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Research article

Reverse fractional integral inclusions and generic $\eta_\hbar$ interval-valued convexity

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Abstract: This paper presents the development of reverse Minkowski and reverse Hölder's fractional integral inclusions. We propose a generic class of $\eta_\hbar$ interval-valued (I.V) convex functions, which unifies various existing classes. Additionally, we obtain a discrete Jensen-type inclusion within this convexity setup. By leveraging this advanced convexity structure together with tempered fractional integral operators, we derive new Hermite–Hadamard (H-H)-type, Fejér-H-H-type, and other fractional inclusions. Moreover, we explore the broader significance of our results, supporting them with graphical visualizations. The applications of our results are demonstrated through average value computations.

Keywords: reverse Hölder's inclusion; reverse Minkowski inclusion; Hermite–Hadamard-type inclusions; tempered fractional integral operators; interval-valued mappings
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1. Introduction

Convex functions have been a part of mathematical theory for over a century with far reaching implications in optimization, statistics, economics, machine learning, artificial intelligence and physics. These functions possess the characteristic that their graph always lies above their secant lines which gives them several significant properties such as being differentiable almost everywhere and having a unique minimum. The significance of convexity extends from pure mathematics to various applied sciences. In recent years, researchers have explored extensions and generalizations of convex functions using innovative techniques. The applicability of convexity in optimization theory, statistics and mathematical analysis stems from its strong geometric intuition.

Inequality theory is an important area of mathematics that studies the properties and uses of inequalities in different fields. It is an essential branch of study in fields such as analysis, geometry, number theory, and optimization, as well as applied fields such as physics and economics. The most important part of inequality theory is convex analysis, which has been instrumental in establishing many classical and modern inequalities. The properties of convex functions constitute a powerful tool for proving and generalizing inequalities in mathematical analysis and statistics. This has led to very significant results, including inequalities of the H-H, Hardy, Simpson, Fejér, and Ostrowski, which have far-reaching applications in both theoretical and applied mathematics. The classical results independently discovered by Hermite and Hadamard constitute a fundamental description of convex functions.

Theorem 1.1. *Consider a convex function $\Omega : [\varepsilon_1, \varepsilon_2] \subset \mathbb{R} \to \mathbb{R}$. The inequality given below holds true*

$$\Omega\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \Omega(r)dr \leq \frac{\Omega(\varepsilon_1) + \Omega(\varepsilon_2)}{2}.$$

For further information and a more in-depth discussion, we refer to [1,2]. In [3], Wu et al. defined the following new categories of g-convex sets and g-convex functions.

Definition 1.2. *Let $\Omega : \mathcal{D} \to \mathbb{R}$ be a function that is both strictly monotone and continuous. A subset $\mathcal{D} \subset \mathbb{R}$ is defined as a g-convex set if*

$$g^{-1}\left(\nu g(r) + (1-\nu)g(t)\right) \in \mathcal{D} \quad \forall\, r, t \in \mathcal{D},\ \nu \in [0,1].$$

Definition 1.3. *Let $\Omega : \mathcal{D} \to \mathbb{R}$ be called a g-convex function with respect to a strictly monotonic and continuous function g if it satisfies the following inequality:*

$$\Omega\left(g^{-1}\left(\nu g(r) + (1-\nu)g(t)\right)\right) \leq \nu\Omega(r) + (1-\nu)\Omega(t), \quad \forall\, r, t \in \mathcal{D},\ \nu \in [0,1].$$

Interval analysis, a specialized branch of set-valued analysis, uses mathematical techniques and concepts from general topology to examine set inequality properties. It is of broad significance in pure and applied sciences, especially in handling interval uncertainty in mathematical and computational models of real-life phenomena. Instead of point variables, it employs interval variables to enhance computational accuracy and minimize errors that could lead to incorrect conclusions. Originally developed to calculate error bounds for numerical solutions of finite state machines, interval analysis has now become a fundamental tool in mathematical and computational modeling.

Ramon E. Moore is considered to be the first person to publish a book [4] on interval analysis in 1966; it has since gained widespread use in engineering and scientific disciplines, particularly for managing uncertainty in structural systems. Beyond these applications, interval analysis plays a vital role in robotics [5], automatic error analysis [6], computer graphics [7], signal processing [8], scientific computing [9], optimization [10], and neural networks [11]. In the article [12], Wolfgang W. Breckner investigated the idea of set-valued convexity.

Definition 1.4. *A set-valued function Ω is said to be convex if it satisfies the following condition:*

$$\Omega\left(\nu\varepsilon_1 + (1-\nu)\varepsilon_2\right) \supset \nu\Omega(\varepsilon_1) + (1-\nu)\Omega(\varepsilon_2),$$

for all $\nu \in [0,1]$ and $\varepsilon_1, \varepsilon_2$ in its domain.

In [13], Elżbieta Sadowska extended H-H integral inequalities to the framework of set-valued functions, as detailed below.

Theorem 1.5. *Assume that Ω is a non-negative continuous convex set-valued function. Then*

$$\Omega\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\supset\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Omega(r)dr\supset\frac{\Omega(\varepsilon_1)+\Omega(\varepsilon_2)}{2}.$$

Let us take another look at the key concept that was analyzed by Moore et al. in [14].

Definition 1.6. *The* I.V *function $\Omega(\nu)=[\Omega_\star(\nu),\Omega^\star(\nu)]$ is defined for ν within I°, the interior of $I\subseteq\mathbb{R}$; it is considered Lebesgue integrable if both $\Omega_\star(\nu)$ and $\Omega^\star(\nu)$ are measurable and Lebesgue integrable over the real interval I°. Additionally, the* I.V *integral of $\int_{\varepsilon_1}^{\varepsilon_2}\Omega(\nu)d\nu$ is expressed as follows:*

$$\int_{\varepsilon_1}^{\varepsilon_2}\Omega(\nu)d\nu=\Big[\int_{\varepsilon_1}^{\varepsilon_2}\Omega_\star(\nu)d\nu,\int_{\varepsilon_1}^{\varepsilon_2}\Omega^\star(\nu)d\nu\Big].$$

Mathematics employs various tools and techniques to interpret physical and natural phenomena. However, since many of these phenomena involve dynamic processes, classical analysis often proves inadequate due to its inherent limitations. One of the most powerful approaches for studying, modeling, and advancing dynamic systems is fractional analysis, which traces its origins back to classical analysis. This field has significantly influenced not only mathematics but also various other disciplines through its effective applications [15,16]. Fractional analysis seeks to enhance mathematical modeling by introducing new fractional derivative and integral operators. Scholars who doubt that power laws are sufficient for modeling actual problems have developed fractional operators that contain the exponential function and its generalization in their kernels. The emerging operators vary in kernel structure, including singularity, locality, and general form. The tempered fractional integral operator is of tremendous worth and has received a significant amount of research interest, as it enjoys many practical advantages over numerous applications. The tempered fractional calculus is an extension of traditional fractional calculus. The tempered fractional integral was initially explored by Buschman [17], while Li et al. [18] and Meerschaert et al. [19] later provided a thorough explanation of the corresponding tempered fractional calculus as follows:

Definition 1.7. *Consider a real interval $[\varepsilon_1,\varepsilon_2]$ with $\chi>0$, $\kappa\geq0$. For a function $\Omega\in L[\varepsilon_1,\varepsilon_2]$, the left and right tempered fractional integral operators are given by:*

$${}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\varepsilon_2)=\frac{1}{\Gamma(\chi)}\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\nu)^{\chi-1}e^{-\kappa(\varepsilon_2-\nu)}\Omega(\nu)d\nu,$$

and

$${}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega(\varepsilon_1)=\frac{1}{\Gamma(\chi)}\int_{\varepsilon_1}^{\varepsilon_2}(\nu-\varepsilon_1)^{\chi-1}e^{-\kappa(\nu-\varepsilon_1)}\Omega(\nu)d\nu,$$

where $\Gamma(\chi)$ is the gamma function.

Let $\mathbb{R}_I$ denote the family of all real intervals and $\mathbb{R}_I^+$ the family of all positive intervals. Now, we present I.V tempered fractional integral operators.

Definition 1.8. *Let* $\Omega : [\varepsilon_1, \varepsilon_2] \subset \mathbb{R} \to \mathbb{R}_I$ *be an* I.V *function expressed as* $\Omega(\nu) = [\Omega_\star(\nu), \Omega^\star(\nu)]$ *where both* $\Omega_\star(\nu)$ *and* $\Omega^\star(\nu)$ *are Riemann integrable over the interval* $[\varepsilon_1, \varepsilon_2]$*. The associated* I.V *left-sided and right-sided tempered fractional integral operators for* Ω *are defined as follows:*

$$^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\varepsilon_2) = \frac{1}{\Gamma(\chi)} \int_{\varepsilon_1}^{\varepsilon_2} (\varepsilon_2 - \nu)^{\chi-1} e^{-\kappa(\varepsilon_2-\nu)} \Omega(\nu) d\nu,$$

and

$$^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega(\varepsilon_1) = \frac{1}{\Gamma(\chi)} \int_{\varepsilon_1}^{\varepsilon_2} (\nu - \varepsilon_1)^{\chi-1} e^{-\kappa(\nu-\varepsilon_1)} \Omega(\nu) d\nu,$$

where $\chi > 0$*,* $\kappa \geq 0$*, and* $\Gamma(\chi)$ *denotes the gamma function. It is clear that*

$$^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\varepsilon_2) = [^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega_\star(\varepsilon_2), {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^\star(\varepsilon_2)],$$

and

$$^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega(\varepsilon_1) = [^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega_\star(\varepsilon_1), {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega^\star(\varepsilon_1)].$$

In [20], Mohammed et al. defined the κ-incomplete gamma function, which is expressed as

Definition 1.9. *The* κ*-incomplete gamma function for the real numbers* $\chi > 0$ *and* $\kappa, r \geq 0$ *is defined as follows:*

$$\gamma_\kappa(\chi, r) = \int_0^r \nu^{\chi-1} e^{-\kappa\nu} d\nu.$$

It simplifies to the incomplete gamma function if $\kappa = 1$

$$\gamma(\chi, r) = \int_0^r \nu^{\chi-1} e^{-\nu} d\nu \quad \chi > 0.$$

The κ-incomplete gamma function satisfies the following properties:

Remark 1.10. *For the real numbers* $\chi > 0$ *and* $\kappa, r \geq 0$*, the following holds:*

$$(i) \quad \gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi, 1) = \int_0^1 \nu^{\chi-1} e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu} d\nu = \frac{1}{(\varepsilon_2 - \varepsilon_1)^\chi} \gamma_\kappa(\chi, \varepsilon_2 - \varepsilon_1).$$

$$(ii) \quad \int_0^1 \gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi, r) dr = \frac{\gamma_\kappa(\chi, \varepsilon_2 - \varepsilon_1)}{(\varepsilon_2 - \varepsilon_1)^\chi} - \frac{\gamma_\kappa(\chi + 1, \varepsilon_2 - \varepsilon_1)}{(\varepsilon_2 - \varepsilon_1)^{\chi+1}}.$$

In recent developments, numerous significant inequalities related to I.V functions have been introduced, including H-H and Ostrowski inequalities. Roman–Flores et al. [21], derived Minkowski's and Beckenbach's inequalities using the Kulisch–Miranker order for I.V functions. In [22], Zhao et al. defined the I.V approximately $\hbar$-convex functions and derived H-H inequalities for them via generalized fractional operators. Budak et al. [23] defined Riemann–Liouville fractional operators for I.V functions and, using these integrals, established H-H and related inequalities. By defining I.V generalized p-convex functions, Kamran et al. [24] were able to formulate the corresponding

H-H inequality. In [25], Shi et al. utilized Riemann–Liouville fractional integrals to prove H-H-type inequalities for I.V-coordinated functions. Sha et al. [26] introduced the I.V K-Riemann integral and K-gH-derivative and established the extended H-H inequalities. Alqudah et al. [27] introduced generalized modified $(p,\hbar)$-convex I.V functions, established H-H inequalities via k-Riemann–Liouville fractional integrals, and derived related inequalities. In [28], Chen et al. introduced I.V generalized $\eta_{\hbar}$-convex functions and established H-H, Jensen, and Ostrowski-type inequalities. Bin–Mohsin and associates [29] explored fractional reverse inequalities related to generic I.V convex functions and their applications.

Extensive research has been conducted on interval analysis, particularly in relation to convexity and various fractional operators. Motivated by these studies and their wide-ranging applications, this work seeks to establish a unified framework that connects existing convexity definitions within interval analysis. We introduced a framework for the I.V generalized $(g,\eta_{\hbar})$ class of convexity. We derive discrete Jensen-type and fractional versions of Minkowski and Hölder's inclusions. By incorporating this convexity concept alongside tempered fractional integral operators, we extend H-H-type, Fejér-H-H-type, and other fractional inclusions in a novel way. Our results are robust and adaptable, allowing for the retrieval of both new and existing inclusions by adjusting parameter values. Moreover, we illustrate that these results serve as natural extensions of the previous findings. To support our conclusion, we present a collection of insightful, non-trivial examples along with graphical representations. Finally, we discuss practical applications of our results in the context of special means.

