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*Research article*

# On some combinatorial identities related to super Catalan matrix

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**Abstract:** In this paper, we define a new matrix $S_n$ constructed by super Catalan numbers. Also, we give Cholesky and LU-decompositions, Hermite normal form, and the determinant of the matrix $S_n$. Moreover, we derive auxiliary results involving some summation formulas via the coefficients of Lucas polynomials and scaled coefficients of Chebyshev polynomials. Additionally, we give a matrix $\acute{S}_n$ by modifying the matrix $S_n$ to deduce a matrix identity related to matrices $S_n$ and $\acute{S}_n$. By using the decomposition method, we give an application of solving a system of linear equations of order $n$ with coefficients $S(m,n)$ and find a general solution.

**Keywords:** super Catalan numbers; LU-decomposition; Cholesky decomposition; Lucas polynomials
**Mathematics Subject Classification:** 15A24, 11B83

## 1. Introduction

Recently, the super Catalan numbers have been the focus of many studies. In 1992, Gessel defined these numbers as

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!} = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{m}}$$

for integers $m,n \geq 0$ [1]. In 1874, before Gessel, E. Catalan observed that the numbers $S(m,n)$ are a generalization of the Catalan numbers by the formula

$$\frac{S(1,n)}{2} = C_n = \frac{1}{n+1}\binom{2n}{n},$$

where $\binom{2n}{n}$ are the central binomial coefficients. Some of the numbers $S(m,n)$ are given in the following Table 1.

**Table 1.** The numbers $S(m,n)$.

| $m/n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 6 | 20 | 70 | 252 | 924 | 3432 | $\cdots$ |
| 1 | 2 | 2 | 4 | 10 | 28 | 84 | 264 | 858 | $\cdots$ |
| 2 | 6 | 4 | 6 | 12 | 28 | 72 | 198 | 572 | $\cdots$ |
| 3 | 20 | 10 | 12 | 20 | 40 | 90 | 220 | 572 | $\cdots$ |
| 4 | 70 | 28 | 28 | 40 | 70 | 140 | 308 | 728 | $\cdots$ |
| 5 | 252 | 84 | 72 | 90 | 140 | 252 | 504 | 1092 | $\cdots$ |
| 6 | 924 | 264 | 198 | 220 | 308 | 504 | 924 | 1848 | $\cdots$ |
| 7 | 3432 | 858 | 572 | 572 | 728 | 1092 | 1848 | 3432 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

From the Table 1, the following can be seen.

$$S(m,n) = S(n,m) \text{ and } S(0,n) = S(n,n) = \binom{2n}{n}.$$

For the sequence $\{S(m,n)\}_{m,n\geq 0}$, many special uses and combinatorial interpretations have been studied by some authors in [2–6]. For example, in [6], Von Szily gave the identity

$$S(m,n) = \sum_k (-1)^k \binom{2m}{m+k}\binom{2n}{n-k}. \tag{1.1}$$

In [7], Prodinger considered the decompositions of the reciprocal super Catalan matrix and formulated them. Then, Kılıç, Akkuş, and Kızılaslan studied a variant of this matrix and also presented the $q$-analogues of the results in [8].

In [9], Lucas polynomials are given by the recurrence

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x),\ n \geq 0;\ L_0(x) = 2,\ L_1(x) = x.$$

The coefficients $\{T(n,k)\}_{n,k\geq 0}$ of the Lucas polynomials can be given in the following triangle.

$$\begin{array}{ccccc}
1 & & & & \\
1 & 2 & & & \\
1 & 3 & & & \\
1 & 4 & 2 & & \\
1 & 5 & 5 & & \\
1 & 6 & 9 & 2 & \\
1 & 7 & 14 & 7 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

These coefficients $T(n,k)$ can be given by the following formula.

$$T(n,k) = \frac{n}{n-k}\binom{n-k}{k}.$$

In [10], Akyüz and Halıcı gave some summation formulas involving the numbers $T(n,k)$ via the matrix method and deduced

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} T(n,k),$$

where $L_n$ is the $n$-th Lucas number, which satisfies the recurrence

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1.$$

The other polynomial we use in this study is the Chebyshev polynomial $T_n(x)$ of the first kind that is a polynomial of degree $n$ and is defined by the relation

$$T_n(x) = \cos n\theta,$$

for $x = \cos\theta$ [11]. In our results, we use a special case of the Chebyshev polynomial

$$2T_{2n}\left(\frac{\sqrt{x}}{2}\right).$$

In the rest of the study, we use the notation $C(n,k)$ for the scaled coefficients of the polynomial $2T_{2n}\left(\frac{\sqrt{x}}{2}\right)$ to prevent the confusions. The coefficients $C(n,k)$ are given by the formula

$$C(n,k) = (-1)^{n+k}\frac{n}{k}\binom{n+k-1}{2k-1}$$

and satisfy the recurrence relation

$$C(n,k) = -2C(n-1,k) + C(n-1,k-1) - C(n-2,k).$$

The numbers $C(n,k)$ can be seen in the following Pascal-like triangle.

