3.4}$$

Proof. The proof is self-evident in view of the identity $S_{R_{(t)}}(x) = \frac{S(x+t)}{S(t)}$, where $S(.)$ given by Eq (2.3).

Corollary 1. According to Eq (3.4), the CDF, PDF, and HRF of $R_{(t)}$ are, respectively, represented by

$$F_{R_{(t)}}(x) = 1-\frac{\left(1-(t+x)^{\lambda}\right)^{\theta}\left(\theta+1-\theta\log\left(1-(t+x)^{\lambda}\right)\right)}{\left(1-t^{\lambda}\right)^{\theta}\left(\theta+1-\theta\log\left(1-t^{\lambda}\right)\right)},\quad x\in(0,1),$$

$$f_{R_{(t)}}(x) = \frac{\lambda(t+x)^{\lambda-1}\left(1-(t+x)^{\lambda}\right)^{\theta-1}\left(\theta^2+\theta-\theta\log\left(1-(t+x)^{\lambda}\right)-1\right)}{\left(1-t^{\lambda}\right)^{\theta}\left(\theta-\log\left(1-t^{\lambda}\right)+1\right)},$$

and

$$h_{R_{(t)}}(x) = \frac{\lambda(t+x)^{\lambda-1}\left(\theta^2+\theta-\theta\log\left(1-(t+x)^{\lambda}\right)-1\right)}{\left(1-(t+x)^{\lambda}\right)\left(\theta+1-\log\left(1-(t+x)^{\lambda}\right)\right)}.$$

Proposition 3. The mean residual life (MRL) function $M(x)$ for the LLE distribution is given by

$$M(x) = \frac{1}{S(x)}\left[\frac{\theta}{\lambda(\theta+1)}\sum_{i=0}^{\infty}(-1)^i\binom{\frac{1}{\lambda}-1}{i}(1+\theta+i)^{-2}\left(1-(1+\theta+i)\log\left(1-x^\lambda\right)\right)\right.$$
$$\left.+(-1)^{1+\theta}\frac{\Gamma(1+\theta)\Gamma(\theta-\frac{1}{\lambda})}{\lambda\Gamma(1-\frac{1}{\lambda})} - \frac{x(1-x^\lambda)^\theta}{1+\theta\lambda}\,{}_2F_1(1,-\theta;1-\theta-\frac{1}{\lambda};\frac{1}{1-x^\lambda})\right],$$

where $S(x)$ is the survival function given by Eq (2.3).
Proof. Applying the following identity as

$$\begin{aligned} M(x) &= \frac{1}{S(x)}\int_x^{+\infty}[1-F(t)]\,dt \\ &= \frac{1}{S(x)}\int_x^1\left[\left(1-t^\lambda\right)^\theta - \frac{\theta}{\theta+1}\left(1-t^\lambda\right)^\theta\log\left(1-t^\lambda\right)\right]dt = \frac{1}{S(x)}I, \end{aligned}$$

where $I = \int_x^1\left[\left(1-t^\lambda\right)^\theta - \frac{\theta}{\theta+1}\left(1-t^\lambda\right)^\theta\log\left(1-t^\lambda\right)\right]dt$.

To solve the previous integral I, let $u = -log(1-t^\lambda)$, then $t = (1-e^{-u})^{\frac{1}{\lambda}}$, and using the expansion of the power series

$$(1-z)^\alpha = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha}{i}z^i,\ |z|<1.$$

After simplifying, this leads to:

$$I = \left[\frac{\theta}{\lambda(\theta+1)}\sum_{i=0}^{\infty}(-1)^i\binom{\frac{1}{\lambda}-1}{i}(1+\theta+i)^{-2}\left(1-(1+\theta+i)\log\left(1-x^\lambda\right)\right)\right.$$
$$\left.+(-1)^{1+\theta}\frac{\Gamma(1+\theta)\Gamma(\theta-\frac{1}{\lambda})}{\lambda\Gamma(1-\frac{1}{\lambda})} - \frac{x(1-x^\lambda)^\theta}{1+\theta\lambda}\,{}_2F_1(1,-\theta;1-\theta-\frac{1}{\lambda};\frac{1}{1-x^\lambda})\right].$$

Substituting the results of the integral into $M(x)$, the desired proof is obtained.

Table 2 provides some numerical values for the MRL for a variety of arbitrary choices of the parameters λ and θ at the time points $t = 0.1, 0.3, 0.5, 0.7, 0.9$. This table shows that the MRL decreases as the time points t increase and as θ increases, but increases with increasing λ.

Figure 3 plots the MRL function for some values of θ and λ. We note that, as the hazard function is increasing (bathtub)-shaped, the MRL has a decreasing (upside-down bathtub) shape.

Table 2. MRL for a variety of parameter values.

λ =1					
$\theta \downarrow$ $t \rightarrow$	0.1	0.3	0.5	0.7	0.9
2	0.36229	0.27522	0.19199	0.11231	0.03625
3.5	0.23195	0.17660	0.12358	0.07261	0.02359
5	0.16915	0.12915	0.09066	0.05346	0.01745
7.5	0.11593	0.08885	0.06261	0.03707	0.01216
9	0.09739	0.07476	0.05277	0.03129	0.01029
$\theta = 2$					
$\lambda \downarrow$ $t \rightarrow$	0.1	0.3	0.5	0.7	0.9
2	0.50463	0.35159	0.22787	0.12470	0.03775
3.5	0.62936	0.43906	0.27448	0.14167	0.03977
5	0.69599	0.49776	0.31247	0.15712	0.04166
7.5	0.75551	0.55560	0.35858	0.17956	0.04465
9	0.77704	0.57706	0.37816	0.19109	0.04637

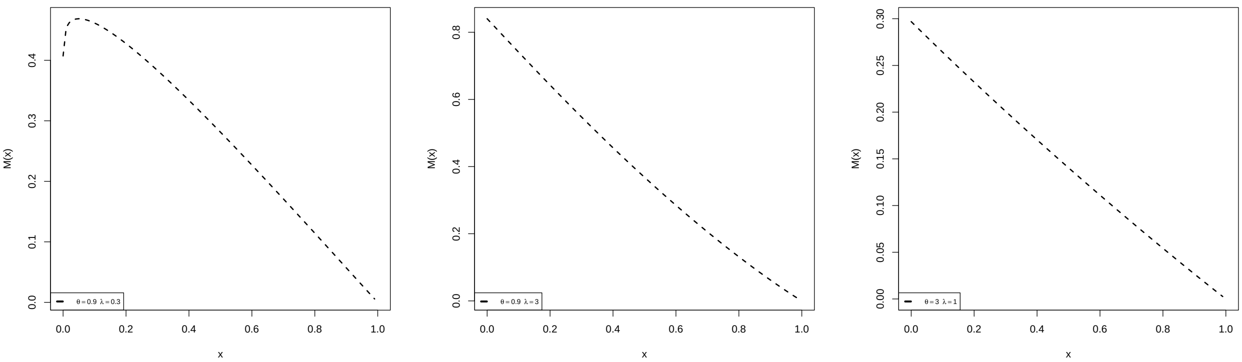

Figure 3. Plots of proposed model MRL for different parametric values.

4. Estimation methods

Parameter estimation is vital in the examination of probability distributions, since precise parameter values are essential for dependable predictions and conclusions. In this section, we study the estimation of our proposed model parameters by different estimation methods.

The maximum likelihood estimation (MLE) [24] is the primary methodology used to estimate the parameters of a statistical distribution. This approach is extensively used in statistical inference, recognized for its advantageous characteristics, such as asymptotic efficiency, consistency, and invariance under transformation.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the PDF of the proposed model; then the log-likelihood function takes the form

$$L = (\theta - 1) \sum_{i=1}^{n} \log\left(1 - x_i^{\lambda}\right) + \sum_{i=1}^{n} \log\left(1 - \log\left(1 - x_i^{\lambda}\right)\right) + (\lambda - 1) \sum_{i=1}^{n} \log(x_i) + n \log\left(\frac{\theta^2 \lambda}{\theta + 1}\right).$$

By maximizing the previous equation, we have the MLE of the LLE distribution parameters.

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be the order statistics of a random sample from LLE distribution. The ordinary least-squares estimation (OLSE) [25] is a prevalent technique for parameter estimation that entails reducing the sum of the squared discrepancies between the observed values and the anticipated values generated by the model. In the LLE model, the least squares estimates of the parameters are derived by minimizing the following function:

$$O = \sum_{i=1}^{n} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2,$$

where F is the CDF of the LLE model.

The Anderson-Darling estimation (ADE) [26] is a prevalent estimation methodology based on goodness-of-fit that assigns more significance to deviations in the tails of the empirical and theoretical CDFs. This property makes it especially advantageous for situations where tail behavior is paramount. For the LLE model, the ADE parameter estimations are derived by minimizing the following function:

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1)[\log F(x_{i:n}) + \log S(x_{i:n})],$$

where S is the survival function of the LLE model.

Weighted least squares estimation (WLSE) [25] is a statistical technique used to estimate the parameters of a probability distribution by allocating varying weights to observations. This method is especially advantageous when some data points are deemed more dependable or informative than others, resulting in more efficient parameter estimations. In the LLE model, the WLSE parameter estimations are derived by minimizing the following function:

$$W = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2.$$

The Cramér–von Mises estimation (CVME) [27] identifies the parameters of a theoretical distribution by minimizing the CVM statistic, which quantifies the divergence between the empirical and proposed CDFs. In the LLE model, the CVME parameter estimates are derived by minimizing the following function:

$$CV = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2.$$

The right-tail Anderson-Darling estimation (RTADE) [26] estimates the parameters of a statistical distribution by prioritizing the right tail of the distribution. This method evaluates the goodness-of-fit in the upper tail by minimizing a modified Anderson-Darling statistic that emphasizes the upper tail characteristics of the CDF. In the LLE model, the RTADE parameter estimations are derived by minimizing the following objective function:

$$R = \frac{n}{2} - 2 \sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log S(x_{i:n}).$$

The left-tail Anderson-Darling estimation (LTADE) [28] is used for its simplicity and efficacy in quantifying the divergence between empirical and theoretical distributions, facilitating parameter estimation. The LTADE of the LLE model parameter estimations is derived by minimizing the following function:

$$L = -\frac{3}{2}n + 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n}).$$

An efficient substitute for the greatest likelihood method is the maximum product of spacings estimation (MPSE) [29]. For the LLE model, the MPSE parameter estimators are derived by maximizing the following function:

$$G = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(A_i),$$

$A_i = F(x_i) - F(x_{i-1})$ denotes the uniform spacings, $F(x_0) = 0$, $F(x_{n+1} = 1)$, and $\sum_{i=1}^{n+1} A_i = 1$.

The percentile estimation (PCE) [30] calculates distribution parameters by matching the quantiles of the empirical distribution function with those of a theoretical distribution. The PCE of LLE model parameter estimations is derived by minimizing the following function:

$$PCE = \sum_{i=1}^{n}[x_{i:n} - Q(p_i)]^2,\ p_i = \frac{i}{n+1},$$

where Q is the quantile function of the LLE model.

5. Numerical simulation

In this section, we study the behavior of LLE model parameters by using the model quantile function (3.3) to simulate data sets and by several estimation methods. We generate $N = 5000$ random samples from the LLE model and then determine average absolute biases ($|BIAS|$) $|BIAS| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}|$, mean square errors (MSEs) $MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta})^2$, mean relative errors (MREs) $MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}|/\boldsymbol{\vartheta}$, average absolute difference (D_{abs}) $D_{abs} = \frac{1}{n\,N}\sum_{i=1}^{N}\sum_{j=1}^{n}|F(x_{ij}|\boldsymbol{\vartheta}) - F(x_{ij}|\widehat{\boldsymbol{\vartheta}})|$, maximum absolute difference (D_{max}) $D_{max} = \frac{1}{N}\sum_{i=1}^{N}\max_{j}|F(x_{ij}|\boldsymbol{\vartheta}) - F(x_{ij}|\widehat{\boldsymbol{\vartheta}})|$ and average squared absolute error (ASAE) $ASAE = \frac{1}{n}\sum_{i=1}^{n}\frac{|x_i - \hat{x}_i|}{x_n - x_1}$, where x_i are the ascending ordered observations, and $\hat{x}_i$ is obtained from the quantile function (3.3) with the estimates plugged in the equation for $p_i = \frac{i}{n+1}$, and $\boldsymbol{\vartheta}$=$\theta,\ \lambda$.

The following tables summarize the simulation results for the suggested model parameters when each of the nine estimating procedures is used. Simulation values for LLE model parameters by several estimation methods are summarized in Tables A1–A5 in the Appendix.

From these tables and Table 3, we conclude the following.

- All the estimators show the property of consistency.
- For all estimate methods, the BIAS of all estimators decreases as n increases.
- For all estimation methods, the MSE of all estimators reduces as n increases.
- The MRE of all estimators reduces as n increases for all estimation techniques.

- The statistics D_{abs}, D_{max}, and ASAE decrease with increasing n for all the methods of estimation.
- D_{abs} is smaller than D_{max} for all the estimation techniques.
- We infer from Table 3 that MPSE is the best (overall score of 38.0) followed by MLE (overall score of 45.0). So we advise researchers to use MPSE if they have data sets from our proposed model.

