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Research article

Convergence behavior of practical iterative schemes for split fixed point problems under fixed and variable stepsize strategies

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Abstract: This work presents a practical iterative algorithm, extending the inertial Mann iteration, for solving split fixed-point problems with demicontractive mappings in real Hilbert spaces. We rigorously establish both its weak and strong convergence under clearly defined parametric conditions. Our methodology utilizes versatile two-step selection techniques with both fixed and variable step sizes. Compelling numerical experiments confirm the algorithm's accuracy and computational efficiency in approximating solutions to these challenging problems.

Keywords: convergence result; numerical method; Hilbert space; demicontractive mapping; split fixed point problem
Mathematics Subject Classification: 47H10, 47J25, 65K15

Abbreviations

This section is dedicated to understanding all the abbreviations used in this manuscript.

SFP	split feasibility problem
BLO	bounded linear operator
SFPP	split fixed-point problem
SCFPP	split common fixed-point problem
IMI	inertial Mann iterative
UFP	unique fixed point
ISM	inverse strongly monotone
VIP	variational inequality problem
SVI	split variational inequalities

1. Introduction

The split feasibility problem (SFP) is a crucial area in functional and nonlinear analysis due to its extensive applications in fields like signal processing, computer-processed tomography, intensity-modulated radiation therapy, and image restoration [1–3]. In 1994, Censor and Elfving [4] introduced the SFP as follows: Find $\vartheta^* \in \mho$ such that

$$\vartheta^* \in C \cap \Im^{-1}(Q),$$

where C and Q are two closed and convex subsets of two Hilbert spaces $\mho$ and $\mho_1$, respectively; $\Im : \mho \to \mho_1$ is a bounded linear operator (BLO); and $\Im^{-1}(Q) = \{\vartheta \in \mho : \Im(\vartheta) \in Q\}$.

In 2002, Byrne's CQ method was one of many innovative approaches developed to solve the SFP [5]. For an arbitrary starting guess $\vartheta_0 \in \mho$, the sequence $\{\vartheta_u\}$ in the CQ technique is described as

$$\vartheta_{u+1} = P_C\left(\vartheta_u - \mu\Im^*(I - P_Q)\Im(\vartheta_u)\right),\ u \geq 0,$$

where P_C and P_Q are the metric projections on C and Q, respectively; $\mu > 0$ is a appropriately selected step size; $\Im^*$ is the adjoint of $\Im$; and I is the identity operator.

The SFP is significantly generalized by including multiple output sets. Let Q_j be a closed and convex subset of Hilbert spaces $\mho_j$ and let $\Im : \mho \to \mho_j$ be a BLO for $j = 1, 2, ..., N$. Then, the SFP with multiple output sets is described as follows: Find $\vartheta^* \in \mho$ such that

$$\vartheta^* \in C \bigcap \left(\bigcap_{j=1}^{N} \Im_j^{-1}(Q_j)\right). \tag{1.1}$$

In 2020, Reich et al. [6] proposed a new iterative strategy for tackling the problem (1.1) as follows: For an arbitrary starting guess $\vartheta_0 \in \mho$, the sequence $\{\vartheta_u\}$ can be iterated by

$$\vartheta_{u+1} = P_C\left(\vartheta_u - \mu \sum_{j=1}^{N} \Im_j^*(I - P_{Q_j})\Im_j\vartheta_u\right),\ u \geq 0,$$

where $\mu > 0$ is a fixed step size. They also established the proposed method's weak and strong convergence. Subsequently, in 2021, Reich and Tuyen [7]introduced a new cyclic projection approach for solving the SFP with multiple output sets.

The split fixed-point problem (SFPP) is an important generalization of the SFP. Let $W_1 : \mho \to \mho$ be an ℓ_1-demicontractive mapping, $W_2 : \mho_1 \to \mho_1$ be an ℓ_2−demicontractive mapping for any $\ell_1, \ell_2 > 0$, and let $\Im : \mho \to \mho_1$ be a BLO. Then, the SFPP is considered as to be

$$\text{Find } \vartheta^* \in Fix(W_1) \text{ such that } \Im\vartheta^* \in Fix(W_2), \tag{1.2}$$

where $Fix\left(W_j\right)$ represents the set of all fixed points of the mapping W_j, $j = 1, 2$.

Moreover, a novel iterative method for solving Problem (1.2) was proposed by Moudafi [8] as follows: For an arbitrary $\vartheta_0 \in \mho$, define

$$\begin{cases} \theta_u = \vartheta_u + \xi\Im^* (W_2 - I) \Im\vartheta_u, \\ \vartheta_{u+1} = (1 - \lambda_u)\theta_u + \lambda_u W_1 \theta_u, \; u \geq 0, \end{cases}$$

where $\xi \in \left(0, \frac{1-\ell_2}{\varkappa}\right)$, $\ell_2 < 1$, $\varkappa$ is the spectral radius of the operator $\Im^*\Im$, and $\lambda_u \in (0, 1)$.

The development of solutions for the split common fixed-point problem (SCFPP) has seen several key contributions. In 2011, Moudafi [9] suggested an iterative technique specifically for quasi-nonextensive mappings. Building on this, Cegielski [10] investigated a broader solution to the SCFPP in 2015. More recently, Padcharoen et al. [11] focused on SCFPPs for demicontractive mappings, proposing a modified iterative strategy to address them.

The inertial methodology originated from the heavy-ball method, an implicit discretization of a second-order time-dynamical system [12, 13]. Polyak [14] introduced the inertial approach as an accelerated scheme for solving smooth convex minimization problems. This method is a two-stage iterative procedure, where each new iteration is defined by the two preceding values. This simple design modification has been consistently shown by numerous authors [15–26] to dramatically enhance the performance and speed of iterative algorithms.

Recently, Wang [27] investigated the SFPP with multiple output sets for demicontractive mappings as follows: Find $\vartheta_0 \in \mho$ such that

$$\vartheta^* \in Fix(W_0) \text{ and } \Im_j\vartheta^* \in Fix\left(W_j\right), \; j = 1, 2, ..., N.$$

Equivalently,

$$\text{Find } \vartheta^* \in Fix(W_0) \bigcap \left(\bigcap_{j=1}^{N} \Im_j^{-1}(Fix\left(W_j\right)) \right), \tag{1.3}$$

where $W_0 : \mho \to \mho$ and $W_j : \mho_j \to \mho_j$ $(j = 1, 2, ..., N)$ are nonlinear mappings. The same author demonstrated the proposed method's weak and strong convergence, then applied these findings to the SFP with multiple output sets. In contrast, Wang's iterative approaches [27] do not incorporate inertial Mann terms.

Consequently, we aimed to enhance iterative algorithms by extending Wang's algorithm through the integration of inertial and Mann terms, thereby making it suitable for split feasibility problems in real Hilbert spaces. Our work includes a detailed theoretical analysis that rigorously proves both weak and strong convergence under specific fixed and variable step size conditions. To demonstrate its practical value, we present comparative numerical experiments showcasing the algorithm's efficiency and superior convergence over the current SFPP solutions.

2. Preliminaries

This section reviews the essential definitions and results that are crucial for analyzing the main theorems. Throughout this paper, we assume that Δ is a non-empty, closed, and convex subset of a real Hilbert space Ω with the inner product $\langle ., . \rangle$ and the norm $\|.\|$. We denote strong convergence by $\longrightarrow$ and weak convergence by $\rightharpoonup$. Furthermore, Θ represents the non-empty solution set of Problem (1.3).

Definition 2.1. *Let Ω be a real Hilbert space. Then, $W : \Omega \to \Omega$ is called*

(i) Nonexpansive mapping on Ω if

$$\|W\rho - W\varpi\| \leq \|\rho - \varpi\|;$$

for all $\rho, \varpi \in \Omega$

(ii) $\ell-$demicontractive mapping if $Fix(W)$ is non-empty and $\ell \in [0, 1)$ exists such that

$$\|W\rho - \varpi\|^2 \leq \|\rho - \varpi\|^2 + \ell \|\rho - W\rho\|^2$$

for all $\rho \in \Omega$ and $\varpi \in Fix(W)$.

Definition 2.2. *Let W be a self-mapping on Ω and $\{\vartheta_u\}$ be a sequence on Ω. Then, $(I - W)$ is demiclosed at zero if $(I - W)\vartheta_u \longrightarrow 0$ and $\vartheta_u \rightharpoonup \vartheta^*$ implies $\vartheta^* = W\vartheta^*$, that is, $\vartheta^* \in Fix(W)$.*

The following lemmas are important in subsequent sections:

Lemma 2.1. *[28] Assume that $W_\lambda = (1 - \lambda) I + \lambda W$, for any $\lambda > 0$. If W is an ℓ-demicontractive mapping, then the assertions below hold:*

(1) $Fix(W)$ is closed and convex.