$$\begin{array}{rrrrrrrr}
2 & & & & & & & \\
-2 & 1 & & & & & & \\
2 & -4 & 1 & & & & & \\
-2 & 9 & -6 & 1 & & & & \\
2 & -16 & 20 & -8 & 1 & & & \\
-2 & 25 & -50 & 35 & -10 & 1 & & \\
2 & -36 & 105 & -112 & 54 & -12 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}.$$

The sequence involving the numbers $C(n,k)$ corresponds to A127677 in OEIS [12]. Moreover, some matrix applications of Chebyshev polynomials can be seen in [13]. It can also be observed that the numbers $C(n,k)$ appear in the Riordan array $\left(\frac{1-x}{1+x}, \frac{x}{(1+x)^2}\right)$ in A110162, OEIS [12].

In this paper, motivated by the studies [7,8], we define a different type of the super Catalan matrix, denoted by $S_n$, and obtain some identities by using this newly defined matrix. We prove our results via auxiliary formulas related to the Chebyshev and Lucas polynomials given above. Furthermore, we also present different matrices to obtain some matrix equalities.

## 2. Some matrix identities related to matrix decompositions

In this section, we introduce a $n \times n$ matrix $S_n$ whose entries are the super Catalan numbers. Then, we give LU-decomposition, Cholesky decomposition, and the Hermite normal form of $S_n$ via some sums involving the numbers $T(n,k)$, $C(n,k)$, and binomial coefficients.

**Definition 1.** *For $1 \le i, j \le n$, the matrix $S_n$ is defined as*

$$S_n = [S(i-1, j-1)] = \begin{bmatrix} S(0,0) & S(0,1) & \dots & S(0,n-1) \\ S(1,0) & S(1,1) & \dots & S(1,n-1) \\ \vdots & \vdots & \ddots & \vdots \\ S(n-1,0) & S(n-1,1) & \dots & S(n-1,n-1) \end{bmatrix}.$$

By virtue of the symmetry of the super Catalan numbers, clearly $S_n$ is a symmetric matrix and has the entries

$$\left\{\binom{2i}{i}\right\}_{0 \le i \le n-1} = \{1, 2, 6, 20, 70, 252, ...\}$$

on the diagonal. For $n = 6, 7$, we get

$$S_6 = \begin{bmatrix} 1 & 2 & 6 & 20 & 70 & 252 \\ 2 & 2 & 4 & 10 & 28 & 84 \\ 6 & 4 & 6 & 12 & 28 & 72 \\ 20 & 10 & 12 & 20 & 40 & 90 \\ 70 & 28 & 28 & 40 & 70 & 140 \\ 252 & 84 & 72 & 90 & 140 & 252 \end{bmatrix} \text{ and } S_7 = \begin{bmatrix} 1 & 2 & 6 & 20 & 70 & 252 & 924 \\ 2 & 2 & 4 & 10 & 28 & 84 & 264 \\ 6 & 4 & 6 & 12 & 28 & 72 & 198 \\ 20 & 10 & 12 & 20 & 40 & 90 & 220 \\ 70 & 28 & 28 & 40 & 70 & 140 & 308 \\ 252 & 84 & 72 & 90 & 140 & 252 & 504 \\ 924 & 264 & 198 & 220 & 308 & 504 & 924 \end{bmatrix},$$

respectively.

We give the following lemmas for later use.

**Lemma 2.1.** *For $n \ge 1$, $j > 1$, we have the followings.*

$$\sum_{k=1}^{n} (-1)^{n+k} \binom{n+k-2}{2k-2} \binom{2k-2}{k-1} = 1, \tag{2.1}$$

$$\sum_{k=1}^{n} (-1)^{n+k} \binom{n+k-2}{2k-2} S(k-1, j-1) = 0. \tag{2.2}$$

*Proof.* We use induction on $n$ to prove the identity (2.1). For $n = 1$, we have

$$\sum_{k=1}^{1} (-1)^{1+k} \binom{k-1}{2k-2} \binom{2k-2}{k-1} = 1,$$

which is clearly true. Now, we suppose for the induction that the identity (2.1) is true for all integers $r \le n$. We claim to show that (2.1) holds for $n+1$. Consider

$$\sum_{k=1}^{n+1} (-1)^{n+k+1} \binom{n+k-1}{2k-2} \binom{2k-2}{k-1} = \sum_{k=0}^{n} (-1)^{n+k} \binom{n+k}{2k} \binom{2k}{k}$$

$$= (-1)^n + \sum_{k=1}^{n} (-1)^{n+k} \frac{(n+k)(n-k+1)}{k^2} \binom{n+k-1}{2k-2} \binom{2k-2}{k-1}.$$

By making some necessary operations, we rewrite the right-hand side of the last equation as

$$(-1)^n + \sum_{k=1}^{n} (-1)^{n+k} \left( (2n-1) \frac{1}{k} + n(n-1) \frac{1}{k^2} + 1 \right) \binom{n+k-2}{2k-2} \binom{2k-2}{k-1}.$$