Table 3. Partial and overall ranks of all the techniques for estimation of the suggested model using various values of θ and λ.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE
$\theta = 0.75, \lambda = 0.5$	30	2.0	3.0	8.0	1.0	6.0	7.0	9.0	4.0	5.0
	75	1.0	3.0	7.0	2.0	6.0	9.0	8.0	4.0	5.0
	150	1.0	3.0	5.5	2.0	7.0	9.0	8.0	4.0	5.5
	250	1.0	3.5	8.0	2.0	6.0	9.0	7.0	3.5	5.0
	400	1.0	2.0	6.0	3.0	7.0	9.0	8.0	4.0	5.0
$\theta = 0.75, \lambda = 1.5$	30	3.0	2.0	8.0	1.0	6.0	7.0	9.0	4.0	5.0
	75	2.0	3.0	9.0	1.0	6.0	7.0	8.0	4.0	5.0
	150	1.0	3.0	8.5	2.0	7.0	8.5	6.0	4.0	5.0
	250	2.0	4.0	9.0	1.0	6.0	7.0	8.0	3.0	5.0
	400	1.0	3.0	9.0	2.0	6.0	7.0	8.0	4.0	5.0
$\theta = 2.5, \lambda = 0.5$	30	4.0	2.0	9.0	1.0	6.0	8.0	5.0	3.0	7.0
	75	2.0	3.0	8.0	1.0	6.0	9.0	5.0	4.0	7.0
	150	2.0	3.0	7.0	1.0	6.0	9.0	5.0	4.0	8.0
	250	2.0	3.0	8.0	1.0	5.5	9.0	5.5	4.0	7.0
	400	2.0	4.0	7.0	1.0	6.0	9.0	5.0	3.0	8.0
$\theta = 1.5, \lambda = 2.5$	30	3.5	2.0	9.0	1.0	7.0	5.0	8.0	3.5	6.0
	75	1.0	3.0	8.0	2.0	7.0	5.0	9.0	4.0	6.0
	150	2.0	3.0	9.0	1.0	7.0	6.0	8.0	4.0	5.0
	250	1.0	4.0	9.0	2.5	6.0	7.0	8.0	2.5	5.0
	400	1.0	3.0	6.0	2.0	9.0	7.5	7.5	4.0	5.0
$\theta = 3, \lambda = 5$	30	4.0	2.0	9.0	1.0	6.0	3.0	8.0	5.0	7.0
	75	1.5	3.0	9.0	1.5	8.0	5.0	7.0	4.0	6.0
	150	2.0	4.0	9.0	1.0	7.0	5.0	8.0	3.0	6.0
	250	1.0	3.0	9.0	2.0	6.0	5.0	7.0	4.0	8.0
	400	1.0	4.0	5.0	2.0	8.0	9.0	7.0	3.0	6.0
$\sum$ Ranks		45.0	75.5	199.0	38.0	163.5	181.0	182.0	93.5	147.5
Overall Rank		2	3	9	1	6	7	8	4	5

6. Real data analysis

This section delves into two real-world data sets to showcase the flexibility of the suggested distribution. The first data set consists of 50 observations on burr (in millimeters) with a hole diameter of 12 mm and a sheet thickness of 3.15 mm. It was presented by Dasgupta [31]. The second set of data included Italian COVID-19 data covering the 111 days from April 1 to July 20, 2020. This statistic is calculated by dividing daily new deaths by daily new cases. Hassan et al. [32] examined it, and it is accessible at `https://covid19.who.int/`.

The recommended distribution is compared to many well-known distributions, including beta (B), log-Lindley (LL) [1], power log-Lindley (PLL) [33], cosine-sine (CS) [3], power logarithmic (PL) [4], reduced Kies (RK) [34], transformed gamma (TG) [35], log-gamma (LG) [36], log-weighted power (LWP) [37], and transmuted power (TP) [37] distributions.

We examined the competing models using a variety of analytical criteria, including some information criterion (IC), such as Akaike IC (AIC), corrected AIC (CAIC), Bayesian IC (BIC), and Hannan–Quinn IC (HQIC), along with numerous goodness-of-fit statistics, such as Cramér–von Mises (CM), Anderson Darling (AD), and Kolmogorov–Smirnov (KS) with its p-value (KS p-value) to identify the best-fitting model for the real-world data sets under consideration.

The analytical measures, as well as MLE and associated standard errors (SEs), are provided in parentheses for the two real data sets in Tables 4 and 5. By analyzing the Burr data, the LLE model emerges as the best model, exhibiting the lowest values across all information criteria, including AIC (-108.147), BIC (-104.323), and HQIC (-106.69). Additionally, its high Kolmogorov–Smirnov (KS) p-value (0.577) suggests no significant deviation from the observed data distribution, while its Anderson–Darling (AD = 0.677) and Cramér–von Mises (CM = 0.104) statistics further confirm its strong fit. B distribution follows as the second-best statistical model, with AIC (-105.213) and BIC (-101.389) values, though its KS p-value is 0.270. Also, for COVID-19 data, the LLE model again proved to be the best fit. It achieves the lowest AIC (-252.324), BIC (-246.905), and HQIC (-250.126) values, along with a high KS p-value (0.645), reinforcing its strong alignment with the data. The B model ranks second, with AIC (-251.038) and BIC (-245.618) scores, though its KS p-value (0.409). From these tables, we can conclude that the LLE model fits better than other models.

Both Figures 4 and 5 depict the estimated PDF, CDF, SF, and P-P plots of the LLE distribution for the respective data sets. These plots are shown in the figures. The results of this analysis demonstrate that the LLE distribution is better in terms of its ability to fit the data sets. The behavior of a log-likelihood function with estimated parameters that presents a unimodal function is seen in Figures 6 and 7, respectively. A visual representation of the existence of estimated LLE model parameters may be seen in Figures 8 and 9, respectively. The TTT plot and the plot of the estimated HRF of the LLE model are shown in Figures 10 and 11, respectively. These figures demonstrate that the HRF of the LLE model has a shape that represents a rising value.

The values of estimates with goodness-of-fit measures of the LLE model for various estimation techniques for the studied data sets are shown in Tables 6 and 7. Figures 12 and 13 depict P-P plots for the LLE model generated by various estimating approaches, as well as estimated PDFs generated by these methods.

Table 4. Goodness-of-fit statistics and MLES for Burr data.

Model	–L	AIC	CAIC	BIC	HQIC	AD	CM	KS	KS p-value	Est. parameters (SEs)
LLE	-56.0733	-108.147	-107.891	-104.323	-106.69	0.676541	0.104132	0.110284	0.577291	$\hat{\theta}$ = 34.0012 (13.9434)
										$\hat{\lambda}$ = 2.07583 (0.255543)
B	-54.6067	-105.213	-104.958	-101.389	-103.757	0.912344	0.153904	0.141461	0.269698	$\hat{a}$ = 2.68257 (0.507179)
										$\hat{b}$ = 13.8658 (2.82802)
LL	-31.8265	-59.653	-59.3977	-55.829	-58.1968	6.56759	1.22677	0.318138	0.0000804383	$\hat{\beta}$ = 1.00664 (0.0762406)
										$\hat{\lambda} = 1 \times 10^{-9}$ (0.129734)
PLL	-31.8265	-57.653	-57.1313	-51.917	-55.4687	6.56759	1.22677	0.318138	0.0000804383	$\hat{\alpha}$ = 1.89694 (0.645369)
										$\hat{\sigma}$ = 0.530664 (0.151324)
										$\hat{\lambda} = 1.03311 \times 10^{-13}$ (1.23461×10^{-7})
CS	-40.649	-79.298	-79.2146	-77.386	-78.5699	4.72157	0.869979	0.261222	0.00217545	$\hat{\lambda}$ = 3.56212 (0.640514)
PL	-31.8265	-57.653	-57.1313	-51.917	-55.4687	6.5676	1.22677	0.318138	0.0000804352	$\hat{\alpha}$ = 0.00664011 (0.100664)
										$\hat{\beta}$ = 0.003714 (27.6384)
										$\hat{\delta} = 2.32994 * 10^6$ (792681)
RK	-11.6763	-21.3526	-21.2692	-19.4405	-20.6244	21.7194	4.86516	0.563342	< 0.000001	$\hat{a}$ = 0.736768 (0.087676)
TG	-53.5138	-105.028	-104.944	-103.116	-104.3	1.23799	0.236507	0.171007	0.107384	$\hat{\sigma}$ = 10.9387 (1.09387)
LG	-31.8265	-61.653	-61.5697	-59.741	-60.9249	6.56759	1.22677	0.318138	0.0000804383	$\hat{\sigma}$ = 1.00664 (0.100664)
LWP	-31.8265	-59.653	-59.3977	-55.829	-58.1968	6.56759	1.22677	0.318138	0.0000804383	$\hat{\alpha}$ = 1.00664 (0.173704)
										$\hat{\lambda}$ = 1.0 (0.511507)
TP	-30.3176	-56.6353	-56.3799	-52.8112	-55.179	6.79842	1.26438	0.320945	0.0000672272	$\hat{\alpha}$ = 0.733619 (0.11218)
										$\hat{\lambda}$ = 1.0 (0.631411)

Table 5. Goodness-of-fit statistics and MLES for COVID-19 data.

Model	–L	AIC	CAIC	BIC	HQIC	AD	CM	KS	KS p-value	Est. parameters (SEs)
LLE	-128.162	-252.324	-252.213	-246.905	-250.126	0.746595	0.123627	0.0701906	0.644856	$\hat{\theta}$ = 38.4416 (10.0158)
										$\hat{\lambda}$ = 2.18146 (0.165322)
B	-127.519	-251.038	-250.926	-245.618	-248.839	1.10027	0.185097	0.084305	0.409216	$\hat{a}$ = 3.30003 (0.422835)
										$\hat{b}$ = 16.5213 (2.25052)
LL	-68.3991	-132.798	-132.687	-127.379	-130.6	17.2227	3.3363	0.323306	< 0.000001	$\hat{\beta}$ = 1.03816 (0.0529068)
										$\hat{\lambda} = 1.08695 \times 10^{-25}$ (0.0841363)
PLL	-68.3991	-130.798	-130.574	-122.67	-127.501	17.2227	3.3363	0.323306	< 0.000001	$\hat{\alpha}$ = 1.26851 (0.285769)
										$\hat{\sigma}$ = 0.818409 (0.15437)
										$\hat{\lambda} = 1 \times 10^{-}16$ (0.0000722543)
CS	-88.0253	-174.051	-174.014	-171.341	-172.952	13.2517	2.52675	0.26402	< 0.000001	$\hat{\lambda}$ = 3.45478 (0.424077)
PL	-68.3972	-130.794	-130.57	-122.666	-127.497	17.2231	3.33639	0.32331	< 0.000001	$\hat{\alpha}$ = 0.0381383 (0.0696753)
										$\hat{\beta}$ = 997.058 (32574.7)
										$\hat{\delta} = 2.80727 \times 10^7$ (6.32432×10^6)
RK	-24.317	-46.6339	-46.5972	-43.9244	-45.5347	51.0508	11.3876	0.555718	< 0.000001	$\hat{a}$ = 0.764765 (0.0608404)
TG	-120.937	-239.874	-239.837	-237.165	-238.775	3.78114	0.72411	0.158281	0.00768437	$\hat{\sigma}$ = 10.6815 (0.716896)
LG	-68.3991	-134.798	-134.762	-132.089	-133.699	17.2227	3.3363	0.323306	< 0.000001	$\hat{\sigma}$ = 1.03816 (0.0696766)
LWP	-68.3991	-132.798	-132.687	-127.379	-130.6	17.2227	3.3363	0.323306	< 0.000001	$\hat{\alpha}$ = 1.03816 (0.113906)
										$\hat{\lambda}$ = 1.0 (0.320189)
TP	-64.8437	-125.687	-125.576	-120.268	-123.489	17.6482	3.39884	0.323654	< 0.000001	$\hat{\alpha}$ = 0.755295 (0.0748657)
										$\hat{\lambda}$ = 1.0 (0.402946)

Table 6. The estimates of the LLE model parameters along with goodness-of-fit statistics for the Burr data using several estimation methods.

	$\hat{\theta}$	$\hat{\lambda}$	AD	CM	KS	KSP
MLE	34.0012	2.07583	0.676541	0.104132	0.110284	0.577291
ADE	26.8383	1.96129	0.596416	0.0886771	0.104963	0.640313
CVME	29.5593	2.03415	0.623349	0.0842043	0.0908235	0.803957
MPSE	26.9395	1.94073	0.630236	0.104738	0.119352	0.47456
OLSE	26.2845	1.96715	0.617316	0.085578	0.0944595	0.763742
PCE	33.1892	2.0819	0.63635	0.0898074	0.0985514	0.71645
RTADE	38.3489	2.18788	0.732502	0.0904329	0.101462	0.682035
WLSE	28.7956	2.00396	0.601098	0.0870169	0.101158	0.685646
LTADE	19.7039	1.79887	0.698907	0.101603	0.105529	0.633563

Table 7. The estimates of the LLE model parameters along with goodness-of-fit statistics for the COVID-19 data using several estimation methods.

	$\hat{\theta}$	$\hat{\lambda}$	AD	CM	KS	KSP
MLE	38.4416	2.18146	0.746595	0.123627	0.0701906	0.644856
ADE	54.0081	2.37885	0.581396	0.067624	0.0551092	0.888868
CVME	80.9959	2.61633	0.806289	0.0457269	0.066144	0.716438
MPSE	32.4742	2.08413	0.955009	0.166131	0.0782152	0.505557
OLSE	75.4469	2.57635	0.737224	0.0463216	0.0643304	0.747863
PCE	37.6368	2.17114	0.766922	0.127283	0.0721846	0.609502
RTADE	58.3461	2.42285	0.589648	0.0602953	0.0558633	0.879062
WLSE	59.9743	2.44119	0.597216	0.0572847	0.0571129	0.861995
LTADE	46.9044	2.30807	0.623002	0.0844193	0.0670848	0.699916

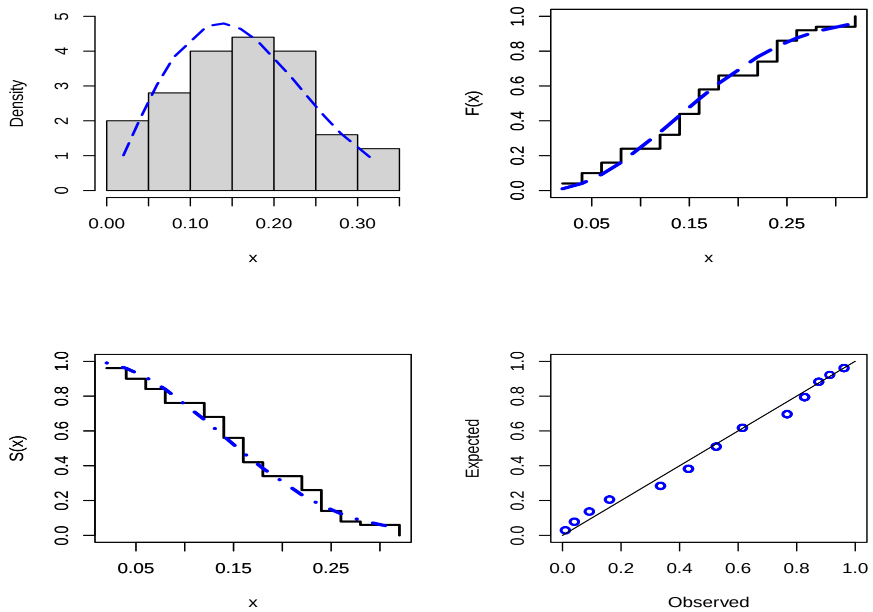

Figure 4. Histogram of the Burr data with the fitted CDF, SF, and P-P plots.

