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Research article

A new class of degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type

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Abstract: In this paper, we consider a new class of degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type, denoted by $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta,\zeta;\rho;\mu)$. We obtain several summation formulae, a recurrence relation, two difference operator formulas, two derivative operator formulas, an implicit summation formula, and a symmetric property for these polynomials. Also, we provide a representation of the degenerate differential operator on the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type. Moreover, we define the degenerate unified Hermite-based Apostol-Stirling polynomials of the second kind and derive some properties of these newly established polynomials. In addition, we prove multifarious correlations, including the new polynomials. Furthermore, we list the first few degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type for some special cases and present data visualizations of zeros forming 2D and 3D structures. Finally, we provide a table covering approximate solutions for the zeros of $\mathcal{W}^{(\alpha)}_{n,3}(\delta,4;3;2)$.

Keywords: Appell-type polynomials; Bernoulli numbers and polynomials; Euler numbers and polynomials; Hermite polynomials; unified polynomials; degenerate exponential function
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1. Introduction

Special functions hold significant importance across various domains of mathematics, physics, engineering, and related disciplines, encompassing topics such as differential equations, mathematical analysis, functional analysis, mathematical physics, and quantum mechanics. Notably, the family of special polynomials represents one of the most useful, widespread, and applicable categories of special functions. Among the most prominent polynomials within the theory of special polynomials are the Bernoulli (see, e.g., [1, 8, 11, 15, 16, 18–22]) and Euler polynomials (see, e.g., [1, 5, 6, 8, 11, 15, 16, 18–22]), and the Hermite polynomials (see, e.g., [2, 3, 10, 12, 18, 23]).

Recently, unified forms of polynomials have been intensely considered, and many of their properties and relations have been investigated in detail. For example, in [1], a unification of the generalized multiparameter Apostol-type Bernoulli, Euler, Fubini, and Genocchi polynomials of higher order was defined, and an explicit formula of these unified generalized polynomials in terms of the Gaussian hypergeometric function, and several symmetry identities was provided. In [5], a novel class of degenerate biparametric Apostol-type polynomials was defined, and diverse algebraic and differential properties associated with these polynomials were derived. In [6], a new class of degenerate unified polynomials covering the Appell-type classical polynomials and their most relevant generalizations was defined, and several of their formulas were examined. In [7], unified Apostol-Bernoulli and Apostol-Euler polynomials were introduced and multifarious properties, some of which were inspired via umbral calculus, were studied. In [17], a degenerate version of the unified Apostol-type Bernoulli, Euler, Genocchi, and Fubini polynomials defined by Acala [1] were considered, and some identities and recurrence relations, a symmetric relation, and summation formulas were given. In [18], Hermite-based unified Apostol-Bernoulli, Euler, and Genocchi polynomials were introduced, and some symmetry identities, explicit closed-form formulae and a finite series relation between this unification and 3d-Hermite polynomials were obtained. In [20], unified Apostol-type Hermite-Bernoulli, Hermite-Euler, and Hermite-Genocchi polynomials were defined, and some correlations with Fubini-Hermite and Bell-Hermite polynomials, implicit summation formulae and symmetric identities were acquired. In [21], the three parametric types of Apostol-type unified Bernoulli-Euler polynomials were introduced, fundamental properties and the partial derivatives were derived, and their zeros, graphical representations, and approximation values for specific parameters were provided.

With the definitions of the first degenerate polynomials and numbers, degenerate Stirling, Bernoulli, and Eulerian numbers, introduced by Carlitz [8, 9], in recent years, the degenerate forms for the many special polynomials have been considerably studied and investigated by many mathematicians [4–6, 11, 13–17, 19], and see also each of the references cited therein. For example, degenerate q-Daehee polynomials with weight α in [4], two novel classes of degenerate biparametric Apostol-type polynomials in [5], degenerate unified polynomials covering the Appell-type classical polynomials and their most relevant generalizations in [6], Gould-Hopper-based fully degenerate poly-Bernoulli polynomials with a q-parameter in [11], degenerate Stirling polynomials of the second kind in [13], fully degenerate Bernoulli polynomials and degenerate Euler polynomials in [15], degenerate Bernoulli and degenerate Euler polynomials in [16], unified degenerate Apostol-type Bernoulli, Euler, Genocchi, and Fubini polynomials in [17], and degenerate Bernoulli polynomials in [19] were defined and many of their properties and applications were investigated in detail.

In this paper, building upon the foundational work of the above studies, particularly [6, 7, 10], we introduce the degenerate unified Bernoulli-Euler Hermite polynomials of the Apostol type and then examine some of their properties and applications. We emphasize the influence of prior research in this domain and introduce novel insights into the algebraic and differential characteristics of these polynomials. Our investigations encompass properties such as recurrence relations, symmetry, and summation formulas. Additionally, we introduce degenerate unified Hermite-based Apostol Stirling polynomials of the second kind, deriving properties analogous to those discussed in the preceding section. Subsequently, we employ the Mathematica program to compute the distribution of zeros of these polynomials.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ represent the set of all natural numbers, the set of all integers, the set of all nonnegative integers, the set of all real numbers, and the set of all complex numbers, respectively.

For $\lambda \in \mathbb{C}$, the λ-falling factorial $(\delta)_{n,\lambda}$ is defined by (see [4–6, 11, 13–17, 19])

$$(\delta)_{n,\lambda} = \begin{cases} \delta(\delta-\lambda)(\delta-2\lambda)\cdots(\delta-(n-1)\lambda), & n = 1, 2, \ldots \\ 1, & n = 0. \end{cases} \tag{2.1}$$

In the case when $\lambda = 1$, the λ-falling factorial simplifies to the conventional falling factorial as follows:

$$(\delta)_{n,1} := (\delta)_n = \delta(\delta-1)\cdots(\delta-n+1) \text{ and } (\delta)_0 = 1.$$

The operator $\Delta_{\lambda,\delta}$, which represents the difference with respect to δ, is defined as follows (cf. [11]):

$$\Delta_{\lambda,\delta} f(\delta) = \frac{1}{\lambda}(f(\delta+\lambda) - f(\delta)), \quad \lambda \neq 0. \tag{2.2}$$

The definition of the degenerate exponential function $e_\lambda^\delta(t)$ is given as follows (see, e.g., [4–6, 11, 13–17, 19]):

$$e_\lambda^\delta(t) = (1+\lambda t)^{\frac{\delta}{\lambda}} \text{ and } e_\lambda^1(t) = e_\lambda(t). \tag{2.3}$$

It is readily seen that $\lim_{\lambda\to 0} e_\lambda^\delta(t) = e^{\delta t}$. From (2.3), we obtain the following relation:

$$e_\lambda^\delta(t) = \sum_{n=0}^{\infty} (\delta)_{n,\lambda} \frac{t^n}{n!}, \tag{2.4}$$

which satisfies the following difference rule

$$\Delta_{\lambda,\delta} e_\lambda^\delta(t) = t e_\lambda^\delta(t). \tag{2.5}$$

Note that $e_\lambda^1(t) := e_\lambda(t)$ and $e_\lambda^\delta(t)\, e_\lambda^\omega(t) = e_\lambda^{\delta+\omega}(t)$.