(2) For each $(\rho, \varpi) \in \Omega \times Fix(W)$, we have

$$\langle \rho - W\rho, \rho - \varpi \rangle \geq \frac{1 - \ell}{2} \|\rho - W\rho\|^2. \tag{2.1}$$

(3) For each $(\rho, \varpi) \in \Omega \times Fix(W)$, we have

$$\|W_\lambda\rho - \varpi\|^2 \leq \|\rho - \varpi\|^2 - \lambda (1 - \lambda - \ell) \|(I - W)\rho\|^2. \tag{2.2}$$

Lemma 2.2. *[29] Assume that $\{\vartheta_{u_k}\}$ is a subsequence of a real sequence $\{\vartheta_u\}$ such that $\vartheta_{u_k} < \vartheta_{u_k+1}$ for all $k \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{\sigma_\ell\} \subset \mathbb{N}$ such that $\lim_{\ell\to\infty} \sigma_\ell = \infty$, and the following conditions hold for all (sufficiently large) values of $\ell \in \mathbb{N}$:*

$$\vartheta_{\sigma_\ell} \leq \vartheta_{\sigma_\ell+1} \text{ and } \vartheta_\ell \leq \vartheta_{\sigma_\ell+1},$$

where σ_ℓ is the greatest number u in the set $\{1, 2, ..., \ell\}$ such that $\vartheta_u < \vartheta_{u+1}$.

Lemma 2.3. *[30] Assume that $\{\zeta_u\}$ is a sequence of non-negative real numbers such that*

$$\zeta_{u+1} \leq (1 - s_u) \zeta_u + s_u r_u, \text{ for all } u \geq 0,$$

where $\{s_u\}$ and $\{r_u\}$ fulfill the conditions below:

(i) $\{s_u\} \subset [0,1]$, $\sum_{u=1}^{\infty} s_u = \infty$;
(ii) $\limsup_{u\to\infty} r_u \leq 0$.

Then, $\lim_{u\to\infty} \zeta_u = 0$.

Lemma 2.4. *[31] Let ∇ be a non-empty subset of Ω and $\{\vartheta_u\} \subset \Omega$ be a sequence such that for all $\vartheta \in \nabla$, $\lim_{u\to\infty} \|\vartheta_u - \vartheta\|$ exists and every sequential weak cluster point of $\{\vartheta_u\}$ is in ∇. Then ϑ_u converges weakly to a point in ∇.*

Lemma 2.5. *[18] Assume that $\{s_u\}, \{r_u\}$ and $\{\tau_u\}$ are three sequences in $[0,\infty)$ such that for all $u \in \mathbb{N}$,*

$$s_{u+1} \leq s_u + r_u (s_u - s_{u+1}) + \tau_u, \text{ and } \sum_{u=1}^{\infty} \tau_u < +\infty.$$

If, for all $u \in \mathbb{N}$, there exists a real number r such that $0 \leq r_u \leq r < 1$, then,

(i) $\sum_{u=1}^{\infty} [s_u - s_{u+1}]_+ < +\infty$, *where* $[\tau]_+ = \max\{0, \tau\}$;
(ii) for some $s^* \in [0,\infty)$, $\lim_{u\to\infty} s_u = s^*$.

3. Weak convergence results

In this section, we lay out the main findings of this work. We start by developing an inertial Mann iterative (IMI) algorithm for the split feasibility problem (SFP) with multiple output sets, first with a fixed step size. We then extend this to an IMI algorithm that incorporates a variable step size.

3.1. An IMI algorithm with a fixed step size

For $\vartheta_0, \vartheta_1 \in \Delta$, construct a sequence $\{\vartheta_u\}$ in the following way:

$$\begin{cases} z_u = \vartheta_u + \alpha_u (\vartheta_u - \vartheta_{u-1}), \\ \omega_u = (1-\kappa_u)\vartheta_u + \kappa_u \Im z_u, \\ \theta_u = W_{0,\lambda}\left(\omega_u - \mu \sum\limits_{j=1}^{N} \Im_j^*(I - W_j)\Im_j \omega_u\right), \\ \vartheta_{u+1} = (1-\varrho_u) z_u + \varrho_u \theta_u, \ u \geq 1, \end{cases} \tag{3.1}$$

where $W_{0,\lambda} = (1-\lambda) I + \lambda W_0$, and $\mu \in \left(0, \frac{\min_{j\in[1,N]}(1-\ell_j)}{\sum_{j=1}^{N} \|\Im_j\|^2}\right)$, $\ell_j < 1$, $\lambda > 0$, $\kappa_u \in (0,1)$ and $\alpha_u \subset [0,\alpha)$ is an increasing sequence with $0 \leq \alpha_u \leq \alpha < 1$ and $\alpha_1 = 0$.

Theorem 3.1. *Let W_j be an ℓ_j-demicontractive mapping that satisfies the demiclosedness property for each $j = 0, 1, ..., N$ and let $\Im$ be a nonexpansive mapping. Choose $\varrho, \gamma_1, \gamma_2, \alpha > 0$ such that*

$$\gamma_2 > \frac{\alpha^2(1+\alpha) + \alpha\gamma_1}{1-\alpha^2}, \text{ and } 0 < \varrho \leq \varrho_u \leq \frac{\gamma_2 - \alpha(\alpha(1+\alpha) + \alpha\gamma_2 + \gamma_1)}{\gamma_2(\alpha(1+\alpha) + \alpha\gamma_2 + \gamma_1)}. \tag{3.2}$$

If $\{\vartheta_u\}$ is a sequence produced by (3.1), then ϑ_u converges weakly to a solution of the problem (1.3).

Proof. First, we show that $\|\vartheta_{u+1} - \vartheta_u\| \to 0$, $u \to \infty$.

For this, let $q \in \Theta = Fix(W_0) \bigcap \left(\bigcap_{j=1}^{N} \mathfrak{I}_j^{-1}(Fix\left(W_j\right)) \right)$ and put $\ell = \min_{j\in[1,N]} \left(1 - \ell_j\right)$ and $\epsilon = \min \left\{ \mu \left(\ell - \mu \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\|^2 \right), \lambda (1 - \lambda - \ell_0) \right\}$, where ℓ_0 is ℓ_0-demicontractive mapping.

Now, from (2.1), we have

$$\begin{aligned} 2\langle \omega_u - q, \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \rangle &= 2\langle \mathfrak{I}_j\omega_u - \mathfrak{I}_j q, (I - W_j)\mathfrak{I}_j\omega_u \rangle \\ &\geq (1 - \ell_j) \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 \\ &\geq \ell \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2, \end{aligned} \tag{3.3}$$

for each $j = 1, 2, ..., N$. Using Cauchy-Schwarz inequality, one has

$$\begin{aligned} \left\| \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \right\|^2 &\leq \left(\sum_{j=1}^{N} \left\|\mathfrak{I}_j^*\right\| \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\| \right)^2 \\ &\leq \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\|^2 \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2. \end{aligned} \tag{3.4}$$

Set $p_u = \omega_u - \mu \sum\limits_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u$. Then by (3.3) and (3.4), we can write

$$\begin{aligned} \|p_u - q\|^2 &= \left\| \omega_u - \mu \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u - q \right\|^2 \\ &= \|\omega_u - q\|^2 + \mu^2 \left\| \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \right\|^2 - 2\mu \sum_{j=1}^{N} \langle \omega_u - q, \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \rangle \\ &\leq \|\omega_u - q\|^2 + \mu^2 \left\| \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \right\|^2 - \ell\mu \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 \\ &\leq \|\omega_u - q\|^2 + \mu^2 \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\|^2 \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 - \ell\mu \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 \\ &= \|\omega_u - q\|^2 + \left(\mu^2 \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\|^2 - \ell\mu \right) \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 \\ &= \|\omega_u - q\|^2 - \mu \left(\ell - \mu \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\|^2 \right) \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2. \end{aligned} \tag{3.5}$$

From (2.2), we have

$$\begin{aligned} \|\theta_u - q\|^2 &= \left\|W_{0,\lambda} p_u - q\right\|^2 \\ &\leq \|p_u - q\|^2 - \lambda (1 - \lambda - \ell_0) \|(I - W_0)\, p_u\|^2. \end{aligned} \tag{3.6}$$

Applying (3.5) in (3.6) and selecting $\epsilon > 0$, we get

$$\|\theta_u - q\|^2 \le \|\omega_u - q\|^2 - \epsilon\left(\|(I - W_0)\, p_u\|^2 + \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2\right). \tag{3.7}$$

It follows from ϑ_{u+1} that

$$\begin{aligned}\|\vartheta_{u+1} - q\|^2 &= \|(1-\varrho_u)z_u + \varrho_u\theta_u - q\|^2\\ &= (1-\varrho_u)\,\|z_u - q\|^2 + \varrho_u\,\|\theta_u - q\|^2 - \varrho_u(1-\varrho_u)\,\|z_u - \theta_u\|^2.\end{aligned}$$

From (3.7), and since $\epsilon > 0$ and $\varrho_u > 0$, we get

$$\begin{aligned}\|\vartheta_{u+1} - q\|^2 &\le (1-\varrho_u)\,\|z_u - q\|\\ &\quad +\varrho_u\left(\|\omega_u - q\|^2 - \epsilon\left(\|(I - W_0)\, p_u\|^2 + \sum_{j=1}^{N}\left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2\right)\right)\\ &\quad -\varrho_u(1-\varrho_u)\,\|z_u - \theta_u\|^2\\ &\le (1-\varrho_u)\,\|z_u - q\| + \varrho_u\,\|\omega_u - q\|^2.\end{aligned} \tag{3.8}$$