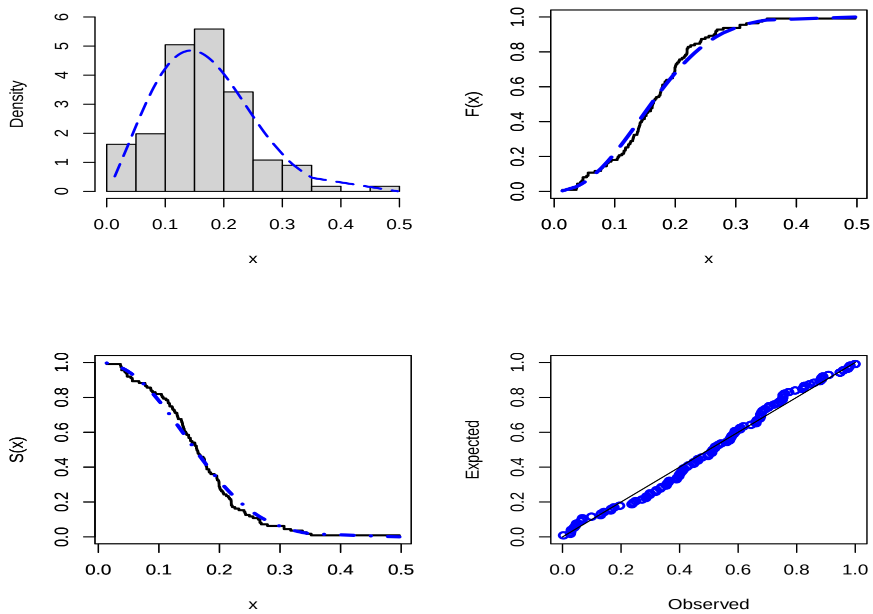

Figure 5. Histogram of the COVID-19 data with the fitted CDF, SF, and P-P plots.

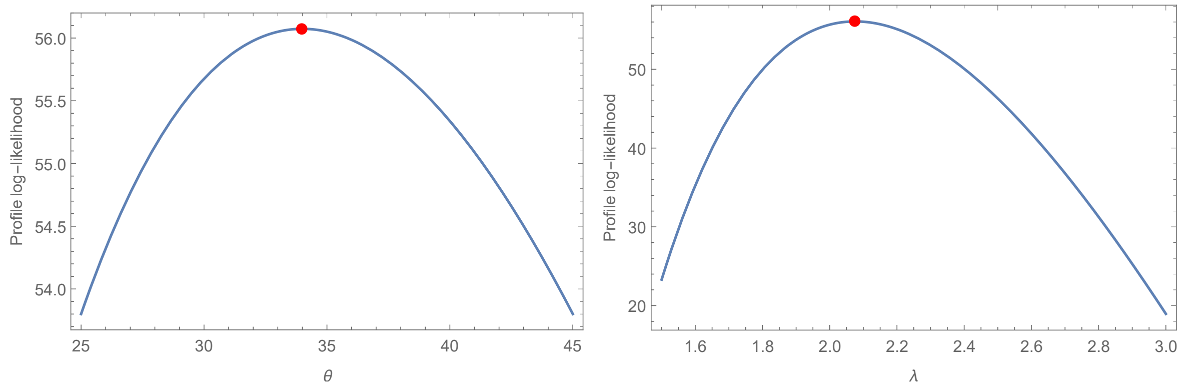

Figure 6. The profile of the log-likelihood functions for θ and λ of the Burr data.

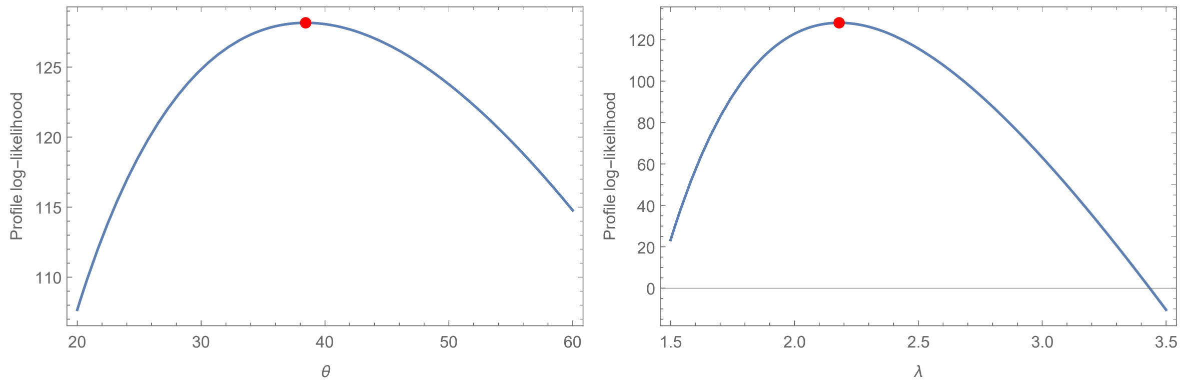

Figure 7. The profile of the log-likelihood functions for θ and λ of the COVID-19 data.

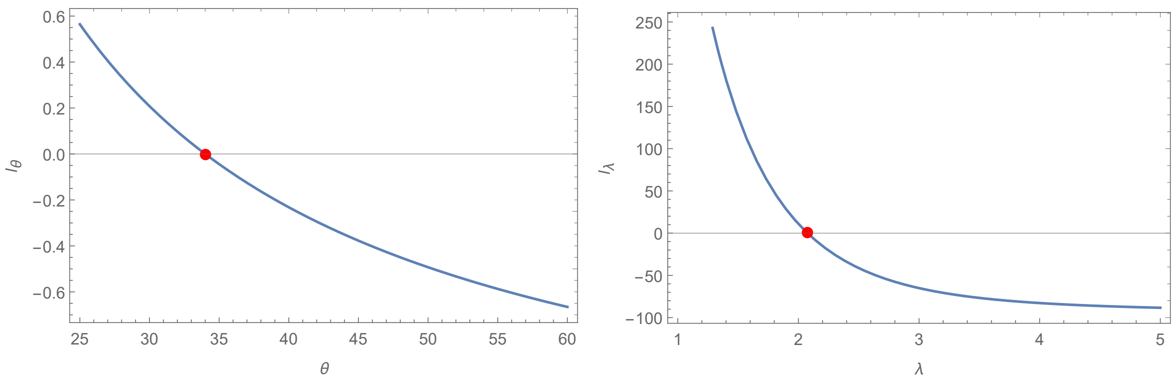

Figure 8. Existence and uniqueness of LLE model parameters for the Burr data.

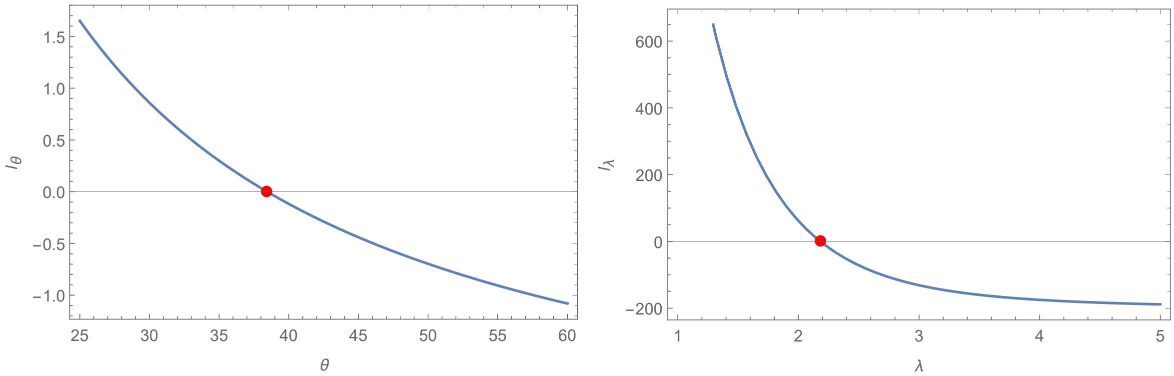

Figure 9. Existence and uniqueness of LLE model parameters for the COVID-19 data.

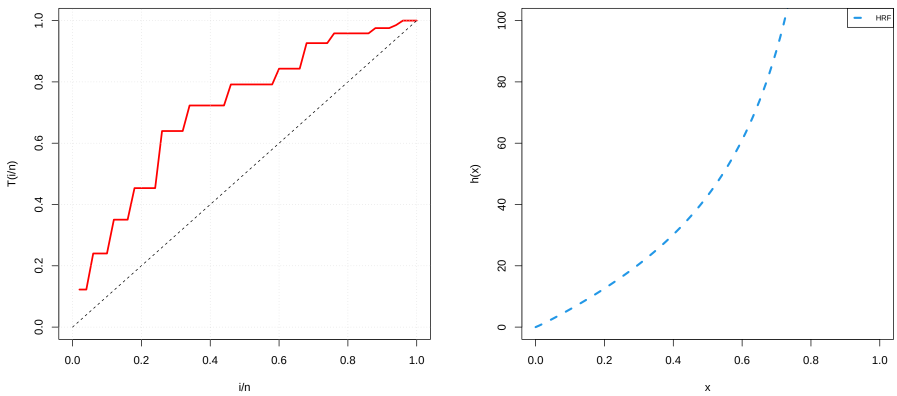

Figure 10. TTT plot and fitted HRF of LLE model for the Burr data.

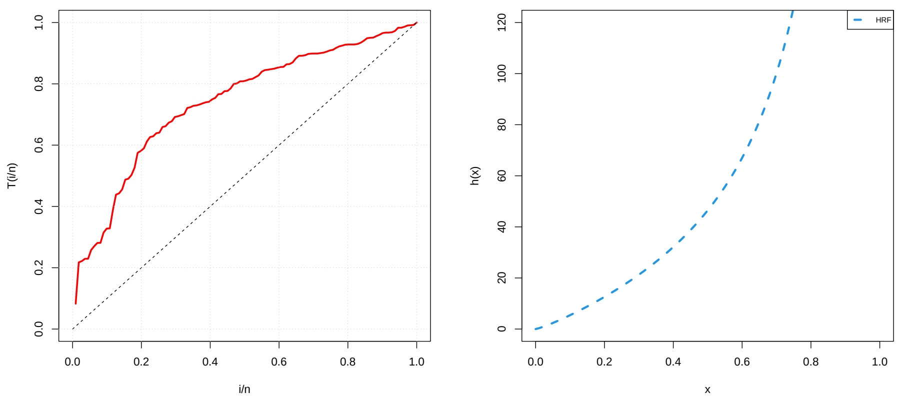

Figure 11. TTT plot and fitted HRF of LLE model for the COVID-19 data.

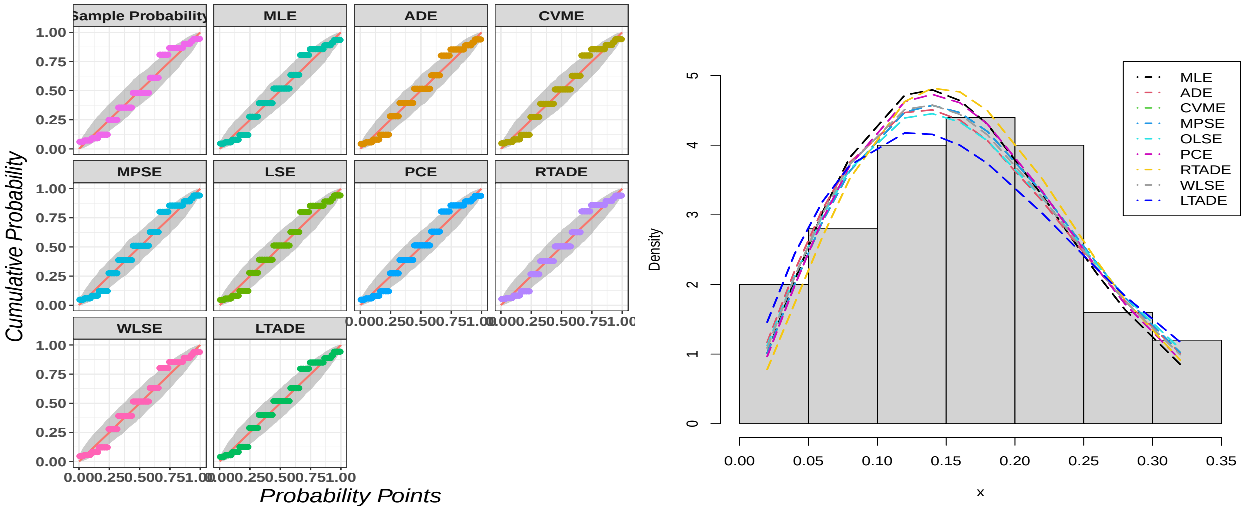

Figure 12. P-P plot and the fitted PDFs of LLE model for Burr data.

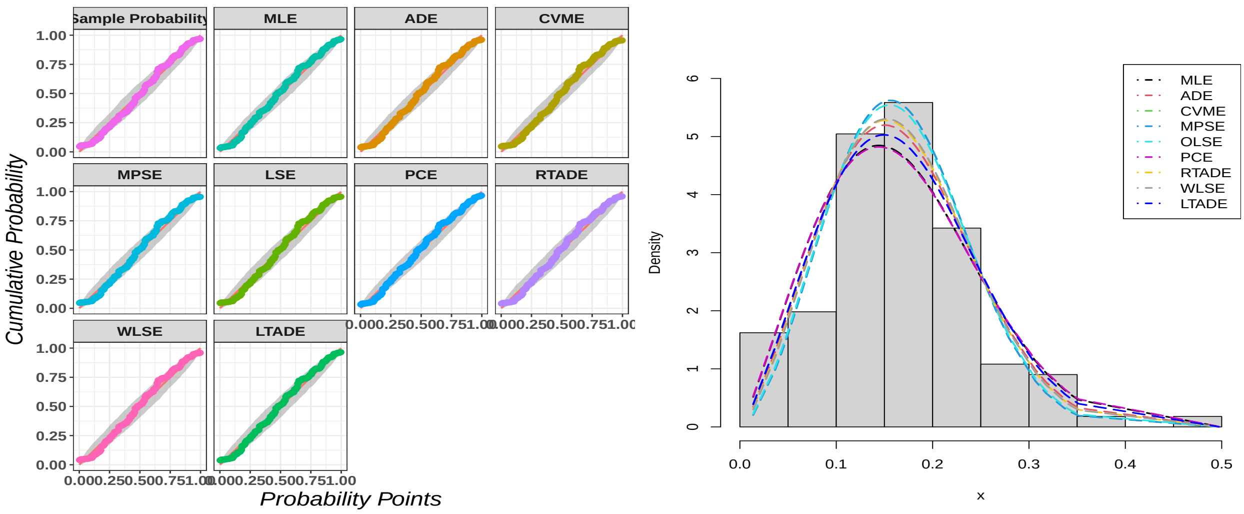

Figure 13. P-P plot and the fitted PDFs of LLE model for COVID-19 data.

7. Conclusions

The study presents the LLE distribution, a new bounded distribution that provides great flexibility for modeling lifetime data. Its main strength is the versatility shapes of its hazard function, which may include increasing, J, and bathtub shapes, making it suitable for a wide range of data sets. Furthermore, the new distribution can be used to define new and well-known distributions with a wide range of parameters and supports. We have derived several key properties of the LLE distribution, such as its moments, entropy, quantile function, and mean residual life function. Numerous estimation techniques have been proposed for estimating the model's parameters. A Monte Carlo simulation is used to assess the accuracy of estimates. In general, we conclude that the maximum product spacing technique estimates the parameters better compared to other estimation methods. Furthermore, when applied to real-world data sets, the LLE distribution demonstrates superior performance compared to existing models based on goodness-of-fit criteria.