Thus, we have

$$\begin{aligned} &\sum_{k=1}^{n+1} (-1)^{n+k+1} \binom{n+k-1}{2k-2} \binom{2k-2}{k-1} \\ &= 1 + (-1)^n + n(n-1) \sum_{k=1}^{n} (-1)^{n+k} \frac{1}{k^2} \binom{n+k-2}{2k-2} \binom{2k-2}{k-1} \\ &+ (2n-1) \sum_{k=1}^{n} (-1)^{n+k} \frac{1}{k} \binom{n+k-2}{2k-2} \binom{2k-2}{k-1}. \end{aligned}$$

Since

$$\sum_{k=1}^{n} (-1)^{n+k} \frac{1}{k} \binom{n+k-2}{2k-2} \binom{2k-2}{k-1} = 0$$

and

$$\sum_{k=1}^{n} (-1)^{n+k} \frac{1}{k^2} \binom{n+k-2}{2k-2} \binom{2k-2}{k-1} = \frac{(-1)^{n+1}}{n(n-1)},$$

we obtain

$$\sum_{k=1}^{n+1} (-1)^{n+k+1} \binom{n+k-1}{2k-2} \binom{2k-2}{k-1} = (-1)^{n+1} + 1 + (-1)^n = 1$$

as required. The identity (2.2) can be shown similarly by using the induction method. □

**Lemma 2.2.** *For the cases* $i > 1,\ j \geq 1$ *and* $i \geq 1,\ j \geq 1$, *we have*

$$\sum_{k=j}^{i} (-1)^{k+1} \binom{2i-2}{i-k} T(2k-2, k-j) = 0 \tag{2.3}$$

*and*

$$\sum_{k=i}^{j} (-1)^{k-1} \binom{2k-2}{k-i} T(2j-2, j-k) = 0 \tag{2.4}$$

*respectively.*

*Proof.* The results can be shown similarly as Lemma 2.1 by using the induction method. □

As a result of Lemma 2.2, the following corollary can be given.

**Corollary 2.1.** *For* $i > 1$*, we have*

$$\sum_{k=-i}^{i}(-1)^{k+1}\binom{2i-2}{i-k} = 0. \tag{2.5}$$

*Proof.* Taking $j = 1$ in (2.3), we have

$$\sum_{k=1}^{i}(-1)^{k+1}\binom{2i-2}{i-k}T(2k-2, k-1) = 0.$$

Since $T(2k-2, k-1) = 2$, we evaluate

$$\begin{aligned}
&\sum_{k=1}^{i}(-1)^{k+1}\binom{2i-2}{i-k}T(2k-2, k-1)\\
&= \binom{2i-2}{i-1} + \sum_{k=2}^{i}(-1)^{k+1}\binom{2i-2}{i-k}T(2k-2, k-1)\\
&= \binom{2i-2}{i-1} + 2\sum_{k=2}^{i}(-1)^{k+1}\binom{2i-2}{i-k}\\
&= \binom{2i-2}{i-1} + \sum_{k=2}^{i}(-1)^{k+1}\binom{2i-2}{i-k} + \sum_{k=-i}^{0}(-1)^{k+1}\binom{2i-2}{i-k}\\
&= \sum_{k=-i}^{i}(-1)^{k+1}\binom{2i-2}{i-k},
\end{aligned}$$

which is the desired result. □

For use in some algebraic applications, including solving systems of linear equations, we give the LU-decomposition of the matrix $S_n$ and the inverses $L^{-1}$ and $U^{-1}$ in the following theorem.

**Theorem 2.1.** *For* $1 \leq i, j \leq n$*,* $n > 1$*,* $S_n$ *admits the LU-decomposition*

$$S_n = LU, \tag{2.6}$$

*where*

$$l_{ij} = \binom{2i-2}{i-j},\ i \geq j,$$

$$u_{ij} = (-1)^{i-1}2^{\lceil \frac{i-1}{i} \rceil}\binom{2j-2}{j-i},\ i \leq j,$$

$$l_{ij}^{-1} = (-1)^{i+j}T(2i-2, i-j),\ i \geq j,$$

$$u_{ij}^{-1} = \frac{(-1)^{i-1}}{2}T(2j-2, j-i),\ i \leq j.$$

*Proof.* To prove $S_n = LU$, we show

$$[LU]_{ij} = \sum_{k=1}^{\min(i,j)} l_{ik}u_{kj} = s_{ij}.$$

By using some summation operations, we have

$$\begin{aligned}\sum_{k=1}^{\min(i,j)} l_{ik}u_{kj} &= l_{i1}u_{1j} + \sum_{k=2}^{\min(i,j)} l_{ik}u_{kj}\\ &= l_{i1}u_{1j} + 2\sum_{k=2}^{\min(i,j)} (-1)^{k-1}\binom{2i-2}{i-k}\binom{2j-2}{j-k}\\ &= l_{i1}u_{1j} + 2\sum_{k=1}^{\min(i-1,j-1)} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1}.\end{aligned}$$