2. Fractional calculus approach to reverse Minkowski and Hölder's inclusions

This section highlights our primary contribution, which is the demonstration of the reverse Minkowski and Hölder's inclusions through the tempered fractional integral operator. We begin by presenting the integral form of the fractional reverse Minkowski inclusion.

Theorem 2.1. *For $\chi > 0$, $\kappa \geq 0$, and $p \geq 1$, consider two* I.V *functions $\Omega, \varphi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}_I^+$ defined as $\Omega(\nu) = [\Omega_\star, \Omega^\star]$, and $\varphi(\nu) = [\varphi_\star, \varphi^\star]$, such that for all $\nu \geq \varepsilon_1$, the conditions ${}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^p(\nu) < \infty$ and ${}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^p(\nu) < \infty$ hold. If for all $u \in [\varepsilon_1, \varepsilon_2]$, the inequalities $0 < s \leq \frac{\Omega_\star(u)}{\varphi_\star(u)} \leq \mathcal{S}$ and $0 < s \leq \frac{\Omega^\star(u)}{\varphi^\star(u)} \leq \mathcal{S}$ are satisfied, then the following relation holds*

$$\left[\frac{1 + s(2 + \mathcal{S})}{(1 + s)(1 + \mathcal{S})}, \frac{1 + \mathcal{S}(2 + s)}{(1 + s)(1 + \mathcal{S})}\right]\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}(\Omega(\nu) + \varphi(\nu))^p\right]^{\frac{1}{p}} \supseteq \left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^p(\nu)\right]^{\frac{1}{p}} + \left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^p(\nu)\right]^{\frac{1}{p}}.$$

Proof. Since $s \leq \frac{\Omega_\star(u)}{\varphi_\star(u)}$, it follows that

$$s^p\,(\Omega_\star(u) + \varphi_\star(u))^p \leq (1 + s)^p \Omega_\star^p(u). \tag{2.1}$$

By multiplying both sides of (2.1) with $\frac{(\nu-u)^{\chi-1}e^{-\kappa(\nu-u)}}{\Gamma(\chi)}$ and integrating over the interval $[\varepsilon_1, \nu]$ with respect to u, we can express it as

$$\begin{aligned}
&\frac{s^p}{\Gamma(\chi)}\int_{\varepsilon_1}^{\nu}(\nu - u)^{\chi-1}e^{-\kappa(\nu-u)}\,(\Omega_\star(u) + \varphi_\star(u))^p\,du\\
&\leq \frac{(1 + s)^p}{\Gamma(\chi)}\int_{\varepsilon_1}^{\nu}(\nu - u)^{\chi-1}e^{-\kappa(\nu-u)}\Omega_\star^p(u)du.
\end{aligned}$$

By relating the above expression to the tempered fractional integral operator, we obtain

$$\frac{s}{1+s}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}(\Omega_\star(\nu)+\eta_\star(\nu))^p\right]^{\frac{1}{p}} \leq \left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^p_\star(\nu)\right]^{\frac{1}{p}}. \tag{2.2}$$

Furthermore, since $\frac{\Omega^\star(u)}{\varphi^\star(u)} \leq \mathcal{S}$, it follows that

$$(1+\mathcal{S})^p\Omega^{\star p}(u) \leq \mathcal{S}^p\left(\Omega^\star(u)+\varphi^\star(u)\right)^p. \tag{2.3}$$

By multiplying both sides of (2.3) by $\frac{(\nu-u)^{\chi-1}e^{-\kappa(\nu-u)}}{\Gamma(\chi)}$ and integrating over the interval $[\varepsilon_1, \nu]$ with respect to u, we can express it as

$$\left[{}^{\chi}\mathcal{I}^{\varphi}_{\varepsilon_1+}\Omega^{\star p}(\nu)\right]^{\frac{1}{p}} \leq \frac{\mathcal{S}}{1+\mathcal{S}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}(\Omega^\star(\nu)+\varphi^\star(\nu))^p\right]^{\frac{1}{p}}. \tag{2.4}$$

Moreover, from (2.2) and (2.4) we can infer that

$$\begin{aligned}
&\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^p(\nu)\right]^{\frac{1}{p}} \\
&= [\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^p_\star(\nu)\right)^{\frac{1}{p}}, \left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\star p}(\nu)\right)^{\frac{1}{p}}] \\
&\subseteq \left[\frac{s}{1+s}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}(\Omega_\star(\nu)+\varphi_\star(\nu))^p\right]^{\frac{1}{p}}, \frac{\mathcal{S}}{1+\mathcal{S}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}(\Omega^\star(\nu)+\varphi^\star(\nu))^p\right]^{\frac{1}{p}}\right] \\
&= \left[\frac{s}{1+s}, \frac{\mathcal{S}}{1+\mathcal{S}}\right]\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\left(\Omega(\nu)+\varphi(\nu)\right)^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{2.5}$$

Since $s \leq \frac{\Omega^\star(u)}{\varphi^\star(u)}$, it follows that

$$(1+s)^p\varphi^{\star p}(u) \leq \left(\Omega^\star(u)+\varphi^\star(u)\right)^p. \tag{2.6}$$

By multiplying both sides of (2.6) by $\frac{(\nu-u)^{\chi-1}e^{-\kappa(\nu-u)}}{\Gamma(\chi)}$ and integrating over the interval $[\varepsilon_1, \nu]$ with respect to u, we can express it as

$$\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^{\star p}(\nu)\right]^{\frac{1}{p}} \leq \frac{1}{1+s}\left[{}^{\chi}\mathcal{I}^{\varphi}_{\varepsilon_1+}\left(\Omega^\star(\nu)+\varphi^\star(\nu)\right)^p\right]^{\frac{1}{p}}. \tag{2.7}$$

Now, given that $\frac{\Omega_\star(u)}{\eta_\star(u)} \leq \mathcal{S}$, we obtain that

$$(\Omega_\star(u)+\eta_\star(u))^p \leq (1+\mathcal{S})^p\varphi^p_\star(u). \tag{2.8}$$

By multiplying both sides of (2.8) by $\frac{(\nu-u)^{\chi-1}e^{-\kappa(\nu-u)}}{\Gamma(\chi)}$ and integrating over the interval $[\varepsilon_1, \nu]$ with respect to u, we can express it as

$$\frac{1}{1+\mathcal{S}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\left(\Omega_\star(\nu)+\varphi_\star(\nu)\right)^p\right]^{\frac{1}{p}} \leq \left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^p_\star(\nu)\right]^{\frac{1}{p}}. \tag{2.9}$$

From (2.7) and (2.9), we deduce that

$$\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^p(\nu)\right]^{\frac{1}{p}}$$

$$= [\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^{p}_{\star}(\nu)\right)^{\frac{1}{p}}, \left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\eta^{\star p}(\nu)\right)^{\frac{1}{p}}]$$
$$\subseteq \left[\frac{1}{1+\mathcal{S}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\left(\Omega_{\star}(\nu)+\varphi_{\star}(\nu)\right)^{p}\right]^{\frac{1}{p}}, \frac{1}{1+s}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\left(\Omega^{\star}(\nu)+\varphi^{\star}(\nu)\right)^{p}\right]^{\frac{1}{p}} \right]$$
$$= \left[\frac{1}{1+\mathcal{S}}, \frac{1}{1+s}\right]\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\left(\Omega(\nu)+\varphi(\nu)\right)^{p}\right]^{\frac{1}{p}}. \tag{2.10}$$

Thus, the proof is completed by adding (2.5) and (2.10). □

We now present the integral form of the fractional reverse Hölder's inclusion.

Theorem 2.2. *For $\chi > 0$ and $\kappa \geq 0$ with $p > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Consider two* I.V *functions $\Omega, \varphi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}^{+}_{I}$ defined as $\Omega(\nu) = [\Omega\star, \Omega^{\star}]$ and $\varphi(\nu) = [\varphi_{\star}, \varphi^{\star}]$. Suppose that for all $\nu > \varepsilon_1$, the conditions ${}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon 1+}\Omega^{p}(\nu) < \infty$ and ${}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon 1+}\varphi^{p}(\nu) < \infty$ hold. If for all $u \in [\varepsilon_1, \varepsilon_2]$, the inequalities $0 < s \leq \frac{\Omega_{\star}(u)}{\varphi_{\star}(u)} \leq \mathcal{S}$ and $0 < s \leq \frac{\Omega^{\star}(u)}{\varphi^{\star}(u)} \leq \mathcal{S}$ are satisfied, then the following relation holds:*

$$\left[\left(\frac{s}{\mathcal{S}}\right)^{\frac{1}{pq}}, \left(\frac{\mathcal{S}}{s}\right)^{\frac{1}{pq}} \right]\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\frac{1}{p}}(\nu)\varphi^{\frac{1}{q}}(\nu)\right] \supseteq \left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\nu)\right]^{\frac{1}{p}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi(\nu)\right]^{\frac{1}{q}}.$$

Proof. Since $s \leq \frac{\Omega_{\star}(u)}{\varphi_{\star}(u)}$, it follows that

$$\Omega_{\star}(u) \geq s^{\frac{1}{q}}\Omega_{\star}^{\frac{1}{p}}(u)\varphi_{\star}^{\frac{1}{q}}(u). \tag{2.11}$$

By multiplying both sides of (2.11) by $\frac{(\nu-u)^{\chi-1}e^{-\kappa(\nu-u)}}{\Gamma(\chi)}$ and integrating over the interval $[\varepsilon_1, \nu]$ with respect to u, we can express it as

$$\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega_{\star}(\nu)\right)^{\frac{1}{p}} \geq s^{\frac{1}{pq}}\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega_{\star}^{\frac{1}{p}}(\nu)\varphi_{\star}^{\frac{1}{q}}(\nu)\right)^{\frac{1}{p}}. \tag{2.12}$$

Now, from $\frac{\Omega^{\star}(u)}{\varphi^{\star}(u)} \leq \mathcal{S}$, we have

$$\Omega^{\star}(u) \leq \mathcal{S}^{\frac{1}{q}}\Omega^{\star\frac{1}{p}}(u)\varphi^{\star\frac{1}{q}}(u). \tag{2.13}$$

By multiplying both sides of (2.13) by $\frac{(\nu-u)^{\chi-1}e^{-\kappa(\nu-u)}}{\Gamma(\chi)}$ and integrating over the interval $[\varepsilon_1, \nu]$ with respect to u, then we obtain

$$\left({}^{\chi}\mathcal{I}^{\varepsilon}_{\varepsilon_1+}\Omega^{\star}(\nu)\right)^{\frac{1}{p}} \leq \mathcal{S}^{\frac{1}{pq}}\left({}^{\chi}\mathfrak{I}^{\kappa}_{\varepsilon_1+}\Omega^{\star\frac{1}{p}}(\nu)\varphi^{\star\frac{1}{q}}(\nu)\right)^{\frac{1}{p}}. \tag{2.14}$$

By combining (2.12) and (2.14), we acquire

$$\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\nu)\right)^{\frac{1}{p}}$$
$$= \left[\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega_{\star}(\nu)\right)^{\frac{1}{p}}, \left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\star}(\nu)\right)^{\frac{1}{p}} \right]$$
$$\subseteq \left[s^{\frac{1}{pq}}\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega_{\star}^{\frac{1}{p}}(\nu)\varphi_{\star}^{\frac{1}{q}}(\nu)\right)^{\frac{1}{p}}, \mathcal{S}^{\frac{1}{pq}}\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\star\frac{1}{p}}(\nu)\varphi^{\star\frac{1}{q}}(\nu)\right)^{\frac{1}{p}} \right]$$

$$= \left[s^{\frac{1}{pq}}, \mathcal{S}^{\frac{1}{pq}}\right] \left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\frac{1}{p}}(\nu)\varphi^{\frac{1}{q}}(\nu)\right)^{\frac{1}{p}}. \quad (2.15)$$

Using identical procedures, we obtain the following bounds for $\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi(\nu)\right)^{\frac{1}{q}}$, as follows

$$\begin{aligned}
&\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi(\nu)\right)^{\frac{1}{q}} \\
&= \left[\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi_{\star}(\nu)\right)^{\frac{1}{q}}, \left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\varphi^{\star}(\nu)\right)^{\frac{1}{q}}\right] \\
&\subseteq \left[\frac{1}{\mathcal{S}^{\frac{1}{pq}}}\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega_{\star}^{\frac{1}{p}}(\nu)\varphi_{\star}^{\frac{1}{q}}(\nu)\right)^{\frac{1}{q}}, \frac{1}{s^{\frac{1}{pq}}}\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\star\frac{1}{p}}(\nu)\varphi^{\star\frac{1}{q}}(\nu)\right)^{\frac{1}{q}}\right] \\
&= \left[\frac{1}{\mathcal{S}^{\frac{1}{pq}}}, \frac{1}{s^{\frac{1}{pq}}}\right]\left({}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega^{\frac{1}{p}}(\nu)\eta^{\frac{1}{q}}(\nu)\right)^{\frac{1}{q}}. \quad (2.16)
\end{aligned}$$

Then, from (2.15) and (2.16), we obtain the required result. □

3. Interval-valued generalized $(g, \eta_\hbar)$ convex functions and their related findings

Let us now present a novel concept, the I.V generalized $(g, \eta_\hbar)$ class of convex functions, which extends several existing convexity classes.