Using the definition of ω_u, and since $\varrho_u > 0$, and $\mathfrak{I}$ is a nonexpansive mapping, we find that

$$\begin{aligned}\|\vartheta_{u+1} - q\|^2 &\le (1-\varrho_u)\,\|z_u - q\| + \varrho_u\left\|(1-\kappa_u)\vartheta_u + \kappa_u\mathfrak{I}z_u - q\right\|^2\\ &\le (1-\varrho_u)\,\|z_u - q\| + \varrho_u(1-\kappa_u)\,\|\vartheta_u - q\| + \kappa_u\varrho_u\left\|\mathfrak{I}z_u - q\right\|^2\\ &\quad -\varrho_u(1-\kappa_u)\kappa_u\left\|\vartheta_u - \mathfrak{I}z_u\right\|^2\\ &= (1-\varrho_u(1-\kappa_u))\,\|z_u - q\|^2 + \varrho_u(1-\kappa_u)\,\|\vartheta_u - q\| - \varrho_u(1-\kappa_u)\kappa_u\left\|\vartheta_u - \mathfrak{I}z_u\right\|^2\\ &\le \|z_u - q\|^2 + \varrho_u\,\|\vartheta_u - q\| - \varrho_u\,\|\vartheta_u - z_u\|^2.\end{aligned} \tag{3.9}$$

From the definition of z_u, we have

$$\begin{aligned}\|z_u - q\|^2 &= \|\vartheta_u + \alpha_u\,(\vartheta_u - \vartheta_{u-1}) - q\|^2\\ &= \|(1+\alpha_u)\vartheta_u - \alpha_u\vartheta_{u-1} - q\|^2\\ &\le (1+\alpha_u)\,\|\vartheta_u - q\|^2 - \alpha_u\,\|\vartheta_{u-1} - q\|^2 + \alpha_u(1+\alpha_u)\,\|\vartheta_u - \vartheta_{u-1}\|^2.\end{aligned} \tag{3.10}$$

Further, we can write

$$\|\vartheta_{u+1} - z_u\|^2 = \|(1-\varrho_u)z_u + \varrho_u\theta_u - z_u\|^2 = \varrho_u^2\,\|\vartheta_u - z_u\|^2. \tag{3.11}$$

Invoking (3.10) and (3.11) in (3.9), we arrive at

$$\begin{aligned}\|\vartheta_{u+1} - q\|^2 &\le (1+\alpha_u)\,\|\vartheta_u - q\|^2 - \alpha_u\,\|\vartheta_{u-1} - q\|^2 + \alpha_u(1+\alpha_u)\,\|\vartheta_u - \vartheta_{u-1}\|^2\\ &\quad +\varrho_u\,\|\vartheta_u - q\| - \frac{\varrho_u}{\varrho_u^2}\,\|\vartheta_{u+1} - z_u\|^2\\ &= (1+\alpha_u+\varrho_u)\,\|\vartheta_u - q\|^2 - \alpha_u\,\|\vartheta_{u-1} - q\|^2 + \alpha_u(1+\alpha_u)\,\|\vartheta_u - \vartheta_{u-1}\|^2\\ &\quad -\frac{1}{\varrho_u}\,\|\vartheta_{u+1} - z_u\|^2.\end{aligned}$$

Equivalently,

$$\begin{aligned} & \|\vartheta_{u+1} - q\|^2 - (1 + \alpha_u + \varrho_u) \|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_{u-1} - q\|^2 \\ \leq \ & -\frac{1}{\varrho_u} \|\vartheta_{u+1} - z_u\|^2 + \alpha_u(1 + \alpha_u) \|\vartheta_u - \vartheta_{u-1}\|^2 . \end{aligned} \tag{3.12}$$

Now, we consider

$$\begin{aligned} \|\vartheta_{u+1} - z_u\|^2 &= \|\vartheta_{u+1} - (\vartheta_u + \alpha_u (\vartheta_u - \vartheta_{u-1}))\|^2 \\ &= \|\vartheta_{u+1} - \vartheta_u\|^2 + \alpha_u^2 \|\vartheta_u - \vartheta_{u-1}\|^2 + 2\alpha_u \langle \vartheta_{u+1} - \vartheta_u, \vartheta_u - \vartheta_{u-1} \rangle \\ &\geq \|\vartheta_{u+1} - \vartheta_u\|^2 + \alpha_u^2 \|\vartheta_u - \vartheta_{u-1}\|^2 \\ &\quad + \alpha_u \left[-\varphi_u \|\vartheta_u - \vartheta_{u+1}\|^2 - \frac{1}{\varphi_u} \|\vartheta_u - \vartheta_{u-1}\|^2 \right] \\ &= (1 - \alpha_u \varphi_u) \|\vartheta_{u+1} - \vartheta_u\|^2 + \left(\alpha_u^2 - \frac{\alpha_u}{\varphi_u} \right) \|\vartheta_u - \vartheta_{u-1}\|^2 . \end{aligned} \tag{3.13}$$

where $\varphi_u = \frac{1}{\alpha_u + \gamma_2 \varrho_u}$. Applying (3.13) in (3.12), and since $1 + \alpha_u + \varrho_u \geq 1 + \alpha_u$, we have

$$\begin{aligned} & \|\vartheta_{u+1} - q\|^2 - (1 + \alpha_u + \varrho_u) \|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_{u-1} - q\|^2 \\ \leq \ & \|\vartheta_{u+1} - q\|^2 - (1 + \alpha_u) \|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_{u-1} - q\|^2 \\ \leq \ & -\frac{(1 - \alpha_u \varphi_u)}{\varrho_u} \|\vartheta_{u+1} - \vartheta_u\|^2 + \eta_u \|\vartheta_u - \vartheta_{u-1}\|^2 . \end{aligned}$$

Hence,

$$\begin{aligned} & \|\vartheta_{u+1} - q\|^2 - (1 + \alpha_u) \|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_{u-1} - q\|^2 \\ \leq \ & -\frac{(1 - \alpha_u \varphi_u)}{\varrho_u} \|\vartheta_{u+1} - \vartheta_u\|^2 + \eta_u \|\vartheta_u - \vartheta_{u-1}\|^2 , \end{aligned} \tag{3.14}$$

where $\eta_u = \alpha_u(1+\alpha_u) + \frac{1}{\varrho_u} \left(\frac{\alpha_u}{\varphi_u} - \alpha_u^2 \right) \geq 0$. For the choice of φ_u, we get $\gamma_2 = \frac{1 - \varphi_u \alpha_u}{\varphi_u \varrho_u}$. Since $0 \leq \alpha_u \leq \alpha < 1$, we have

$$\begin{aligned} \eta_u &= \alpha_u(1 + \alpha_u) + \frac{1}{\varrho_u} \left(\frac{\alpha_u}{\varphi_u} - \alpha_u^2 \right) \leq \alpha(1 + \alpha) + \frac{1}{\varrho_u} \left(\frac{\alpha_u - \alpha_u^2 \varphi_u}{\varphi_u} \right) \\ &= \alpha(1 + \alpha) + \alpha_u \left(\frac{1 - \alpha_u \varphi_u}{\varrho_u \varphi_u} \right) \leq \alpha(1 + \alpha) + \alpha \gamma_2, \text{ for all } u \geq 0. \end{aligned}$$

Define the sequences $\{\Phi_u\}$ and $\{\Lambda_u\}$ by

$$\Phi_u = \|\vartheta_u - q\|^2 , \ \Lambda_u = \Phi_u - \alpha_u \Phi_{u-1} + \eta_u \|\vartheta_u - \vartheta_{u-1}\|^2 , \text{ for all } u \geq 1.$$

Because $\Phi_u \geq 0$ and $\{\alpha_u\}$ is non-decreasing for all $u \in \mathbb{N}$, we have

$$\begin{aligned} \Lambda_{u+1} - \Lambda_u &= \Phi_{u+1} - \alpha_{u+1} \Phi_u + \eta_{u+1} \|\vartheta_{u+1} - \vartheta_u\|^2 - \Phi_u + \alpha_u \Phi_{u-1} - \eta_u \|\vartheta_u - \vartheta_{u-1}\|^2 \\ &= \Phi_{u+1} - (1 + \alpha_u) \Phi_u + \alpha_u \Phi_{u-1} + \eta_{u+1} \|\vartheta_u - \vartheta_{u+1}\|^2 - \eta_u \|\vartheta_u - \vartheta_{u-1}\|^2 . \end{aligned}$$

From (3.13), we get

$$\Lambda_{u+1} - \Lambda_u \leq \left(\eta_{u+1} - \frac{(1-\alpha_u\varphi_u)}{\varrho_u}\right)\|\vartheta_u - \vartheta_{u+1}\|^2 . \tag{3.15}$$

Now, we claim that

$$\eta_{u+1} - \frac{(1-\alpha_u\varphi_u)}{\varrho_u} \leq -\gamma_1. \tag{3.16}$$