On the other hand, observing

$$\begin{aligned}\sum_{k=1}^{\min(i-1,j-1)} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1} &= \sum_{k=-\min(i-1,j-1)}^{-1} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1}\\ &= \sum_{k=-\min(i-1,j-1)}^{0} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1}\\ &\quad -\binom{2i-2}{i-1}\binom{2j-2}{j-1}\end{aligned}$$

and by using $l_{i1}u_{1j} = \binom{2i-2}{i-1}$, the last equation gives us

$$\begin{aligned}\sum_{k=1}^{\min(i,j)} l_{ik}u_{kj} &= \sum_{k=1}^{\min(i-1,j-1)} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1}\\ &\quad + \sum_{k=-\min(i-1,j-1)}^{0} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1}\\ &= \sum_{k=-\min(i-1,j-1)}^{\min(i-1,j-1)} (-1)^{k}\binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1}.\end{aligned}$$

It follows from the Szily's identity (1.1) that

$$\sum_{k=1}^{\min(i,j)} l_{ik}u_{kj} = S(i-1, j-1) = s_{ij}.$$

Thus, we prove $S_n = LU$ as claimed.

Now, we show $LL^{-1} = I_n$, where $I_n$ is the identity matrix of order $n$. Clearly for $i < j$,

$$\left[LL^{-1}\right]_{ij} = 0$$

and for $i \geq j$,

$$\left[LL^{-1}\right]_{ij} = (-1)^j \sum_{k=j}^{i} (-1)^k \binom{2i-2}{i-k} T(2k-2, k-j).$$

The following cases can be observed for the last summation.

Case $i = j$:

$$\begin{aligned}\left[LL^{-1}\right]_{ii} &= \sum_{k=i}^{i} l_{ik} l_{kj}^{-1} \\ &= (-1)^{2i} \binom{2i-2}{0} T(2i-2, 0) \\ &= 1.\end{aligned}$$

The case $i > j \geq 1$ follows from Lemma 2.2, which gives $\left[LL^{-1}\right]_{ij} = 0$.

Now, we show the fact that $UU^{-1} = I_n$, where

$$\left[UU^{-1}\right]_{ij} = \frac{(-1)^{i-1}}{2} \sum_{k=i}^{j} (-1)^{k-1} 2^{\lceil \frac{i-1}{i} \rceil} \binom{2k-2}{k-i} T(2j-2, j-k),$$

for $j \geq i$. From the last equation, we can clearly observe the case $i = j = 1$ as

$$\left[UU^{-1}\right]_{11} = u_{11} u_{11}^{-1} = 1.$$

The other cases can be handled as follows.

Case $i = j$, $i \neq 1$:

$$\left[UU^{-1}\right]_{ii} = \sum_{k=i}^{i} u_{ik} u_{ki}^{-1} = 1.$$

Case $j > i \geq 1$:

By using the Lemma 2.2, we have

$$\left[UU^{-1}\right]_{ij} = (-1)^{i-1} \sum_{k=i}^{j} (-1)^{k-1} \binom{2k-2}{k-i} T(2j-2, j-k) = 0.$$

Thus, the proof is completed. □

It would be useful to notice that $L$ is a lower triangular matrix whose diagonal entries are 1. Hence, we can consider Theorem 2.1 as Doolittle's decomposition, and consequently we have that $\det L = 1$.

In the following corollary, we evaluate the determinant $\det(S_n)$ via the matrices $L$ and $U$ in Theorem 2.1.

**Corollary 2.2.** *For $n \geq 1$, we have the following:*

$$\det S_n = (-1)^{\lceil \frac{n-1}{2} \rceil} 2^{n-1}. \tag{2.7}$$

*Proof.* The proof is clear by calculating the following:

$$\det S_n = \det L \det U. \qquad \square$$

In the following theorem, we give the Cholesky decomposition of $S_n$ via the lower triangular matrix $C$ of order $n$.

**Theorem 2.2.** *For $1 \le i, j \le n$, $n > 1$, $S_n$ admits the Cholesky decomposition*

$$S_n = CC^T, \tag{2.8}$$

*where*

$$c_{ij} = \mathbf{i}^{\frac{1+(-1)^j}{2}} \binom{2i-2}{i-j} \left(\sqrt{2}\right)^{\left\lceil \frac{j-1}{j} \right\rceil},\ i \ge j,$$

$$c_{ij}^{-1} = \frac{(-1)^{i-1}}{\sqrt{2}} \mathbf{i}^{\frac{1+(-1)^i}{2}} C(i-1, j-1),\ i > 1, j \ge 1,\ c_{11}^{-1} = 1.$$

*Proof.* We claim that

$$\left[CC^T\right]_{ij} = \sum_{k=1}^{\min(i,j)} c_{ik} c_{jk} = s_{ij}.$$

For this purpose, we rewrite the sum and make necessary rearrangements as follows.