Definition 3.1. *Consider a function $\hbar : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ that is non-negative and satisfies $(0,1) \subseteq [\varepsilon_1, \varepsilon_2]$ with $\hbar \neq 0$. Additionally, let $\eta : J_1 \times J_1 \to J_2$ be a bifunction for appropriate $J_1, J_2 \subseteq \mathbb{R}$. A function $\Omega : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}_I^+$ is called an* I.V *generalized $(g, \eta_\hbar)$ convex function if it satisfies the condition $\Omega(r) = [\Omega_\star(r), \Omega^\star(r)]$ and*

$$\Omega\left((\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right) \supseteq \Omega(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega(\varepsilon_1), \Omega(\varepsilon_2)\right), \quad (3.1)$$

$\forall\, r, t \in [\varepsilon_1, \varepsilon_2]$ and $\nu \in [0, 1]$.

Remark 3.2. *The fourteen special cases of Definition 3.1, which establish its connection to existing literature, are outlined as follows:*

(i) By choosing $g(r) = r$, Definition 3.1 simplifies to the definition of I.V *generalized $\eta_\hbar$-convex function given in [28, Definition 9], that is,*

$$\Omega\left(\nu\varepsilon_1 + (1-\nu)\varepsilon_2\right) \supseteq \Omega(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega(\varepsilon_1), \Omega(\varepsilon_2)\right).$$

(ii) Setting $g(r) = r^p$, Definition 3.1 simplifies to the definition of I.V *generalized modified (p,ħ) convex function given in [27, Defintion 4.6], that is,*

$$\Omega\left(\left(\nu\varepsilon_1^p + (1-\nu)\varepsilon_2^p\right)^{\frac{1}{p}}\right) \supseteq \Omega(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega(\varepsilon_1), \Omega(\varepsilon_2)\right).$$

(iii) Setting $g(r) = r^p$ and $\hbar(\nu) = \nu$, Definition 3.1 simplifies to the definition of the I.V *generalized p-convex function given in [24, Definition 6], that is,*

$$\Omega\left(\left(\nu\varepsilon_1^p + (1-\nu)\varepsilon_2^p\right)^{\frac{1}{p}}\right) \supseteq \Omega(\varepsilon_2) + \nu\eta\left(\Omega(\varepsilon_1), \Omega(\varepsilon_2)\right).$$

(iv) *By choosing* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *Definition 3.1 reduces to the definition of generic modified* $\hbar$*-convexity for* I.V *functions, which is given by*

$$\Omega\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \supseteq \hbar(\nu)\Omega(\varepsilon_1) + (1-\hbar(\nu))\Omega(\varepsilon_2).$$

(v) *By choosing* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ *and* $g(r) = r$, *Definition 3.1 reduces to the definition of modified* $\hbar$*-convexity for* I.V *function, which is given by*

$$\Omega(\nu\varepsilon_1 + (1-\nu)\varepsilon_2) \supseteq \hbar(\nu)\Omega(\varepsilon_1) + (1-\hbar(\nu))\Omega(\varepsilon_2).$$

(vi) *By choosing* $\hbar(\nu) = \nu$, *Definition 3.1 simplifies to the definition of* I.V *generalized* (g, η) *convex function, which is expressed as*

$$\Omega\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \supseteq \Omega(\varepsilon_2) + \nu\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)).$$

(vii) *By choosing* $g(r) = r$ *and* $\hbar(\nu) = \nu$, *Definition 3.1 reduces to the definition of* I.V *generalized* η*-convex function which is expressed as*

$$\Omega(\nu\varepsilon_1 + (1-\nu)\varepsilon_2) \supseteq \Omega(\varepsilon_2) + \nu\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)).$$

(viii) *By choosing* $\hbar(\nu) = \nu^s$, $s \in (0,1)$, $g(r) = r$ *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *Definition 3.1 simplifies to the definition of the* s*-convex* I.V *function of the first kind, which is given by*

$$\Omega(\nu\varepsilon_1 + (1-\nu)\varepsilon_2) \supseteq \nu^s\Omega(\varepsilon_1) + (1-\nu^s)\Omega(\varepsilon_2).$$

(ix) *By choosing* $\hbar(\nu) = \nu$, $g(r) = r$, *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *Definition 3.1 simplifies to the definition of* I.V *convex function, which is given by*

$$\Omega(\nu\varepsilon_1 + (1-\nu)\varepsilon_2) \supseteq \nu\Omega(\varepsilon_1) + (1-\nu)\Omega(\varepsilon_2).$$

(x) *By choosing* $g(r) = \frac{1}{r}$, *Definition 3.1 simplifies to the definition of a generalized harmonically* $\eta_\hbar$*-convex* I.V *function, which is expressed as*

$$\Omega\left(\frac{\varepsilon_1\varepsilon_2}{\nu\varepsilon_2 + (1-\nu)\varepsilon_1}\right) \supseteq \Omega(\varepsilon_2) + \hbar(\nu)\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)).$$

(xi) *By choosing* $g(r) = \frac{1}{r}$ *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *Definition 3.1 simplifies to the definition of a harmonically modified* $\hbar$*-convex* I.V *function, which is expressed as*

$$\Omega\left(\frac{\varepsilon_1\varepsilon_2}{\nu\varepsilon_2 + (1-\nu)\varepsilon_1}\right) \supseteq \hbar(\nu)\Omega(\varepsilon_1) + (1-\hbar(\nu))\Omega(\varepsilon_2).$$

(xiii) *By choosing* $g(r) = \frac{1}{r}$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\hbar(\nu) = \nu^s$ $s \in (0,1)$, *Definition 3.1 simplifies to the definition of the harmonically* s*-convex* I.V *function of the first kind, which is given by*

$$\Omega\left(\frac{\varepsilon_1\varepsilon_2}{\nu\varepsilon_2 + (1-\nu)\varepsilon_1}\right) \supseteq \nu^s\Omega(\varepsilon_1) + (1-\nu^s)\Omega(\varepsilon_2).$$

(xiv) *By choosing* $g(r) = \frac{1}{r}$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\hbar(\nu) = \nu$, *Definition 3.1 simplifies to the definition of* I.V *harmonically convex function, which is given by*

$$\Omega\left(\frac{\varepsilon_1\varepsilon_2}{\nu\varepsilon_2 + (1-\nu)\varepsilon_1}\right) \supseteq \nu\Omega(\varepsilon_1) + (1-\nu)\Omega(\varepsilon_2).$$

For convenience, we use the notations $SIGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R}_I^+)$, $SIGV((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R}_I^+)$, $SGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$, and $SGV((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$ to represent the family of I.V generalized $(g,\eta_\hbar)$ convex functions, I.V generalized $(g,\eta_\hbar)$ concave functions, generalized $(g,\eta_\hbar)$ convex functions, and generalized $(g,\eta_\hbar)$ concave functions, respectively.

Theorem 3.3. *Suppose* $\Omega : [\varepsilon_1,\varepsilon_2] \to \mathbb{R}_I^+$ *be an* I.V *function defined as* $\Omega = [\Omega_\star, \Omega^\star]$ *where* $\Omega_\star \leq \Omega^\star$. *Then,* $\Omega \in SIGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R}_I^+) \Leftrightarrow \Omega_\star \in SGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$ *and* $\Omega^\star \in SGV((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$.

Proof. Suppose that $\Omega \in SIGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R}_I^+)$, $r,t \in [\varepsilon_1,\varepsilon_2]$, and $\nu \in [0,1]$, then we have

$$\Omega\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \supseteq \Omega(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega(\varepsilon_1), \Omega(\varepsilon_2)\right).$$

This implies that

$$\begin{aligned} &[\Omega_\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right), \Omega^\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right)] \\ &\supseteq [\Omega_\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega_\star(\varepsilon_1), \Omega_\star(\varepsilon_2)\right), \Omega^\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega^\star(\varepsilon_1), \Omega^\star(\varepsilon_2)\right)]. \end{aligned} \tag{3.2}$$

From (3.2), we have

$$\Omega_\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \leq \Omega_\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega_\star(\varepsilon_1), \Omega_\star(\varepsilon_2)\right), \tag{3.3}$$

and

$$\Omega^\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \geq \Omega^\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega^\star(\varepsilon_1), \Omega^\star(\varepsilon_2)\right). \tag{3.4}$$

From inequalities (3.3) and (3.4), it indicates that $\Omega_\star \in SGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$ and $\Omega^\star \in SGV((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$.
Conversely, suppose that $\Omega_\star \in SGX((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$ and $\Omega^\star \in SGV((g,\eta_\hbar),[\varepsilon_1,\varepsilon_2],\mathbb{R})$. Then we have

$$\Omega_\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \leq \Omega_\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega_\star(\varepsilon_1), \Omega_\star(\varepsilon_2)\right),$$

and

$$\Omega^\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) \geq \Omega^\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega^\star(\varepsilon_1), \Omega^\star(\varepsilon_2)\right).$$

This implies that

$$\begin{aligned} &[\Omega_\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right), \Omega^\star\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right)] \\ &\supseteq [\Omega_\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega_\star(\varepsilon_1), \Omega_\star(\varepsilon_2)\right), \Omega^\star(\varepsilon_2) + \hbar(\nu)\eta\left(\Omega^\star(\varepsilon_1), \Omega^\star(\varepsilon_2)\right)]. \end{aligned} \tag{3.5}$$

Thus, the result is obtained. □

4. A discrete Jensen-type inclusion for interval-valued generalized $(g, \eta_\hbar)$ convex functions

Now, we derive a discrete Jensen-type inclusion for I.V generalized $(g, \eta_\hbar)$ convexity.

Theorem 4.1. *Let $\Omega \in SIGX((g, \eta_\hbar), [\varepsilon_1, \varepsilon_2], \mathbb{R}_I^+)$ and assume that the function η is nonnegative, nondecreasing, and sublinear in the first variable. Suppose we define $L_z = \sum_{z=1}^{m} \beta_z$ for $z = 1, ..., m$ with $L_m = 1$. Then, the following inclusion holds*

$$\Omega\left(g^{-1}\left(\sum_{z=1}^{m} \alpha_z g(r_z)\right)\right) \supseteq \Omega(r_m) + \left(\sum_{z=1}^{m-1} \hbar(L_z)\eta_\Omega(r_z, r_{z+1}, ..., r_m)\right), \tag{4.1}$$

where $\eta_\Omega(r_z, r_{z+1}, ..., r_m) = \eta(\eta_\Omega(r_z, r_{z+1}, ..., r_m), \Omega(r_m))$ and for all $r \in [\varepsilon_1, \varepsilon_2]$, $\eta_\Omega(r) = \Omega(r)$.