According to the choice of φ_u, one has

$$\begin{aligned}
\eta_{u+1} - \frac{(1-\alpha_u\varphi_u)}{\varrho_u} &\leq -\gamma_1 \\
&\Leftrightarrow \varrho_u\left(\eta_{u+1}+\gamma_1\right) + \left(\alpha_u\varphi_u - 1\right) \leq 0 \\
&\Leftrightarrow \varrho_u\left(\eta_{u+1}+\gamma_1\right) + \left(\frac{\alpha_u}{\alpha_u+\gamma_2\varrho_u} - 1\right) \leq 0 \\
&\Leftrightarrow \varrho_u\left(\eta_{u+1}+\gamma_1\right) - \frac{\gamma_2\varrho_u}{\alpha_u+\gamma_2\varrho_u} \leq 0 \\
&\Leftrightarrow \left(\alpha_u+\gamma_2\varrho_u\right)\left(\eta_{u+1}+\gamma_1\right) - \gamma_2 \leq 0 \\
&\Leftrightarrow \left(\alpha_u+\gamma_2\varrho_u\right)\left(\eta_{u+1}+\gamma_1\right) \leq \gamma_2.
\end{aligned}$$

Now,

$$\begin{aligned}
\left(\alpha_u+\gamma_2\varrho_u\right)\left(\eta_{u+1}+\gamma_1\right) &\leq \left(\alpha+\gamma_2\varrho_u\right)\left(\alpha(1+\alpha)+\alpha\gamma_2+\gamma_1\right) \\
&\leq \gamma_2,
\end{aligned}$$

where the final inequality is derived by the bound ϱ_u. Thus, from (3.15) and (3.16), we obtain that

$$\Lambda_{u+1} - \Lambda_u \leq -\gamma_1\|\vartheta_u - \vartheta_{u+1}\|^2, \text{ for all } u \geq 1. \tag{3.17}$$

It follows from the monotonicity of $\{\Lambda_u\}$ and the boundedness of $\{\alpha_u\}$ that

$$-\alpha\Phi_{u-1} < \Phi_u - \alpha\Phi_{u-1} \leq \Lambda_u \leq \Lambda_1, \text{ for all } u \geq 1,$$

that is,

$$\Phi_u \leq \Lambda_1 + \alpha\Phi_{u-1}. \tag{3.18}$$

Hence,

$$\Phi_u \leq \alpha^u\Phi_0 + \Lambda_1\sum_{j=1}^{u-1}\alpha^j \leq \alpha^u\Phi_0 + \frac{\Lambda_1}{1-\alpha}, \text{ for all } u \geq 1. \tag{3.19}$$

It is clear that $\Phi_1 = \Lambda_1$ (since $\alpha_1 = 0$). According to (3.17)-(3.19), one can write

$$\begin{aligned}
\gamma_1\sum_{u=0}^{\infty}\|\vartheta_u - \vartheta_{u+1}\|^2 &\leq \Lambda_u - \Lambda_{u+1} \leq \Lambda_1 - \Lambda_{u+1} \\
&\leq \Lambda_1 + \alpha\Phi_u \leq \Lambda_1 + \alpha^{u+1}\Phi_0 + \frac{\alpha\Lambda_1}{1-\alpha}, \text{ for all } u \geq 1.
\end{aligned}$$

This yields

$$\sum_{u=0}^{\infty}\|\vartheta_u - \vartheta_{u+1}\|^2 < +\infty. \tag{3.20}$$

Therefore,

$$\|\vartheta_u - \vartheta_{u+1}\| \to 0 \text{ as } u \to \infty. \tag{3.21}$$

It follows from (3.13) that

$$\begin{aligned}\|\vartheta_{u+1} - z_u\|^2 &= \|\vartheta_{u+1} - (\vartheta_u + \alpha_u (\vartheta_u - \vartheta_{u-1}))\|^2 \\ &= \|\vartheta_{u+1} - \vartheta_u\|^2 + \alpha_u^2 \|\vartheta_u - \vartheta_{u-1}\|^2 + 2\alpha_u \langle \vartheta_{u+1} - \vartheta_u, \vartheta_u - \vartheta_{u-1}\rangle \\ &\to 0 \text{ as } u \to \infty. \end{aligned} \tag{3.22}$$

Now, for $q \in \Theta$, it follows from (3.13), (3.14), (3.20), and $\alpha_u \varphi_u < 1$ that $\lim_{n\to\infty} \|\vartheta_u - q\|$ exists. Hence, the sequence $\{\vartheta_u\}$ is bounded. Let ϑ^* be a weak cluster point of $\{\vartheta_u\}$. From (3.8) and $\kappa_u \in (0,1)$, we have

$$\begin{aligned} &\varrho_u \epsilon \left(\|(I - W_0) p_u\|^2 + \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j \omega_u\right\|^2 \right) \\ \leq\ & \varrho_u \|\omega_u - q\|^2 + (1 - \varrho_u) \|z_u - q\|^2 - \|\vartheta_{u+1} - q\|^2 \\ =\ & \varrho_u \left\|(1 - \kappa_u) z_u + \kappa_u \mathfrak{I} z_u - q\right\|^2 + (1 - \varrho_u) \|z_u - q\| - \|\vartheta_{u+1} - q\|^2 \\ \leq\ & \varrho_u (1 - \kappa_u) \|z_u - q\|^2 + \kappa_u \varrho_u \left\|\mathfrak{I} z_u - q\right\|^2 - \varrho_u (1 - \kappa_u)\kappa_u \left\|\mathfrak{I} z_u - z_u\right\|^2 \\ & +(1 - \varrho_u) \|z_u - q\| - \|\vartheta_{u+1} - q\|^2 \\ \leq\ & \|z_u - q\|^2 - \|\vartheta_{u+1} - q\|^2 \\ =\ & \|\vartheta_{u+1} - z_u\| (\|z_u - q\| - \|\vartheta_{u+1} - q\|). \end{aligned}$$

As $u \to \infty$ and using the definition of ϱ_u, we have

$$\lim_{u\to\infty} \|(I - W_0) p_u\| = 0 \text{ and } \lim_{u\to\infty} \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j \omega_u\right\| = 0. \tag{3.23}$$

Now, from (3.21) and (3.22), we have

$$\|z_u - \vartheta_u\| \leq \|z_u - \vartheta_{u+1}\| + \|\vartheta_{u+1} - \vartheta_u\| \to 0 \text{ as } u \to \infty. \tag{3.24}$$

Because $\vartheta_{u_k} \rightharpoonup \vartheta^*$ and $\|z_u - \vartheta_u\| \to 0$ as $u \to \infty$, we conclude that $z_{u_j} \rightharpoonup \vartheta^*$. Thus, $\left\{\mathfrak{I}_j \omega_{u_k}\right\} \rightharpoonup \left\{\mathfrak{I}_j \vartheta^*\right\}$. Since $(I - W_j)$ is demiclosed at 0 and by (3.22), we infer that $\mathfrak{I}_j \vartheta^* \in Fix\left(W_j\right)$ for all $j = 0, 1, ..., N$.

Further, by using (3.23) and (3.24), we get

$$\begin{aligned} \|p_u - \vartheta_u\| &= \left\|\omega_u - \mu \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j \omega_u - \vartheta_u\right\| \\ &\leq \|\omega_u - \vartheta_u\| + \mu \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\| \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j \omega_u\right\| \\ &= \left\|(1 - \kappa_u) z_u + \kappa_u \mathfrak{I} z_u - \vartheta_u\right\| + \mu \sum_{j=1}^{N} \left\|\mathfrak{I}_j\right\| \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j \omega_u\right\| \end{aligned}$$

$$\leq \quad \kappa_u \|z_u - \vartheta_u\| + \mu \sum_{j=1}^{N} \left\|\Im_j\right\| \sum_{j=1}^{N} \left\|(I - W_j)\Im_j \omega_u\right\| \to 0, \text{ as } u \to \infty.$$

Since $\vartheta_{u_k} \rightharpoonup \vartheta^*$, then $p_u \rightharpoonup \vartheta^*$. By the demiclosedness of $(I - W_0)$ at 0, then $\vartheta^* \in Fix(W_0)$. Thus, $\vartheta^* \in \Theta$. According to Lemma 2.4, we conclude that $\{\vartheta_u\} \rightharpoonup \vartheta^* \in \Theta$. □

3.2. An IMI algorithm with variable step size

For $\vartheta_0, \vartheta_1 \in \Delta$, build a sequence $\{\vartheta_u\}$ as follows:

$$\begin{cases} z_u = \vartheta_u + \alpha_u (\vartheta_u - \vartheta_{u-1}), \\ \omega_u = (1 - \kappa_u) z_u + \kappa_u \Im z_u, \\ \theta_u = W_{0,\lambda} \left(\omega_u - \mu_u \sum\limits_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u\right), \\ \vartheta_{u+1} = (1 - \varrho_u) z_u + \varrho_u \theta_u, \ u \geq 1, \end{cases} \tag{3.25}$$

where $W_{0,\lambda} = (1 - \lambda) I + \lambda W_0$, $\lambda > 0$, $\kappa_u \in (0, 1)$ and

$$\mu = \begin{cases} 0, & \text{if } \left\|\sum\limits_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u\right\| = 0, \\ \dfrac{\min_{j \in [1,N]} \left(1 - \ell_j\right) \sum\limits_{j=1}^{N} \left\|(I - W_j) \Im_j \omega_u\right\|^2}{2 \left\|\sum\limits_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u\right\|^2}, & \text{otherwise.} \end{cases}$$

Furthermore, $\alpha_u \subset [0, \infty)$ is an increasing sequence with $0 \leq \alpha_u \leq \alpha < 1$ and $\alpha_1 = 0$.