$$\begin{aligned}\sum_{k=1}^{\min(i,j)} c_{ik} c_{jk} &= \sum_{k=1}^{\min(i,j)} 2^{\left\lceil \frac{k-1}{k} \right\rceil} \mathbf{i}^{\left(1+(-1)^k\right)} \binom{2i-2}{i-k}\binom{2j-2}{j-k} \\ &= 2\sum_{k=2}^{\min(i,j)} (-1)^{k+1} \binom{2i-2}{i-k}\binom{2j-2}{j-k} + \binom{2i-2}{i-1}\binom{2j-2}{j-1} \\ &= 2\sum_{k=1}^{\min(i-1,j-1)} (-1)^k \binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1} + \binom{2i-2}{i-1}\binom{2j-2}{j-1}.\end{aligned}$$

Similar to the proof of Theorem 2.1, we obtain from Szily's identity that

$$\begin{aligned}\sum_{k=1}^{\min(i,j)} c_{ik} c_{jk} &= \sum_{k=-\min(i-1,j-1)}^{\min(i-1,j-1)} (-1)^k \binom{2i-2}{i-k-1}\binom{2j-2}{j-k-1} \\ &= S(i-1, j-1) = s_{ij}\end{aligned}$$

as claimed.

Now, we prove $CC^{-1} = I_n$. For the case $i < j$, using the sum

$$\left[CC^{-1}\right]_{ij} = \sum_{k=1}^{n} c_{ik} c_{kj}^{-1},$$

we observe $\left[CC^{-1}\right]_{ij} = 0$ and $\left[CC^{-1}\right]_{11} = 1$. To prove our claim, we need to examine the following three cases. For the cases $i = j \ne 1$ and $i > j$, $j > 1$, we have

$$\left[CC^{-1}\right]_{ii} = \mathbf{i}^{\left(1+(-1)^i\right)} (-1)^{i-1} C(i-1, i-1)$$

$$
\begin{aligned}
&= \mathbf{i}^{\left(2i-1+(-1)^{i}\right)} C(i-1, i-1) \\
&= C(i-1, i-1) = 1
\end{aligned}
$$

and

$$
\left[CC^{-1}\right]_{ij} = \sum_{k=j}^{i} \binom{2i-2}{i-k} \left(\sqrt{2}\right)^{\left\lceil \frac{k-1}{k} \right\rceil - 1} C(k-1, j-1),
$$

respectively. From the assumption of the case $i > j$, $j > 1$, clearly $k > 1$ is observed. Then, we obtain

$$
\begin{aligned}
\left[CC^{-1}\right]_{ij} &= \sum_{k=j}^{i} \binom{2i-2}{i-k} C(k-1, j-1) \\
&= \sum_{k=j}^{i} (-1)^{k+j} \binom{2i-2}{i-k} \frac{k-1}{j-1} \binom{k+j-3}{2j-3} \\
&= (-1)^{j} \sum_{k=j}^{i} (-1)^{k} \binom{2i-2}{i-k} T(2k-2, k-j).
\end{aligned}
$$

By virtue of Lemma 2.2, we have $\left[CC^{-1}\right]_{ij} = 0$. For the last case, $i > j$, $j = 1$, we obtain

$$
\begin{aligned}
\left[CC^{-1}\right]_{i1} &= \sum_{k=1}^{i} c_{ik} c_{k1}^{-1} \\
&= c_{i1} c_{11}^{-1} + \sum_{k=2}^{i} c_{ik} c_{k1}^{-1} \\
&= \binom{2i-2}{i-1} + \sum_{k=2}^{i} \mathbf{i}^{\left(1+(-1)^{k}\right)} (-1)^{k-1} \binom{2i-2}{i-k} \left(\sqrt{2}\right)^{\left\lceil \frac{k-1}{k} \right\rceil - 1} C(k-1, 0) \\
&= \binom{2i-2}{i-1} + 2 \sum_{k=2}^{i} (-1)^{k-1} \binom{2i-2}{i-k} \\
&= \sum_{k=-i}^{i} (-1)^{k-1} \binom{2i-2}{i-k}.
\end{aligned}
$$

From the Corollary 2.1, we obtain $\left[CC^{-1}\right]_{i1} = 0$ as required. Thus, the proof is done. □

In the following theorem, we give the Hermite normal form $H$ of $S_n$ that could be used in solving linear systems, cryptographic applications, and abstract algebra in future studies.