Proof. Given that η possesses the properties of being nonnegative, nondecreasing, and sublinear in the first variable, we deduce from (4.1) that

$$\Omega\Big(g^{-1}\Big(\sum_{z=1}^{m} \beta_z g(r_z)\Big)\Big) \supseteq \Omega(r_m) + \hbar(L_{m-1})\eta\Big(\Omega\Big(g^{-1}\Big(\sum_{z=1}^{m-1} \frac{\beta_z}{L_{m-1}} g(r_z)\Big)\Big), \Omega(r_m)\Big)$$

$$\left[\Omega_\star\Big(g^{-1}\Big(\sum_{z=1}^{m} \beta_z g(r_z)\Big)\Big), \Omega^\star\Big(g^{-1}\Big(\sum_{z=1}^{m} \beta_z g(r_z)\Big)\Big)\right]$$

$$\supseteq\Big[\Omega_\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega_\star\Big(g^{-1}\Big(\sum_{z=1}^{m-1} \frac{\beta_z}{L_{m-1}} g(r_z)\Big)\Big), \Omega_\star(r_m)\Big),$$

$$\Omega^\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega^\star\Big(g^{-1}\Big(\sum_{z=1}^{m-1} \frac{\beta_z}{L_{m-1}} g(r_z)\Big)\Big), \Omega^\star(r_m)\Big)\Big]$$

$$=\Big[\Omega_\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega_\star\Big(g^{-1}\Big(\frac{L_{m-2}}{L_{m-1}} \sum_{z=1}^{m-2} \frac{\beta_z}{L_{m-2}} g(r_z) + \frac{\beta_{m-1}}{L_{m-1}} g(r_{m-1})\Big)\Big), \Omega_\star(r_m)\Big),$$

$$\Omega^\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega^\star\Big(g^{-1}\Big(\frac{L_{m-2}}{L_{m-1}} \sum_{z=1}^{m-2} \frac{\beta_z}{L_{m-2}} g(r_z) + \frac{\beta_{m-1}}{L_{m-1}} g(r_{m-1})\Big)\Big), \Omega^\star(r_m)\Big)\Big]$$

$$\supseteq\Big[\Omega_\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega_\star(r_{m-1}) + \hbar\left(\frac{L_{m-2}}{L_{m-1}}\right)$$

$$\times \eta\Big(\Omega_\star\Big(g^{-1}\Big(\sum_{z=1}^{m-2} \frac{\beta_z}{l_{m-2}} g(r_z)\Big)\Big), \Omega_\star(r_{m-1})\Big), \Omega_\star(r_m)\Big),$$

$$\Omega^\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega^\star(r_{m-1}) + \hbar\left(\frac{L_{m-2}}{L_{m-1}}\right)$$

$$\times \eta\Big(\Omega^\star\Big(g^{-1}\Big(\sum_{z=1}^{m-2} \frac{\alpha_z}{L_{m-2}} g(r_z)\Big)\Big), \Omega^\star(r_{m-1})\Big), \Omega^\star(r_m)\Big)\Big]$$

$$\supseteq\Big[\Omega_\star(r_m) + \hbar(L_{m-1})\eta\Big(\Omega_\star(r_{m-1}), \Omega_\star(r_m)\Big)$$

$$+ \hbar(L_{m-2})\eta\Big(\eta\Big(\Omega_\star\Big(g^{-1}\Big(\sum_{z=1}^{m-2} \frac{\beta_z}{L_{m-2}} g(r_z)\Big)\Big), \Omega_\star(r_{m-1})\Big), \Omega_\star(r_m)\Big),$$

$$
\begin{aligned}
&\Omega^{\star}(r_m) + \hbar(L_{m-1})\eta\Big(\Omega^{\star}(r_{m-1}), \Omega^{\star}(r_m)\Big)\\
&\quad + \hbar(L_{m-2})\eta\Big(\eta\Big(\Omega^{\star}\Big(g^{-1}\Big(\sum_{z=1}^{m-2}\frac{\beta_z}{L_{m-2}}g(r_z)\Big)\Big), \Omega^{\star}(r_{m-1})\Big), \Omega^{\star}(r_m)\Big)\Big]\\
&\supseteq ....\\
&\supseteq\Big[\Omega_{\star}(r_m) + \hbar(L_{m-1})\eta(\Omega_{\star}(r_{m-1}), \Omega_{\star}(r_m)) + \hbar(L_{m-2})\eta(\eta(\Omega_{\star}(r_{m-2}), \Omega_{\star}(r_{m-1})), \Omega_{\star}(r_m))\\
&\quad + ... + \hbar(L_1)\eta(\eta(...\Omega_{\star}(r_1), \Omega_{\star}(r_2)), \Omega_{\star}(r_3)), ..., \Omega_{\star}(r_{m-1})), \Omega_{\star}(r_m)),\\
&\quad \Omega^{\star}(r_m) + \hbar(L_{m-1})\eta(\Omega^{\star}(r_{m-1}), \Omega^{\star}(r_m)) + \hbar(L_{m-2})\eta(\eta(\Omega^{\star}(r_{m-2}), \Omega^{\star}(r_{m-1})), \Omega^{\star}(r_m))\\
&\quad + ... + \hbar(L_1)\eta(\eta(...\Omega^{\star}(r_1), \Omega^{\star}(r_2)), \Omega^{\star}(r_3)), ..., \Omega^{\star}(r_{m-1})), \Omega^{\star}(r_m))\Big]\\
&=\Big[\Omega_{\star}(r_m) + \hbar(L_{m-1})\eta_{\Omega_{\star}}(r_{m-1}, r_m) + \hbar(L_{m-2})\eta_{\Omega_{\star}}(r_{m-2}, r_{m-1}, r_m)\\
&\quad + ... + \hbar(L_1)\eta_{\Omega_{\star}}(r_1, r_2, ..., r_{m-1}, r_m),\\
&\quad \Omega^{\star}(r_m) + \hbar(L_{m-1})\eta_{\Omega^{\star}}(r_{m-1}, r_m) + \hbar(L_{m-2})\eta_{\Omega^{\star}}(r_{m-2}, r_{m-1}, r_m)\\
&\quad + ... + \hbar(L_1)\eta_{\Omega^{\star}}(r_1, r_2, ..., r_{m-1}, r_m)\Big]\\
&=\Omega(r_m) + \sum_{z=1}^{m-1}\hbar(L_z)\eta_{\Omega}(r_z, r_{z+1}, ..., r_m).
\end{aligned}
$$

Hence, the result is obtained. □

Remark 4.2. *By choosing $g(r) = r$ in Theorem 4.1, we obtain a Jensen-type inclusion for* I.V *generalized $\eta_\hbar$ convexity, as given in [28, Theorem 6].*

5. Inclusions involving tempered fractional integral operators and interval-valued generalized $(g, \eta_\hbar)$ convex functions

This section emphasizes our primary contributions in deriving H-H-type, Fejér-H-H-type, and other fractional inclusions through the tempered fractional integral operators and generalized $(g, \eta_\hbar)$ I.V convexity. To set the stage, we first introduce a unified I.V H-H-type inclusion presented in the following theorem.

Theorem 5.1. *Suppose $\Omega \in SIGX((g, \eta_\hbar), [\varepsilon_1, \varepsilon_2], \mathbb{R}_I^+)$, then the following relation holds*

$$
\begin{aligned}
&\Omega\Big(g^{-1}\Big(\frac{g(\varepsilon_1) + g(\varepsilon_2)}{2}\Big)\Big) - \mathfrak{Q}\\
&\supseteq \frac{\Gamma(\chi)}{2\,(g(\varepsilon_2) - g(\varepsilon_1))^{\chi}\,\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi, 1)}\\
&\times \Big[{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\circ g^{-1}(g(\varepsilon_1))\Big]\\
&\supseteq \frac{\Omega(\varepsilon_1) + \Omega(\varepsilon_2)}{2} + \frac{\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) + \eta(\Omega(\varepsilon_2), \Omega(\varepsilon_1))}{2\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi, 1)}\\
&\times \int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\hbar(\nu)d\nu,
\end{aligned}
$$

where

$$\mathfrak{Q} = \frac{\hbar(\frac{1}{2})}{2\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi,1)}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}$$
$$\Big\{\eta\Big(\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))),\Omega(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)))\Big)$$
$$+\eta\Big(\Omega(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))),\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))\Big)\Big\}\,d\nu,$$

and $\chi > 0$*,* $\kappa \geq 0$*, and* $\forall\, r,t \in [\varepsilon_1,\varepsilon_2]$.

Proof. Since Ω is I.V generalized $(g,\eta_\hbar)$ convex function, it follows that for $\nu = \frac{1}{2}$, we obtain

$$\Omega\left(g^{-1}\left(\frac{g(r)+g(t)}{2}\right)\right) \supseteq \Omega(t) + \hbar\left(\frac{1}{2}\right)\eta\Big(\Omega(r),\Omega(t)\Big), \tag{5.1}$$

for all $r,t \in [\varepsilon_1,\varepsilon_2]$. By selecting $r = g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))$ and $t = g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))$ in (5.1), we obtain

$$\begin{aligned}\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right) \supseteq{}& \Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\\ &+\hbar\left(\frac{1}{2}\right)\eta\Big(\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right),\\ &\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Big).\end{aligned} \tag{5.2}$$

By selecting $r = g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))$ and $t = g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))$ in (5.1), we have

$$\begin{aligned}\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right) \supseteq{}& \Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\\ &+\hbar\left(\frac{1}{2}\right)\eta\Big(\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1))+\nu g(\varepsilon_2))\right),\\ &\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\Big).\end{aligned} \tag{5.3}$$

Add (5.2) and (5.3), then multiplying the result by $\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}$ and integrating from 0 to 1 with respect to ν, we obtain

$$\begin{aligned}&\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu - \frac{\hbar\left(\frac{1}{2}\right)}{2}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\\ &\times\Big\{\eta\Big(\Omega(g^{-1}((1-\nu)g(\varepsilon_1))+\nu g(\varepsilon_2)),\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))\Big)\\ &+\eta\Big(\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right),\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Big)\Big\}\,d\nu\\ &\supseteq \frac{1}{2}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Big\{\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\\ &+\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Big\}\,d\nu.\end{aligned} \tag{5.4}$$

Now

$$\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu$$
$$= \Big[\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega_\star\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu,$$
$$\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega^\star\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu\Big]$$
$$= \Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi,1). \tag{5.5}$$

Also

$$\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\left\{\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\right.$$
$$\left.+\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\right\}d\nu$$
$$= \Big[\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega_\star\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)d\nu$$
$$+\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega_\star\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)d\nu,$$
$$\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega^\star\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)d\nu$$
$$+\int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega^\star\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)d\nu\Big]$$
$$= \Big[\frac{1}{(g(\varepsilon_2)-g(\varepsilon_1))^\chi}\int_{g(\varepsilon_1)}^{g(\varepsilon_2)}(g(\varepsilon_2)-y)^{\chi-1}e^{-\kappa(g(\varepsilon_2)-y)}\Omega_\star(g^{-1}(y))dy$$
$$+\frac{1}{(g(\varepsilon_2)-g(\varepsilon_1))^\chi}\int_{g(\varepsilon_1)}^{g(\varepsilon_2)}(y-g(\varepsilon_1))^{\chi-1}e^{-\kappa(y-g(\varepsilon_1))}\Omega_\star(g^{-1}(y))dy,$$
$$\frac{1}{(g(\varepsilon_2)-g(\varepsilon_1))^\chi}\int_{g(\varepsilon_1)}^{g(\varepsilon_2)}(g(\varepsilon_2)-y)^{\chi-1}e^{-\kappa(g(\varepsilon_2)-y)}\Omega^\star(g^{-1}(y))dy$$
$$+\frac{1}{(g(\varepsilon_2)-g(\varepsilon_1))^\chi}\int_{g(\varepsilon_1)}^{g(\varepsilon_2)}(y-g(\varepsilon_1))^{\chi-1}e^{-\kappa(y-g(\varepsilon_1))}\Omega^\star(g^{-1}(y))dy\Big]$$
$$= \frac{\Gamma(\chi)}{(g(\varepsilon_2)-g(\varepsilon_1))^\chi}\Big[{}^\chi\mathcal{I}^\kappa_{g(\varepsilon_1)+}\Omega\circ g^{-1}(g(\varepsilon_2))+{}^\chi\mathcal{I}^\kappa_{g(\varepsilon_2)-}\Omega\circ g^{-1}(g(\varepsilon_1))\Big]. \tag{5.6}$$

Now, by using (5.5) and (5.6) in (5.4), we obtain

$$\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi,1)-\frac{\hbar(\frac{1}{2})}{2}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}$$
$$\times\Big\{\eta\Big(\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1))+\nu g(\varepsilon_2)\right),\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\Big)$$
$$+\eta\Big(\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))),\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Big)\Big\}d\nu$$

$$\supseteq \frac{\Gamma(\chi)}{2(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}\Big[{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\circ g^{-1}(g(\varepsilon_1))\Big].$$

Now, by applying the definition of I.V generalized $(g,\eta_\hbar)$ convex function

$$\Omega\left(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)\right)\supseteq \Omega(\varepsilon_2)+\hbar(\nu)\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2)), \tag{5.7}$$

and

$$\Omega\left(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)\right)\supseteq \Omega(\varepsilon_1)+\hbar(\nu)\eta(\Omega(\varepsilon_2),\Omega(\varepsilon_1)). \tag{5.8}$$

Adding (5.7) and (5.8), we get

$$\begin{aligned}&\Omega\left(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)\right)+\Omega\left(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)\right)\\&\supseteq \Omega(\varepsilon_1)+\Omega(\varepsilon_2)+\hbar(\nu)\{\eta(\Omega(\varepsilon_2),\Omega(\varepsilon_1))+\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2))\}.\end{aligned} \tag{5.9}$$

By multiplying both sides of (5.9) by $\frac{\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}}{2}$ and integrating with respect to ν over the interval [0,1], the desired relation is obtained. □