In future work, we plan to develop a regression model based on the proposed distribution for an upcoming study. This extension will incorporate categorical data and employ both classical and Bayesian estimation techniques. We will investigate the proposed distribution as a bivariate extension and use probabilistic machine learning techniques for analyzing the COVID-19 data. By extensively comparing our model against existing regression and machine learning methodologies, we hope to assess its performance and adaptability in real-world data sets.

Author contributions

All authors have worked equally to write and review the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability statement

The datasets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare no conflicts of interest.

Appendix

Table A1. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max} and ASAE for ($\theta = 0.75,\ \lambda = 0.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
30	BIAS	$\hat{\theta}$	$0.12184^{\{2\}}$	$0.12218^{\{3\}}$	$0.1448^{\{8\}}$	$0.11728^{\{1\}}$	$0.13132^{\{5\}}$	$0.14796^{\{9\}}$	$0.13832^{\{6\}}$	$0.12579^{\{4\}}$	$0.13897^{\{7\}}$
		$\hat{\lambda}$	$0.16252^{\{3\}}$	$0.16193^{\{2\}}$	$0.20944^{\{8\}}$	$0.15137^{\{1\}}$	$0.18142^{\{6\}}$	$0.18735^{\{7\}}$	$0.22675^{\{9\}}$	$0.17415^{\{5\}}$	$0.17185^{\{4\}}$
	MSE	$\hat{\theta}$	$0.02593^{\{2\}}$	$0.02631^{\{3\}}$	$0.04334^{\{9\}}$	$0.02142^{\{1\}}$	$0.03022^{\{5\}}$	$0.03744^{\{7\}}$	$0.03867^{\{8\}}$	$0.02867^{\{4\}}$	$0.03562^{\{6\}}$
		$\hat{\lambda}$	$0.04895^{\{2\}}$	$0.04967^{\{3\}}$	$0.0982^{\{8\}}$	$0.03498^{\{1\}}$	$0.06171^{\{7\}}$	$0.06077^{\{5\}}$	$0.12125^{\{9\}}$	$0.06109^{\{6\}}$	$0.05687^{\{4\}}$
	MRE	$\hat{\theta}$	$0.16245^{\{2\}}$	$0.1629^{\{3\}}$	$0.19307^{\{8\}}$	$0.15637^{\{1\}}$	$0.1751^{\{5\}}$	$0.19728^{\{9\}}$	$0.18443^{\{6\}}$	$0.16771^{\{4\}}$	$0.18529^{\{7\}}$
		$\hat{\lambda}$	$0.32505^{\{3\}}$	$0.32387^{\{2\}}$	$0.41888^{\{8\}}$	$0.30273^{\{1\}}$	$0.36285^{\{6\}}$	$0.3747^{\{7\}}$	$0.4535^{\{9\}}$	$0.3483^{\{5\}}$	$0.34371^{\{4\}}$
	D_{abs}		$0.04457^{\{1\}}$	$0.0467^{\{3\}}$	$0.04912^{\{7.5\}}$	$0.04803^{\{5\}}$	$0.04851^{\{6\}}$	$0.05287^{\{9\}}$	$0.04912^{\{7.5\}}$	$0.04497^{\{2\}}$	$0.04773^{\{4\}}$
	D_{max}		$0.07202^{\{1\}}$	$0.07409^{\{3\}}$	$0.08077^{\{7\}}$	$0.07501^{\{4\}}$	$0.07798^{\{6\}}$	$0.08377^{\{9\}}$	$0.08087^{\{8\}}$	$0.07278^{\{2\}}$	$0.07684^{\{5\}}$
	ASAE		$0.03913^{\{9\}}$	$0.03273^{\{6\}}$	$0.03117^{\{3\}}$	$0.0386^{\{8\}}$	$0.02992^{\{2\}}$	$0.02961^{\{1\}}$	$0.03523^{\{7\}}$	$0.03211^{\{5\}}$	$0.03188^{\{4\}}$
	$\sum Ranks$		$25^{\{2\}}$	$28^{\{3\}}$	$66.5^{\{8\}}$	$23^{\{1\}}$	$48^{\{6\}}$	$63^{\{7\}}$	$69.5^{\{9\}}$	$37^{\{4\}}$	$45^{\{5\}}$
75	BIAS	$\hat{\theta}$	$0.07157^{\{1\}}$	$0.07244^{\{2\}}$	$0.08666^{\{8\}}$	$0.07359^{\{3\}}$	$0.08255^{\{6\}}$	$0.09351^{\{9\}}$	$0.08133^{\{5\}}$	$0.07835^{\{4\}}$	$0.08481^{\{7\}}$
		$\hat{\lambda}$	$0.09528^{\{2\}}$	$0.1013^{\{3\}}$	$0.11588^{\{6\}}$	$0.09478^{\{1\}}$	$0.11614^{\{7\}}$	$0.12016^{\{8\}}$	$0.12868^{\{9\}}$	$0.1065^{\{5\}}$	$0.10399^{\{4\}}$
	MSE	$\hat{\theta}$	$0.00842^{\{2\}}$	$0.00913^{\{3\}}$	$0.01264^{\{7\}}$	$0.00823^{\{1\}}$	$0.01115^{\{6\}}$	$0.01399^{\{9\}}$	$0.01089^{\{5\}}$	$0.00983^{\{4\}}$	$0.01269^{\{8\}}$
		$\hat{\lambda}$	$0.01566^{\{2\}}$	$0.01789^{\{3\}}$	$0.02402^{\{8\}}$	$0.01377^{\{1\}}$	$0.02252^{\{6\}}$	$0.02328^{\{7\}}$	$0.03035^{\{9\}}$	$0.01929^{\{5\}}$	$0.01927^{\{4\}}$
	MRE	$\hat{\theta}$	$0.09543^{\{1\}}$	$0.09658^{\{2\}}$	$0.11554^{\{8\}}$	$0.09812^{\{3\}}$	$0.11007^{\{6\}}$	$0.12468^{\{9\}}$	$0.10845^{\{5\}}$	$0.10447^{\{4\}}$	$0.11308^{\{7\}}$
		$\hat{\lambda}$	$0.19056^{\{2\}}$	$0.20259^{\{3\}}$	$0.23175^{\{6\}}$	$0.18956^{\{1\}}$	$0.23228^{\{7\}}$	$0.24032^{\{8\}}$	$0.25737^{\{9\}}$	$0.213^{\{5\}}$	$0.20798^{\{4\}}$
	D_{abs}		$0.02792^{\{1\}}$	$0.02995^{\{3\}}$	$0.03199^{\{6\}}$	$0.02842^{\{2\}}$	$0.0324^{\{8\}}$	$0.03311^{\{9\}}$	$0.03215^{\{7\}}$	$0.03009^{\{4\}}$	$0.03056^{\{5\}}$
	D_{max}		$0.04483^{\{1\}}$	$0.04769^{\{3\}}$	$0.05205^{\{7\}}$	$0.04543^{\{2\}}$	$0.052^{\{6\}}$	$0.0534^{\{9\}}$	$0.05261^{\{8\}}$	$0.04847^{\{4\}}$	$0.04934^{\{5\}}$
	ASAE		$0.02455^{\{8\}}$	$0.02024^{\{4\}}$	$0.01925^{\{3\}}$	$0.02513^{\{9\}}$	$0.0191^{\{2\}}$	$0.01867^{\{1\}}$	$0.02201^{\{7\}}$	$0.02066^{\{6\}}$	$0.02034^{\{5\}}$
	$\sum Ranks$		$20^{\{1\}}$	$26^{\{3\}}$	$59^{\{7\}}$	$23^{\{2\}}$	$54^{\{6\}}$	$69^{\{9\}}$	$64^{\{8\}}$	$41^{\{4\}}$	$49^{\{5\}}$
150	BIAS	$\hat{\theta}$	$0.04892^{\{1\}}$	$0.04939^{\{2\}}$	$0.05897^{\{7\}}$	$0.05145^{\{3\}}$	$0.05896^{\{6\}}$	$0.06823^{\{9\}}$	$0.05591^{\{5\}}$	$0.05213^{\{4\}}$	$0.05963^{\{8\}}$
		$\hat{\lambda}$	$0.06579^{\{2\}}$	$0.07042^{\{3\}}$	$0.07763^{\{6\}}$	$0.06391^{\{1\}}$	$0.0835^{\{7\}}$	$0.087^{\{9\}}$	$0.08602^{\{8\}}$	$0.07195^{\{4\}}$	$0.07493^{\{5\}}$
	MSE	$\hat{\theta}$	$0.00388^{\{1\}}$	$0.00409^{\{3\}}$	$0.00557^{\{6\}}$	$0.00403^{\{2\}}$	$0.00577^{\{7\}}$	$0.00727^{\{9\}}$	$0.00507^{\{5\}}$	$0.00449^{\{4\}}$	$0.00586^{\{8\}}$
		$\hat{\lambda}$	$0.00736^{\{2\}}$	$0.00794^{\{3\}}$	$0.01016^{\{6\}}$	$0.00611^{\{1\}}$	$0.01112^{\{7\}}$	$0.01198^{\{8\}}$	$0.0123^{\{9\}}$	$0.0085^{\{4\}}$	$0.00971^{\{5\}}$
	MRE	$\hat{\theta}$	$0.06523^{\{1\}}$	$0.06585^{\{2\}}$	$0.07863^{\{7\}}$	$0.06861^{\{3\}}$	$0.07862^{\{6\}}$	$0.09098^{\{9\}}$	$0.07454^{\{5\}}$	$0.06951^{\{4\}}$	$0.07951^{\{8\}}$
		$\hat{\lambda}$	$0.13157^{\{2\}}$	$0.14084^{\{3\}}$	$0.15527^{\{6\}}$	$0.12781^{\{1\}}$	$0.16701^{\{7\}}$	$0.174^{\{9\}}$	$0.17204^{\{8\}}$	$0.1439^{\{4\}}$	$0.14985^{\{5\}}$
	D_{abs}		$0.01948^{\{1\}}$	$0.02061^{\{4\}}$	$0.02191^{\{8\}}$	$0.02039^{\{2\}}$	$0.02182^{\{6\}}$	$0.02426^{\{9\}}$	$0.02184^{\{7\}}$	$0.02042^{\{3\}}$	$0.0216^{\{5\}}$
	D_{max}		$0.03126^{\{1\}}$	$0.03288^{\{4\}}$	$0.03546^{\{6\}}$	$0.03243^{\{2\}}$	$0.03555^{\{7\}}$	$0.03936^{\{9\}}$	$0.03565^{\{8\}}$	$0.03282^{\{3\}}$	$0.03501^{\{5\}}$
	ASAE		$0.01741^{\{8\}}$	$0.01454^{\{6\}}$	$0.01343^{\{1\}}$	$0.018^{\{9\}}$	$0.01365^{\{3\}}$	$0.01352^{\{2\}}$	$0.01525^{\{7\}}$	$0.01442^{\{5\}}$	$0.01438^{\{4\}}$
	$\sum Ranks$		$19^{\{1\}}$	$30^{\{3\}}$	$53^{\{5.5\}}$	$24^{\{2\}}$	$56^{\{7\}}$	$73^{\{9\}}$	$62^{\{8\}}$	$35^{\{4\}}$	$53^{\{5.5\}}$
250	BIAS	$\hat{\theta}$	$0.03803^{\{1\}}$	$0.04018^{\{2\}}$	$0.04655^{\{8\}}$	$0.0412^{\{4\}}$	$0.04605^{\{7\}}$	$0.05181^{\{9\}}$	$0.0429^{\{5\}}$	$0.04072^{\{3\}}$	$0.04439^{\{6\}}$
		$\hat{\lambda}$	$0.04844^{\{1\}}$	$0.05514^{\{5\}}$	$0.06128^{\{6\}}$	$0.0529^{\{2\}}$	$0.06214^{\{7\}}$	$0.06886^{\{9\}}$	$0.06654^{\{8\}}$	$0.0548^{\{4\}}$	$0.05385^{\{3\}}$
	MSE	$\hat{\theta}$	$0.00231^{\{1\}}$	$0.00261^{\{3\}}$	$0.00346^{\{8\}}$	$0.00253^{\{2\}}$	$0.00335^{\{7\}}$	$0.00411^{\{9\}}$	$0.00307^{\{5\}}$	$0.00266^{\{4\}}$	$0.00314^{\{6\}}$
		$\hat{\lambda}$	$0.0039^{\{1\}}$	$0.005^{\{5\}}$	$0.00622^{\{7\}}$	$0.00426^{\{2\}}$	$0.00619^{\{6\}}$	$0.00735^{\{9\}}$	$0.00722^{\{8\}}$	$0.00472^{\{4\}}$	$0.00461^{\{3\}}$
	MRE	$\hat{\theta}$	$0.05071^{\{1\}}$	$0.05357^{\{2\}}$	$0.06206^{\{8\}}$	$0.05493^{\{4\}}$	$0.0614^{\{7\}}$	$0.06908^{\{9\}}$	$0.0572^{\{5\}}$	$0.05429^{\{3\}}$	$0.05918^{\{6\}}$
		$\hat{\lambda}$	$0.09688^{\{1\}}$	$0.11029^{\{5\}}$	$0.12256^{\{6\}}$	$0.1058^{\{2\}}$	$0.12428^{\{7\}}$	$0.13772^{\{9\}}$	$0.13308^{\{8\}}$	$0.1096^{\{4\}}$	$0.1077^{\{3\}}$
	D_{abs}		$0.01538^{\{1\}}$	$0.01639^{\{4\}}$	$0.01731^{\{8\}}$	$0.01613^{\{2\}}$	$0.01713^{\{5\}}$	$0.01858^{\{9\}}$	$0.01715^{\{6\}}$	$0.01632^{\{3\}}$	$0.01725^{\{7\}}$
	D_{max}		$0.02453^{\{1\}}$	$0.02625^{\{4\}}$	$0.02795^{\{8\}}$	$0.02584^{\{2\}}$	$0.02777^{\{6\}}$	$0.03047^{\{9\}}$	$0.0278^{\{7\}}$	$0.02609^{\{3\}}$	$0.02748^{\{5\}}$
	ASAE		$0.01347^{\{8\}}$	$0.01099^{\{4\}}$	$0.01051^{\{3\}}$	$0.01376^{\{9\}}$	$0.01034^{\{1\}}$	$0.01043^{\{2\}}$	$0.01187^{\{7\}}$	$0.01153^{\{6\}}$	$0.01104^{\{5\}}$
	$\sum Ranks$		$16^{\{1\}}$	$34^{\{3.5\}}$	$62^{\{8\}}$	$29^{\{2\}}$	$53^{\{6\}}$	$74^{\{9\}}$	$59^{\{7\}}$	$34^{\{3.5\}}$	$44^{\{5\}}$
400	BIAS	$\hat{\theta}$	$0.02851^{\{1\}}$	$0.03104^{\{2\}}$	$0.03578^{\{6\}}$	$0.0313^{\{4\}}$	$0.03696^{\{8\}}$	$0.04162^{\{9\}}$	$0.03542^{\{5\}}$	$0.03112^{\{3\}}$	$0.03607^{\{7\}}$
		$\hat{\lambda}$	$0.0392^{\{1\}}$	$0.04244^{\{3\}}$	$0.04781^{\{6\}}$	$0.04104^{\{2\}}$	$0.04843^{\{7\}}$	$0.05471^{\{9\}}$	$0.05205^{\{8\}}$	$0.04342^{\{4\}}$	$0.04456^{\{5\}}$
	MSE	$\hat{\theta}$	$0.00129^{\{1\}}$	$0.00148^{\{2\}}$	$0.00201^{\{6\}}$	$0.00151^{\{3\}}$	$0.00212^{\{8\}}$	$0.00273^{\{9\}}$	$0.00197^{\{5\}}$	$0.00153^{\{4\}}$	$0.00203^{\{7\}}$
		$\hat{\lambda}$	$0.00244^{\{1\}}$	$0.00281^{\{3\}}$	$0.00369^{\{7\}}$	$0.00254^{\{2\}}$	$0.00359^{\{6\}}$	$0.0048^{\{9\}}$	$0.00445^{\{8\}}$	$0.00289^{\{4\}}$	$0.00321^{\{5\}}$
	MRE	$\hat{\theta}$	$0.03801^{\{1\}}$	$0.04138^{\{2\}}$	$0.04771^{\{6\}}$	$0.04174^{\{4\}}$	$0.04928^{\{8\}}$	$0.05549^{\{9\}}$	$0.04723^{\{5\}}$	$0.04149^{\{3\}}$	$0.0481^{\{7\}}$
		$\hat{\lambda}$	$0.0784^{\{1\}}$	$0.08489^{\{3\}}$	$0.09562^{\{6\}}$	$0.08208^{\{2\}}$	$0.09687^{\{7\}}$	$0.10943^{\{9\}}$	$0.1041^{\{8\}}$	$0.08685^{\{4\}}$	$0.08912^{\{5\}}$
	D_{abs}		$0.0119^{\{1\}}$	$0.01272^{\{3\}}$	$0.0137^{\{7\}}$	$0.01257^{\{2\}}$	$0.01359^{\{6\}}$	$0.01465^{\{9\}}$	$0.01403^{\{8\}}$	$0.01294^{\{4\}}$	$0.01339^{\{5\}}$
	D_{max}		$0.01903^{\{1\}}$	$0.02034^{\{3\}}$	$0.0221^{\{7\}}$	$0.02007^{\{2\}}$	$0.02208^{\{6\}}$	$0.02378^{\{9\}}$	$0.02283^{\{8\}}$	$0.02071^{\{4\}}$	$0.02159^{\{5\}}$
	ASAE		$0.0107^{\{8\}}$	$0.00891^{\{6\}}$	$0.00835^{\{2\}}$	$0.0108^{\{9\}}$	$0.00841^{\{3\}}$	$0.00826^{\{1\}}$	$0.00935^{\{7\}}$	$0.00889^{\{5\}}$	$0.0087^{\{4\}}$
	$\sum Ranks$		$16^{\{1\}}$	$27^{\{2\}}$	$53^{\{6\}}$	$30^{\{3\}}$	$59^{\{7\}}$	$73^{\{9\}}$	$62^{\{8\}}$	$35^{\{4\}}$	$50^{\{5\}}$