**Theorem 2.3.** *For $1 \le i, j \le n$, the following equation is true.*

$$
H = AS_n. \tag{2.9}
$$

*Here the diagonal matrix $H$ and the unitary matrix $A$ are given by*

$$
h_{ij} = \left\lceil \frac{2i-1}{i} \right\rceil, \; i = j,
$$

*and*

$$A = 2S_n^{-1} - B$$

*respectively, where the entries of the matrix B are*

$$b_{ij} = (-1)^{n+j} \left\lceil \frac{2-i}{i} \right\rceil \binom{n+j-2}{2j-2}.$$

*Proof.* Before we start the proof, we must emphasize that $H$ is a diagonal matrix. We start the proof by letting $g_{ij}$ and $d_{ij}$ be the elements of the matrices $BS_n$ and $AS_n$, respectively. Since $AS_n = 2I - BS_n$, it is simple to prove

$$g_{ij} = \begin{cases} 1, & i = j = 1, \\ 0, & \text{other}, \end{cases}$$

that implies

$$h_{ij} = d_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j = 1, \\ 2, & \text{other}. \end{cases}$$

We prove our claim by considering the cases of $i$ and $j$ as followings:

For the case $i = j$, we can write

$$g_{ii} = \sum_{k=1}^{n} b_{ik} s_{ki}.$$

Also, from the definition of the matrix $B$, we obtain

$$g_{ii} = \sum_{k=1}^{n} b_{ik} s_{ki} = 0,$$

for $i > 1$. The case $i = j = 1$ follows from Lemma 2.1 that

$$\begin{aligned} g_{11} &= \sum_{k=1}^{n} b_{1k} s_{k1} \\ &= \sum_{k=1}^{n} (-1)^{n+k} \binom{n+k-2}{2k-2} \binom{2k-2}{k-1} \\ &= 1. \end{aligned}$$

In the case $i \neq j$, $i > 1$, from the definition of $b_{ij}$, we clearly have

$$g_{ij} = \sum_{k=1}^{n} b_{ik} s_{kj} = 0.$$

The last case $i = 1$, follows from Lemma 2.1 as

$$g_{1j} = \sum_{k=1}^{n} b_{1k} s_{kj}$$

$$= \sum_{k=1}^{n} (-1)^{n+k} \binom{n+k-2}{2k-2} S(k-1, j-1) = 0.$$

Thus, we prove that $H = AS_n$ as claimed. On the other hand, we will show that $A$ is a unitary matrix. From the equation $H = AS_n$, one can write

$$A = HS_n^{-1}$$

and

$$\begin{aligned} \det(A) &= \det(H)\det(S_n^{-1}) \\ &= 2^{n-1}(-1)^{-\lceil \frac{n-1}{2} \rceil} 2^{-n+1} \\ &= (-1)^{\lceil \frac{n-1}{2} \rceil} \end{aligned}$$

which is the desired result. □

In addition to the results given before, we give a matrix identity in the next theorem. For this purpose, we define an $n \times n$ matrix $\acute{S}_n$ constructed by deleting the $1st$ row and the $nth$ column of the matrix $S_{n+1}$ as follows:

$$\acute{S}_n = \left[\acute{s}_{ij}\right]_{1 \le i,j \le n} = [S(i, j-1)] = \left[s_{(i+1)j}\right] = \begin{bmatrix} S(1,0) & S(1,1) & \dots & S(1,n-1) \\ S(2,0) & S(2,1) & \dots & S(2,n-1) \\ \vdots & \vdots & \ddots & \vdots \\ S(n,0) & S(n,1) & \dots & S(n,n-1) \end{bmatrix}.$$

We also present an $n \times n$ matrix $M$

$$M = \begin{bmatrix} \frac{T(2n,n-1)}{2} & -\frac{T(2n,n-2)}{2} & \frac{T(2n,n-3)}{2} & \dots & (-1)^{n+1}\frac{T(2n,0)}{2} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix},$$

whose entries are given by

$$m_{ij} = \begin{cases} \frac{(-1)^{j+1}}{2} T(2n, n-j), & i = 1, \\ 1, & i = j+1, \\ 0, & \text{other}. \end{cases}$$

In the following theorem, we give a matrix equation that involves the super Catalan matrix.

**Theorem 2.4.** *For $n \ge 1$, we have*

$$S_n = M\acute{S}_n. \tag{2.10}$$

*Proof.* We have two cases for the proof of this theorem. In the first case, $i = 1$, we write $M\acute{S}_n$ as

$$\left[M\acute{S}_n\right]_{1j} = \sum_{k=1}^{n} m_{1k}\acute{s}_{kj}$$

$$= \frac{1}{2}\sum_{k=1}^{n}(-1)^{k+1}T(2n, n-k)S(k, j-1)$$
$$= \frac{1}{2}\left(\sum_{k=0}^{n}(-1)^{k+1}T(2n, n-k)S(k, j-1) + T(2n, n)S(0, j-1)\right)$$
$$= \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}T(2n, n-k)S(k, j-1) + S(0, j-1).$$

Then, from Lemma 2.1, we conclude the case $i = 1$ as

$$\left[M\acute{S}_n\right]_{1j} = S(0, j-1) = s_{1j}.$$

Now, in the second case, $i \neq 1$, we have

$$\left[M\acute{S}_n\right]_{ij} = \sum_{k=1}^{n} m_{ik}\acute{s}_{kj}$$
$$= \sum_{k=1}^{i-2} m_{ik}\acute{s}_{kj} + m_{i(i-1)}\acute{s}_{(i-1)j} + \sum_{k=i}^{n} m_{ik}\acute{s}_{kj}.$$

Since $m_{ik} = 0$, for $k \neq i-1$, we conclude that

$$\left[M\acute{S}_n\right]_{ij} = m_{i(i-1)}\acute{s}_{(i-1)j} = \acute{s}_{(i-1)j} = s_{ij}$$

as claimed. Thus, the proof is completed. □

## 3. Solution of a system of linear equations involving super Catalan nmbers

LU-decomposition is one of the fundamental methods to solve systems of linear equations in linear algebra and an essential tool in various engineering applications. We solve a system of linear equations in the following calculations.