The classical Bernoulli polynomials, denoted as $B_n(\delta)$, and the degenerate Bernoulli polynomials, represented as $B_{n,\lambda}(\delta)$, are defined as follows:

$$\sum_{n=0}^{\infty} B_n(\delta) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{\delta t} \text{ and } \sum_{n=0}^{\infty} B_{n,\lambda}(\delta) \frac{t^n}{n!} = \frac{t}{e_\lambda(t) - 1} e_\lambda^\delta(t), \; |t| < 2\pi.$$

The classical Euler polynomials, denoted as $E_n(\delta)$, and the degenerate Euler polynomials, represented as $E_{n,\lambda}(\delta)$, are defined as follows:

$$\sum_{n=0}^{\infty} E_n(\delta)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{\delta t} \text{ and } \sum_{n=0}^{\infty} E_{n,\lambda}(\delta)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}e_\lambda^{\delta}(t), \ \ |t| < \pi.$$

For a comprehensive understanding of the diverse applications of Bernoulli and Euler polynomials, one may consult the references [1, 8, 11, 15, 16, 18–22]. The degenerate Stirling polynomials of the second kind are defined as follows (cf. [8, 13]):

$$\sum_{n=0}^{\infty} S_{2,\lambda}(n,k:\delta)\frac{t^n}{n!} = \frac{(e_\lambda(t)-1)^k}{k!}e_\lambda^{\delta}(t). \tag{2.6}$$

Taking $\delta = 0$ in (2.6) yields $S_{2,\lambda}(n,k:0) := S_{2,\lambda}(n,k)$, called degenerate Stirling numbers of the second kind. The degenerate Stirling numbers of the second kind are also given by

$$(\delta)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(\delta)_k. \tag{2.7}$$

The two-variable Hermite polynomial of the Gould-Hopper type [2, 3, 10, 12, 18, 23] is revisited by employing the following formula:

$$H_n(\delta,\zeta) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{\zeta^r\delta^{n-2r}}{r!(n-2r)!},$$

where $[\frac{n}{2}]$ is the truncated part of $\frac{n}{2}$. These polynomials satisfy the following generating equation:

$$e^{\delta t+\zeta t^2} = \sum_{n=0}^{\infty} H_n(\delta,\zeta)\frac{t^n}{n!}.$$

It is to be noted that

$$H_0(\delta,\zeta) = 1 \text{ and } H_n(2\delta,-1) = H_n(\delta),$$

where $H_n(\delta)$ are the ordinary Hermite polynomials; see [2]. By applying the derivative of order r to the Hermite polynomials, we have the following equality:

$$\frac{\partial^r}{\partial\delta^r}H_n(\delta,\zeta) = \frac{n!}{(n-r)!}H_{n-r}(\delta,\zeta),$$

see [23]. This family of polynomials solves the following difference equation:

$$\frac{\partial}{\partial\zeta}H_n(\delta,\zeta) = \frac{\partial^2}{\partial\delta^2}H_n(\delta,\zeta).$$

The above equation is called the heat equation. Taking $\zeta = 0$ for $H_n(\delta,\zeta)$, we obtain

$$H_n(\delta,0) = \delta^n.$$

Let $\lambda \in \mathbb{R}\backslash\{0\}$. The degenerate Hermite polynomials $H_{n,\lambda}(\delta,\zeta)$ are defined below:

$$\sum_{n=0}^{\infty} H_{n,\lambda}(\delta,\zeta)\frac{t^n}{n!} = e_\lambda^{\delta}(t)e_\lambda^{\zeta}(t^2). \tag{2.8}$$

In [10, 11, 22], the polynomials $H_{n,\lambda}(\delta,\zeta)$ were examined for their varied applications and characteristics.

3. A novel class of degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type

In this part, we aim to introduce a degenerate form of unified Bernoulli-Euler Hermite polynomials of Apostol type in (3.2) and to derive some of its properties and relationships.

In recent years, Belbachir et al. [7] introduced and analyzed a category of polynomials termed the unified Bernoulli-Euler polynomials of Apostol type, characterized by the following power series:

$$\frac{2-\mu+\frac{\mu}{2}t}{\rho e^{t}+(1-\mu)}e^{\delta t}=\sum_{n=0}^{\infty}\mathcal{W}_{n}(\delta;\rho;\mu)\frac{t^{n}}{n!}, \tag{3.1}$$

where

$$\left|\ln\left(\frac{\rho}{1-\mu}\right)+t\right|<2\pi, \quad for \quad 1>\mu\geq 0,$$

and

$$\left|\ln\left(\frac{\rho}{\mu-1}\right)+t\right|<\pi, \quad \text{otherwise.}$$

These polynomials provide some families of polynomials in the literature according to the specific values of their parameters. Determinantal representation of Bernoulli-Euler polynomials of Apostol type, generalized Raabe's Theorem, some explicit formulas, derivation, and integration representations, and identities inspired via umbral calculus for the unified Bernoulli-Euler polynomials of Apostol type were investigated in [7]. Then, Bedoya et al. [6] conducted a study on the degenerate version of these polynomials. Utilizing the generating function, they analyzed numerous algebraic properties of this polynomial and provided several graphical examples.

Very recently, Diaz et al. [10] defined the unified Bernoulli-Euler Hermite polynomials of Apostol type, which are provided by

$$\left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e^{t}+(1-\mu)}\right)^{\alpha}e^{\delta t+\zeta t^{2}}=\sum_{n=0}^{\infty}\mathcal{W}_{n}^{(\alpha)}\left(\delta,\zeta;\rho;\mu\right)\frac{t^{n}}{n!}. \tag{3.2}$$

These polynomials were studied and analyzed in detail in [10]. Also, applications of the monomiality principle were considered, and many interesting formulas were investigated.

We are now ready to give our primary definition.

Definition 3.1. *Let $\mu\in\mathbb{R}^{+}-\{1\}$, $\rho\in\mathbb{R}^{+}$ and $\alpha\in\mathbb{C}$. We define the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type via the following exponential generating function:*

$$\left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_{\lambda}(t)+(1-\mu)}\right)^{\alpha}e_{\lambda}^{\delta}(t)e_{\lambda}^{\zeta}(t^{2})=\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)\frac{t^{n}}{n!}, \tag{3.3}$$

where

$$\left|\ln\left(\frac{\rho}{1-\mu}\right)+t\right|<2\pi, \quad for \quad 1>\mu\geq 0,$$

and

$$\left|\ln\left(\frac{\rho}{\mu-1}\right)+t\right|<\pi, \quad otherwise.$$

Throughout the paper, when the superscript $\alpha = 1$ is omitted, we denote

$$\mathcal{W}_{n,\lambda}(\delta, \zeta; \rho; \mu) := \mathcal{W}^{(1)}_{n,\lambda}(\delta, \zeta; \rho; \mu).$$

Here, we examine some special circumstances of the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$.

Remark 3.2. *Choosing $\lambda \to 0$ and $\alpha = 1$ in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the usual unified Bernoulli-Euler Hermite polynomials of Apostol type in (3.2).*

Remark 3.3. *Choosing $\lambda \to 0 = \zeta = \alpha - 1$ in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the degenerate unified Bernoulli-Euler numbers of Apostol type in (3.1).*

Remark 3.4. *Choosing $\zeta = 0$ in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the degenerate unified Bernoulli-Euler polynomials of Apostol type $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta; \rho; \mu)$ given by*

$$\left(\frac{2 - \mu + \frac{\mu}{2}t}{\rho e_\lambda(t) + (1 - \mu)}\right)^\alpha e^\delta_\lambda(t) = \sum_{n=0}^{\infty} \mathcal{W}^{(\alpha)}_{n,\lambda}(\delta; \rho; \mu)\frac{t^n}{n!}.$$

Remark 3.5. *Replacing δ by 0 in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the degenerate Apostol type unified Bernoulli-Euler polynomials of $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$, which is a new family of polynomials, as follows:*

$$\left(\frac{2 - \mu + \frac{\mu}{2}t}{\rho e_\lambda(t) + (1 - \mu)}\right)^\alpha e^\zeta_\lambda(t^2) = \sum_{n=0}^{\infty} \mathcal{W}^{(\alpha)}_{n,\lambda}(0, \zeta; \rho; \mu)\frac{t^n}{n!}. \tag{3.4}$$

Remark 3.6. *Choosing $\zeta = \delta = 0$ in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the degenerate unified Bernoulli-Euler numbers of Apostol type $\mathcal{W}^{(\alpha)}_{n,\lambda}(\rho; \mu)$ given by*

$$\left(\frac{2 - \mu + \frac{\mu}{2}t}{\rho e_\lambda(t) + (1 - \mu)}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{W}^{(\alpha)}_{n,\lambda}(\rho; \mu)\frac{t^n}{n!}.$$

Remark 3.7. *Replacing μ by 0 in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the degenerate Hermite based Apostol-Euler polynomials $\mathcal{E}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho)$ of order α as follows (cf. [17, 22]):*

$$\left(\frac{2}{\rho e_\lambda(t) + 1}\right)^\alpha e^\delta_\lambda(t)e^\zeta_\lambda(t^2) = \sum_{n=0}^{\infty} \mathcal{E}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho)\frac{t^n}{n!}.$$

Remark 3.8. *Replacing μ by 2 in (3.3), the polynomials $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ become the degenerate Hermite based Apostol-Bernoulli polynomials $\mathcal{B}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho)$ of order α as follows (cf. [17, 22]):*

$$\left(\frac{t}{\rho e_\lambda(t) - 1}\right)^\alpha e^\delta_\lambda(t)e^\zeta_\lambda(t^2) = \sum_{n=0}^{\infty} \mathcal{B}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho)\frac{t^n}{n!}.$$

Now, we examine several properties of $\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta, \zeta; \rho; \mu)$ by the following consecutive theorems with their proofs.