Table A2. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max} and ASAE for ($\theta = 0.75,\ \lambda = 1.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
30	BIAS	$\hat{\theta}$	$0.12189^{\{3\}}$	$0.12017^{\{2\}}$	$0.14201^{\{7\}}$	$0.11857^{\{1\}}$	$0.13448^{\{6\}}$	$0.16977^{\{9\}}$	$0.12837^{\{5\}}$	$0.12563^{\{4\}}$	$0.14658^{\{8\}}$
		$\hat{\lambda}$	$0.49599^{\{3\}}$	$0.48719^{\{2\}}$	$0.6076^{\{8\}}$	$0.44902^{\{1\}}$	$0.58918^{\{7\}}$	$0.52093^{\{4\}}$	$0.6365^{\{9\}}$	$0.55008^{\{6\}}$	$0.52123^{\{5\}}$
	MSE	$\hat{\theta}$	$0.02685^{\{3\}}$	$0.02442^{\{2\}}$	$0.03945^{\{7\}}$	$0.02137^{\{1\}}$	$0.0304^{\{6\}}$	$0.04586^{\{9\}}$	$0.02931^{\{4\}}$	$0.02961^{\{5\}}$	$0.04198^{\{8\}}$
		$\hat{\lambda}$	$0.50886^{\{4\}}$	$0.43367^{\{2\}}$	$0.8201^{\{8\}}$	$0.28979^{\{1\}}$	$0.66307^{\{7\}}$	$0.47839^{\{3\}}$	$0.86516^{\{9\}}$	$0.65125^{\{6\}}$	$0.57285^{\{5\}}$
	MRE	$\hat{\theta}$	$0.16251^{\{3\}}$	$0.16022^{\{2\}}$	$0.18935^{\{7\}}$	$0.15809^{\{1\}}$	$0.17931^{\{6\}}$	$0.22636^{\{9\}}$	$0.17116^{\{5\}}$	$0.1675^{\{4\}}$	$0.19544^{\{8\}}$
		$\hat{\lambda}$	$0.33066^{\{3\}}$	$0.3248^{\{2\}}$	$0.40507^{\{8\}}$	$0.29935^{\{1\}}$	$0.39279^{\{7\}}$	$0.34729^{\{4\}}$	$0.42433^{\{9\}}$	$0.36672^{\{6\}}$	$0.34748^{\{5\}}$
	D_{abs}		$0.04565^{\{2\}}$	$0.04595^{\{3\}}$	$0.04984^{\{6\}}$	$0.04534^{\{1\}}$	$0.05003^{\{7\}}$	$0.05778^{\{9\}}$	$0.05075^{\{8\}}$	$0.04831^{\{4\}}$	$0.04866^{\{5\}}$
	D_{max}		$0.07357^{\{3\}}$	$0.07336^{\{2\}}$	$0.08137^{\{7\}}$	$0.07229^{\{1\}}$	$0.08061^{\{6\}}$	$0.09181^{\{9\}}$	$0.08201^{\{8\}}$	$0.07751^{\{4\}}$	$0.07856^{\{5\}}$
	ASAE		$0.03576^{\{8\}}$	$0.03209^{\{5\}}$	$0.03285^{\{6\}}$	$0.03436^{\{7\}}$	$0.03181^{\{4\}}$	$0.02953^{\{1\}}$	$0.03894^{\{9\}}$	$0.03165^{\{3\}}$	$0.02978^{\{2\}}$
	$\sum Ranks$		$32^{\{3\}}$	$22^{\{2\}}$	$64^{\{8\}}$	$15^{\{1\}}$	$56^{\{6\}}$	$57^{\{7\}}$	$66^{\{9\}}$	$42^{\{4\}}$	$51^{\{5\}}$
75	BIAS	$\hat{\theta}$	$0.07125^{\{2\}}$	$0.07255^{\{3\}}$	$0.08359^{\{7\}}$	$0.07078^{\{1\}}$	$0.08073^{\{6\}}$	$0.10486^{\{9\}}$	$0.07972^{\{5\}}$	$0.07568^{\{4\}}$	$0.08443^{\{8\}}$
		$\hat{\lambda}$	$0.28586^{\{1\}}$	$0.29971^{\{4\}}$	$0.35679^{\{8\}}$	$0.28683^{\{2\}}$	$0.33927^{\{7\}}$	$0.31644^{\{6\}}$	$0.368^{\{9\}}$	$0.30382^{\{5\}}$	$0.2978^{\{3\}}$
	MSE	$\hat{\theta}$	$0.00852^{\{2\}}$	$0.0087^{\{3\}}$	$0.01178^{\{7\}}$	$0.00768^{\{1\}}$	$0.01078^{\{5\}}$	$0.0173^{\{9\}}$	$0.01092^{\{6\}}$	$0.0096^{\{4\}}$	$0.01179^{\{8\}}$
		$\hat{\lambda}$	$0.1412^{\{2\}}$	$0.16193^{\{4\}}$	$0.23442^{\{8\}}$	$0.12177^{\{1\}}$	$0.19288^{\{7\}}$	$0.165^{\{6\}}$	$0.238^{\{9\}}$	$0.16356^{\{5\}}$	$0.14692^{\{3\}}$
	MRE	$\hat{\theta}$	$0.095^{\{2\}}$	$0.09674^{\{3\}}$	$0.11146^{\{7\}}$	$0.09437^{\{1\}}$	$0.10763^{\{6\}}$	$0.13981^{\{9\}}$	$0.10629^{\{5\}}$	$0.1009^{\{4\}}$	$0.11258^{\{8\}}$
		$\hat{\lambda}$	$0.19057^{\{1\}}$	$0.19981^{\{4\}}$	$0.23786^{\{8\}}$	$0.19122^{\{2\}}$	$0.22618^{\{7\}}$	$0.21096^{\{6\}}$	$0.24534^{\{9\}}$	$0.20254^{\{5\}}$	$0.19853^{\{3\}}$
	D_{abs}		$0.02929^{\{4\}}$	$0.02862^{\{2\}}$	$0.03167^{\{8\}}$	$0.02874^{\{3\}}$	$0.03061^{\{5\}}$	$0.03603^{\{9\}}$	$0.03084^{\{7\}}$	$0.02838^{\{1\}}$	$0.03083^{\{6\}}$
	D_{max}		$0.04667^{\{4\}}$	$0.04592^{\{3\}}$	$0.05128^{\{8\}}$	$0.0457^{\{1\}}$	$0.04952^{\{5\}}$	$0.05766^{\{9\}}$	$0.05033^{\{7\}}$	$0.0458^{\{2\}}$	$0.0496^{\{6\}}$
	ASAE		$0.02095^{\{8\}}$	$0.01885^{\{4\}}$	$0.01933^{\{6\}}$	$0.0205^{\{7\}}$	$0.01913^{\{5\}}$	$0.01779^{\{2\}}$	$0.02181^{\{9\}}$	$0.01881^{\{3\}}$	$0.01749^{\{1\}}$
	$\sum Ranks$		$26^{\{2\}}$	$30^{\{3\}}$	$67^{\{9\}}$	$19^{\{1\}}$	$53^{\{6\}}$	$65^{\{7\}}$	$66^{\{8\}}$	$33^{\{4\}}$	$46^{\{5\}}$
150	BIAS	$\hat{\theta}$	$0.0483^{\{1\}}$	$0.05262^{\{3\}}$	$0.05872^{\{6\}}$	$0.04926^{\{2\}}$	$0.05895^{\{7\}}$	$0.07323^{\{9\}}$	$0.05366^{\{4\}}$	$0.05418^{\{5\}}$	$0.05964^{\{8\}}$
		$\hat{\lambda}$	$0.20019^{\{2\}}$	$0.21886^{\{4\}}$	$0.25274^{\{9\}}$	$0.19074^{\{1\}}$	$0.23903^{\{7\}}$	$0.23428^{\{6\}}$	$0.25247^{\{8\}}$	$0.21515^{\{3\}}$	$0.22356^{\{5\}}$
	MSE	$\hat{\theta}$	$0.00367^{\{1\}}$	$0.00441^{\{3\}}$	$0.0057^{\{8\}}$	$0.00374^{\{2\}}$	$0.0055^{\{6\}}$	$0.00843^{\{9\}}$	$0.00457^{\{4.5\}}$	$0.00457^{\{4.5\}}$	$0.00564^{\{7\}}$
		$\hat{\lambda}$	$0.06507^{\{2\}}$	$0.07799^{\{4\}}$	$0.10965^{\{9\}}$	$0.05676^{\{1\}}$	$0.0938^{\{7\}}$	$0.08428^{\{6\}}$	$0.1037^{\{8\}}$	$0.07285^{\{3\}}$	$0.08185^{\{5\}}$
	MRE	$\hat{\theta}$	$0.0644^{\{1\}}$	$0.07016^{\{3\}}$	$0.07829^{\{6\}}$	$0.06567^{\{2\}}$	$0.0786^{\{7\}}$	$0.09763^{\{9\}}$	$0.07154^{\{4\}}$	$0.07224^{\{5\}}$	$0.07952^{\{8\}}$
		$\hat{\lambda}$	$0.13346^{\{2\}}$	$0.14591^{\{4\}}$	$0.16849^{\{9\}}$	$0.12716^{\{1\}}$	$0.15935^{\{7\}}$	$0.15619^{\{6\}}$	$0.16831^{\{8\}}$	$0.14343^{\{3\}}$	$0.14904^{\{5\}}$
	D_{abs}		$0.01975^{\{1\}}$	$0.02085^{\{3\}}$	$0.022^{\{6\}}$	$0.02011^{\{2\}}$	$0.02234^{\{7\}}$	$0.02524^{\{9\}}$	$0.02164^{\{5\}}$	$0.02122^{\{4\}}$	$0.02244^{\{8\}}$
	D_{max}		$0.03161^{\{1\}}$	$0.03361^{\{3\}}$	$0.03574^{\{6\}}$	$0.03195^{\{2\}}$	$0.03609^{\{7\}}$	$0.04079^{\{9\}}$	$0.03533^{\{5\}}$	$0.03424^{\{4\}}$	$0.0362^{\{8\}}$
	ASAE		$0.01428^{\{8\}}$	$0.01287^{\{3\}}$	$0.01316^{\{6\}}$	$0.01426^{\{7\}}$	$0.01313^{\{5\}}$	$0.01217^{\{2\}}$	$0.01552^{\{9\}}$	$0.01288^{\{4\}}$	$0.01212^{\{1\}}$
	$\sum Ranks$		$19^{\{1\}}$	$30^{\{3\}}$	$65^{\{8.5\}}$	$20^{\{2\}}$	$60^{\{7\}}$	$65^{\{8.5\}}$	$55.5^{\{6\}}$	$35.5^{\{4\}}$	$55^{\{5\}}$
250	BIAS	$\hat{\theta}$	$0.038^{\{2\}}$	$0.04068^{\{3\}}$	$0.04678^{\{8\}}$	$0.03764^{\{1\}}$	$0.04374^{\{6\}}$	$0.05645^{\{9\}}$	$0.0429^{\{5\}}$	$0.04083^{\{4\}}$	$0.04509^{\{7\}}$
		$\hat{\lambda}$	$0.15265^{\{2\}}$	$0.16646^{\{5\}}$	$0.19402^{\{8\}}$	$0.14525^{\{1\}}$	$0.18255^{\{7\}}$	$0.17907^{\{6\}}$	$0.19658^{\{9\}}$	$0.16418^{\{3\}}$	$0.16548^{\{4\}}$
	MSE	$\hat{\theta}$	$0.00232^{\{2\}}$	$0.00254^{\{3\}}$	$0.00347^{\{8\}}$	$0.00218^{\{1\}}$	$0.00301^{\{6\}}$	$0.00512^{\{9\}}$	$0.00297^{\{5\}}$	$0.00257^{\{4\}}$	$0.00326^{\{7\}}$
		$\hat{\lambda}$	$0.03793^{\{2\}}$	$0.04374^{\{5\}}$	$0.06083^{\{8\}}$	$0.03399^{\{1\}}$	$0.05229^{\{7\}}$	$0.05074^{\{6\}}$	$0.06245^{\{9\}}$	$0.04171^{\{3\}}$	$0.04364^{\{4\}}$
	MRE	$\hat{\theta}$	$0.05067^{\{2\}}$	$0.05424^{\{3\}}$	$0.06237^{\{8\}}$	$0.05019^{\{1\}}$	$0.05832^{\{6\}}$	$0.07527^{\{9\}}$	$0.0572^{\{5\}}$	$0.05444^{\{4\}}$	$0.06012^{\{7\}}$
		$\hat{\lambda}$	$0.10177^{\{2\}}$	$0.11098^{\{5\}}$	$0.12934^{\{8\}}$	$0.09683^{\{1\}}$	$0.1217^{\{7\}}$	$0.11938^{\{6\}}$	$0.13106^{\{9\}}$	$0.10945^{\{3\}}$	$0.11032^{\{4\}}$
	D_{abs}		$0.01541^{\{1\}}$	$0.01657^{\{4\}}$	$0.01736^{\{7\}}$	$0.01566^{\{2\}}$	$0.01667^{\{5\}}$	$0.01961^{\{9\}}$	$0.01762^{\{8\}}$	$0.01613^{\{3\}}$	$0.01712^{\{6\}}$
	D_{max}		$0.02456^{\{1\}}$	$0.02647^{\{4\}}$	$0.0283^{\{7\}}$	$0.02483^{\{2\}}$	$0.02705^{\{5\}}$	$0.03162^{\{9\}}$	$0.02842^{\{8\}}$	$0.02593^{\{3\}}$	$0.02736^{\{6\}}$
	ASAE		$0.01078^{\{7\}}$	$0.00962^{\{4\}}$	$0.00988^{\{6\}}$	$0.01096^{\{8\}}$	$0.00987^{\{5\}}$	$0.00917^{\{2\}}$	$0.01183^{\{9\}}$	$0.00956^{\{3\}}$	$0.0091^{\{1\}}$
	$\sum Ranks$		$21^{\{2\}}$	$36^{\{4\}}$	$68^{\{9\}}$	$18^{\{1\}}$	$54^{\{6\}}$	$65^{\{7\}}$	$67^{\{8\}}$	$30^{\{3\}}$	$46^{\{5\}}$
400	BIAS	$\hat{\theta}$	$0.02937^{\{1\}}$	$0.03182^{\{3\}}$	$0.03584^{\{8\}}$	$0.03043^{\{2\}}$	$0.03434^{\{6\}}$	$0.04528^{\{9\}}$	$0.03338^{\{5\}}$	$0.03208^{\{4\}}$	$0.0353^{\{7\}}$
		$\hat{\lambda}$	$0.11734^{\{1\}}$	$0.12632^{\{3\}}$	$0.15358^{\{8\}}$	$0.11739^{\{2\}}$	$0.14174^{\{7\}}$	$0.13972^{\{6\}}$	$0.15647^{\{9\}}$	$0.13489^{\{4\}}$	$0.1363^{\{5\}}$
	MSE	$\hat{\theta}$	$0.0014^{\{1\}}$	$0.00159^{\{3\}}$	$0.00207^{\{8\}}$	$0.00145^{\{2\}}$	$0.0019^{\{6\}}$	$0.00326^{\{9\}}$	$0.00181^{\{5\}}$	$0.00161^{\{4\}}$	$0.002^{\{7\}}$
		$\hat{\lambda}$	$0.02217^{\{2\}}$	$0.02525^{\{3\}}$	$0.03904^{\{8\}}$	$0.0221^{\{1\}}$	$0.03295^{\{7\}}$	$0.03112^{\{6\}}$	$0.04091^{\{9\}}$	$0.02795^{\{4\}}$	$0.02917^{\{5\}}$
	MRE	$\hat{\theta}$	$0.03917^{\{1\}}$	$0.04243^{\{3\}}$	$0.04778^{\{8\}}$	$0.04057^{\{2\}}$	$0.04579^{\{6\}}$	$0.06038^{\{9\}}$	$0.04451^{\{5\}}$	$0.04278^{\{4\}}$	$0.04706^{\{7\}}$
		$\hat{\lambda}$	$0.07823^{\{1\}}$	$0.08421^{\{3\}}$	$0.10239^{\{8\}}$	$0.07826^{\{2\}}$	$0.09449^{\{7\}}$	$0.09315^{\{6\}}$	$0.10431^{\{9\}}$	$0.08992^{\{4\}}$	$0.09087^{\{5\}}$
	D_{abs}		$0.01218^{\{1\}}$	$0.01272^{\{3\}}$	$0.01351^{\{7\}}$	$0.01256^{\{2\}}$	$0.01342^{\{6\}}$	$0.01568^{\{9\}}$	$0.01363^{\{8\}}$	$0.01328^{\{4\}}$	$0.01336^{\{5\}}$
	D_{max}		$0.01947^{\{1\}}$	$0.02035^{\{3\}}$	$0.02201^{\{7\}}$	$0.01995^{\{2\}}$	$0.02168^{\{6\}}$	$0.0253^{\{9\}}$	$0.02211^{\{8\}}$	$0.02124^{\{4\}}$	$0.02158^{\{5\}}$
	ASAE		$0.00844^{\{8\}}$	$0.00761^{\{3\}}$	$0.00802^{\{6\}}$	$0.00834^{\{7\}}$	$0.00767^{\{5\}}$	$0.00728^{\{2\}}$	$0.0091^{\{9\}}$	$0.00762^{\{4\}}$	$0.00713^{\{1\}}$
	$\sum Ranks$		$17^{\{1\}}$	$27^{\{3\}}$	$68^{\{9\}}$	$22^{\{2\}}$	$56^{\{6\}}$	$65^{\{7\}}$	$67^{\{8\}}$	$36^{\{4\}}$	$47^{\{5\}}$