Consider the system of linear equations

$$\begin{aligned} S(0,0)x_1 + S(0,1)x_2 + ... + S(0, n-1)x_n &= b_1 \\ S(1,0)x_1 + S(1,1)x_2 + ... + S(1, n-1)x_n &= b_2 \\ &\vdots \\ S(n-1,0)x_1 + S(n-1,1)x_2 + ... + S(n-1, n-1)x_n &= b_n. \end{aligned}$$

By using the LU-decomposition, we have

$$L(Ux) = b,$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Let $Ux = c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

We first solve the equation $Lc = b$:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{2n-2}{n-1} & \binom{2n-2}{n-2} & \dots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{3.1}$$

as follows. We obtain from Eq (3.1) that

$$\begin{aligned} c_1 &= b_1 \\ c_2 &= b_2 - 2b_1 \\ c_3 &= b_3 - 4b_2 + 2b_1 \\ c_4 &= b_4 - 6b_3 + 9b_2 - 2b_1 \\ &\vdots \\ c_n &= \sum_{k=1}^{n} b_k C(n-1, k-1), \end{aligned}$$

that is,

$$c = Ux = \left[ \sum_{k=1}^{i} b_k C(i-1, k-1) \right]_{1<i\leq n}, \ c_1 = b_1.$$

Now, we solve the equation $Ux = c$:

$$\begin{bmatrix} 1 & 2 & \dots & \binom{2n-2}{n-1} \\ 0 & -2 & \dots & -2\binom{2n-2}{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^{n+1}2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Then, we obtain

$$\begin{aligned} x_n &= \frac{(-1)^{n+1}}{2} c_n \\ x_{n-1} &= \frac{(-1)^n}{2} \left( c_{n-1} + (2n-2)\, c_n \right) \\ x_{n-2} &= \frac{(-1)^{n+1}}{2} \left( c_{n-2} + (2n-4)\, c_{n-1} + \frac{2n-2}{2n-4} \binom{2n-4}{2} c_n \right) \\ x_{n-3} &= \frac{(-1)^n}{2} \left( c_{n-3} + \frac{2n-6}{2n-7} \binom{2n-7}{1} c_{n-2} + \frac{2n-4}{2n-6} \binom{2n-6}{2} c_{n-1} + \frac{2n-2}{2n-5} \binom{2n-5}{3} c_n \right) \\ &\vdots \\ x_1 &= \frac{1}{2} \sum_{k=1}^{n} c_k T(2k-2, k-1) = \sum_{k=1}^{n} c_k \end{aligned}$$

which implies

$$x = \left[ \frac{(-1)^{i+1}}{2} \sum_{k=i}^{n} c_k T(2k-2, k-i) \right]_{1 \le i \le n}.$$

By substituting the $c_k$, we get the solution as

$$\begin{aligned} x &= \left[ \frac{(-1)^{i+1}}{2} \sum_{1 \le r \le k, i \le k \le n} b_r T(2k-2, k-i) C(k-1, r-1) \right]_{1 < i \le n}, \\ x_1 &= b_1 + \sum_{1 \le r \le k, 2 \le k \le n} b_r C(k-1, r-1). \end{aligned} \tag{3.2}$$

## 4. Numerical examples

Some numerical examples of the results obtained in this study are given in this section.

**Example 1.** *For $n = 5$, as an example of Theorems 2.1 and 2.2, we have:*

$$S_5 = \begin{bmatrix} 1 & 2 & 6 & 20 & 70 \\ 2 & 2 & 4 & 10 & 28 \\ 6 & 4 & 6 & 12 & 28 \\ 20 & 10 & 12 & 20 & 40 \\ 70 & 28 & 28 & 40 & 70 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \\ 20 & 15 & 6 & 1 & 0 \\ 70 & 56 & 28 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 & 20 & 70 \\ 0 & -2 & -8 & -30 & -112 \\ 0 & 0 & 2 & 12 & 56 \\ 0 & 0 & 0 & -2 & -16 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -2 & 9 & -6 & 1 & 0 \\ 2 & -16 & 20 & -8 & 1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -2 & -\frac{9}{2} & -8 \\ 0 & 0 & \frac{1}{2} & 3 & 10 \\ 0 & 0 & 0 & -\frac{1}{2} & -4 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