Theorem 3.9. *The following implicit summation formula holds for $n \in \mathbb{N}_0$:*

$$\mathcal{W}_{n,\lambda}^{(\alpha+\beta)}(\delta_1+\delta_2,\zeta_1+\zeta_2;\rho;\mu)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta_1,\zeta_1;\rho;\mu)\,\mathcal{W}_{n-k,\lambda}^{(\beta)}(\delta_2,\zeta_2;\rho;\mu). \tag{3.5}$$

Proof. It is readily seen from (3.3) that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}^{(\alpha+\beta)}(\delta_1+\delta_2,\zeta_1+\zeta_2;\rho;\mu)\frac{t^n}{n!}&=\left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\alpha+\beta}e_\lambda^{\delta_1+\delta_2}(t)e_\lambda^{\zeta_1+\zeta_2}(t^2)\\&=\left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\alpha}e_\lambda^{\delta_1}(t)e_\lambda^{\zeta_1}(t^2)\left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\beta}e_\lambda^{\delta_2}(t)e_\lambda^{\zeta_2}(t^2)\\&=\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta_1,\zeta_1;\rho;\mu)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}^{(\beta)}(\delta_2,\zeta_2;\rho;\mu)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta_1,\zeta_1;\rho;\mu)\,\mathcal{W}_{n-k,\lambda}^{(\beta)}(\delta_2,\zeta_2;\rho;\mu)\frac{t^n}{n!},\end{aligned}$$

which gives the alleged result (3.5). □

Corollary 3.10. *Some special cases of the addition formula given above are provided as follows:*

$$\begin{aligned}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta_1+\delta_2,\zeta;\rho;\mu)&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{n-k,\lambda}^{(\alpha)}(\delta_1,\zeta;\rho;\mu)\,(\delta_2)_{k,\lambda},\\\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta_1+\delta_2,\zeta_1+\zeta_2;\rho;\mu)&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta_1,\zeta_1;\rho;\mu)\,H_{n-k}(\delta_2,\zeta_2),\\\mathcal{W}_{n,\lambda}^{(\alpha+\beta)}(\delta,\zeta;\rho;\mu)&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)\,\mathcal{W}_{n-k,\lambda}^{(\beta)}(\rho;\mu),\\\mathcal{W}_{n,\lambda}^{(\alpha+1)}(\delta_1+\delta_2,\zeta_1+\zeta_2;\rho;\mu)&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta_1,\zeta_1;\rho;\mu)\,\mathcal{W}_{n-k,\lambda}(\delta_2,\zeta_2;\rho;\mu).\end{aligned}$$

The following theorem gives a recurrence relation for the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type.

Theorem 3.11. *The following relation*

$$(2-\mu)\,H_{n,\lambda}(\delta,\zeta)+\frac{\mu n}{2}H_{n-1,\lambda}(\delta,\zeta)=\rho\mathcal{W}_{n,\lambda}(\delta+1,\zeta;\rho;\mu)+(1-\mu)\,\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)$$

with

$$(2-\mu)=\rho\mathcal{W}_{0,\lambda}(\delta+1,\zeta;\rho;\mu)+(1-\mu)\,\mathcal{W}_{0,\lambda}(\delta,\zeta;\rho;\mu),$$

holds for $n \in \mathbb{N}$.

Proof. It can be written by Definition 3.1 that

$$\Lambda=\left(2-\mu+\frac{\mu}{2}t\right)e_{\lambda}^{\delta}(t)e_{\lambda}^{\zeta}(t^{2})=(\rho e_{\lambda}(t)+(1-\mu))\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^{n}}{n!}.$$

So, we observe that

$$\begin{aligned}\Lambda &= \left(2-\mu+\frac{\mu}{2}t\right)e_{\lambda}^{\delta}(t)e_{\lambda}^{\zeta}(t^{2})\\ &= \sum_{n=0}^{\infty}(2-\mu)H_{n,\lambda}(\delta,\zeta)\frac{t^{n}}{n!}+\frac{\mu}{2}\sum_{n=0}^{\infty}H_{n,\lambda}(\delta,\zeta)\frac{t^{n+1}}{n!}\\ &= (2-\mu)+\sum_{n=0}^{\infty}\frac{2-\mu}{n+1}H_{n+1,\lambda}(\delta,\zeta)\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty}\frac{\mu}{2}H_{n,\lambda}(\delta,\zeta)\frac{t^{n+1}}{n!}\\ &= (2-\mu)+\sum_{n=0}^{\infty}\left(\frac{2-\mu}{n+1}H_{n+1,\lambda}(\delta,\zeta)+\frac{\mu}{2}H_{n,\lambda}(\delta,\zeta)\right)\frac{t^{n+1}}{n!},\end{aligned}$$

and also

$$\begin{aligned}\Lambda &= (\rho e_{\lambda}(t)+(1-\mu))\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^{n}}{n!}\\ &= \rho\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta+1,\zeta;\rho;\mu)\frac{t^{n}}{n!}+(1-\mu)\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^{n}}{n!}\\ &= \sum_{n=0}^{\infty}(\rho\mathcal{W}_{n,\lambda}(\delta+1,\zeta;\rho;\mu)+(1-\mu)\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu))\frac{t^{n}}{n!}.\end{aligned}$$

In addition, we get

$$\Lambda=\sum_{n=0}^{\infty}\left(\rho\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}(\delta,\zeta;\rho;\mu)(1)_{n-k,\lambda}+(1-\mu)\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\right)\frac{t^{n}}{n!}.$$

Therefore, the proof is completed. □

Corollary 3.12. *Another representation of Theorem 3.11 is given by:*

$$\begin{aligned}&(2-\mu)H_{n,\lambda}(\delta,\zeta)+\frac{\mu n}{2}H_{n-1,\lambda}(\delta,\zeta)\\ &= \rho\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,\lambda}(\delta,\zeta;\rho;\mu)(1)_{n-k,\lambda}+(1-\mu)\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\end{aligned}$$

with

$$\mathcal{W}_{0,\lambda}(\delta,\zeta;\rho;\mu)=\frac{(2-\mu)}{(\rho+1-\mu)}.$$

Remark 3.13. *The formula in Theorem 3.11 is an extension of the well-known formulas for Bernoulli and Euler polynomials provided by (cf. [18, 20])*

$$\delta^n = \frac{B_{n+1}(\delta+1) - B_{n+1}(\delta)}{n+1} \text{ and } \delta^n = \frac{E_{n+1}(\delta+1) + E_{n+1}(\delta)}{2}.$$

Two different operator rules for $\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu)$ are as follows.

Theorem 3.14. *The polynomials $\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \lambda; \rho; \mu)$ fulfill the following difference operator formulas for $n \in \mathbb{N}$:*

$$\Delta_{\lambda,\delta}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) = n\,\mathcal{W}_{n-1,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu),$$

and

$$\Delta_{\lambda,\zeta}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) = n(n-1)\,\mathcal{W}_{n-2,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu).$$

Proof. Based on the following difference properties

$$\Delta_{\lambda,\delta} e_\lambda^\delta(t) e_\lambda^\zeta(t^2) = t e_\lambda^\delta(t) e_\lambda^\zeta(t^2),$$

and

$$\Delta_{\lambda,\zeta} e_\lambda^\delta(t) e_\lambda^\zeta(t^2) = t^2 e_\lambda^\delta(t) e_\lambda^\zeta(t^2),$$

the proof is done. □

We now provide derivative operator properties for the polynomials $\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu)$ as follows.