Corollary 5.2. *Now, we present innovative findings in connection with Theorem 5.1.*

(i) By choosing $g(r)=r$ and $\hbar(\nu)=\nu$ in Theorem 5.1, we get

$$\begin{aligned}&\Omega\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)-\frac{1}{4\gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi,1)}\\&\times\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}\Big\{\eta\Big(\Omega(\nu\varepsilon_1+(1-\nu)\varepsilon_2),\Omega((1-\nu)\varepsilon_1+\nu\varepsilon_2)\Big)\\&+\eta\Big(\Omega((1-\nu)\varepsilon_1+\nu\varepsilon_2),\Omega(\nu\varepsilon_1+(1-\nu)\varepsilon_2)\Big)\Big\}d\nu\\&\supseteq \frac{\Gamma(\chi)}{2(\varepsilon_2-\varepsilon_1)^{\chi}\gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi,1)}\Big[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\varepsilon_2)+{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega(\varepsilon_1)\Big]\\&\supseteq \frac{\Omega(\varepsilon_1)+\Omega(\varepsilon_2)}{2}+\frac{\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2))+\eta(\Omega(\varepsilon_2),\Omega(\varepsilon_1))\gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi+1,1)}{2\gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi,1)}.\end{aligned}$$

(ii) By choosing $g(r)=r$, $\hbar(\nu)=\nu$, and $\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2))=\Omega(\varepsilon_1)-\Omega(\varepsilon_2)$ in Theorem 5.1, we obtain

$$\begin{aligned}&\Omega\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\supseteq \frac{\Gamma(\chi)}{2(\varepsilon_2-\varepsilon_1)^{\chi}\gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi,1)}\Big[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega(\varepsilon_2)+{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega(\varepsilon_1)\Big]\\&\supseteq \frac{\Omega(\varepsilon_1)+\Omega(\varepsilon_2)}{2}.\end{aligned}$$

(iii) By choosing $g(r)=r^p$, $p\geq -1$ and $\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2))=\Omega(\varepsilon_1)-\Omega(\varepsilon_2)$ in Theorem 5.1, we obtain

$$\begin{aligned}&\Omega\left(\left(\frac{\varepsilon_1^p+\varepsilon_2^p}{2}\right)^{\frac{1}{p}}\right)\supseteq \frac{\Gamma(\chi)}{2\left(\varepsilon_2^p-\varepsilon_1^p\right)^{\chi}\gamma_{\kappa(\varepsilon_2^p-\varepsilon_1^p)}(\chi,1)}\Big[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1^p+}\Omega\circ\mathcal{K}(\varepsilon_2^p)+{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2^p-}\Omega\circ\mathcal{K}(\varepsilon_1^p)\Big]\\&\supseteq \frac{\Omega(\varepsilon_1)+\Omega(\varepsilon_2)}{2},\end{aligned}$$

where $\mathcal{K}(r)=r^{\frac{1}{p}}$.

Remark 5.3. *We now examine Theorem 5.1 in the context of related work in the literature.*

(i) By selecting $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, and $\kappa = 0$, Theorem 5.1 simplifies to [29, Theorem 6, Case (i)].

(ii) By selecting $g(r) = r$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, and $\kappa = 0$, Theorem 5.1 simplifies to [23, Theorem 3.4].

(iii) By selecting $g(r) = r^p$, $p \geq -1$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, and $\kappa = 0$, Theorem 5.1 simplifies to [29, Theorem 6, Case (iv)].

Example 5.4. *Consider $\Omega(\nu) = [\nu^2, 8 - e^\nu]$, assuming that all the conditions of Theorem 5.1 are met, where the interval $[\varepsilon_1, \varepsilon_2]$ is defined as $[0, 2]$, and the functions $g(r) = r$, $\hbar(\nu) = \nu$, and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ are specified. If $\chi = 2$ and $\kappa = 2$ are chosen, we obtain the following results:*

$$\frac{1}{\Gamma(\chi)}\Omega(1) = [1,\ 5.28172],$$

$$\frac{1}{2(2)^\chi\Gamma(\chi)\gamma_{2\kappa}(\chi, 1)}\Big[\int_0^2 \Big((2-\nu)^{\chi-1}e^{-\kappa(2-\nu)} + \nu^{\chi-1}e^{-\kappa\nu}\Big)\Omega(\nu)\, d\nu\Big] = [1.25806,\ 4.91607],$$

$$\frac{\Omega(0) + \Omega(2)}{2\Gamma(\chi)} = [2,\ 3.80547].$$

Figures 1 and 2 illustrate Theorem 5.1 by depicting the lower and upper functions across the left, middle, and right segments, labeled as LLF, LUF, MLF, MUF, RLF, and RUF, respectively. These graphical representations offer a clear and effective illustration of the key concepts and results established in Theorem 5.1.

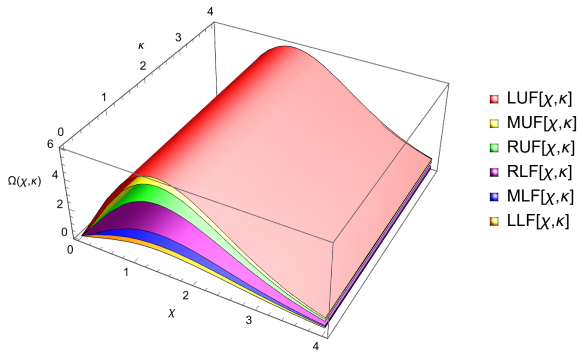

Figure 1. Theorem 5.1 offers a graph representation valid for the ranges $0 \leq \chi \leq 4$ and $0 \leq \kappa \leq 4$.

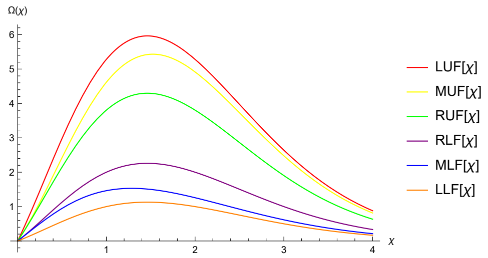

Figure 2. Theorem 5.1 provides a graph representation applicable to the range $0 \leq \chi \leq 4$ and for $\kappa = 2$.

We now present a table summarizing Theorem 5.1, which highlights its main components and facilitates clearer understanding. Table 1 provides clear evidence of the validity of Theorem 5.1 for varying values of χ.

Table 1. A comparative analysis of every part of Theorem 5.1.

χ	$LUF[\chi]$	$LLF[\chi]$	$MUF[\chi]$	$MLF[\chi]$	$RUF[\chi]$	$RLF[\chi]$
0.5	2.9799	0.5642	2.4219	0.9477	2.1470	1.1284
1.5	5.9598	1.1284	5.4306	1.4998	4.2940	2.2568
2.5	3.9732	0.7523	3.7281	0.9255	2.8627	1.5045
3.5	1.5893	0.3009	1.4831	0.3756	1.1451	0.6018

This example clearly demonstrates the validity of the H-H-*type inclusion in the* I.V *generalized* $(g, \eta_\hbar)$ *function context, emphasizing its strong reliability and broad applicability.*

Theorem 5.5. *Suppose* $\Omega \in SIGX((g, \eta_\hbar), [\varepsilon_1, \varepsilon_2], \mathbb{R}_I^+)$ *and let* $\aleph : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ *be a symmetric function with respect to* $\frac{\varepsilon_1+\varepsilon_2}{2}$. *Then, the following relation holds:*

$$\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\aleph \circ g^{-1}(g(\varepsilon_1))\right\} - \mathcal{W}$$
$$\supseteq \left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph \circ g^{-1}(g(\varepsilon_1))\right\}$$
$$\supseteq \frac{\Omega(\varepsilon_1)+\Omega(\varepsilon_2)}{2}\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\aleph \circ g^{-1}(g(\varepsilon_1))\right\}$$
$$+ \frac{\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2)) + \eta(\Omega(\varepsilon_2),\Omega(\varepsilon_1))\,(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}{\Gamma(\chi)}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\hbar(\nu)$$
$$\times \aleph(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\,d\nu,$$

where

$$\mathcal{W} = \frac{\hbar\left(\frac{1}{2}\right)(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}{\Gamma(\chi)}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))$$
$$\Big\{\eta\Big(\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))),\Omega(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)))\Big)$$
$$+\eta\Big(\Omega(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))),\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))\Big)\Big\}\,d\nu,$$

and $\chi > 0$, $\kappa \geq 0$ *and* $\forall\, r, t \in [\varepsilon_1, \varepsilon_2]$.

Proof. Since Ω is an I.V generalized $(g, \eta_\hbar)$ convex function, setting $\nu = \frac{1}{2}$ yields the following relation:

$$\Omega\left(g^{-1}\left(\frac{g(r)+g(t)}{2}\right)\right) \supseteq \Omega(t) + \hbar\left(\frac{1}{2}\right)\eta\Big(\Omega(r), \Omega(t)\Big), \tag{5.10}$$

for all $r, t \in [\varepsilon_1, \varepsilon_2]$. Substituting $r = g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))$ and $t = g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))$ in (5.10), we obtain

$$\begin{aligned}\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right) \supseteq\ & \Omega\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right) \\ & + \hbar\left(\frac{1}{2}\right)\eta\Big(\Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right), \\ & \Omega\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)\Big).\end{aligned} \tag{5.11}$$

By choosing $r = g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))$ and $t = g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))$ in (5.10), we obtain

$$\begin{aligned}\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right) \supseteq\ & \Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right) \\ & + \hbar\left(\frac{1}{2}\right)\eta\Big(\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)) + \nu g(\varepsilon_2))\right), \\ & \Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)\Big).\end{aligned} \tag{5.12}$$

Adding (5.11) and (5.12), then multiplying the result by $\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu} \times$ $\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)$ and integrating from 0 to 1 with respect to ν, we obtain

$$\begin{aligned}& 2\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\Big(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\Big)\Omega\Big(g^{-1}\Big(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\Big)\Big)d\nu \\ & - \hbar\left(\frac{1}{2}\right)\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right) \\ & \times\Big\{\eta\Big(\Omega(g^{-1}((1-\nu)g(\varepsilon_1)) + \nu g(\varepsilon_2)), \Omega(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)))\Big) \\ & + \eta\Big(\Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right), \Omega\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)\Big)\Big\}\, d\nu \\ \supseteq & \int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)\times \\ & \Big\{\Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right) + \Omega\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)\Big\}\, d\nu.\end{aligned}$$

By utilizing the fact that

$$\aleph\left(g^{-1}\left(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2)\right)\right) = \aleph\left(g^{-1}\left((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2)\right)\right),$$

$$\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu$$

$$+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu$$
$$-\hbar\left(\frac{1}{2}\right)\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)$$
$$\times\Big\{\eta\Big(\Omega(g^{-1}((1-\nu)g(\varepsilon_1))+\nu g(\varepsilon_2)),\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))\Big)$$
$$+\eta\Big(\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right),\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Big)\Big\}d\nu$$
$$\supseteq\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\aleph\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)d\nu$$
$$+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)d\nu. \tag{5.13}$$

Now,

$$\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu$$
$$+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)d\nu$$
$$=\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\frac{\Gamma(\chi)}{(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}$$
$$\times\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\aleph\circ g^{-1}(g(\varepsilon_2))+{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\aleph\circ g^{-1}(g(\varepsilon_1))\right\}. \tag{5.14}$$

and

$$\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\aleph\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)d\nu$$
$$+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)d\nu$$
$$=\frac{\Gamma(\chi)}{(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph\circ g^{-1}(g(\varepsilon_2))+{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph\circ g^{-1}(g(\varepsilon_1))\right\}. \tag{5.15}$$

By applying (5.14) and (5.15) to (5.13), we obtain the first part of inclusion. To derive the second part, we multiply (5.9) by $\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)$ and integrate over the interval [0,1] with respect to ν, then we obtain

$$\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)\aleph\left(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2))\right)d\nu$$
$$+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\right)d\nu$$
$$\supseteq\frac{\Omega(\varepsilon_1)+\Omega(\varepsilon_2)}{2}\left\{\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph(g^{-1}\,(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))d\nu\right.$$
$$\left.+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\aleph(g^{-1}\,((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2))\,d\nu\right\}$$

$$+ [\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) + \eta(\Omega(\varepsilon_2), \Omega(\varepsilon_1))] \int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu} \hbar(\nu) \times \aleph(g^{-1}(\nu g(\varepsilon_1) + (1-\nu) g(\varepsilon_2)) \, d\nu,$$

A few straightforward calculations lead to the desired inequality. □

Corollary 5.6. *Now, we present innovative findings in connection with Theorem 5.5.*