*and*

$$\begin{aligned} S_5 &= \begin{bmatrix} 1 & 2 & 6 & 20 & 70 \\ 2 & 2 & 4 & 10 & 28 \\ 6 & 4 & 6 & 12 & 28 \\ 20 & 10 & 12 & 20 & 40 \\ 70 & 28 & 28 & 40 & 70 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & i\sqrt{2} & 0 & 0 & 0 \\ 6 & 4i\sqrt{2} & \sqrt{2} & 0 & 0 \\ 20 & 15i\sqrt{2} & 6\sqrt{2} & i\sqrt{2} & 0 \\ 70 & 56i\sqrt{2} & 28\sqrt{2} & 8i\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 & 20 & 70 \\ 0 & i\sqrt{2} & 4i\sqrt{2} & 15i\sqrt{2} & 56i\sqrt{2} \\ 0 & 0 & \sqrt{2} & 6\sqrt{2} & 28\sqrt{2} \\ 0 & 0 & 0 & i\sqrt{2} & 8i\sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \end{aligned}$$

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ i\sqrt{2} & -\frac{1}{2}i\sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & -2\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ i\sqrt{2} & -\frac{9}{2}i\sqrt{2} & 3i\sqrt{2} & -\frac{1}{2}i\sqrt{2} & 0 \\ \sqrt{2} & -8\sqrt{2} & 10\sqrt{2} & -4\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix},$$

*respectively.*

**Example 2.** *For $n = 6$, Theorem 2.3 can be seen by a simple calculation:*

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 15 & -35 & 28 & -9 & 1 \\ 30 & -435 & 980 & -756 & 234 & -25 \\ -70 & 980 & -2135 & 1596 & -480 & 50 \\ 56 & -756 & 1596 & -1162 & 342 & -35 \\ -18 & 234 & -480 & 342 & -99 & 10 \\ 2 & -25 & 50 & -35 & 10 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 & 20 & 70 & 252 \\ 2 & 2 & 4 & 10 & 28 & 84 \\ 6 & 4 & 6 & 12 & 28 & 72 \\ 20 & 10 & 12 & 20 & 40 & 90 \\ 70 & 28 & 28 & 40 & 70 & 140 \\ 252 & 84 & 72 & 90 & 140 & 252 \end{bmatrix},$$

*where*

$$A = \begin{bmatrix} -1 & 15 & -35 & 28 & -9 & 1 \\ 30 & -435 & 980 & -756 & 234 & -25 \\ -70 & 980 & -2135 & 1596 & -480 & 50 \\ 56 & -756 & 1596 & -1162 & 342 & -35 \\ -18 & 234 & -480 & 342 & -99 & 10 \\ 2 & -25 & 50 & -35 & 10 & -1 \end{bmatrix}$$
$$= 2\begin{bmatrix} -1 & 15 & -35 & 28 & -9 & 1 \\ 15 & -\frac{435}{2} & 490 & -378 & 117 & -\frac{25}{2} \\ -35 & 490 & -\frac{2135}{2} & 798 & -240 & 25 \\ 28 & -378 & 798 & -581 & 171 & -\frac{35}{2} \\ -9 & 117 & -240 & 171 & -\frac{99}{2} & 5 \\ 1 & -\frac{25}{2} & 25 & -\frac{35}{2} & 5 & -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} -1 & 15 & -35 & 28 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Example 3.** *An example of Theorem 2.4 can be given as*

$$S_5 = \begin{bmatrix} 1 & 2 & 6 & 20 & 70 \\ 2 & 2 & 4 & 10 & 28 \\ 6 & 4 & 6 & 12 & 28 \\ 20 & 10 & 12 & 20 & 40 \\ 70 & 28 & 28 & 40 & 70 \end{bmatrix} = \begin{bmatrix} \frac{25}{2} & -25 & \frac{35}{2} & -5 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 & 10 & 28 \\ 6 & 4 & 6 & 12 & 28 \\ 20 & 10 & 12 & 20 & 40 \\ 70 & 28 & 28 & 40 & 70 \\ 252 & 84 & 72 & 90 & 140 \end{bmatrix}.$$

## 5. Conclusions

As an integer number sequence, the super Catalan number sequence is one of the most studied contents in number theory, combinatorics, graph theory, etc, due to being a generalization of Catalan numbers and related to binomial coefficients. Moreover, some matrix applications of this sequence can be found in linear algebra. In this article, we have studied the matrix decompositions related to super Catalan numbers and showed the relation between these numbers, Lucas polynomials, and Chebyshev polynomials. We have obtained some auxiliary summation formulas. By using the decomposition method, we have given the solution of a system of linear equations. Decompositions and Hermite normal form can be used in applications of engineering, numerical analysis, cryptography, etc. In addition, the methods and identities in this paper can be extended to q-calculus in future studies.

## Author contributions

Serpil Halıcı: Supervision, conceptualization, project administration, investigation, writing-review and editing; Zehra Betül Gür: Formal analysis, investigation, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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