Theorem 3.15. *The derivative operator formulas for the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type hold for $n \in \mathbb{N}$:*

$$\frac{\partial}{\partial \delta}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) = \sum_{k=0}^{n-1} \mathcal{W}_{n-(k+1),\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) \frac{n!\,(-\lambda)^k}{(n-(k+1))!\,(k+1)}, \tag{3.6}$$

and

$$\frac{\partial}{\partial \zeta}\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) = \sum_{k=0}^{\lfloor \frac{n}{2}-1 \rfloor} \mathcal{W}_{n-2(k+1),\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) \frac{n!\,(-\lambda)^{2k}}{(n-2(k+1))!\,(2k+1)}. \tag{3.7}$$

Proof. By applying the usual partial derivative operator to both sides of (3.3) with respect to δ and ζ, respectively, we then derive

$$\begin{aligned}
\frac{\partial}{\partial \delta}\left(\sum_{n=0}^{\infty} \mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) \frac{t^n}{n!}\right) &= \left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t) + (1-\mu)}\right)^\alpha e_\lambda^\zeta(t^2) \frac{\partial}{\partial \delta}(1+\lambda t)^{\frac{\delta}{\lambda}} \\
&= \left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t) + (1-\mu)}\right)^\alpha e_\lambda^\zeta(t^2)(1+\lambda t)^{\frac{\delta}{\lambda}} \ln(1+\lambda t)^{\frac{1}{\lambda}} \\
&= \sum_{n=0}^{\infty} \frac{\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu)}{n!} t^n \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n+1} t^{n+1} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{W}_{n-k,\lambda}^{(\alpha)}(\delta, \zeta; \rho; \mu) \frac{(-\lambda)^k}{(n-k)!\,(k+1)} t^{n+1},
\end{aligned}$$

and

$$\frac{\partial}{\partial \zeta}\left(\sum_{n=0}^{\infty} \mathcal{W}_{n,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!}\right) = \left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\alpha} e_\lambda^{\delta}(t)\frac{\partial}{\partial \zeta}\left(1+\lambda t^2\right)^{\frac{\zeta}{\lambda}}$$
$$= \left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\alpha} e_\lambda^{\delta}(t)\left(1+\lambda t^2\right)^{\frac{\zeta}{\lambda}}\ln\left(1+\lambda t^2\right)^{\frac{1}{\lambda}}$$
$$= \sum_{n=0}^{\infty}\frac{\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)}{n!}t^n\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n+1}t^{2n+2}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}\mathcal{W}_{n-2k,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)\frac{(\lambda)^{2k}}{(n-2k)!\,(2k+1)}t^{n+2},$$

which mean the assertions. □

We readily observe from (2.3) that

$$\frac{\partial}{\partial t}e_\lambda^{\delta}(t) = \frac{\partial}{\partial t}(1+\lambda t)^{\frac{\delta}{\lambda}} = \delta(1+\lambda t)^{\frac{\delta-\lambda}{\lambda}} = \delta e_\lambda^{\delta-\lambda}(t) \tag{3.8}$$

and

$$\frac{\partial}{\partial t}e_\lambda^{\zeta}(t^2) = \frac{\partial}{\partial t}\left(1+\lambda t^2\right)^{\frac{\zeta}{\lambda}} = 2t\zeta\left(1+\lambda t^2\right)^{\frac{\zeta-\lambda}{\lambda}} = 2t\zeta e_\lambda^{\zeta-\lambda}(t^2). \tag{3.9}$$

We give the following theorem, including a recursive formula for $\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)$.

Theorem 3.16. *For $n>1$, we have*

$$\mathcal{W}_{n+1,\lambda}(\delta,\zeta;\rho;\mu) = \frac{\mu}{2(2-\mu)}(1-n)\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)$$
$$-\frac{\rho}{2-\mu}\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{n-k,\lambda}(\delta+1-\lambda,\zeta;\rho;\mu)\mathcal{W}_{k,\lambda}(\rho;\mu)$$
$$+\delta\mathcal{W}_{n,\lambda}(\delta-\lambda,\zeta;\rho;\mu)+n\frac{\mu\delta}{2(2-\mu)}\mathcal{W}_{n-1,\lambda}(\delta-\lambda,\zeta;\rho;\mu)$$
$$+n2\zeta\mathcal{W}_{n-1,\lambda}(\delta,\zeta-\lambda;\rho;\mu)+n(n-1)\frac{\mu\zeta}{2-\mu}\mathcal{W}_{n-2,\lambda}(\delta,\zeta-\lambda;\rho;\mu).$$

Proof. By applying the operator $\frac{\partial}{\partial t}$ to both sides of (3.3), we then derive using (3.8) and (3.9) that

$$\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\mathcal{W}_{n+1,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!}$$
$$=\frac{\partial}{\partial t}\left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}e_\lambda^{\delta}(t)e_\lambda^{\zeta}(t^2)\right)$$
$$=e_\lambda^{\delta}(t)e_\lambda^{\zeta}(t^2)\frac{\frac{\mu}{2}(\rho e_\lambda(t)+(1-\mu))-\left(2-\mu+\frac{\mu}{2}t\right)\rho e_\lambda^{1-\lambda}(t)}{(\rho e_\lambda(t)+(1-\mu))^2}$$
$$+\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\left(\delta e_\lambda^{\delta-\lambda}(t)e_\lambda^{\zeta}(t^2)+2t\zeta e_\lambda^{\delta}(t)e_\lambda^{\zeta-\lambda}(t)\right)$$

$$=\frac{\mu}{2}\frac{e_\lambda^\delta(t)e_\lambda^\zeta(t^2)}{\rho e_\lambda(t)+(1-\mu)}-\rho\frac{\left(2-\mu+\frac{\mu}{2}t\right)e_\lambda^{\delta+1-\lambda}(t)e_\lambda^\zeta(t^2)}{(\rho e_\lambda(t)+(1-\mu))^2}$$
$$+\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\delta e_\lambda^{\delta-\lambda}(t)e_\lambda^\zeta(t^2)$$
$$+\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}2t\zeta e_\lambda^\delta(t)e_\lambda^{\zeta-\lambda}(t).$$

Then, we get

$$\begin{aligned}
&\left(2-\mu+\frac{\mu}{2}t\right)\sum_{n=0}^{\infty}\mathcal{W}_{n+1,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!}\\
=\;&\frac{\mu}{2}\frac{\left(2-\mu+\frac{\mu}{2}t\right)e_\lambda^\delta(t)e_\lambda^\zeta(t^2)}{\rho e_\lambda(t)+(1-\mu)}-\rho\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}e_\lambda^{\delta+1-\lambda}(t)e_\lambda^\zeta(t^2)\\
&+\delta\left(2-\mu+\frac{\mu}{2}t\right)\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}e_\lambda^{\delta-\lambda}(t)e_\lambda^\zeta(t^2)+2t\zeta\left(2-\mu+\frac{\mu}{2}t\right)\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}e_\lambda^\delta(t)e_\lambda^{\zeta-\lambda}(t)\\
=\;&\frac{\mu}{2}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!}-\rho\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta+1-\lambda,\zeta;\rho;\mu)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\rho;\mu)\frac{t^n}{n!}\\
&+\left(2\delta-\mu\delta+\frac{\mu\delta}{2}t\right)\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta-\lambda,\zeta;\rho;\mu)\frac{t^n}{n!}\\
&+\left(4t\zeta-2t\zeta\mu+\mu\zeta t^2\right)\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta-\lambda;\rho;\mu)\frac{t^n}{n!}\\
=\;&\frac{\mu}{2}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!}-\rho\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{n-k,\lambda}(\delta+1-\lambda,\zeta;\rho;\mu)\mathcal{W}_{k,\lambda}(\rho;\mu)\frac{t^n}{n!}\\
&+(2\delta-\mu\delta)\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta-\lambda,\zeta;\rho;\mu)\frac{t^n}{n!}+\frac{\mu\delta}{2}\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta-\lambda,\zeta;\rho;\mu)\frac{t^{n+1}}{n!}\\
&+(4\zeta-2\zeta\mu)\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta-\lambda;\rho;\mu)\frac{t^{n+1}}{n!}+\mu\zeta\sum_{n=0}^{\infty}\mathcal{W}_{n,\lambda}(\delta,\zeta-\lambda;\rho;\mu)\frac{t^{n+2}}{n!},
\end{aligned}$$

which means the assertion. □

The following theorem provides a summation formula including the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type and degenerate Stirling numbers of the second kind.