(i) By choosing $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ in Theorem 5.5, we get

$$\begin{aligned}
&\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1) + g(\varepsilon_2)}{2}\right)\right)\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\aleph \circ g^{-1}(g(\varepsilon_1))\right\} \\
&\supseteq \left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph \circ g^{-1}(g(\varepsilon_1))\right\} \\
&\supseteq \frac{\Omega(\varepsilon_1) + \Omega(\varepsilon_2)}{2}\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\aleph \circ g^{-1}(g(\varepsilon_1))\right\}.
\end{aligned}$$

(ii) By choosing $g(r) = r$, $\hbar(\nu) = \nu$, and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ in Theorem 5.5, we get

$$\begin{aligned}
&\Omega\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)\left\{{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\aleph(\varepsilon_2) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\aleph(\varepsilon_1)\right\} \supseteq \left\{{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega\aleph(\varepsilon_2) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega\aleph(\varepsilon_1)\right\} \\
&\supseteq \frac{\Omega(\varepsilon_1) + \Omega(\varepsilon_2)}{2}\left\{{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\aleph(\varepsilon_2) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\aleph(\varepsilon_1)\right\}.
\end{aligned}$$

(iii) By choosing $g(r) = r^p$, $p \geq -1$ and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ in Theorem 5.5, we obtain

$$\begin{aligned}
&\Omega\left(\left(\frac{\varepsilon_1^p + \varepsilon_2^p}{2}\right)^{\frac{1}{p}}\right)\left\{{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1^p+}\aleph \circ \mathcal{K}(\varepsilon_2^p) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2^p-}\aleph \circ \mathcal{K}(\varepsilon_1^p)\right\} \\
&\supseteq \left\{{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1^p+}\Omega\aleph \circ \mathcal{K}(\varepsilon_2^p) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2^p-}\Omega\aleph \circ \mathcal{K}(\varepsilon_1^p)\right\} \\
&\supseteq \frac{\Omega(\varepsilon_1) + \Omega(\varepsilon_2)}{2}\left\{{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1^p+}\aleph \circ \mathcal{K}(\varepsilon_2^p) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2^p-}\aleph \circ \mathcal{K}(\varepsilon_1^p)\right\},
\end{aligned}$$

where $\mathcal{K}(r) = r^{\frac{1}{p}}$.

Remark 5.7. *We now examine Theorem 5.5 in the context of related work in the literature.*

(i) By selecting $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, and $\kappa = 0$, Theorem 5.5 simplifies to [29, Theorem 7, Case (i)].

(ii) By selecting $g(r) = r$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, and $\kappa = 0$, Theorem 5.5 simplifies to [29, Theorem 7, Case (ii)].

(iii) By selecting $g(r) = r^p$, $p \geq -1$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, and $\kappa = 0$, Theorem 5.5 simplifies to [29, Theorem 7, Case (iv)].

Example 5.8. *Assuming that all the conditions of Theorem 5.5 are fulfilled, let us consider the function $\Omega : [0, 2] \to \mathbb{R}_I$ is defined by $\Omega(\nu) = [\nu^2, 8 - e^{\nu}]$ along with $g(r) = r$, $\hbar(\nu) = \nu$ and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$. The function $\aleph(\nu)$ is symmetric with respect to $\frac{\varepsilon_1+\varepsilon_2}{2}$ satisfying the* *condition* $\aleph(\varepsilon_1 + \varepsilon_2 - r) = \aleph(r)$, *for* $r \in [0, 2]$. *Specifically,* $\aleph(\nu)$ *is defined as*

$$\aleph(\nu) = \begin{cases} \nu, & \text{if } \nu \in [0,1] \\ 2-\nu, & \text{if } \nu \in [1,2]. \end{cases}$$

Now, taking $\chi = 2$ *and* $\kappa = 2$, *we obtain the following result:*

$$\frac{\Omega(1)}{\Gamma(\chi)}\left[\int_0^2 \left((2-\nu)^{\chi-1}e^{-\kappa(2-\nu)} + \nu^{\chi-1}e^{-\kappa\nu}\right)\aleph(\nu)\,d\nu\right] = [0.256803,\ 1.35636],$$
$$\frac{1}{\Gamma(\chi)}\left[\int_0^2 \left((2-\nu)^{\chi-1}e^{-\kappa(2-\nu)} + \nu^{\chi-1}e^{-\kappa\nu}\right)\Omega\aleph(\nu)\,d\nu\right] = [0.294319,\ 1.30385],$$
$$\frac{\Omega(0)+\Omega(2)}{2\Gamma(\chi)}\left[\int_0^2 \left((2-\nu)^{\chi-1}e^{-\kappa(2-\nu)} + \nu^{\chi-1}e^{-\kappa\nu}\right)\aleph(\nu)\,d\nu\right] = [0.513606,\ 0.977256].$$

Figures 3 and 4 illustrate Theorem 5.5 by depicting the lower and upper functions across the left, middle, and right segments, labeled as LLF, LUF, MLF, MUF, RLF, and RUF, respectively. These graphical representations offer a clear and effective illustration of the key concepts and results established in Theorem 5.5.

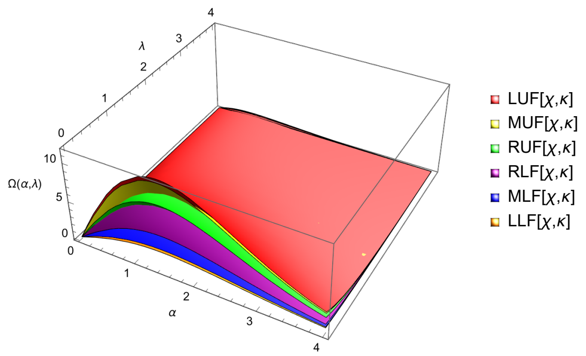

Figure 3. Theorem 5.5 offers a graph representation valid for the ranges $0 \le \chi \le 4$ and $0 \le \kappa \le 4$.

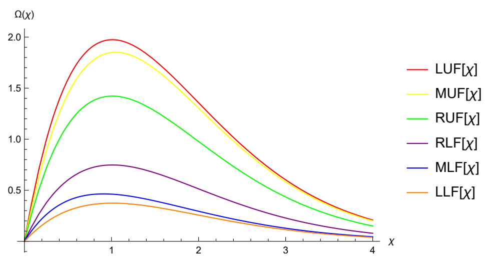

Figure 4. Theorem 5.5 provides a graph representation applicable to the range $0 \le \chi \le 4$ and $\kappa = 2$.

We now present a table summarizing Theorem 5.5, which highlights its main components and facilitates clearer understanding. Table 2 provides clear evidence of the validity of Theorem 5.5 for varying values of χ.

Table 2. A comparative analysis of every part of Theorem 5.5.

χ	$LUF[\chi]$	$LLF[\chi]$	$MUF[\chi]$	$MLF[\chi]$	$RUF[\chi]$	$RLF[\chi]$
0.5	1.5866	0.3004	1.4462	0.3989	1.1431	0.6008
1.5	1.7701	0.3351	1.6858	0.3951	1.2754	0.6703
2.5	0.9387	0.1777	0.9064	0.2008	0.6763	0.3554
3.5	0.3634	0.0688	0.3509	0.0777	0.2618	0.1376

This example clearly demonstrates the validity of Theorem 5.5, emphasizing its strong reliability and broad applicability.

Theorem 5.9. *Suppose* $\Omega, \aleph \in SIGX((g, \eta_\hbar), [\varepsilon_1, \varepsilon_2], \mathbb{R}_I^+)$*, then the following relation holds:*

$$\frac{\Gamma(\chi)}{(g(\varepsilon_2) - g(\varepsilon_1))^\chi}\Big[{}^\chi\mathcal{I}^\kappa_{g(\varepsilon_1)+}\Omega\aleph \circ g^{-1}(g(\varepsilon_2)) + {}^\chi\mathcal{I}^\kappa_{g(\varepsilon_2)-}\Omega\aleph \circ g^{-1}(g(\varepsilon_1))\Big]$$
$$\supseteq \int_0^1 \nu^{\chi-1} e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\mathcal{P}(\nu) + \mathcal{Q}(\nu)]d\nu,$$

where

$$\mathcal{P}(\nu) = [\Omega(\varepsilon_2) + \hbar(\nu)\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2))][\aleph(\varepsilon_2) + \hbar(\nu)\eta(\aleph(\varepsilon_1), \aleph(\varepsilon_2))],$$
$$\mathcal{Q}(\nu) = [\Omega(\varepsilon_1) + \hbar(\nu)\eta(\Omega(\varepsilon_2), \Omega(\varepsilon_1))][\aleph(\varepsilon_1) + \hbar(\nu)\eta(\aleph(\varepsilon_2), \aleph(\varepsilon_1))].$$

Proof. Since $\Omega, \aleph \in SIGX((g, \eta_\hbar), [\varepsilon_1, \varepsilon_2], \mathbb{R}_I^+)$, then we have

$$\Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right) \supseteq \Omega(\varepsilon_2) + \hbar(\nu)\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)), \tag{5.16}$$

and

$$\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right) \supseteq \aleph(\varepsilon_2) + \hbar(\nu)\eta(\aleph(\varepsilon_1), \aleph(\varepsilon_2)). \tag{5.17}$$

Multiplying (5.16) and (5.17), we obtain

$$\Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)$$
$$\supseteq [\Omega(\varepsilon_2) + \hbar(\nu)\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2))][\aleph(\varepsilon_2) + \hbar(\nu)\eta(\aleph(\varepsilon_1), \aleph(\varepsilon_2))]. \tag{5.18}$$

Similarly,

$$\Omega\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)$$
$$\supseteq [\Omega(\varepsilon_1) + \hbar(\nu)\eta(\Omega(\varepsilon_2), \Omega(\varepsilon_1))][\aleph(\varepsilon_1) + \hbar(\nu)\eta(\aleph(\varepsilon_2), \aleph(\varepsilon_1))]. \tag{5.19}$$

By summing (5.18) and (5.19), we obtain

$$\Omega\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)\aleph\left(g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))\right)$$
$$+ \Omega\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)\aleph\left(g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))\right)$$

$$\supseteq \mathcal{P}(\nu) + \mathcal{Q}(\nu). \tag{5.20}$$

Multiplying (5.20) by $\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}$ and integrating from 0 to 1 with respect to ν, we get

$$\begin{aligned}
&\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)\right)\aleph\left(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)\right)d\nu\\
&+\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\Omega\left(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)\right)\aleph\left(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)\right)d\nu\\
&\supseteq \int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\mathcal{P}(\nu)+\mathcal{Q}(\nu)]d\nu.
\end{aligned}$$

Now, by applying the tempered fractional integral operators, we obtain

$$\begin{aligned}
&\frac{\Gamma(\chi)}{(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph\circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph\circ g^{-1}(g(\varepsilon_1))\right]\\
&\supseteq \int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\mathcal{P}(\nu)+\mathcal{Q}(\nu)]d\nu.
\end{aligned}$$

This completes the proof. □

Corollary 5.10. *Now, we present innovative findings in connection with Theorem 5.9.*

(i) By choosing $\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2)) = \Omega(\varepsilon_1)-\Omega(\varepsilon_2)$ in Theorem 5.9, we obtain

$$\begin{aligned}
&\frac{\Gamma(\chi)}{(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph\circ g^{-1}(g(\varepsilon_2)) + {}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph\circ g^{-1}(g(\varepsilon_1))\right]\\
&\supseteq M(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\hbar^2(\nu)+(1-\hbar(\nu))^2]d\nu\\
&+2N(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\hbar(\nu)(1-\hbar(\nu))]d\nu,
\end{aligned}$$

where

$$M(\varepsilon_1,\varepsilon_2) = \Omega(\varepsilon_1)\aleph(\varepsilon_1)+\Omega(\varepsilon_2)\aleph(\varepsilon_2), \tag{5.21}$$
$$N(\varepsilon_1,\varepsilon_2) = \Omega(\varepsilon_1)\aleph(\varepsilon_2)+\Omega(\varepsilon_2)\aleph(\varepsilon_1). \tag{5.22}$$

(ii) By choosing $g(r)=r$, $\hbar(\nu)=\nu$, and $\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2)) = \Omega(\varepsilon_1)-\Omega(\varepsilon_2)$ in Theorem 5.9, we get

$$\begin{aligned}
&\frac{\Gamma(\chi)}{(\varepsilon_2-\varepsilon_1)^{\chi}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega\aleph(\varepsilon_2) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega\aleph(\varepsilon_1)\right]\\
&\supseteq M(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu^2+(1-\nu)^2]d\nu\\
&+2N(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu(1-\nu)]d\nu,
\end{aligned}$$

where $M(\varepsilon_1,\varepsilon_2)$ and $N(\varepsilon_1,\varepsilon_2)$ are defined in (5.21) and (5.22), respectively.