Theorem 3.2. *Let W_j be an ℓ_j-demicontractive mapping and satisfies the demiclosedness principle for each $j = 0, 1, ..., N$, and let $\Im$ be a nonexpansive mapping. Choose $\varrho, \gamma_1, \gamma_2 > 0$ such that the assumptions of (3.2) hold. If $\{\vartheta_u\}$ is a sequence generated by (3.25), then $\vartheta_u \rightharpoonup \vartheta^* \in \Theta$.*

Proof. First, we claim that $\|\vartheta_{u+1} - \vartheta_u\| \to 0$, $u \to \infty$. For this regard, let $q \in \Theta$. Set $\ell = \min_{j \in [1,N]} \left(1 - \ell_j\right)$ and $\epsilon = \lambda (1 - \lambda - \ell_0)$, for each $j = 1, 2, \cdots N$.

Now, from the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left\|\sum_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u\right\|^2 &\leq \left(\sum_{j=1}^{N} \left\|\Im_j\right\| \left\|(I - W_j) \Im_j \omega_u\right\|\right)^2 \\ &\leq \sum_{j=1}^{N} \left\|\Im_j\right\|^2 \sum_{j=1}^{N} \left\|(I - W_j) \Im_j \omega_u\right\|^2. \end{aligned}$$

Set $p_u = \omega_u - \mu_u \sum\limits_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u$. Then by (2.1), we have

$$\|p_u - q\|^2 = \left\|\omega_u - \mu_u \sum_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u - q\right\|^2$$

$$
\begin{aligned}
&= \|\omega_u - q\|^2 + \mu_u^2 \left\| \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \right\|^2 - 2\mu_u \sum_{j=1}^{N} \langle \omega_u - q, \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \rangle \\
&\leq \|\omega_u - q\|^2 + \mu_u^2 \left\| \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \right\|^2 - \ell\mu_u \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2.
\end{aligned}
$$

Applying the definition of μ_u, we find that

$$
\begin{aligned}
\|p_u - q\|^2 &\leq \|\omega_u - q\|^2 + \mu_u \left(\mu_u \left\| \sum_{j=1}^{N} \mathfrak{I}_j^*(I - W_j)\mathfrak{I}_j\omega_u \right\|^2 - \ell \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 \right) \\
&= \|\omega_u - q\|^2 + \mu_u \left(\frac{\ell \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2}{2} - \ell \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2 \right) \\
&= \|\omega_u - q\|^2 - \frac{\mu_u \ell}{2} \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2. \qquad (3.26)
\end{aligned}
$$

It is simple to demonstrate that the inequality above holds when $\mu_u = 0$. Combining (2.2) with (3.26), we can write

$$
\|\theta_u - q\|^2 \leq \|\omega_u - q\|^2 - \epsilon \|(I - W_0)\, p_u\|^2 - \frac{\mu_u \ell}{2} \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2. \qquad (3.27)
$$

From the definition of ω_u, we have

$$
\begin{aligned}
\|\omega_u - q\|^2 &= \left\|(1 - \kappa_u)z_u + \kappa_u \mathfrak{I} z_u - q\right\|^2 \\
&\leq (1 - \kappa_u)\|z_u - q\|^2 + \kappa_u \left\|\mathfrak{I} z_u - q\right\|^2 - \kappa_u(1 - \kappa_u)\left\|z_u - \mathfrak{I} z_u\right\|^2 \\
&\leq \|z_u - q\|^2. \qquad (3.28)
\end{aligned}
$$

Applying (3.28) in (3.27), we get

$$
\|\theta_u - q\|^2 \leq \|z_u - q\|^2 - \epsilon \|(I - W_0)\, p_u\|^2 - \frac{\mu_u \ell}{2} \sum_{j=1}^{N} \left\|(I - W_j)\mathfrak{I}_j\omega_u\right\|^2. \qquad (3.29)
$$

In the basis of definition of ϑ_{u+1}, one has

$$
\begin{aligned}
\|\vartheta_{u+1} - q\|^2 &= \|(1 - \varrho_u)z_u + \varrho_u\theta_u - q\|^2 \\
&= (1 - \varrho_u)\|z_u - q\|^2 + \varrho_u\|\theta_u - q\|^2 - \varrho_u(1 - \varrho_u)\|z_u - \theta_u\|^2. \qquad (3.30)
\end{aligned}
$$

From (3.29) in (3.30), we get

$$
\|\vartheta_{u+1} - q\|^2 \leq \|z_u - q\|^2
$$

$$-\varrho_u\left(\epsilon\,\|(I-W_0)\,p_u\|^2-\frac{\mu_u\ell}{2}\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2\right)$$
$$-\varrho_u(1-\varrho_u)\,\|z_u-\theta_u\|^2\,. \tag{3.31}$$

Since $\varrho_u>0$ and $\epsilon>0$, we have

$$\begin{aligned}\mu_u &= \frac{\ell\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2}{2\left\|\sum_{j=1}^{N}\mathfrak{I}_j^*(I-W_j)\mathfrak{I}_j\omega_u\right\|^2}\geq\frac{\ell}{2}\frac{\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2}{\left(\sum_{j=1}^{N}\left\|\mathfrak{I}_j^*(I-W_j)\mathfrak{I}_j\omega_u\right\|\right)^2}\\ &\geq \frac{\ell}{2}\frac{\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2}{\left(\sum_{j=1}^{N}\left\|\mathfrak{I}_j\right\|\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|\right)^2}\geq\frac{\ell}{2}\frac{\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2}{\sum_{j=1}^{N}\left\|\mathfrak{I}_j\right\|^2\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2}\\ &= \frac{\ell}{2\sum_{j=1}^{N}\left\|\mathfrak{I}_j\right\|^2}>0.\end{aligned}$$

Hence, from (3.31), we conclude that

$$\|\vartheta_{u+1}-q\|^2\leq\|z_u-q\|^2-\varrho_u(1-\varrho_u)\,\|z_u-\theta_u\|^2\,.$$

Similar to Theorem 3.1, we find that

$$\sum_{u=0}^{\infty}\|\vartheta_u-\vartheta_{u+1}\|^2<+\infty. \tag{3.32}$$

Therefore,

$$\|\vartheta_u-\vartheta_{u+1}\|\to 0 \text{ as } u\to\infty, \tag{3.33}$$

and

$$\begin{aligned}\|\vartheta_{u+1}-z_u\|^2 &= \|\vartheta_{u+1}-(\vartheta_u+\alpha_u\,(\vartheta_u-\vartheta_{u-1}))\|^2\\ &= \|\vartheta_{u+1}-\vartheta_u\|^2+\alpha_u^2\,\|\vartheta_u-\vartheta_{u-1}\|^2+2\alpha_u\langle\vartheta_{u+1}-\vartheta_u,\vartheta_u-\vartheta_{u-1}\rangle\\ &\to 0 \text{ as } u\to\infty.\end{aligned} \tag{3.34}$$

Using the same computations as in the formulation of Theorem 3.1; Eqs (3.13), (3.14), (3.32), and by Lemma 2.5, we determine that $\lim_{n\to\infty}\|\vartheta_u-q\|$ exists. Hence, the sequence $\{\vartheta_u\}$ is bounded.

Now, let ϑ^* be a weak cluster point of $\{\vartheta_u\}$. From (3.31), we get

$$\|\vartheta_{u+1}-q\|^2\leq\|z_u-q\|^2-\varrho_u\left(\epsilon\,\|(I-W_0)\,p_u\|^2-\frac{\mu_u\ell}{2}\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2\right),$$

which implies that

$$\begin{aligned} & \varrho_u \left(\epsilon \left\| (I - W_0)\, p_u \right\|^2 - \frac{\mu_u \ell}{2} \sum_{j=1}^{N} \left\| (I - W_j)\mathfrak{I}_j \omega_u \right\|^2 \right) \\ \leq \;& \| z_u - q \|^2 - \| \vartheta_{u+1} - q \|^2 \\ = \;& \| \vartheta_{u+1} - z_u \| \left(\| z_u - q \| - \| \vartheta_{u+1} - q \| \right). \end{aligned}$$

It follows from (3.34) and the definition of ϱ_u that

$$\lim_{u\to\infty} \| (I - W_0)\, p_u \| = 0 \text{ and } \lim_{u\to\infty} \sum_{j=1}^{N} \left\| (I - W_j)\mathfrak{I}_j \omega_u \right\| = 0. \tag{3.35}$$

Now, from (3.33) and (3.34), we have

$$\| z_u - \vartheta_u \| \leq \| z_u - \vartheta_{u+1} \| + \| \vartheta_{u+1} - \vartheta_u \| \to 0 \text{ as } u \to \infty. \tag{3.36}$$

Because $\vartheta_{u_k} \rightharpoonup \vartheta^*$ and $\| z_u - \vartheta_u \| \to 0$ as $u \to \infty$, we conclude that $z_{u_j} \rightharpoonup \vartheta^*$. Thus, $\left\{ \mathfrak{I}_j \omega_{u_k} \right\} \rightharpoonup \left\{ \mathfrak{I}_j \vartheta^* \right\}$. Since $(I - W_j)$ is demiclosed at 0 and by (3.36), we infer that $\mathfrak{I}_j \vartheta^* \in Fix\left(W_j\right)$ for all $j = 0, 1, ..., N$.