Theorem 3.17. *The following summation formula*

$$\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)=\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{l}(\delta)_k\,\mathcal{W}_{n-l,\lambda}^{(\alpha)}(0,\zeta;\rho;\mu)\,S_{2,\lambda}(l,k)$$

holds for $n\in\mathbb{N}_0$.

Proof. By applying Definition 3.1 together with Eqs (2.6) and (2.7), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!} &= \left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\alpha} e^{\zeta}_{\lambda}(t^2)\,(e_\lambda(t)-1+1)^{\delta}\\ &= \left(\frac{2-\mu+\frac{\mu}{2}t}{\rho e_\lambda(t)+(1-\mu)}\right)^{\alpha} e^{\zeta}_{\lambda}(t^2)\sum_{k=0}^{\infty}(\delta)_k\frac{(e_\lambda(t)-1)^k}{k!}\\ &= \sum_{k=0}^{\infty}(\delta)_k\sum_{n=0}^{\infty}\mathcal{W}^{(\alpha)}_{n,\lambda}(0,\zeta;\rho;\mu)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(\delta)_k\sum_{l=0}^{n}\binom{n}{l}\mathcal{W}^{(\alpha)}_{n-l,\lambda}(0,\zeta;\rho;\mu)\,S_{2,\lambda}(l,k)\frac{t^n}{n!},\end{aligned}$$

which gives the asserted result. □

We note that the following series manipulation formulas hold (*cf.* [20]):

$$\sum_{N=0}^{\infty}f(N)\frac{(\delta+\zeta)^N}{N!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{\delta^n}{n!}\frac{\zeta^m}{m!} \tag{3.10}$$

and

$$\sum_{k,l=0}^{\infty}B(l,k)=\sum_{k=0}^{\infty}\sum_{l=0}^{k}B(l,k-l). \tag{3.11}$$

We give the following theorem.

Theorem 3.18. *The following implicit summation formula*

$$\mathcal{W}^{(\alpha)}_{k+l,\lambda}(\delta,\zeta;\rho;\mu)=\sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(\delta-z)_{n+m,\lambda}\ \mathcal{W}^{(\alpha)}_{k+l-m-n,\lambda}(z,\zeta;\rho;\mu) \tag{3.12}$$

holds.

Proof. Upon setting t by $t+u$ in (3.3), we derive

$$\left(\frac{2-\mu+\frac{\mu}{2}(t+u)}{\rho e_\lambda(t+u)+(1-\mu)}\right)^{\alpha} e^{\zeta}_{\lambda}((t+u)^2)=e^{-z}_{\lambda}(t+u)\sum_{k,l=0}^{\infty}\mathcal{W}^{(\alpha)}_{k+l,\lambda}(z,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!}.$$

Again, by replacing z by δ in the last equation and using (3.10), we get

$$\left(\frac{2-\mu+\frac{\mu}{2}(t+u)}{\rho e_\lambda(t+u)+(1-\mu)}\right)^{\alpha} e^{\zeta}_{\lambda}((t+u)^2)=e^{-\delta}_{\lambda}(t+u)\sum_{k,l=0}^{\infty}\mathcal{W}^{(\alpha)}_{k+l,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!}.$$

By the last two equations, we obtain

$$\sum_{k,l=0}^{\infty}\mathcal{W}^{(\alpha)}_{k+l,\lambda}(\delta,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!}=e^{\delta-z}_{\lambda}(t+u)\sum_{k,l=0}^{\infty}\mathcal{W}^{(\alpha)}_{k+l,\lambda}(z,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!},$$

which yields

$$\sum_{k,l=0}^{\infty} \mathcal{W}_{k+l,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{n,m=0}^{\infty}(\delta-z)_{n+m,\lambda}\frac{t^n}{n!}\frac{u^m}{m!} \times \sum_{k,l=0}^{\infty} \mathcal{W}_{k+l,\lambda}^{(\alpha)}(z,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!},$$

Utilizing (3.11), we acquire

$$\sum_{k,l=0}^{\infty} \mathcal{W}_{k+l,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l}\frac{(\delta-z)_{n+m,\lambda}\ \mathcal{W}_{k+l-m-n,\lambda}^{(\alpha)}(z,\zeta;\rho;\mu)}{n!m!(k-n)!(l-m)!}t^k u^l,$$

which implies the asserted result (3.12). □

Corollary 3.19. *Letting $l = 0$ in (3.12), the following implicit summation formula holds:*

$$\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu) = \sum_{n=0}^{k}\binom{k}{n}(\delta-z)_{n,\lambda}\ \mathcal{W}_{k-n,\lambda}^{(\alpha)}(z,\zeta;\rho;\mu).$$

Corollary 3.20. *Upon setting $l = 0$ and replacing δ by $\delta + z$ in (3.12), we attain*

$$\mathcal{W}_{k,\lambda}^{(\alpha)}(\delta+z,\zeta;\rho;\mu) = \sum_{n=0}^{k}\binom{k}{n}(\delta)_{n,\lambda}\ \mathcal{W}_{k-n,\lambda}^{(\alpha)}(z,\zeta;\rho;\mu).$$

Now, we give the following theorem.

Theorem 3.21. *The following symmetric identity*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{n-k,b\lambda}^{(\alpha)}(b\delta,b\zeta;\rho;\mu)\,\mathcal{W}_{k,a\lambda}^{(\alpha)}(a\delta,a\zeta;\rho;\mu)\,a^{n-k}b^k \\ =\ &\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,b\lambda}^{(\alpha)}(b\delta,b\zeta;\rho;\mu)\,\mathcal{W}_{n-k,a\lambda}^{(\alpha)}(a\delta,a\zeta;\rho;\mu)\,a^k b^{n-k}\end{aligned} \tag{3.13}$$

holds for $a, b \in \mathbb{R}$ and $n \geq 0$.

Proof. Assume that

$$\Psi = \left(\frac{\left(2-\mu+\frac{\mu}{2}at\right)\left(2-\mu+\frac{\mu}{2}bt\right)}{(\rho e_{b\lambda}(at)+(1-\mu))(\rho e_{a\lambda}(bt)+(1-\mu))}\right)^{\alpha} e_{\lambda}^{2\delta}(abt)e_{\lambda}^{\zeta}(b(at)^2)e_{\lambda}^{\zeta}(a(bt)^2).$$

Then, the expression for Ψ is symmetric in a and b, and we derive the following two expansions of Ψ:

$$\begin{aligned}\Psi &= \sum_{n=0}^{\infty}\mathcal{W}_{n,b\lambda}^{(\alpha)}(b\delta,b\zeta;\rho;\mu)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\mathcal{W}_{n,a\lambda}^{(\alpha)}(a\delta,a\zeta;\rho;\mu)\frac{(bt)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{n-k,b\lambda}^{(\alpha)}(b\delta,b\zeta;\rho;\mu)\,\mathcal{W}_{k,a\lambda}^{(\alpha)}(a\delta,a\zeta;\rho;\mu)\,a^{n-k}b^k\frac{t^n}{n!}\end{aligned}$$

and similarly

$$\Psi = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}_{k,b\lambda}^{(\alpha)}(b\delta,b\zeta;\rho;\mu)\,\mathcal{W}_{n-k,a\lambda}^{(\alpha)}(a\delta,a\zeta;\rho;\mu)\,a^k b^{n-k}\frac{t^n}{n!},$$

which gives the desired result (3.13). □

The degenerate differential operator is introduced by Kim et al. [14] as follows:

$$\left(\delta\frac{d}{d\delta}\right)_{m,\lambda}=\left(\delta\frac{d}{d\delta}\right)\left(\delta\frac{d}{d\delta}-\lambda\right)\left(\delta\frac{d}{d\delta}-2\lambda\right)\cdots\left(\delta\frac{d}{d\delta}-(m-1)\lambda\right). \tag{3.14}$$

By (3.14), we have

$$\left(\delta\frac{d}{d\delta}\right)_{m,\lambda}\delta^{n}=(n)_{m,\lambda}\,\delta^{n}.$$

Let f be a formal power series written as $f(\delta)=\sum_{n=0}^{\infty}a_n\delta^n$ and $k\geq 0$. Then, the degenerate differential operator of this series is given by

$$\left(\delta\frac{d}{d\delta}\right)_{m,\lambda}f(\delta)=\sum_{n=0}^{\infty}(n)_{m,\lambda}\,a_n\delta^{n}.$$

Kim et al. [14] showed that a degenerate differential operator plays an important role in boson operators. Here, we focus on a representation of the degenerate differential operator on the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type as follows.