(iii) *By choosing* $g(r) = r^p$, $p \geq -1$, $\hbar(\nu) = \nu$, *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ *in Theorem 5.9, we obtain*

$$\begin{aligned}&\frac{\Gamma(\chi)}{\left(\varepsilon_2^p - \varepsilon_1^p\right)^{\chi}}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1^p+}\Omega\aleph \circ \mathcal{K}(\varepsilon_2^p) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2^p-}\Omega\aleph \circ \mathcal{K}(\varepsilon_1^p)\right]\\ &\supseteq M(\varepsilon_1, \varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2^p-\varepsilon_1^p)\nu}[\nu^2 + (1-\nu)^2]d\nu\\ &+ 2N(\varepsilon_1, \varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2^p-\varepsilon_1^p)\nu}[\nu(1-\nu)]d\nu,\end{aligned}$$

where $\mathcal{K}(r) = r^{\frac{1}{p}}$. *Additionally,* $M(\varepsilon_1, \varepsilon_2)$ *and* $N(\varepsilon_1, \varepsilon_2)$ *are defined in (5.21) and (5.22), respectively.*

Remark 5.11. *We now examine Theorem 5.9 in the context of related work in the literature.*

(i) *By selecting* $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\kappa = 0$, *Theorem 5.9 simplifies to [29, Theorem 8, Case (i)].*
(ii) *By selecting* $g(r) = r$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\kappa = 0$, *Theorem 5.9 simplifies to [29, Theorem 8, Case (ii)].*
(iii) *By selecting* $g(r) = r^p$, $p \geq -1$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\kappa = 0$, *Theorem 5.9 simplifies to [29, Theorem 8, Case (iv)].*

Example 5.12. *Assuming that all the conditions of Theorem 5.9 are fulfilled, let us consider the functions* $\Omega, \aleph : [0, 2] \to \mathbb{R}_I$, *defined by* $\Omega(\nu) = [\nu^2, 8 - e^{\nu}]$ *and* $\aleph(\nu) = [\nu^2, 3 - \nu^2]$ *with* $g(r) = r$, $\hbar(\nu) = \nu$ *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$. *Taking* $\chi = 2$ *and* $\kappa = 2$, *we obtain the following result:*

$$\begin{aligned}&\frac{1}{(2)^{\chi}\Gamma(\chi)}\left[\int_0^2 \left((2-\nu)^{\chi-1}e^{-\kappa(2-\nu)} + \nu^{\chi-1}e^{-\kappa\nu}\right)\Omega\aleph(\nu)\,d\nu\right] = [0.304016,\ 1.15565],\\ &\frac{1}{\Gamma(\chi)}\left[M(\varepsilon_1, \varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu^2 + (1-\nu)^2]d\nu\right.\\ &\left.+ 2N(\varepsilon_1, \varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu(1-\nu)]d\nu\right] = [0.571423,\ 0.61934].\end{aligned}$$

Figures 5 and 6 illustrate Theorem 5.9 by depicting the lower and upper functions across the left and right segments, labeled as LLF, LUF, RLF, and RUF, respectively. These graphical representations offer a clear and effective illustration of the key concepts and results established in Theorem 5.9.

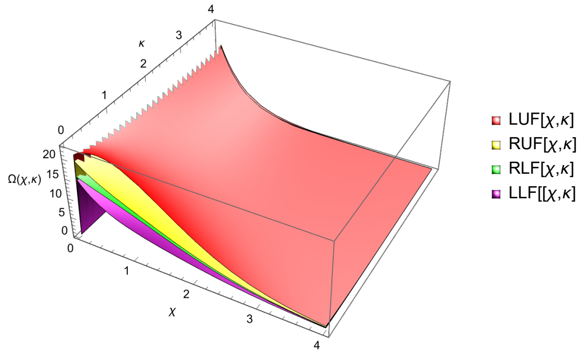

Figure 5. Theorem 5.9 offers a graph representation valid for the ranges $0 \leq \chi \leq 3$ and $0 \leq \kappa \leq 3$.

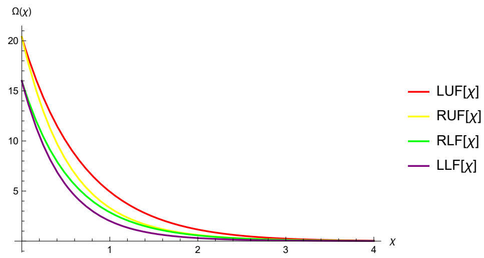

Figure 6. Theorem 5.9 provides a graph representation applicable to the range $0 \leq \chi \leq 3$ and $\kappa = 2$.

We now present a table summarizing Theorem 5.9, which highlights its main components and facilitates clearer understanding. Table 3 provides clear evidence of the validity of Theorem 5.9 for varying values of χ*.*

Table 3. A comparative analysis of every part of Theorem 5.9.

χ	$LUF[\chi]$	$LLF[\chi]$	$RUF[\chi]$	$RLF[\chi]$
0.5	9.9885	5.6105	8.1100	6.6874
1.5	2.4135	0.7541	1.4092	1.2680
2.5	0.5384	0.1315	0.2783	0.2595
3.5	0.1065	0.0273	0.0562	0.0520

This example clearly demonstrates the validity of Theorem 5.9, emphasizing its strong reliability and broad applicability.

Theorem 5.13. *Suppose* $\Omega, \aleph \in SIGX((g, \eta_\hbar), [\varepsilon_1, \varepsilon_2], \mathbb{R}_I^+)$*, then the following relation holds*

$$\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\aleph\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi,1)$$
$$\supseteq \frac{\Gamma(\chi)}{4\,(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}\left\{{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph\circ g^{-1}(g(\varepsilon_2))+{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph\circ g^{-1}(g(\varepsilon_1))\right\}$$

$$+\frac{1}{4}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\left[N(\varepsilon_1,\varepsilon_2)+\hbar(\nu)\mathcal{G}+\hbar^2(\nu)\mathcal{U}\right]d\nu$$
$$+\frac{\hbar(\frac{1}{2})}{4}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}\mathcal{B}d\nu+\frac{\hbar^2(\frac{1}{2})}{4}\int_0^1 \nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_2))\nu}\mathcal{C}d\nu,$$

where

$$\begin{aligned}\mathcal{B}=&\Big(\Omega\left(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)\right)+\Omega\left(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)\right)\Big)\times\\&\Big\{\eta\Big(\aleph(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)),\aleph(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right))\Big)\\&+\eta\Big(\aleph(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)),\aleph(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right))\Big)\Big\}\\&+\Big(\aleph\left(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)\right)+\aleph\left(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)\right)\Big)\times\\&\Big\{\eta\Big(\Omega(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)),\Omega(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right))\Big)\\&+\eta\Big(\Omega(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)),\Omega(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right))\Big)\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}=&\Big\{\eta\Big(\aleph(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)),\aleph(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right))\Big)\\&+\eta\Big(\aleph(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)),\aleph(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right))\Big)\Big\}\times\\&\Big\{\eta\Big(\Omega(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right)),\Omega(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right))\Big)\\&+\eta\Big(\Omega(g^{-1}\left((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)\right)),\Omega(g^{-1}\left(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)\right))\Big)\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{G}=\Omega(\varepsilon_1)\eta(\aleph(\varepsilon_1),\aleph(\varepsilon_2))+\Omega(\varepsilon_2)\eta(\aleph(\varepsilon_2),\aleph(\varepsilon_1))+\aleph(\varepsilon_1)\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2))\\+\aleph(\varepsilon_2)\eta(\Omega(\varepsilon_2),\Omega(\varepsilon_1)),\end{aligned}$$

$$\mathcal{U}=\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2))\eta(\aleph(\varepsilon_2),\aleph(\varepsilon_1))+\eta(\Omega(\varepsilon_2),\Omega(\varepsilon_1))\eta(\aleph(\varepsilon_1),\aleph(\varepsilon_2)),$$

and $N(\varepsilon_1,\varepsilon_2)$ *is given by* (5.22).

Proof. By the I.V generalized $(g,\eta_\hbar)$ convexity of Ω and $\aleph$, we have

$$\begin{aligned}&\Omega\left(g^{-1}\left(\frac{g(r)+g(t)}{2}\right)\right)\aleph\left(g^{-1}\left(\frac{g(r)+g(t)}{2}\right)\right)\\&\supseteq\Big(\frac{1}{2}[\Omega(r)+\Omega(t)]+\frac{\hbar(\frac{1}{2})}{2}\{\eta(\Omega(r),\Omega(t))+\eta(\Omega(t),\Omega(r))\}\Big)\\&\times\Big(\frac{1}{2}[\aleph(r)+\aleph(t)]+\frac{\hbar(\frac{1}{2})}{2}\{\eta(\aleph(r),\aleph(t))+\eta(\aleph(t),\aleph(r))\}\Big)\\&=\frac{1}{4}[\Omega(r)\aleph(t)+\Omega(t)\aleph(r)]+\frac{1}{4}[\Omega(r)\aleph(r)+\Omega(t)\aleph(t)]\\&+\frac{\hbar(\frac{1}{2})}{4}\{[\Omega(r)+\Omega(t)][\eta(\aleph(r),\aleph(t))+\eta(\aleph(t),\aleph(r))]\end{aligned}$$

$$+\frac{\hbar(\frac{1}{2})}{4}\{[\aleph(r)+\aleph(t)][\eta(\Omega(r),\Omega(t))+\eta(\Omega(t),\Omega(r))]$$

$$+\frac{\hbar^2(\frac{1}{2})}{4}[\eta(\Omega(r),\Omega(t))+\eta(\Omega(t),\Omega(r))][\eta(\aleph(r),\aleph(t))+\eta(\aleph(t),\aleph(r))]. \tag{5.23}$$

Using $r = g^{-1}(\nu g(\varepsilon_1) + (1-\nu)g(\varepsilon_2))$ and $t = g^{-1}((1-\nu)g(\varepsilon_1) + \nu g(\varepsilon_2))$, along with the indicated convexity, we have

$$\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\aleph\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)$$
$$\supseteq \frac{1}{4}\Big[\Omega(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))\aleph(g^{-1}(\nu g(\varepsilon_1)+(1-\nu)g(\varepsilon_2)))$$
$$+\Omega(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)))\aleph(g^{-1}((1-\nu)g(\varepsilon_1)+\nu g(\varepsilon_2)))\Big]$$
$$+\frac{1}{4}[N(\varepsilon_1,\varepsilon_2)+\hbar(\nu)\mathcal{G}(\varepsilon_1,\varepsilon_2)+\hbar^2(w)\mathcal{U}(\varepsilon_1,\varepsilon_2)]+\frac{\hbar(\frac{1}{2})}{4}\mathcal{B}+\frac{\hbar^2(\frac{1}{2})}{4}\mathcal{C}.$$

After multiplying the last inclusion by $\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}$ and integrating from 0 to 1 with respect to ν, we obtain the desired inclusion. □

Corollary 5.14. *Now, we present innovative findings in connection with Theorem 5.13.*

(i) By choosing $\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2)) = \Omega(\varepsilon_1)-\Omega(\varepsilon_2)$ in Theorem 5.13, we get

$$\Omega\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\aleph\left(g^{-1}\left(\frac{g(\varepsilon_1)+g(\varepsilon_2)}{2}\right)\right)\gamma_{\kappa(g(\varepsilon_2)-g(\varepsilon_1))}(\chi,1)$$
$$\supseteq \frac{\Gamma(\chi)}{4\,(g(\varepsilon_2)-g(\varepsilon_1))^{\chi}}\Big[{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_1)+}\Omega\aleph\circ g^{-1}(g(\varepsilon_2))+{}^{\chi}\mathcal{I}^{\kappa}_{g(\varepsilon_2)-}\Omega\aleph\circ g^{-1}(g(\varepsilon_1))\Big]$$
$$+\frac{1}{4}\Big[2M(\varepsilon_1,\varepsilon_2)\int_0^1\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\hbar(\nu)(1-\hbar(\nu))]d\nu$$
$$+N(\varepsilon_1,\varepsilon_2)\int_0^1\nu^{\chi-1}e^{-\kappa(g(\varepsilon_2)-g(\varepsilon_1))\nu}[\hbar^2(\nu)+(1-\hbar(\nu))^2]d\nu\Big],$$

where $M(\varepsilon_1,\varepsilon_2)$ and $N(\varepsilon_1,\varepsilon_2)$ are defined in (5.21) and (5.22), respectively.