Further, by using (3.35) and (3.36), we get

$$\begin{aligned} \| p_u - \vartheta_u \| &= \left\| \omega_u - \mu_u \sum_{j=1}^{N} \mathfrak{I}_j^* (I - W_j) \mathfrak{I}_j \omega_u - \vartheta_u \right\| \\ &\leq \| \omega_u - \vartheta_u \| + \mu_u \sum_{j=1}^{N} \left\| \mathfrak{I}_j \right\| \sum_{j=1}^{N} \left\| (I - W_j)\mathfrak{I}_j \omega_u \right\| \\ &= \left\| (1 - \kappa_u) z_u + \kappa_u \mathfrak{I} z_u - \vartheta_u \right\| + \mu_u \sum_{j=1}^{N} \left\| \mathfrak{I}_j \right\| \sum_{j=1}^{N} \left\| (I - W_j)\mathfrak{I}_j \omega_u \right\| \\ &= \kappa_u \| z_u - \vartheta_u \| + \mu_u \sum_{j=1}^{N} \left\| \mathfrak{I}_j \right\| \sum_{j=1}^{N} \left\| (I - W_j)\mathfrak{I}_j \omega_u \right\| \to 0, \text{ as } u \to \infty. \end{aligned}$$

Since $\vartheta_{u_k} \rightharpoonup \vartheta^*$, then $p_u \rightharpoonup \vartheta^*$. By the demiclosedness of $(I - W_0)$ at 0, then $\vartheta^* \in Fix\,(W_0)$. Thus, $\vartheta^* \in \Theta$. According to Lemma 2.4, we conclude that $\{\vartheta_u\} \rightharpoonup \vartheta^* \in \Theta$. □

4. Strong convergence results

This section is dedicated to the development of several new IMI algorithms for solving the split feasibility problem (SFPP), for which we establish strong convergence results.

4.1. An IMI algorithm with a fixed step size

For $\vartheta_0, \vartheta_1 \in \Delta$, construct a sequence $\{\vartheta_u\}$ in the following way:

$$\begin{cases} z_u = \vartheta_u + \alpha_u (\vartheta_u - \vartheta_{u-1}), \\ \omega_u = (1-\kappa_u) z_u + \kappa_u \Im z_u, \\ \theta_u = W_{0,\lambda}\left(\omega_u - \mu \sum\limits_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u\right), \\ \vartheta_{u+1} = (1-\zeta_u)\theta_u + \zeta_u G(\vartheta_u), \ u \geq 1, \end{cases} \tag{4.1}$$

where $W_{0,\lambda} = (1-\lambda) I + \lambda W_0$ and $\mu \in \left(0, \frac{\min_{j \in [1,N]}(1-\ell_j)}{\sum_{j=1}^{N} \|\Im_j\|^2}\right)$, $\lambda > 0$, and G is a contraction mapping with a comparison factor of σ. Assume that the sequences $\{\zeta_u\}$, $\{\kappa_u\}$, and $\{\alpha_u\}$ fulfill the axioms below:

(A_1) $\{\zeta_u\} \subset [0,1]$, $\{\kappa_u\} \subset (0,1)$, $\lim_{u\to\infty} \zeta_u = 0 = \lim_{u\to\infty} \kappa_u$, and $\sum\limits_{u=1}^{\infty} \zeta_u = \infty$,

(A_2) $\{\alpha_u\} \subset [0, \alpha)$ for some $\alpha > 0$ with $\lim_{u\to\infty} \frac{\alpha_u}{\zeta_u} \|\vartheta_u - \vartheta_{u-1}\| = 0$.

Theorem 4.1. *Let W_j be an ℓ_j-demicontractive mapping and satisfies the demiclosedness principle for each $j = 0, 1, ..., N$ and $\Im$ be a nonexpansive mapping. If $\{\vartheta_u\}$ is a sequence generated by (4.1), then $\vartheta_u \longrightarrow q$, where q is the unique fixed point (UFP) of the contraction $P_\Theta G$.*

Proof. Let $p_u = \omega_u - \mu \sum\limits_{j=1}^{N} \Im_j^* (I - W_j) \Im_j \omega_u$, and let q be a UFP of the contraction $P_\Theta G$. Then, following the lines of the deduction of Eq (3.7) in Theorem 3.1, we have

$$\|\theta_u - q\|^2 \leq \|\omega_u - q\|^2 - \epsilon\left(\|(I - W_0) p_u\|^2 + \sum_{j=1}^{N} \left\|(I - W_j)\Im_j \omega_u\right\|^2\right). \tag{4.2}$$

Now, we divide the rest of the proof into the following three steps:

Step 1. Prove that $\{\vartheta_u\}$ is bounded. Based on (3.7), we deduce that

$$\|\theta_u - q\| \leq \|\omega_u - q\|, \tag{4.3}$$

and

$$\begin{aligned} \|\omega_u - q\| &= \left\|(1-\kappa_u) z_u + \kappa_u \Im z_u - q\right\| \\ &\leq (1-\kappa_u)\|z_u - q\| + \kappa_u \left\|\Im z_u - \Im q\right\| \\ &= \|z_u - q\| \\ &= \|\vartheta_u + \alpha_u(\vartheta_u - \vartheta_{u-1}) - q\| \\ &\leq \|\vartheta_u - z\| + \zeta_u \frac{\alpha_u}{\zeta_u}\|\vartheta_u - \vartheta_{u-1}\|. \end{aligned}$$

As $\frac{\alpha_u}{\zeta_u}\|\vartheta_u - \vartheta_{u-1}\| \to 0$ as $u \to \infty$, there is $\eth_1 > 0$ such that $\frac{\alpha_u}{\zeta_u}\|\vartheta_u - \vartheta_{u-1}\| \leq \eth_1$. Thus, one has

$$\|\omega_u - q\| \leq \|\vartheta_u - z\| + \zeta_u \eth_1. \tag{4.4}$$

From the definition of ϑ_{u+1}, and using (4.2) and (4.4), we can write

$$\begin{aligned}
\|\vartheta_{u+1} - q\| &= \|(1-\zeta_u)\theta_u + \zeta_u G(\vartheta_u) - q\| \\
&\leq (1-\zeta_u)\|\theta_u - q\| + \zeta_u \|G(\vartheta_u) - q\| \\
&\leq (1-\zeta_u)\|\omega_u - q\| + \zeta_u \|G(\vartheta_u) - G(q)\| + \zeta_u \|G(q) - q\| \\
&\leq (1-\zeta_u)\left[\|\vartheta_u - z\| + \zeta_u \eth_1\right] + \zeta_u \sigma \|\vartheta_u - q\| + \zeta_u \|G(q) - q\| \\
&= (1-\zeta_u(1-\sigma))\|\vartheta_u - z\| + \zeta_u\left[\|G(q) - q\| + (1-\zeta_u)\eth_1\right] \\
&\leq (1-\zeta_u(1-\sigma))\|\vartheta_u - z\| + \zeta_u(1-\sigma)\left[\frac{\|G(q) - q\| + (1-\zeta_u)\eth_1}{1-\sigma}\right] \\
&\leq \max\left\{\|\vartheta_u - z\|, \frac{\|G(q) - q\| + \eth_1}{1-\sigma}\right\} \\
&\leq \cdots \leq \max\left\{\|\vartheta_0 - z\|, \frac{\|G(q) - q\| + \eth_1}{1-\sigma}\right\}.
\end{aligned}$$

Therefor, $\{\vartheta_u\}$ is bounded.