Thus, we state the following theorem.

Theorem 3.22. *The following operator formula holds for $n\in\mathbb{N}$:*

$$\left(\delta\frac{\partial}{\partial\delta}\right)_{m,\lambda}\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta,\zeta;\rho;\mu)=\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=0}^{l}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\,S_{2,\lambda}(k,l)\,S_1(l,j)\,(j)_{m,\lambda}\,\delta^{j},$$

where the symbol $S_1(l,j)$ is the (signed) Stirling numbers of the first kind, cf. [14] defined by

$$\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}=\frac{(\log(1+t))^j}{j!}\ \textit{for}\ j\geq 0.$$

Proof. By Corollary 3.10 and Eq (2.7), we have

$$\begin{aligned}\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta,\zeta;\rho;\mu)&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\,(\delta)_{k,\lambda}\\&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\sum_{l=0}^{k}S_{2,\lambda}(k,l)\,(\delta)_{l}\\&=\sum_{k=0}^{n}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\sum_{l=0}^{k}S_{2,\lambda}(k,l)\sum_{j=0}^{l}S_1(l,j)\,\delta^{j}.\end{aligned}$$

Therefore, from the above computations and Eq (3.14), we observe

$$\begin{aligned}\left(\delta\frac{\partial}{\partial\delta}\right)_{m,\lambda}\mathcal{W}^{(\alpha)}_{n,\lambda}(\delta,\zeta;\rho;\mu)&=\left(\delta\frac{\partial}{\partial\delta}\right)_{m,\lambda}\left\{\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=0}^{l}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\,S_{2,\lambda}(k,l)\,S_1(l,j)\,\delta^{j}\right\}\\&=\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=0}^{l}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\,S_{2,\lambda}(k,l)\,S_1(l,j)\left\{\left(\delta\frac{\partial}{\partial\delta}\right)_{m,\lambda}\delta^{j}\right\}\\&=\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=0}^{l}\binom{n}{k}\mathcal{W}^{(\alpha)}_{n-k,\lambda}(0,\zeta;\rho;\mu)\,S_{2,\lambda}(k,l)\,S_1(l,j)\,(j)_{m,\lambda}\,\delta^{j},\end{aligned}$$

which is the claimed result. □

4. Further remarks

In this part, we aim to define degenerate unified Hermite-based Apostol-Stirling polynomials of the second kind and to derive some of their properties and relations.

Definition 4.1. *Let $\mu, \rho \in \mathbb{R}$. We define degenerate unified Hermite-based Apostol-Stirling polynomials of the second kind via the following exponential generating function:*

$$\frac{(\rho e_\lambda(t) + (1-\mu))^k}{k!} e_\lambda^\delta(t) e_\lambda^\zeta(t^2) = \sum_{n=0}^{\infty} S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta)\frac{t^n}{n!}. \tag{4.1}$$

Here, we examine some special circumstances of the polynomials $S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta)$.

Remark 4.2. *Choosing $\zeta = \delta = 0$ in (4.1), the polynomials $S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta)$ become the degenerate unified Apostol-Stirling numbers of the second kind, which is a new extension of the numbers in (2.7), as follows:*

$$\frac{(\rho e_\lambda(t) + (1-\mu))^k}{k!} = \sum_{n=0}^{\infty} S_{2,\lambda}(n,k;\rho;\mu)\frac{t^n}{n!}. \tag{4.2}$$

Remark 4.3. *Choosing $\zeta = 0$ in (4.1), the polynomials $S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta)$ become the degenerate unified Apostol-Stirling polynomials of the second kind, which is a new extension of the polynomials in (2.6) given by*

$$\frac{(\rho e_\lambda(t) + (1-\mu))^k}{k!} e_\lambda^\delta(t) \sum_{n=0}^{\infty} S_{2,\lambda}(n,k;\rho;\mu:\delta)\frac{t^n}{n!}. \tag{4.3}$$

Remark 4.4. *Choosing $\lambda \to 0$ in (4.1), the polynomials $S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta)$ become the unified Hermite-based Apostol-Stirling polynomials of the second kind, which is a new family of polynomials, as follows:*

$$\frac{(\rho e^t + (1-\mu))^k}{k!} e^{\delta t + \zeta t^2} = \sum_{n=0}^{\infty} S_2(n,k;\rho;\mu:\delta,\zeta)\frac{t^n}{n!}.$$

The following theorem gives some properties of the degenerate unified Hermite-based Apostol-Stirling polynomials of the second kind.

Theorem 4.5. *The following properties*

$$\begin{aligned} &S_{2,\lambda}(n,k_1+k_2;\rho;\mu:\delta_1+\delta_2,\zeta_1+\zeta_2) \\ &= \frac{k_1!k_2!}{(k_1+k_2)!}\sum_{l=0}^{n}\binom{n}{l} S_{2,\lambda}(l,k_1;\rho;\mu:\delta_1,\zeta_1) S_{2,\lambda}(n-l,k_2;\rho;\mu:\delta_2,\zeta_2), \end{aligned}$$

$$S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = \sum_{l=0}^{n}\binom{n}{l} S_{2,\lambda}(l,k;\rho;\mu) H_{n-l,\lambda}(\delta,\zeta),$$

$$S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = n!\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{S_{2,\lambda}(l,k;\rho;\mu:\delta)(\zeta)_\lambda^{n-2l}}{l!(n-2l)!},$$

$$\Delta_{\lambda,\delta} S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = n\, S_{2,\lambda}(n-1,k;\rho;\mu:\delta,\zeta),$$

$$\Delta_{\lambda,\zeta} S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = n(n-1) S_{2,\lambda}(n-2,k;\rho;\mu:\delta,\zeta),$$

$$\frac{\partial}{\partial\delta} S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = \sum_{s=0}^{n-1} S_{2,\lambda}(n-1-k,k;\rho;\mu:\delta,\zeta) \frac{n!(-\lambda)^k}{(n-(k+1))!(k+1)},$$

$$\frac{\partial}{\partial\zeta} S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = \sum_{s=0}^{\lfloor \frac{n}{2}-1 \rfloor} \frac{S_{2,\lambda}(n,-2(k+1);\rho;\mu:\delta,\zeta) n!(-\lambda)^{2k}}{(n-2(k+1))!(2k+1)}$$

are valid for $k > 0$ *and* $n > 1$.

Proof. The proofs of the claimed formulas in the theorem can be done similarly to those of Theorems 3.9, 3.14, and 3.15. Therefore, we omit them. □

We now aim to provide some connections between the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type in (3.3) and the degenerate unified Hermite-based Apostol-Stirling polynomials of the second kind in (4.1).

Theorem 4.6. *The following formula*

$$\sum_{l=0}^{k} \binom{k}{l} (2-\mu)^{k-l} \left(\frac{\mu}{2}\right)^l \frac{H_{n-l,\lambda}(\delta,\zeta)}{(n-l)!} = \frac{k!}{n!} \sum_{m=0}^{n} \binom{n}{m} S_{2,\lambda}(m,k;\rho;\mu) \mathcal{W}_{n-m,\lambda}^{(k)}(\delta,\zeta;\rho;\mu) \tag{4.4}$$

is valid for $n \in \mathbb{N}_0$ *and* $k \in \mathbb{N}$.