(ii) By choosing $g(r) = r$, $\hbar(\nu) = \nu$, and $\eta(\Omega(\varepsilon_1),\Omega(\varepsilon_2)) = \Omega(\varepsilon_1)-\Omega(\varepsilon_2)$ in Theorem 5.13, we get

$$\Omega\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\aleph\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)$$
$$\supseteq \frac{\Gamma(\chi)}{4\,(\varepsilon_2-\varepsilon_1)^{\chi}\,\gamma_{\kappa(\varepsilon_2-\varepsilon_1)}(\chi,1)}\Big[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1+}\Omega\aleph(\varepsilon_2)+{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2-}\Omega\aleph(\varepsilon_1)\Big]$$
$$+\frac{1}{4}\Big[2M(\varepsilon_1,\varepsilon_2)\int_0^1\nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu(1-\nu)]d\nu$$
$$+N(\varepsilon_1,\varepsilon_2)\int_0^1\nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu^2+(1-\nu)^2]d\nu\Big],$$

where $M(\varepsilon_1,\varepsilon_2)$ and $N(\varepsilon_1,\varepsilon_2)$ are defined in (5.21) and (5.22), respectively.

(iii) *By choosing* $g(r) = r^p$, $p \geq -1$, $\hbar(\nu) = \nu$, *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ *in Theorem 5.13, we obtain*

$$\begin{aligned}
&\Omega\left(\left(\frac{\varepsilon_1^p + \varepsilon_2^p}{2}\right)^{\frac{1}{p}}\right)\aleph\left(\left(\frac{\varepsilon_1^p + \varepsilon_2^p}{2}\right)^{\frac{1}{p}}\right)\\
&\supseteq \frac{\Gamma(\chi)}{4\left(\varepsilon_2^p - \varepsilon_1^p\right)^{\chi}\gamma_{\kappa(\varepsilon_2^p-\varepsilon_1^p)}(\chi,1)}\left[{}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_1^p+}\Omega\aleph\circ\mathcal{K}(\varepsilon_2^p) + {}^{\chi}\mathcal{I}^{\kappa}_{\varepsilon_2^p-}\Omega\aleph\circ\mathcal{K}(\varepsilon_1^p)\right]\\
&+ \frac{1}{4}\Big[2M(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2^p-\varepsilon_1^p)\nu}[\nu(1-\nu)]d\nu\\
&+ N(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2^p-\varepsilon_1^p)\nu}[\nu^2+(1-\nu)^2]d\nu\Big],
\end{aligned}$$

where $\mathcal{K}(r) = r^{\frac{1}{p}}$. *Additionally,* $M(\varepsilon_1,\varepsilon_2)$ *and* $N(\varepsilon_1,\varepsilon_2)$ *are defined in (5.21) and (5.22), respectively.*

Remark 5.15. *We now examine Theorem 5.13 in the context of related work in the literature.*

(i) *By selecting* $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\kappa = 0$, *Theorem 5.13 simplifies to [29, Theorem 9, Case (i)].*

(ii) *By selecting* $g(r) = r$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\kappa = 0$, *Theorem 5.13 simplifies to [29, Theorem 9, Case (ii)].*

(iii) *By selecting* $g(r) = r^p$, $p \geq -1$, $\hbar(\nu) = \nu$, $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, *and* $\kappa = 0$, *Theorem 5.13 simplifies to [29, Theorem 9, Case (iii)].*

Example 5.16. *Assuming that all the conditions of Theorem 5.13 are fulfilled, let us consider the functions* $\Omega, \aleph : [0,2] \to \mathbb{R}_I$, *defined by* $\Omega(\nu) = [\nu^2, 8 - e^{\nu}]$ *and* $\aleph(\nu) = [\nu^2, 3 - \nu^2]$ *with* $g(r) = r$, $\hbar(\nu) = \nu$, *and* $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$. *Taking* $\chi = 2$ *and* $\kappa = 2$, *we obtain the following result:*

$$\begin{aligned}
&\frac{1}{\Gamma(\chi)}\Omega(1)\aleph(1) = [1,\ 10.5634],\\
&+ \frac{1}{4(2)^{\chi}\Gamma(\chi)\gamma_{2\kappa}(\chi,1)}\Big[\int_0^2\Big((2-\nu)^{\chi-1}e^{-\kappa(2-\nu)} + \nu^{\chi-1}e^{-\kappa\nu}\Big)\Omega\aleph(\nu)\,d\nu\Big]\\
&\frac{1}{4\Gamma(\chi)\gamma_{2\kappa}(\chi,1)}\Big[2M(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu(1-\nu)]d\nu\\
&+ N(\varepsilon_1,\varepsilon_2)\int_0^1 \nu^{\chi-1}e^{-\kappa(\varepsilon_2-\varepsilon_1)\nu}[\nu^2+(1-\nu)^2]d\nu\Big] = [2.82255,\ 6.167].
\end{aligned}$$

Figures 7 and 8 illustrate Theorem 5.13 by depicting the lower and upper functions across the left and right segments, labeled as LLF, LUF, RLF, and RUF, respectively. These graphical representations offer a clear and effective illustration of the key concepts and results established in Theorem 5.13.

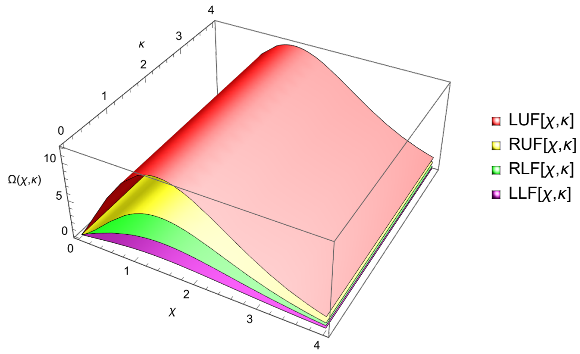

Figure 7. Theorem 5.13 offers a graph representation valid for the ranges $0 \leq \chi \leq 4$ and $0 \leq \kappa \leq 4$.

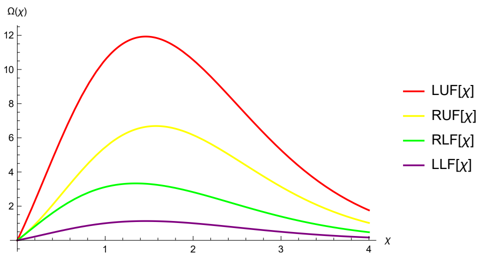

Figure 8. Theorem 5.13 provides a graph representation applicable to the range $0 \leq \chi \leq 4$ and $\kappa = 2$.

We now present a table summarizing Theorem 5.13, which highlights its main components and facilitates clearer understanding. Table 4 provides clear evidence of the validity of Theorem 5.13 for varying values of χ.

Table 4. A comparative analysis of every part of Theorem 5.13.

χ	$LUF[\chi]$	$LLF[\chi]$	$RUF[\chi]$	$RLF[\chi]$
0.5	5.9598	0.5462	2.6794	1.9515
1.5	11.9196	1.1284	6.6697	3.2977
2.5	7.9464	0.7523	4.7182	2.0956
3.5	3.1786	0.3009	1.8696	0.8468

This example clearly demonstrates the validity of Theorem 5.13, emphasizing its strong reliability and broad applicability.

6. Applications to means

This section investigates the interplay among specialized means through the inclusion of the generic class of I.V generalized convex functions in the context of H-H-type inclusion.

(i) The arithmetic mean

$$\mathrm{A}(\varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1 + \varepsilon_2}{2}.$$

(ii) The geometric mean

$$\mathrm{G}(\varepsilon_1, \varepsilon_2) = \sqrt{\varepsilon_1 \varepsilon_2}.$$

(iii) The logarithmic mean

$$\mathrm{L}(\varepsilon_1, \varepsilon_2) = \frac{\varepsilon_2 - \varepsilon_1}{\ln(\varepsilon_2) - \ln(\varepsilon_1)}, \quad \varepsilon_2 \neq \varepsilon_2, r \quad \varepsilon_1 \varepsilon_2 \neq 0.$$

(iv) The generalized log-mean

$$\mathrm{L}_i(\varepsilon_1, \varepsilon_2) = \left[\frac{\varepsilon_2^{i+1} - \varepsilon_1^{i+1}}{(i+1)(\varepsilon_2 - \varepsilon_1)}\right]^{\frac{1}{i}}; i \in \mathbb{Z} \setminus \{-1, 0\}.$$

Proposition 6.1. *For ε_1, $\varepsilon_2 > 0$, we have*

$$\begin{aligned}\left[\mathrm{A}^2(\varepsilon_1, \varepsilon_2),\ -\mathrm{A}^2(\varepsilon_1, \varepsilon_2) + 8\right] &\supseteq \left[\mathrm{L}_2^2(\varepsilon_1, \varepsilon_2),\ -\mathrm{L}_2^2(\varepsilon_1, \varepsilon_2) + 8\right] \\ &\supseteq \left[\mathrm{A}(\varepsilon_1^2, \varepsilon_2^2),\ -\mathrm{A}(\varepsilon_1^2, \varepsilon_2^2) + 8\right].\end{aligned}$$

Proof. This outcome is derived by applying Theorem 5.1 under the specific substitution $\Omega(\nu) = [\nu^2,\ 8-\nu^2]$ and choosing $\hbar(\nu) = \nu$, $g(r) = r$, and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$, where $\chi = 1$ and $\kappa = 0$. □

Proposition 6.2. *For ε_1, $\varepsilon_2 > 0$, we have*

$$\begin{aligned}\left[\mathrm{G}^2(\varepsilon_1, \varepsilon_2),\ -\mathrm{G}^2(\varepsilon_1, \varepsilon_2) + 8\right] &\supseteq \left[\mathrm{L}(\varepsilon_1^2, \varepsilon_2^2),\ -\mathrm{L}(\varepsilon_1^2, \varepsilon_2^2) + 8\right] \\ &\supseteq \left[\mathrm{A}(\varepsilon_1^2, \varepsilon_2^2),\ -\mathrm{A}(\varepsilon_1^2, \varepsilon_2^2) + 8\right].\end{aligned}$$

Proof. This outcome is derived by applying Theorem 5.1 under the specific substitution $\Omega(\nu) = [\nu^2,\ 8-\nu^2]$ and choosing $\hbar(\nu) = \nu$, $g(r) = \ln r$, and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ where $\chi = 1$ and $\kappa = 0$. □

Proposition 6.3. *For ε_1, $\varepsilon_2 > 0$, we have*

$$\begin{aligned}\left[\frac{1}{e^{\mathrm{A}(\varepsilon_1, \varepsilon_2)}},\ -\frac{1}{e^{\mathrm{A}(\varepsilon_1, \varepsilon_2)}} + 2\right] &\supseteq \left[\frac{\mathrm{L}(\varepsilon_1, \varepsilon_2)}{\mathrm{G}^2(\varepsilon_1, \varepsilon_2)},\ -\frac{\mathrm{L}(\varepsilon_1, \varepsilon_2)}{\mathrm{G}^2(\varepsilon_1, \varepsilon_2)} + 2\right] \\ &\supseteq \left[\frac{\mathrm{A}(\varepsilon_1, \varepsilon_2)}{\mathrm{G}^2(\varepsilon_1, \varepsilon_2)},\ -\frac{\mathrm{A}(\varepsilon_1, \varepsilon_2)}{\mathrm{G}^2(\varepsilon_1, \varepsilon_2)} + 2\right].\end{aligned}$$

Proof. This outcome is derived by applying Theorem 5.1 under the specific substitution $\Omega(\nu) = [\frac{1}{\nu},\ 2-\frac{1}{\nu}]$ for $\nu > 1$ and choosing $\hbar(\nu) = \nu$, $g(r) = \ln r$ and $\eta(\Omega(\varepsilon_1), \Omega(\varepsilon_2)) = \Omega(\varepsilon_1) - \Omega(\varepsilon_2)$ where $\chi = 1$ and $\kappa = 0$. □

7. Conclusions

In contemporary research, the study of inequality has rapidly progressed, particularly with a focus on the convex properties of functions within fractional domains. Interval analysis models uncertainty using interval variables to improve accuracy in computations. It is widely used in engineering, robotics, optimization, and neural networks for reliable results. This study presents a comprehensive unification of classical concepts by introducing I.V generalized $(g, \eta_\hbar)$ convex function, which generalizes H-H-type, Fejér-H-H-type, and other fractional inclusions through tempered fractional operators. By tuning parameters, we derive unified and extended results supported by illustrative examples, graphical representations and applications to means. These outcomes contribute significantly to deriving bounds for special functions, including modified Beta and Bessel functions, among others. This idea aims to be applied across various frameworks, including time-scale calculus, coordinates systems, fuzzy and quantum calculus. The approach and recent advancements outlined in the paper are intended to spark interest and motivate further research in this area.

Author contributions

Muhammad Samraiz: Investigation, methodology, writing original draft, writing-review & editing; Somia Zafar: Conceptualization, data curation, formal analysis, writing original draft; Muath Awadalla: Conceptualization, supervision, validation, writing-review & editing; Hajer Zaway: Resources, software, writing-review & editing. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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