Step 2. Claim that

$$e_{u+1} \leq (1-c_u)e_u + c_u r_u,$$

where $e_u = \|\vartheta_u - q\|^2$, $c_u = \zeta_u\left(1-\sigma^2\right)$, and

$$r_u = \left(\frac{(1-\zeta_u)\alpha_u}{(1-\sigma^2)\zeta_u}\right)\|\vartheta_u - \vartheta_{u-1}\|\eth_2 + \frac{2}{1-\sigma^2}\langle G(q) - q, \vartheta_{u+1} - q\rangle.$$

Now, from the definition of ϑ_{u+1}, we have

$$\begin{aligned}
\|\vartheta_{u+1} - q\|^2 &= \|\zeta_u G(\vartheta_u) + (1-\zeta_u)\theta_u - q\|^2 \\
&\leq \|\zeta_u(G(\vartheta_u) - G(q)) + (1-\zeta_u)(\theta_u - q)\|^2 + 2\zeta_u\langle G(q) - q, \vartheta_{u+1} - q\rangle \\
&\leq \zeta_u \|G(\vartheta_u) - G(q)\|^2 + (1-\zeta_u)\|\theta_u - q\|^2 + 2\zeta_u\langle G(q) - q, \vartheta_{u+1} - q\rangle.
\end{aligned}$$

Applying (3.7), we can write

$$\begin{aligned}
&\|\vartheta_{u+1} - q\|^2 \\
\leq\ & \zeta_u\sigma^2\|\vartheta_u - q\|^2 + 2\zeta_u\langle G(q) - q, \vartheta_{u+1} - q\rangle \\
&+(1-\zeta_u)\left[\|\omega_u - q\|^2 - \epsilon\left(\|(I-W_0)p_u\|^2 + \sum_{j=1}^{N}\left\|(I-W_j)\Im_j\omega_u\right\|^2\right)\right].
\end{aligned} \tag{4.5}$$

From the definition of ω_u, we have

$$\begin{aligned}
\|\omega_u - q\|^2 &\leq \|z_u - q\|^2 \\
&= \|\vartheta_u + \alpha_u(\vartheta_u - \vartheta_{u-1}) - q\|^2 \\
&= \|\vartheta_u - q\|^2 + \alpha_u^2\|\vartheta_u - \vartheta_{u-1}\|^2 + 2\alpha_u\langle\vartheta_u - q, \vartheta_u - \vartheta_{u-1}\rangle \\
&\leq \|\vartheta_u - q\|^2 + \alpha_u^2\|\vartheta_u - \vartheta_{u-1}\|^2 + 2\alpha_u\|\vartheta_u - q\|\|\vartheta_u - \vartheta_{u-1}\| \\
&\leq \|\vartheta_u - q\|^2 + \alpha_u\|\vartheta_u - \vartheta_{u-1}\|\eth_2,
\end{aligned} \tag{4.6}$$

where

$$\eth_2 = \sup_u \{2 \|\vartheta_u - q\| + \alpha_u \|\vartheta_u - \vartheta_{u-1}\|\}.$$

Applying (4.6) in (4.7), we have

$$\begin{aligned}
& \|\vartheta_{u+1} - q\|^2 \\
\leq\ & \zeta_u \sigma^2 \|\vartheta_u - q\|^2 + 2\zeta_u \langle G(q) - q, \vartheta_{u+1} - q\rangle \\
& +(1-\zeta_u)\Big[\|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_u - \vartheta_{u-1}\| \eth_2 \\
& -\epsilon\Big(\|(I - W_0)\, p_u\|^2 + \sum_{j=1}^{N} \big\|(I - W_j)\Im_j \omega_u\big\|^2\Big)\Big] \\
=\ & \Big(1 - \zeta_u\big(1-\sigma^2\big)\Big)\|\vartheta_u - q\|^2 + \zeta_u\big(1-\sigma^2\big)\Big[\frac{(1-\zeta_u)\alpha_u}{(1-\sigma^2)\,\zeta_u}\|\vartheta_u - \vartheta_{u-1}\| \eth_2 \\
& -\Big(\frac{\epsilon(1-\zeta_u)}{(1-\sigma^2)\,\zeta_u}\Big\{\|(I - W_0)\, p_u\|^2 + \sum_{j=1}^{N} \big\|(I - W_j)\Im_j \omega_u\big\|^2\Big\}\Big) \\
& + \frac{2}{1-\sigma^2}\langle G(q) - q, \vartheta_{u+1} - q\rangle\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\vartheta_{u+1} - q\|^2 \leq\ & \Big(1 - \zeta_u\big(1-\sigma^2\big)\Big)\|\vartheta_u - q\|^2 \\
& + \zeta_u\big(1-\sigma^2\big)\Big[\frac{(1-\zeta_u)\alpha_u}{(1-\sigma^2)\,\zeta_u}\|\vartheta_u - \vartheta_{u-1}\| \eth_2 \\
& + \frac{2}{1-\sigma^2}\langle G(q) - q, \vartheta_{u+1} - q\rangle\Big].
\end{aligned} \tag{4.7}$$

Step 3. Show that $\vartheta_u \longrightarrow q$. To accomplish this step, we present the following two cases:

(I) Assume that there is $U \in \mathbb{N}$ such that

$$\|\vartheta_{u+1} - q\|^2 \leq \|\vartheta_u - q\|^2, \text{ for all } u \geq U.$$

It follows that $\lim_{u\to\infty} \|\vartheta_u - q\|^2$ exists. First, we prove that

$$\lim_{u\to\infty} \|\vartheta_{u+1} - \vartheta_u\| = 0.$$

From the definition of ϑ_{u+1}, we have

$$\begin{aligned}
\|\vartheta_{u+1} - q\|^2 &= \|\zeta_u (G(\vartheta_u) - q) + (1-\zeta_u)(\theta_u - q)\|^2 \\
&= \zeta_u \|G(\vartheta_u) - q\|^2 + (1-\zeta_u)\|\theta_u - q\|^2 - \zeta_u(1-\zeta_u)\|G(\vartheta_u) - \theta_u\|^2.
\end{aligned}$$

Applying (4.2) in (4.6), we get

$$\|\vartheta_{u+1} - q\|^2 \leq \zeta_u \|G(\vartheta_u) - q\|^2$$

$$
\begin{aligned}
&+(1-\zeta_u)\Big[\|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_u - \vartheta_{u-1}\| \eth_2 \\
&-\epsilon\left(\|(I-W_0)\,p_u\|^2 + \sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2\right)\Big] \\
&-\zeta_u(1-\zeta_u)\,\|G(\vartheta_u)-\theta_u\|^2 \\
\leq\ & \zeta_u\,\|G(\vartheta_u)-q\|^2 \\
&+(1-\zeta_u)\Big[\|\vartheta_u - q\|^2 + \alpha_u \|\vartheta_u - \vartheta_{u-1}\| \eth_2 \\
&-\epsilon\left(\|(I-W_0)\,p_u\|^2 + \sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2\right)\Big].
\end{aligned}
$$

Hence, we can write

$$
\begin{aligned}
&(1-\zeta_u)\epsilon\left(\|(I-W_0)\,p_u\|^2 + \sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2\right) \\
\leq\ & \zeta_u\,\|G(\vartheta_u)-q\|^2 + (1-\zeta_u)\,\|\vartheta_u - q\|^2 - \|\vartheta_{u+1} - q\|^2 \\
&+\zeta_u\left(\frac{\alpha_u}{\zeta_u}\,\|\vartheta_u - \vartheta_{u-1}\|\,\eth_2\right).
\end{aligned}
$$

Because $\lim_{u\to\infty}\zeta_u = 0$, $\lim_{u\to\infty}\frac{\alpha_u}{\zeta_u}\|\vartheta_u - \vartheta_{u-1}\| = 0$, and $\lim_{u\to\infty}\|\vartheta_u - q\|^2$ exists, we have

$$
\epsilon\left(\|(I-W_0)\,p_u\|^2 + \sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\|^2\right) \to 0 \text{ as } u\to\infty.
$$

Since $\epsilon > 0$, we obtain

$$
\lim_{u\to\infty}\|(I-W_0)\,p_u\| = 0, \text{ and } \lim_{u\to\infty}\sum_{j=1}^{N}\left\|(I-W_j)\mathfrak{I}_j\omega_u\right\| = 0. \tag{4.8}
$$

Next, consider $W_0 p_u = h_u$. So,

$$
\|p_u - h_u\| = \|p_u - W_0 p_u\| = \|(I-W_0)\,p_u\| \to 0 \text{ as } u\to\infty.
$$

By the definition of p_u and using (4.8), one has

$$
\begin{aligned}
\|p_u - \omega_u\| &= \left\|\omega_u - \mu\sum_{j=1}^{N}\mathfrak{I}_j^*(I-W_j)\mathfrak{I}_j\omega_u - \omega_u\right\| \\
&\leq \mu\sum_{j=1}^{N}\left\|\mathfrak{I}_j^*(I-W_j)\mathfrak{I}_j\omega_u\right\| \to 0 \text{ as } u\to\infty.
\end{aligned}
$$

From (4.1), and by the axioms (A_1) and (A_2), we have

$$
\|\omega_u - \vartheta_u\| = \left\|(1-\kappa_u)z_u + \kappa_u\mathfrak{I}z_u - \vartheta_u\right\|
$$

$$
\begin{aligned}
&\leq (1-\kappa_u)\|z_u - \vartheta_u\| + \kappa_u \left\|\Im z_u - \vartheta_u\right\| \\
&= (1-\kappa_u)\zeta_u \frac{\alpha_u}{\zeta_u} \|\vartheta_u - \vartheta_{u-1}\| + \kappa_u \left\|\Im z_u - \vartheta_u\right\| \\
&\to 0 \text{ as } u \to \infty.
\end{aligned}
$$