Proof. By means of Definitions 3.1 and 4.1, based on the following equality

$$\left(2-\mu+\frac{\mu}{2}t\right)^k e_\lambda^\delta(t) e_\lambda^\zeta(t^2) = k! \frac{(\rho e_\lambda(t)+(1-\mu))^k}{k!} \sum_{n=0}^{\infty} \mathcal{W}_{n,\lambda}^{(k)}(\delta,\zeta;\rho;\mu) \frac{t^n}{n!},$$

we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{l=0}^{k} \binom{k}{l} (2-\mu)^{k-l} \left(\frac{\mu}{2}\right)^l H_{n,\lambda}(\delta,\zeta) \frac{t^{n+l}}{n!} &= k! \sum_{n=0}^{\infty} S_{2,\lambda}(n,k;\rho;\mu) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{W}_{n,\lambda}^{(k)}(\delta,\zeta;\rho;\mu) \frac{t^n}{n!} \\ &= k! \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} S_{2,\lambda}(m,k;\rho;\mu) \mathcal{W}_{n-m,\lambda}^{(k)}(\delta,\zeta;\rho;\mu) \frac{t^n}{n!},\end{aligned}$$

which means the claimed result (4.4). □

Theorem 4.7. *The following summation formula*

$$S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) = \sum_{l=0}^{k} \frac{(2-\mu)^{k-l}}{(n-l)!(k-l)!l!} \left(\frac{\mu}{2}\right)^l \mathcal{W}_{n-l,\lambda}^{(-k)}(\delta,\zeta;\rho;\mu) \tag{4.5}$$

holds for non-negative integers k *and* n *with* $n \geq k$.

Proof. By Definitions 3.1 and 4.1, we have

$$\sum_{n=0}^{\infty} S_{2,\lambda}(n,k;\rho;\mu:\delta,\zeta) \frac{t^n}{n!} = \frac{(\rho e_\lambda(t)+(1-\mu))^k}{k!} e_\lambda^\delta(t) e_\lambda^\zeta(t^2)$$

$$= \frac{1}{k!}\sum_{n=0}^{\infty} \mathcal{W}_{n,\lambda}^{(-k)}(\delta,\zeta;\rho;\mu)\frac{t^n}{n!}\left(2-\mu+\frac{\mu}{2}t\right)^k$$
$$= \frac{1}{k!}\sum_{n=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l}(2-\mu)^{k-l}\left(\frac{\mu}{2}\right)^l \mathcal{W}_{n,\lambda}^{(-k)}(\delta,\zeta;\rho;\mu)\frac{t^{n+l}}{n!},$$

which implies the claimed result (4.5). □

5. Distribution of zeros and graphical representations

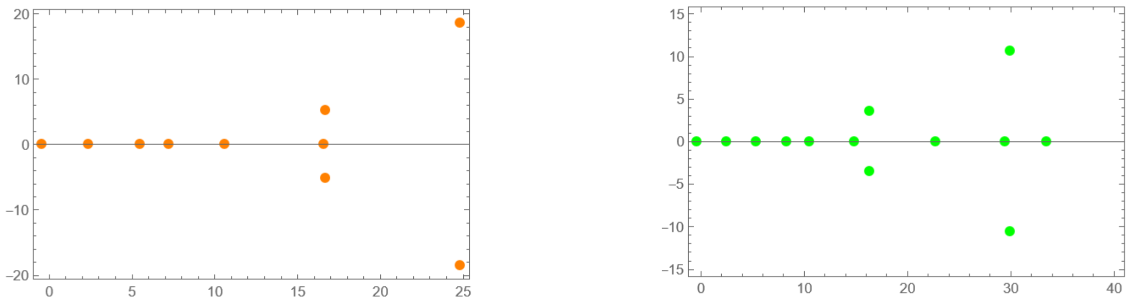
This section demonstrates how numerical analysis can be employed to confirm theoretical predictions and uncover new and interesting patterns in the zeros of certain members of the newly introduced hybrid polynomial family. Here, we contribute to the field by giving the presentation of the first few values of the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type $\mathcal{W}_{n,3}^{(1)}(\delta,\zeta;3;2)$. These values are not only a practical reference but also help to establish a foundation for further research and exploration.

The first few polynomials of $\mathcal{W}_{n,3}^{(1)}(\delta,\zeta;3;2)$ are as follows:

$$\begin{aligned}
\mathcal{W}_{0,3}^{(1)}(\delta,\zeta;3;2) &= 1,\\
\mathcal{W}_{1,3}^{(1)}(\delta,\zeta;3;2) &= 1+\delta,\\
\mathcal{W}_{2,3}^{(1)}(\delta,\zeta;3;2) &= \delta^2-\delta+2\zeta-\frac{4}{3},\\
\mathcal{W}_{3,3}^{(1)}(\delta,\zeta;3;2) &= \delta^3-6\delta^2+6\delta\zeta+6\zeta+6,\\
\mathcal{W}_{4,3}^{(1)}(\delta,\zeta;3;2) &= \delta^4-14\delta^3+(55+12\zeta)\delta^2-(7+2\zeta)\delta-\frac{4}{3}(34+39\zeta-9\zeta^2),\\
\mathcal{W}_{5,3}^{(1)}(\delta,\zeta;3;2) &= \delta^5-25\delta^4+\left(\frac{635}{3}+20\zeta\right)\delta^3-15(45+8\zeta)\delta^2+\left(\frac{1462}{3}-8\zeta+60\zeta^2\right)\delta\\
&\quad +60(8-\zeta7+\zeta^2),\\
\mathcal{W}_{6,3}^{(1)}(\delta,\zeta;3;2) &= \delta^6-39\delta^5+5(113+6\zeta)\delta^4-15(247+28\zeta)\delta^3+2(5177+555\zeta+9\zeta^2)\delta^2\\
&\quad -12(598+60\zeta+15\zeta^2)\delta-\frac{40}{3}(490-114\zeta+99\zeta^2-9\zeta^3),\\
\mathcal{W}_{7,3}^{(1)}(\delta,\zeta;3;2) &= \delta^7-56\delta^6+14(88+3\zeta)\delta^5-210(64+65\zeta)\delta^4+\frac{7}{3}(32071+3270\zeta+180\zeta^2)\delta^3\\
&\quad -42(4532+495\zeta+60\zeta^2)\delta^2+\frac{4}{3}(95831+21966\zeta-4095\zeta^2+630\zeta^3)\delta\\
&\quad +280(388+99\zeta-18\zeta^2+3\zeta^3).
\end{aligned}$$

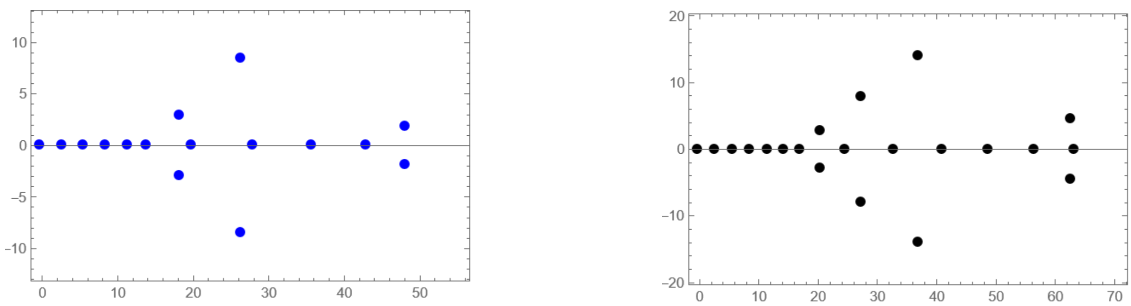
To analyze the zero distributions of the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type, we utilize the software Wolfram Mathematica. We present graphical representations of these zero distributions by assigning specific values to the polynomial coefficients: $\{\alpha\to 1,\ \lambda\to 3,\ \rho\to 3\ \text{and}\ \zeta\to 4\}$.

Here, we give values for $n = 10, 15, 20, 25$, and have provided the graphical representations in Figures 1 and 2 forming 2D structures, enhancing the understanding of the numerical data and facilitating a more intuitive grasp of the concepts discussed. In addition, we present perspectives for visualizing the zeros (or stacks of zeros) of degenerate unified Bernoulli-Euler Hermite polynomials of the Apostol type, as illustrated in Figures 3 and 4. We have provided the graphical representations in Figures 3 and 4 forming 3D structures, enhancing the understanding of the numerical data and facilitating a more intuitive grasp of the concepts discussed. Using computational techniques, approximate solutions for the zeros of $\mathcal{W}^{(1)}_{n,3}(\delta, 4; 3; 2)$ for $1 \leq n \leq 12$ are obtained and summarized in Table 1.