Table A3. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max} and ASAE for ($\theta = 2.5,\ \lambda = 0.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
30	BIAS	$\hat{\theta}$	0.59224[4]	0.53657[2]	0.79148[9]	0.50748[1]	0.68103[6]	0.69683[7]	0.66514[5]	0.56666[3]	0.71424[8]
		$\hat{\lambda}$	0.09257[1]	0.09556[3]	0.12293[8]	0.09478[2]	0.11101[6]	0.14797[9]	0.11409[7]	0.09905[4]	0.10365[5]
	MSE	$\hat{\theta}$	0.93712[5]	0.54809[2]	1.46737[9]	0.41186[1]	1.20138[8]	0.90187[4]	1.08785[6]	0.6484[3]	1.1995[7]
		$\hat{\lambda}$	0.01501[3]	0.01484[2]	0.0271[8]	0.01349[1]	0.02123[6]	0.03538[9]	0.0239[7]	0.01604[4]	0.01814[5]
	MRE	$\hat{\theta}$	0.2369[4]	0.21463[2]	0.31659[9]	0.20299[1]	0.27241[6]	0.27873[7]	0.26606[5]	0.22666[3]	0.2857[8]
		$\hat{\lambda}$	0.18514[1]	0.19112[3]	0.24586[8]	0.18956[2]	0.22202[6]	0.29594[9]	0.22818[7]	0.19809[4]	0.2073[5]
	D_{abs}		0.04696[3]	0.04783[4]	0.04999[7.5]	0.04669[2]	0.04969[6]	0.06095[9]	0.04938[5]	0.04629[1]	0.04999[7.5]
	D_{max}		0.07541[3]	0.07618[4]	0.08278[8]	0.07349[1]	0.07996[6]	0.09881[9]	0.07991[5]	0.07449[2]	0.08192[7]
	ASAE		0.0351[8]	0.03198[4]	0.03343[6]	0.03402[7]	0.03254[5]	0.02969[1]	0.02987[2]	0.03136[3]	*NaN*[9]
	$\sum Ranks$		32[4]	26[2]	72.5[9]	18[1]	55[6]	64[8]	49[5]	27[3]	61.5[7]
75	BIAS	$\hat{\theta}$	0.33485[2]	0.3359[3]	0.39949[8]	0.31397[1]	0.39491[6]	0.40262[9]	0.37396[5]	0.34979[4]	0.39913[7]
		$\hat{\lambda}$	0.0575[1]	0.05941[3]	0.07039[8]	0.05922[2]	0.06713[6]	0.09043[9]	0.06989[7]	0.06098[4]	0.06275[5]
	MSE	$\hat{\theta}$	0.20406[4]	0.19819[2]	0.3022[9]	0.15014[1]	0.27721[7]	0.2579[6]	0.25291[5]	0.19882[3]	0.28693[8]
		$\hat{\lambda}$	0.00553[2]	0.00586[3]	0.0082[8]	0.00531[1]	0.00734[6]	0.01275[9]	0.00807[7]	0.00605[4]	0.00645[5]
	MRE	$\hat{\theta}$	0.13394[2]	0.13436[3]	0.15979[8]	0.12559[1]	0.15796[6]	0.16105[9]	0.14959[5]	0.13992[4]	0.15965[7]
		$\hat{\lambda}$	0.115[1]	0.11883[3]	0.14077[8]	0.11844[2]	0.13426[6]	0.18086[9]	0.13977[7]	0.12196[4]	0.12549[5]
	D_{abs}		0.02798[1]	0.02943[3]	0.03091[6]	0.02902[2]	0.03115[7]	0.0382[9]	0.03085[5]	0.03004[4]	0.03128[8]
	D_{max}		0.0453[1]	0.04749[3]	0.05069[8]	0.04628[2]	0.05068[7]	0.06178[9]	0.05022[5]	0.04863[4]	0.05066[6]
	ASAE		0.01994[8]	0.01844[4]	0.0191[6]	0.01954[7]	0.01868[5]	0.01745[2]	0.01695[1]	0.01827[3]	0.02243[9]
	$\sum Ranks$		22[2]	27[3]	69[8]	19[1]	56[6]	71[9]	47[5]	34[4]	60[7]
150	BIAS	$\hat{\theta}$	0.22233[2]	0.2263[3]	0.26514[7]	0.22176[1]	0.25986[6]	0.27912[9]	0.256[5]	0.23922[4]	0.27659[8]
		$\hat{\lambda}$	0.04059[1]	0.04156[3]	0.04799[7]	0.04069[2]	0.04655[6]	0.06479[9]	0.0499[8]	0.04299[4]	0.04511[5]
	MSE	$\hat{\theta}$	0.0832[2]	0.08331[3]	0.11269[6]	0.07581[1]	0.1127[7]	0.12509[8]	0.11154[5]	0.0926[4]	0.12622[9]
		$\hat{\lambda}$	0.00266[2]	0.00282[3]	0.0037[7]	0.00248[1]	0.00353[6]	0.00648[9]	0.00403[8]	0.0029[4]	0.0033[5]
	MRE	$\hat{\theta}$	0.08893[2]	0.09052[3]	0.10606[7]	0.0887[1]	0.10394[6]	0.11165[9]	0.1024[5]	0.09569[4]	0.11063[8]
		$\hat{\lambda}$	0.08118[1]	0.08312[3]	0.09599[7]	0.08138[2]	0.0931[6]	0.12958[9]	0.0998[8]	0.08598[4]	0.09022[5]
	D_{abs}		0.02106[3]	0.02097[2]	0.02203[7]	0.01999[1]	0.02171[5]	0.02743[9]	0.0219[6]	0.02114[4]	0.02242[8]
	D_{max}		0.03355[2]	0.03357[3]	0.03598[7]	0.0321[1]	0.0351[5]	0.04419[9]	0.03565[6]	0.03411[4]	0.03649[8]
	ASAE		0.01304[8]	0.01237[4]	0.01267[5]	0.013[7]	0.01268[6]	0.01187[2]	0.01131[1]	0.01226[3]	0.01478[9]
	$\sum Ranks$		23[2]	27[3]	60[7]	17[1]	53[6]	73[9]	52[5]	35[4]	65[8]
250	BIAS	$\hat{\theta}$	0.16551[1]	0.18315[3]	0.20938[7]	0.16859[2]	0.20671[6]	0.23042[9]	0.19725[5]	0.18332[4]	0.21209[8]
		$\hat{\lambda}$	0.03144[2]	0.03281[4]	0.0377[7]	0.03127[1]	0.03701[6]	0.05142[9]	0.03791[8]	0.03273[3]	0.03526[5]
	MSE	$\hat{\theta}$	0.04431[1]	0.05412[3]	0.07175[8]	0.04627[2]	0.06887[6]	0.08538[9]	0.06249[5]	0.05428[4]	0.07122[7]
		$\hat{\lambda}$	0.00155[2]	0.00172[3]	0.00227[8]	0.00154[1]	0.00217[6]	0.00414[9]	0.00224[7]	0.00173[4]	0.00191[5]
	MRE	$\hat{\theta}$	0.0662[1]	0.07326[3]	0.08375[7]	0.06743[2]	0.08268[6]	0.09217[9]	0.0789[5]	0.07333[4]	0.08484[8]
		$\hat{\lambda}$	0.06288[2]	0.06562[4]	0.07539[7]	0.06255[1]	0.07403[6]	0.10283[9]	0.07582[8]	0.06547[3]	0.07051[5]
	D_{abs}		0.0164[4]	0.016[2]	0.01704[6]	0.01572[1]	0.01738[7]	0.02034[9]	0.01751[8]	0.01626[3]	0.01694[5]
	D_{max}		0.02612[3]	0.02572[2]	0.0278[6]	0.02512[1]	0.02819[7]	0.03335[9]	0.02841[8]	0.02626[4]	0.02761[5]
	ASAE		0.00985[7]	0.00923[3]	0.0097[6]	0.00993[8]	0.00961[5]	0.00897[2]	0.00858[1]	0.00942[4]	0.01105[9]
	$\sum Ranks$		23[2]	27[3]	62[8]	19[1]	55[5.5]	74[9]	55[5.5]	33[4]	57[7]
400	BIAS	$\hat{\theta}$	0.13222[2]	0.14534[4]	0.16506[7]	0.13084[1]	0.16162[6]	0.17442[9]	0.15216[5]	0.14533[3]	0.16932[8]
		$\hat{\lambda}$	0.02399[1]	0.02613[4]	0.02936[7]	0.02479[2]	0.02915[6]	0.0391[9]	0.03008[8]	0.02569[3]	0.02756[5]
	MSE	$\hat{\theta}$	0.02813[2]	0.03336[3]	0.04412[7]	0.02706[1]	0.04149[6]	0.04788[8]	0.03577[5]	0.03482[4]	0.04899[9]
		$\hat{\lambda}$	0.00092[1]	0.00106[4]	0.00137[7]	0.00095[2]	0.00133[6]	0.00236[9]	0.00139[8]	0.00103[3]	0.00125[5]
	MRE	$\hat{\theta}$	0.05289[2]	0.05814[4]	0.06602[7]	0.05233[1]	0.06465[6]	0.06977[9]	0.06086[5]	0.05813[3]	0.06773[8]
		$\hat{\lambda}$	0.04798[1]	0.05227[4]	0.05872[7]	0.04958[2]	0.0583[6]	0.07819[9]	0.06017[8]	0.05138[3]	0.05513[5]
	D_{abs}		0.0127[3]	0.01337[4]	0.01373[7]	0.01254[2]	0.01367[6]	0.01591[9]	0.01358[5]	0.0124[1]	0.01381[8]
	D_{max}		0.02025[3]	0.02148[4]	0.02234[7]	0.02002[1]	0.02221[6]	0.02601[9]	0.02199[5]	0.02005[2]	0.02245[8]
	ASAE		0.00767[7]	0.00718[4]	0.00742[6]	0.00768[8]	0.00733[5]	0.00677[2]	0.00668[1]	0.0071[3]	0.00863[9]
	$\sum Ranks$		22[2]	35[4]	62[7]	20[1]	53[6]	73[9]	50[5]	25[3]	65[8]