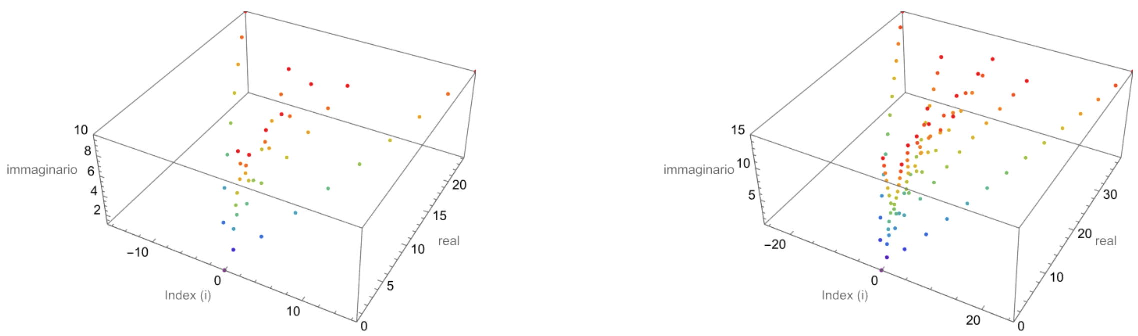
(a) Zeros of $\mathcal{W}^{(1)}_{10,3}(\delta, 4; 3; 2)$ **(b)** Zeros of $\mathcal{W}^{(1)}_{15,3}(\delta, 4; 3; 2)$

Figure 1. Solutions of $\mathcal{W}^{(1)}_{10,3}(\delta, 4; 3; 2) = 0$ and $\mathcal{W}^{(1)}_{15,3}(\delta, 4; 3; 2) = 0$.



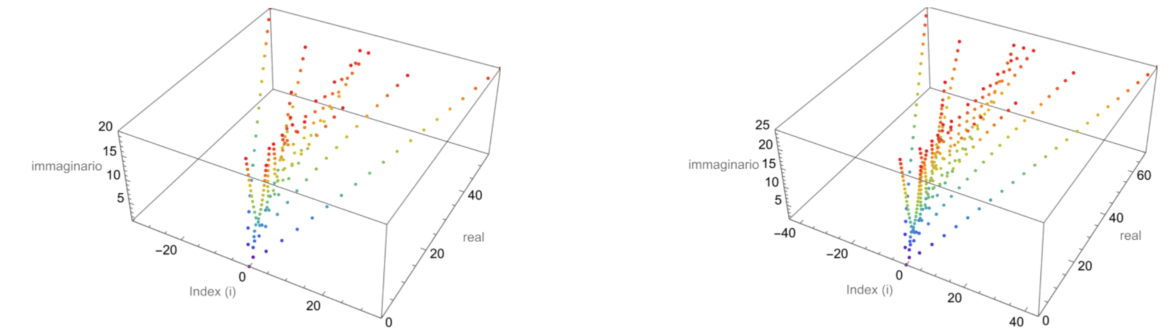
(a) Zeros of $\mathcal{W}^{(1)}_{20,3}(\delta, 4; 3; 2)$ **(b)** Zeros of $\mathcal{W}^{(1)}_{25,3}(\delta, 4; 3; 2)$

Figure 2. Solutions of $\mathcal{W}^{(1)}_{20,3}(\delta, 4; 3; 2) = 0$ and $\mathcal{W}^{(1)}_{25,3}(\delta, 4; 3; 2) = 0$.



(a) Data visualization of zeros of $\mathcal{W}^{(1)}_{10,3}(\delta, 4; 3; 2)$ **(b)** Data visualization of zeros of $\mathcal{W}^{(1)}_{15,3}(\delta, 4; 3; 2)$

Figure 3. Data visualizations of solutions of $\mathcal{W}^{(1)}_{10,3}(\delta, 4; 3; 2) = 0$ and $\mathcal{W}^{(1)}_{15,3}(\delta, 4; 3; 2) = 0$.



(a) Data visualization of zeros of $\mathcal{W}^{(1)}_{20,3}(\delta, 4; 3; 2)$ **(b)** Data visualization of zeros of $\mathcal{W}^{(1)}_{25,3}(\delta, 4; 3; 2)$

Figure 4. Data visualizations of solutions of $\mathcal{W}^{(1)}_{20,3}(\delta, 4; 3; 2) = 0$ and $\mathcal{W}^{(1)}_{25,3}(\delta, 4; 3; 2) = 0$.

Table 1. Approximate solutions of $\mathcal{W}^{(1)}_{n,3}(\delta, 4; 3; 2) = 0$.

degree n	p
2	1.5
3	3 - 2.54951 i, 3 + 2.54951 i
4	1.69392, 5.90304 - 4.81202i, 5.90304 + 4.81202i
5	2.28077, 3.88264, 8.9183 - 6.90466 i, 8.9183 + 6.90466 i
6	2.19868, 5.67809 - 0.530573i, 5.67809 + 0.530573i, 11.9726 - 8.92153i, 11.9726 + 8.92153i
7	2.48384, 4.30623, 8.56533 - 1.67771 i, 8.56533 + 1.67771i, 15.0396 - 10.8902 i, 15.0396 + 10.8902i
8	2.96993, 3.85507, 7.73906, 11.3582 - 2.47035 i 11.3582 + 2.47035 i, 18.1097 - 12.827 i, 18.1097 + 12.827 i
9	3.56348 - 0.357688i, 3.56348 + 0.357688i, 7.09115, 11.3699, 14.0265 - 3.21759i, 14.0265 + 3.21759i, 21.1795 - 14.7414 i, 21.1795 + 14.7414i
10	3.67911 - 0.707093 i, 3.67911 + 0.707093 i, 7.27041, 9.91189, 15.2993, 16.5824 - 4.06856 i, 16.5824 + 4.06856i, 24.2477 - 16.6394i, 24.2477 + 16.6394i
11	3.73795 - 0.99663 i, 3.73795 + 0.99663i, 7.75139, 9.56992, 13.0992, 19.1551, 19.1602 - 5.05697 i, 19.1602 + 5.05697i, 27.314 - 18.5247i, 27.314 + 18.5247i
12	3.79612 - 1.23935 i, 3.79612 + 1.23935i, 7.70397, 11.1245, 11.3053, 16.6233, 21.7952 - 6.0913i, 21.7952 + 6.0913i, 22.8034, 30.3784 - 20.3997i, 30.3784 + 20.3997 i

6. Conclusions

In this paper, a new class of degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type, denoted by $\mathcal{W}_{n,\lambda}^{(\alpha)}(\delta,\zeta;\rho;\mu)$, was considered in Definition 3.1, and then, diverse properties were derived. Some summation formulas in Theorems 3.9 and 3.17 and Corollaries 3.10, 3.12, 3.19, and 3.20 a relation in Theorem 3.11; two difference operator formulas in Theorem 3.14; two derivative operator formulas in Theorem 3.15; a recursive formula in Theorem 3.16; an implicit summation formula in Theorem 3.18; and a symmetric property in Theorem 3.21 for the new polynomials were provided with their proofs. Also, a representation of the degenerate differential operator on the degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type was given in Theorem 3.22. Moreover, the degenerate unified Hermite-based Apostol-Stirling polynomials of the second kind were defined in Definition 4.1, and some properties of these newly established polynomials were derived in Theorem 4.5. In addition, we prove multifarious correlations, including the two new polynomials in Theorems 4.6. and 4.7. Furthermore, the first few degenerate unified Bernoulli-Euler Hermite polynomials of Apostol type for some special cases were listed and data visualizations of zeros forming 2D and 3D structures were presented in Section 5. Finally, a table covering approximate solutions for the zeros of $\mathcal{W}_{n,3}^{(\alpha)}(\delta,4;3;2)$ is provided in Table 1.

In conclusion, the introduction and investigation of mixed polynomials represent a significant milestone in the field of mathematics and science, promoting novel examination avenues and applications in different disciplines. It is crucial to continue exploring and collaborating to fully realize their potential and understand their broader implications.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest.

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