Table A4. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max} and ASAE for ($\theta = 1.5,\ \lambda = 2.5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
30	BIAS	$\hat{\theta}$	$0.31205^{\{5\}}$	$0.28797^{\{2\}}$	$0.36528^{\{9\}}$	$0.2627^{\{1\}}$	$0.32315^{\{7\}}$	$0.30908^{\{4\}}$	$0.31708^{\{6\}}$	$0.2981^{\{3\}}$	$0.34766^{\{8\}}$
		$\hat{\lambda}$	$0.60201^{\{3\}}$	$0.58168^{\{2\}}$	$0.72669^{\{9\}}$	$0.54472^{\{1\}}$	$0.64768^{\{7\}}$	$0.61131^{\{4\}}$	$0.7037^{\{8\}}$	$0.61967^{\{5\}}$	$0.64001^{\{6\}}$
	MSE	$\hat{\theta}$	$0.18647^{\{5\}}$	$0.15848^{\{2\}}$	$0.29815^{\{9\}}$	$0.10518^{\{1\}}$	$0.20049^{\{7\}}$	$0.16435^{\{3\}}$	$0.196^{\{6\}}$	$0.17409^{\{4\}}$	$0.25877^{\{8\}}$
		$\hat{\lambda}$	$0.64782^{\{5\}}$	$0.57602^{\{2\}}$	$0.97353^{\{9\}}$	$0.45924^{\{1\}}$	$0.69197^{\{6\}}$	$0.58713^{\{3\}}$	$0.88667^{\{8\}}$	$0.63025^{\{4\}}$	$0.72525^{\{7\}}$
	MRE	$\hat{\theta}$	$0.20803^{\{5\}}$	$0.19198^{\{2\}}$	$0.24352^{\{9\}}$	$0.17513^{\{1\}}$	$0.21543^{\{7\}}$	$0.20605^{\{4\}}$	$0.21139^{\{6\}}$	$0.19873^{\{3\}}$	$0.23177^{\{8\}}$
		$\hat{\lambda}$	$0.2408^{\{3\}}$	$0.23267^{\{2\}}$	$0.29067^{\{9\}}$	$0.21789^{\{1\}}$	$0.25907^{\{7\}}$	$0.24452^{\{4\}}$	$0.28148^{\{8\}}$	$0.24787^{\{5\}}$	$0.256^{\{6\}}$
	D_{abs}		$0.04464^{\{1\}}$	$0.04658^{\{3\}}$	$0.04915^{\{6\}}$	$0.04608^{\{2\}}$	$0.04947^{\{7\}}$	$0.05109^{\{9\}}$	$0.04955^{\{8\}}$	$0.04725^{\{5\}}$	$0.04711^{\{4\}}$
	D_{max}		$0.07257^{\{2\}}$	$0.07446^{\{3\}}$	$0.08159^{\{9\}}$	$0.07243^{\{1\}}$	$0.07933^{\{6\}}$	$0.08087^{\{8\}}$	$0.0805^{\{7\}}$	$0.07573^{\{4\}}$	$0.07711^{\{5\}}$
	ASAE		$0.03577^{\{8\}}$	$0.03163^{\{2\}}$	$0.03261^{\{6\}}$	$0.03509^{\{7\}}$	$0.03235^{\{5\}}$	$0.03138^{\{1\}}$	$0.03647^{\{9\}}$	$0.03207^{\{4\}}$	$0.03164^{\{3\}}$
	$\sum Ranks$		$37^{\{3.5\}}$	$20^{\{2\}}$	$75^{\{9\}}$	$16^{\{1\}}$	$59^{\{7\}}$	$40^{\{5\}}$	$66^{\{8\}}$	$37^{\{3.5\}}$	$55^{\{6\}}$
75	BIAS	$\hat{\theta}$	$0.15878^{\{1\}}$	$0.17387^{\{3\}}$	$0.21284^{\{9\}}$	$0.16368^{\{2\}}$	$0.20443^{\{7\}}$	$0.1958^{\{5\}}$	$0.19786^{\{6\}}$	$0.17819^{\{4\}}$	$0.20739^{\{8\}}$
		$\hat{\lambda}$	$0.33259^{\{1\}}$	$0.36004^{\{3\}}$	$0.4236^{\{8\}}$	$0.3525^{\{2\}}$	$0.40766^{\{7\}}$	$0.37308^{\{4\}}$	$0.45133^{\{9\}}$	$0.37989^{\{5\}}$	$0.38641^{\{6\}}$
	MSE	$\hat{\theta}$	$0.04257^{\{2\}}$	$0.05247^{\{3\}}$	$0.08505^{\{9\}}$	$0.04254^{\{1\}}$	$0.06837^{\{7\}}$	$0.05972^{\{5\}}$	$0.06774^{\{6\}}$	$0.05381^{\{4\}}$	$0.07785^{\{8\}}$
		$\hat{\lambda}$	$0.18341^{\{1\}}$	$0.21491^{\{3\}}$	$0.30242^{\{8\}}$	$0.18558^{\{2\}}$	$0.26585^{\{7\}}$	$0.21737^{\{4\}}$	$0.3343^{\{9\}}$	$0.2399^{\{5\}}$	$0.25395^{\{6\}}$
	MRE	$\hat{\theta}$	$0.10585^{\{1\}}$	$0.11591^{\{3\}}$	$0.14189^{\{9\}}$	$0.10912^{\{2\}}$	$0.13629^{\{7\}}$	$0.13053^{\{5\}}$	$0.13191^{\{6\}}$	$0.11879^{\{4\}}$	$0.13826^{\{8\}}$
		$\hat{\lambda}$	$0.13304^{\{1\}}$	$0.14402^{\{3\}}$	$0.16944^{\{8\}}$	$0.141^{\{2\}}$	$0.16306^{\{7\}}$	$0.14923^{\{4\}}$	$0.18053^{\{9\}}$	$0.15195^{\{5\}}$	$0.15456^{\{6\}}$
	D_{abs}		$0.02803^{\{1\}}$	$0.02919^{\{2\}}$	$0.03114^{\{6\}}$	$0.02975^{\{3\}}$	$0.03205^{\{8\}}$	$0.03068^{\{5\}}$	$0.03227^{\{9\}}$	$0.03035^{\{4\}}$	$0.03132^{\{7\}}$
	D_{max}		$0.04501^{\{1\}}$	$0.047^{\{2\}}$	$0.05103^{\{7\}}$	$0.0471^{\{3\}}$	$0.05183^{\{8\}}$	$0.04939^{\{5\}}$	$0.05253^{\{9\}}$	$0.04872^{\{4\}}$	$0.05089^{\{6\}}$
	ASAE		$0.02054^{\{8\}}$	$0.01878^{\{4\}}$	$0.01946^{\{6\}}$	$0.02047^{\{7\}}$	$0.0189^{\{5\}}$	$0.01869^{\{3\}}$	$0.02152^{\{9\}}$	$0.01846^{\{1\}}$	$0.01864^{\{2\}}$
	$\sum Ranks$		$17^{\{1\}}$	$26^{\{3\}}$	$70^{\{8\}}$	$24^{\{2\}}$	$63^{\{7\}}$	$40^{\{5\}}$	$72^{\{9\}}$	$36^{\{4\}}$	$57^{\{6\}}$
150	BIAS	$\hat{\theta}$	$0.1202^{\{3\}}$	$0.12005^{\{2\}}$	$0.14451^{\{9\}}$	$0.11879^{\{1\}}$	$0.14013^{\{6\}}$	$0.14197^{\{7\}}$	$0.13488^{\{5\}}$	$0.12479^{\{4\}}$	$0.14212^{\{8\}}$
		$\hat{\lambda}$	$0.24036^{\{1\}}$	$0.24907^{\{3\}}$	$0.29489^{\{8\}}$	$0.24262^{\{2\}}$	$0.28803^{\{7\}}$	$0.26228^{\{6\}}$	$0.29683^{\{9\}}$	$0.24992^{\{4\}}$	$0.26135^{\{5\}}$
	MSE	$\hat{\theta}$	$0.02364^{\{3\}}$	$0.02326^{\{2\}}$	$0.03515^{\{9\}}$	$0.02162^{\{1\}}$	$0.03165^{\{7\}}$	$0.03077^{\{6\}}$	$0.03026^{\{5\}}$	$0.02532^{\{4\}}$	$0.03242^{\{8\}}$
		$\hat{\lambda}$	$0.09501^{\{2\}}$	$0.09944^{\{3\}}$	$0.14204^{\{8\}}$	$0.09005^{\{1\}}$	$0.13011^{\{7\}}$	$0.10872^{\{6\}}$	$0.14525^{\{9\}}$	$0.10084^{\{4\}}$	$0.10548^{\{5\}}$
	MRE	$\hat{\theta}$	$0.08013^{\{3\}}$	$0.08003^{\{2\}}$	$0.09634^{\{9\}}$	$0.0792^{\{1\}}$	$0.09342^{\{6\}}$	$0.09464^{\{7\}}$	$0.08992^{\{5\}}$	$0.08319^{\{4\}}$	$0.09475^{\{8\}}$
		$\hat{\lambda}$	$0.09615^{\{1\}}$	$0.09963^{\{3\}}$	$0.11796^{\{8\}}$	$0.09705^{\{2\}}$	$0.11521^{\{7\}}$	$0.10491^{\{6\}}$	$0.11873^{\{9\}}$	$0.09997^{\{4\}}$	$0.10454^{\{5\}}$
	D_{abs}		$0.02017^{\{1\}}$	$0.0211^{\{4\}}$	$0.02243^{\{8\}}$	$0.02051^{\{2\}}$	$0.02226^{\{7\}}$	$0.02259^{\{9\}}$	$0.02207^{\{5\}}$	$0.02083^{\{3\}}$	$0.02223^{\{6\}}$
	D_{max}		$0.03247^{\{1\}}$	$0.03374^{\{4\}}$	$0.03659^{\{9\}}$	$0.03266^{\{2\}}$	$0.03618^{\{7\}}$	$0.03632^{\{8\}}$	$0.03578^{\{5\}}$	$0.03344^{\{3\}}$	$0.036^{\{6\}}$
	ASAE		$0.01405^{\{7\}}$	$0.01278^{\{3\}}$	$0.01299^{\{6\}}$	$0.01416^{\{8\}}$	$0.01293^{\{5\}}$	$0.01265^{\{2\}}$	$0.01441^{\{9\}}$	$0.01288^{\{4\}}$	$0.0125^{\{1\}}$
	$\sum Ranks$		$22^{\{2\}}$	$26^{\{3\}}$	$74^{\{9\}}$	$20^{\{1\}}$	$59^{\{7\}}$	$57^{\{6\}}$	$61^{\{8\}}$	$34^{\{4\}}$	$52^{\{5\}}$
250	BIAS	$\hat{\theta}$	$0.08701^{\{1\}}$	$0.09819^{\{4\}}$	$0.10842^{\{7\}}$	$0.09329^{\{2\}}$	$0.10276^{\{6\}}$	$0.1115^{\{9\}}$	$0.10195^{\{5\}}$	$0.09656^{\{3\}}$	$0.11095^{\{8\}}$
		$\hat{\lambda}$	$0.18215^{\{1\}}$	$0.20168^{\{5\}}$	$0.22227^{\{8\}}$	$0.19515^{\{2\}}$	$0.21391^{\{7\}}$	$0.20956^{\{6\}}$	$0.23211^{\{9\}}$	$0.19644^{\{3\}}$	$0.19769^{\{4\}}$
	MSE	$\hat{\theta}$	$0.0122^{\{1\}}$	$0.01547^{\{4\}}$	$0.01971^{\{9\}}$	$0.01371^{\{2\}}$	$0.01724^{\{6\}}$	$0.01957^{\{8\}}$	$0.01688^{\{5\}}$	$0.01528^{\{3\}}$	$0.01951^{\{7\}}$
		$\hat{\lambda}$	$0.05244^{\{1\}}$	$0.06382^{\{4\}}$	$0.07846^{\{8\}}$	$0.05955^{\{2\}}$	$0.0718^{\{7\}}$	$0.06888^{\{6\}}$	$0.08712^{\{9\}}$	$0.06397^{\{5\}}$	$0.06214^{\{3\}}$
	MRE	$\hat{\theta}$	$0.05801^{\{1\}}$	$0.06546^{\{4\}}$	$0.07228^{\{7\}}$	$0.06219^{\{2\}}$	$0.0685^{\{6\}}$	$0.07433^{\{9\}}$	$0.06797^{\{5\}}$	$0.06437^{\{3\}}$	$0.07397^{\{8\}}$
		$\hat{\lambda}$	$0.07286^{\{1\}}$	$0.08067^{\{5\}}$	$0.08891^{\{8\}}$	$0.07806^{\{2\}}$	$0.08556^{\{7\}}$	$0.08383^{\{6\}}$	$0.09284^{\{9\}}$	$0.07858^{\{3\}}$	$0.07908^{\{4\}}$
	D_{abs}		$0.01534^{\{1\}}$	$0.01646^{\{3\}}$	$0.01736^{\{8\}}$	$0.01656^{\{4\}}$	$0.01664^{\{6\}}$	$0.01659^{\{5\}}$	$0.01682^{\{7\}}$	$0.01641^{\{2\}}$	$0.01739^{\{9\}}$
	D_{max}		$0.02461^{\{1\}}$	$0.02646^{\{4\}}$	$0.02824^{\{9\}}$	$0.02636^{\{3\}}$	$0.02699^{\{5\}}$	$0.02718^{\{6\}}$	$0.02737^{\{7\}}$	$0.02635^{\{2\}}$	$0.02809^{\{8\}}$
	ASAE		$0.01055^{\{7\}}$	$0.00971^{\{4\}}$	$0.00986^{\{6\}}$	$0.01061^{\{8\}}$	$0.00976^{\{5\}}$	$0.00951^{\{2\}}$	$0.01104^{\{9\}}$	$0.00957^{\{3\}}$	$0.00944^{\{1\}}$
	$\sum Ranks$		$15^{\{1\}}$	$37^{\{4\}}$	$70^{\{9\}}$	$27^{\{2.5\}}$	$55^{\{6\}}$	$57^{\{7\}}$	$65^{\{8\}}$	$27^{\{2.5\}}$	$52^{\{5\}}$
400	BIAS	$\hat{\theta}$	$0.06997^{\{1\}}$	$0.07379^{\{3\}}$	$0.08321^{\{6\}}$	$0.07247^{\{2\}}$	$0.08356^{\{7\}}$	$0.08755^{\{9\}}$	$0.0807^{\{5\}}$	$0.07732^{\{4\}}$	$0.08508^{\{8\}}$
		$\hat{\lambda}$	$0.14482^{\{1\}}$	$0.15344^{\{4\}}$	$0.17084^{\{7\}}$	$0.14935^{\{2\}}$	$0.17273^{\{8\}}$	$0.1612^{\{6\}}$	$0.18095^{\{9\}}$	$0.15805^{\{5\}}$	$0.15219^{\{3\}}$
	MSE	$\hat{\theta}$	$0.00768^{\{1\}}$	$0.00869^{\{3\}}$	$0.01111^{\{6\}}$	$0.0081^{\{2\}}$	$0.01112^{\{7\}}$	$0.0122^{\{9\}}$	$0.01059^{\{5\}}$	$0.00958^{\{4\}}$	$0.01198^{\{8\}}$
		$\hat{\lambda}$	$0.03265^{\{1\}}$	$0.03774^{\{3\}}$	$0.0463^{\{7\}}$	$0.0346^{\{2\}}$	$0.04822^{\{8\}}$	$0.04042^{\{5\}}$	$0.05272^{\{9\}}$	$0.04066^{\{6\}}$	$0.03915^{\{4\}}$
	MRE	$\hat{\theta}$	$0.04665^{\{1\}}$	$0.0492^{\{3\}}$	$0.05547^{\{6\}}$	$0.04831^{\{2\}}$	$0.05571^{\{7\}}$	$0.05837^{\{9\}}$	$0.0538^{\{5\}}$	$0.05154^{\{4\}}$	$0.05672^{\{8\}}$
		$\hat{\lambda}$	$0.05793^{\{1\}}$	$0.06138^{\{4\}}$	$0.06834^{\{7\}}$	$0.05974^{\{2\}}$	$0.06909^{\{8\}}$	$0.06448^{\{6\}}$	$0.07238^{\{9\}}$	$0.06322^{\{5\}}$	$0.06088^{\{3\}}$
	D_{abs}		$0.01271^{\{2\}}$	$0.01277^{\{4\}}$	$0.01347^{\{6\}}$	$0.01275^{\{3\}}$	$0.01379^{\{9\}}$	$0.01375^{\{7\}}$	$0.01312^{\{5\}}$	$0.01259^{\{1\}}$	$0.01376^{\{8\}}$
	D_{max}		$0.02024^{\{1\}}$	$0.02048^{\{4\}}$	$0.02183^{\{6\}}$	$0.02036^{\{2\}}$	$0.02226^{\{9\}}$	$0.02224^{\{8\}}$	$0.02139^{\{5\}}$	$0.02045^{\{3\}}$	$0.02215^{\{7\}}$
	ASAE		$0.0082^{\{7\}}$	$0.00752^{\{4\}}$	$0.00764^{\{6\}}$	$0.00831^{\{8\}}$	$0.00762^{\{5\}}$	$0.00745^{\{2\}}$	$0.00847^{\{9\}}$	$0.0075^{\{3\}}$	$0.00728^{\{1\}}$
	$\sum Ranks$		$16^{\{1\}}$	$32^{\{3\}}$	$57^{\{6\}}$	$25^{\{2\}}$	$68^{\{9\}}$	$61^{\{7.5\}}$	$61^{\{7.5\}}$	$35^{\{4\}}$	$50^{\{5\}}$

Table A5. Simulation values of BIAS, MSE, MRE, D_{abs}, D_{max} and ASAE for ($\theta = 3,\ \lambda = 5$).

n	Est.	Est. Par.	MLE	ADE	CVME	MPSE	LSE	PCE	RTADE	WLSE	LTADE
30	BIAS	$\hat{\theta}$	0.807[5]	0.73493[2]	0.92649[9]	0.63608[1]	0.85126[7]	0.75372[3]	0.83375[6]	0.8051[4]	0.9095[8]
		$\hat{\lambda}$	0.94782[3]	0.95162[4]	1.11005[9]	0.90387[1]	1.02115[7]	0.94398[2]	1.07895[8]	1.0173[6]	0.97877[5]
	MSE	$\hat{\theta}$	1.42918[4]	1.15781[2]	1.90497[8]	0.64258[1]	1.50639[6]	1.31966[3]	1.62384[7]	1.46748[5]	2.33066[9]
		$\hat{\lambda}$	1.52273[4]	1.51862[3]	2.15989[9]	1.21432[1]	1.70326[6]	1.37581[2]	2.10267[8]	1.76607[7]	1.68247[5]
	MRE	$\hat{\theta}$	0.269[5]	0.24498[2]	0.30883[9]	0.21203[1]	0.28375[7]	0.25124[3]	0.27792[6]	0.26837[4]	0.30317[8]
		$\hat{\lambda}$	0.18956[3]	0.19032[4]	0.22201[9]	0.18077[1]	0.20423[7]	0.1888[2]	0.21579[8]	0.20346[6]	0.19575[5]
	D_{abs}		0.04598[1]	0.04627[2]	0.04907[7]	0.04658[3]	0.04779[4]	0.05092[9]	0.04829[5]	0.04862[6]	0.04953[8]
	D_{max}		0.07479[3]	0.07416[2]	0.08112[9]	0.07371[1]	0.07775[4]	0.08067[7]	0.0786[6]	0.07827[5]	0.08098[8]
	ASAE		0.0358[8]	0.03234[4]	0.03359[6]	0.034[7]	0.03272[5]	0.03191[2]	0.03583[9]	0.03179[1]	0.0322[3]
	$\sum$ *Ranks*		36[4]	25[2]	75[9]	17[1]	53[6]	33[3]	63[8]	44[5]	59[7]
75	BIAS	$\hat{\theta}$	0.41682[2]	0.43487[3]	0.53868[9]	0.3974[1]	0.4823[7]	0.45692[5]	0.46715[6]	0.45242[4]	0.49726[8]
		$\hat{\lambda}$	0.54597[1]	0.57692[3]	0.66627[9]	0.5644[2]	0.64969[8]	0.5826[4]	0.63576[7]	0.58392[5]	0.60256[6]
	MSE	$\hat{\theta}$	0.29895[2]	0.32772[3]	0.53633[9]	0.23612[1]	0.38907[6]	0.33056[4]	0.39609[7]	0.37192[5]	0.42337[8]
		$\hat{\lambda}$	0.48609[2]	0.55452[4]	0.74241[9]	0.48454[1]	0.65827[7]	0.52416[3]	0.66627[8]	0.56191[5]	0.57098[6]
	MRE	$\hat{\theta}$	0.13894[2]	0.14496[3]	0.17956[9]	0.13247[1]	0.16077[7]	0.15231[5]	0.15572[6]	0.15081[4]	0.16575[8]
		$\hat{\lambda}$	0.10919[1]	0.11538[3]	0.13325[9]	0.11288[2]	0.12994[8]	0.11652[4]	0.12715[7]	0.11678[5]	0.12051[6]
	D_{abs}		0.02776[1]	0.0297[2]	0.03115[9]	0.02981[3]	0.03107[8]	0.03104[6.5]	0.03024[5]	0.02987[4]	0.03104[6.5]
	D_{max}		0.04505[1]	0.04767[3]	0.05115[9]	0.04749[2]	0.05034[7]	0.0497[6]	0.04896[5]	0.04837[4]	0.05051[8]
	ASAE		0.02006[8]	0.01816[1]	0.01912[6]	0.01969[7]	0.01869[4]	0.01896[5]	0.02111[9]	0.01851[2]	0.0186[3]
	$\sum$ *Ranks*		20[1.5]	25[3]	78[9]	20[1.5]	62[8]	42.5[5]	60[7]	38[4]	59.5[6]
150	BIAS	$\hat{\theta}$	0.28653[2]	0.30651[4]	0.34294[8]	0.27438[1]	0.33761[7]	0.32815[6]	0.32042[5]	0.30498[3]	0.35537[9]
		$\hat{\lambda}$	0.38595[2]	0.4048[3]	0.44673[8]	0.3754[1]	0.44412[7]	0.41449[4]	0.45702[9]	0.42101[5]	0.43061[6]
	MSE	$\hat{\theta}$	0.13509[2]	0.15802[4]	0.20504[8]	0.11275[1]	0.18525[7]	0.16966[5]	0.17902[6]	0.15066[3]	0.21722[9]
		$\hat{\lambda}$	0.23798[2]	0.27233[4]	0.3244[8]	0.22684[1]	0.31601[7]	0.26547[3]	0.34181[9]	0.27699[5]	0.2946[6]
	MRE	$\hat{\theta}$	0.09551[2]	0.10217[4]	0.11431[8]	0.09146[1]	0.11254[7]	0.10938[6]	0.10681[5]	0.10166[3]	0.11846[9]
		$\hat{\lambda}$	0.07719[2]	0.08096[3]	0.08935[8]	0.07508[1]	0.08882[7]	0.0829[4]	0.0914[9]	0.0842[5]	0.08612[6]
	D_{abs}		0.02052[2]	0.0218[8]	0.02149[5]	0.02037[1]	0.02228[9]	0.02144[4]	0.02157[7]	0.02069[3]	0.02154[6]
	D_{max}		0.03294[2]	0.03502[5]	0.0352[8]	0.03242[1]	0.03618[9]	0.03473[4]	0.03513[6]	0.03353[3]	0.03517[7]
	ASAE		0.01333[8]	0.0123[2]	0.01267[5]	0.01325[7]	0.01258[4]	0.01268[6]	0.01362[9]	0.01235[3]	0.01227[1]
	$\sum$ *Ranks*		24[2]	37[4]	66[9]	15[1]	64[7]	42[5]	65[8]	33[3]	59[6]
250	BIAS	$\hat{\theta}$	0.21694[2]	0.21994[3]	0.25856[8]	0.21157[1]	0.2559[6]	0.25597[7]	0.23466[5]	0.23001[4]	0.27978[9]
		$\hat{\lambda}$	0.29357[1]	0.29738[3]	0.35492[9]	0.2951[2]	0.33413[7]	0.32885[5]	0.34102[8]	0.31799[4]	0.33199[6]
	MSE	$\hat{\theta}$	0.07779[2]	0.08026[3]	0.10809[8]	0.06923[1]	0.10153[7]	0.10117[6]	0.09143[5]	0.08502[4]	0.12682[9]
		$\hat{\lambda}$	0.13914[1]	0.14317[3]	0.19456[9]	0.14064[2]	0.17729[7]	0.1698[5]	0.18679[8]	0.15734[4]	0.17661[6]
	MRE	$\hat{\theta}$	0.07231[2]	0.07331[3]	0.08619[8]	0.07052[1]	0.0853[6]	0.08532[7]	0.07822[5]	0.07667[4]	0.09326[9]
		$\hat{\lambda}$	0.05871[1]	0.05948[3]	0.07098[9]	0.05902[2]	0.06683[7]	0.06577[5]	0.0682[8]	0.0636[4]	0.0664[6]
	D_{abs}		0.01566[1]	0.01636[5]	0.01724[8]	0.01586[2]	0.0169[7]	0.01606[3]	0.01671[6]	0.0161[4]	0.01749[9]
	D_{max}		0.02514[1]	0.02612[4]	0.02807[8]	0.02528[2]	0.02751[7]	0.02621[5]	0.02706[6]	0.02605[3]	0.02858[9]
	ASAE		0.00957[7]	0.00917[2.5]	0.00931[5]	0.00976[8]	0.00924[4]	0.00933[6]	0.01013[9]	0.00917[2.5]	0.00912[1]
	$\sum$ *Ranks*		18[1]	29.5[3]	72[9]	21[2]	58[6]	49[5]	60[7]	33.5[4]	64[8]
400	BIAS	$\hat{\theta}$	0.16705[1]	0.18148[4]	0.204[6]	0.16757[2]	0.20727[7]	0.20851[8]	0.19912[5]	0.17232[3]	0.21447[9]
		$\hat{\lambda}$	0.22515[1]	0.24644[4]	0.2738[7]	0.23023[2]	0.27843[8]	0.26425[6]	0.29221[9]	0.23635[3]	0.25369[5]
	MSE	$\hat{\theta}$	0.04609[2]	0.05067[4]	0.06832[7]	0.04426[1]	0.06826[6]	0.07029[8]	0.06259[5]	0.04737[3]	0.07416[9]
		$\hat{\lambda}$	0.08039[1]	0.09399[4]	0.11884[7]	0.08733[3]	0.12072[8]	0.11065[6]	0.13193[9]	0.08721[2]	0.10189[5]
	MRE	$\hat{\theta}$	0.05568[1]	0.06049[4]	0.068[6]	0.05586[2]	0.06909[7]	0.0695[8]	0.06637[5]	0.05744[3]	0.07149[9]
		$\hat{\lambda}$	0.04503[1]	0.04929[4]	0.05476[7]	0.04605[2]	0.05569[8]	0.05285[6]	0.05844[9]	0.04727[3]	0.05074[5]
	D_{abs}		0.0127[2]	0.01276[4]	0.01318[5]	0.01234[1]	0.01343[7]	0.01356[9]	0.01337[6]	0.01275[3]	0.0135[8]
	D_{max}		0.02022[2]	0.02062[4]	0.02149[5]	0.01968[1]	0.02191[7]	0.02206[9]	0.02179[6]	0.02043[3]	0.02205[8]
	ASAE		0.00743[8]	0.00702[3]	0.00717[5.5]	0.0074[7]	0.00717[5.5]	0.00709[4]	0.00781[9]	0.007[2]	0.00689[1]
	$\sum$ *Ranks*		19[1]	35[4]	55.5[5]	21[2]	63.5[8]	64[9]	63[7]	25[3]	59[6]

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