Further, from the results above, we can write

$$
\lim_{u\to\infty} \|\theta_u - h_u\| = \lim_{u\to\infty} \|(1-\lambda)\, p_u + \lambda h_u - h_u\| = (1-\lambda) \lim_{u\to\infty} \|p_u - h_u\| = 0,
$$

$$
\lim_{u\to\infty} \|h_u - \vartheta_u\| \leq \lim_{u\to\infty} \|h_u - p_u\| + \lim_{u\to\infty} \|p_u - \omega_u\| + \lim_{u\to\infty} \|\omega_u - \vartheta_u\| = 0,
$$

and

$$
\lim_{u\to\infty} \|\theta_u - \vartheta_u\| \leq \lim_{u\to\infty} \|\theta_u - h_u\| + \lim_{u\to\infty} \|h_u - \vartheta_u\| = 0.
$$

Now, by the definition of ϑ_{u+1}, one has

$$
\begin{aligned}
\|\vartheta_{u+1} - \vartheta_u\| &= \|(1-\zeta_u)\theta_u + \zeta_u G(\vartheta_u) - \vartheta_u\| \\
&\leq \zeta_u \|G(\vartheta_u) - \vartheta_u\| + (1-\zeta_u)\|\theta_u - \vartheta_u\|.
\end{aligned}
$$

As $\lim_{u\to\infty} \zeta_u = 0$ and $\lim_{u\to\infty} \|\theta_u - \vartheta_u\| = 0$, we have

$$
\lim_{u\to\infty} \|\vartheta_{u+1} - \vartheta_u\| = 0.
$$

The boundedness of $\{\vartheta_u\}$ implies that there is a subsequence $\{\vartheta_{u_k}\} \subset \{\vartheta_u\}$ such that $\vartheta_{u_k} \rightharpoonup \vartheta^*$ as $k \to \infty$.
Now, we shall show that $\vartheta^* \in \Theta$. For this, we have

$$
\begin{aligned}
\left\|p_{u_k} - \vartheta_{u_k}\right\| &= \left\|\omega_{u_k} - \mu \sum_{j=1}^{N} \Im_j^*(I - W_j)\Im_j \omega_{u_k} - \vartheta_{u_k}\right\| \\
&\leq \left\|\omega_{u_k} - \vartheta_{u_k}\right\| + \mu \sum_{j=1}^{N} \left\|\Im_j\right\| \sum_{j=1}^{N} \left\|(I - W_j)\Im_j \omega_{u_k}\right\|.
\end{aligned}
$$

Because $\lim_{k\to\infty} \left\|\omega_{u_k} - \vartheta_{u_k}\right\| = 0$, and $\lim_{j\to\infty} \sum_{j=1}^{N} \left\|(I - W_j)\Im_j \omega_{u_k}\right\| = 0$, we get $\lim_{j\to\infty} \left\|p_{u_k} - \vartheta_{u_k}\right\| = 0$. Hence, $p_{u_k} \rightharpoonup \vartheta_{u_k}$ and $\omega_{u_k} \rightharpoonup \vartheta_{u_k}$. Since $\left\{\Im_j \omega_{u_k}\right\} \rightharpoonup \left\{\Im_j \vartheta^*\right\}$, $\lim_{u\to\infty} \left\|(I - W_j)\Im_j \omega_u\right\| = 0$ and $(I - W_j)$ is demiclosed at 0, then $\Im_j \vartheta^* \in Fix\left(W_j\right)$ for all $j = 0, 1, ..., N$, which implies that $\vartheta^* \in \cap_{j=1}^{N} \Im_j^{-1}\left(Fix\left(W_j\right)\right)$. Moreover, the demiclosedness of $(I - W_0)$ at 0 leads to $\vartheta^* \in Fix(W_0)$. Hence,

$$
\vartheta^* \in Fix(W_0) \cap \left(\cap_{j=1}^{N} \Im_j^{-1}\left(Fix\left(W_j\right)\right)\right).
$$

As a result, utilizing the projection's attributes, we obtain

$$
\begin{aligned}
&\lim_{u\to\infty} \sup\langle G(q) - q, \vartheta_u - q\rangle \\
\leq\ &\lim_{u\to\infty} \sup\langle G(q) - P_\Theta G(q), \vartheta_{u_k} - P_\Theta G(q)\rangle
\end{aligned}
$$

$$= \langle G(q) - P_\Theta G(q), \vartheta^* - P_\Theta G(q)\rangle.$$

Hence,

$$\begin{aligned} & \lim_{u\to\infty} \sup\langle G(q) - q, \vartheta_{u+1} - q\rangle \\ \leq\ & \lim_{u\to\infty} \sup\langle G(q) - P_\Theta G(q), \vartheta_{u+1} - \vartheta_u\rangle + \lim_{u\to\infty} \sup\langle G(q) - P_\Theta G(q), \vartheta_u - P_\Theta G(q)\rangle \\ =\ & \langle G(q) - P_\Theta G(q), \vartheta^* - P_\Theta G(q)\rangle \leq 0. \end{aligned}$$

By combining Lemma 2.3 with the result of Step 2, we may deduce that $\lim_{u\to\infty} \|\vartheta_u - q\| = 0$. Hence $\{\vartheta_u\} \longrightarrow z$. This finishes the proof.

(II) There is a subsequence $\{\left\|\vartheta_{u_k} - q\right\|\} \subset \{\|\vartheta_u - q\|\}$ such that

$$\left\|\vartheta_{u_k} - q\right\| \leq \left\|\vartheta_{u_k+1} - q\right\|, \text{ for all } k \in \mathbb{N}.$$

From Lemma 2.2, there is a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and

$$\left\|\vartheta_{m_k} - q\right\| \leq \left\|\vartheta_{m_k+1} - q\right\| \text{ and } \|\vartheta_k - q\| \leq \left\|\vartheta_{m_k+1} - q\right\| \text{ for all } k \in \mathbb{N}. \tag{4.9}$$

Using the same rationale as in Case (I), we have

$$\lim_{k\to\infty} \left\|\vartheta_{m_k+1} - \vartheta_{m_k}\right\| = 0.$$

Analogously, we get

$$\lim_{k\to\infty} \sup \sup\langle G(q) - q, \vartheta_{m_k+1} - q\rangle \leq 0. \tag{4.10}$$

Applying (4.7) and the condition $\|\vartheta_k - q\| \leq \left\|\vartheta_{m_k+1} - q\right\|$, we have

$$\begin{aligned} \left\|\vartheta_{m_k+1} - q\right\|^2 \leq\ & \frac{(1-\zeta_{m_k})}{(1-\sigma^2)} \frac{\alpha_{m_k}}{\zeta_{m_k}} \left\|\vartheta_{m_k} - \vartheta_{m_k-1}\right\| \partial_2 \\ & + \frac{2}{1-\sigma^2}\langle G(q) - q, \vartheta_{m_k+1} - q\rangle. \end{aligned} \tag{4.11}$$

Because $\sigma \in [0, 1)$, $\lim_{k\to\infty} \frac{\alpha_{m_k}}{\zeta_{m_k}} \left\|\vartheta_{m_k} - \vartheta_{m_k-1}\right\| = 0$ and $\{\left\|\vartheta_{m_k} - z\right\|\}$ is bounded, then by (4.10) and (4.11), one has

$$\lim_{k\to\infty} \left\|\vartheta_{m_k+1} - q\right\| = 0. \tag{4.12}$$

Combining (4.9) and (4.12), we have

$$\lim_{k\to\infty} \|\vartheta_k - q\| \leq \lim_{k\to\infty} \left\|\vartheta_{m_k+1} - q\right\| = 0.$$

Hence, $\vartheta_u \longrightarrow z$. This finishes the proof.

□

4.2. An IMI algorithm with a variable step size

For $\vartheta_0, \vartheta_1 \in \Delta$, construct a sequence $\{\vartheta_u\}$ in the following way:

$$\begin{cases} z_u = \vartheta_u + \alpha_u (\vartheta_u - \vartheta_{u-1}), \\ \omega_u = (1-\kappa_u) z_u + \kappa_u \Im z_u, \\ \theta_u = W_{0,\lambda}\left(\omega_u - \mu_u \sum_{j=1}^{N} \Im_j^*(I - W_j)\Im_j \omega_u\right), \\ \vartheta_{u+1} = (1-\zeta_u)\theta_u + \zeta_u G(\vartheta_u), \; u \geq 1, \end{cases} \tag{4.13}$$

where $W_{0,\lambda} = (1-\lambda) I + \lambda W_0$, $\lambda > 0$, $\kappa_u \in (0,1)$, G is a contraction mapping with a comparison factor σ and

$$\mu = \begin{cases} 0, & \text{if } \left\| \sum_{j=1}^{N} \Im_j^*(I - W_j)\Im_j \omega_u \right\| = 0, \\ \dfrac{\min_{j\in[1,N]}(1-\ell_j) \sum_{j=1}^{N} \left\|(I-W_j)\Im_j\omega_u\right\|^2}{2\left\| \sum_{j=1}^{N} \Im_j^*(I-W_j)\Im_j\omega_u \right\|^2}, & \text{otherwise.} \end{cases}$$

Moreover, we assume that the sequences $\{\zeta_u\}$, $\{\kappa_u\}$, and $\{\alpha_u\}$ satisfy the axioms (A_1) and (A_2).

Theorem 4.2. *Let W_j be an ℓ_j-demicontractive mapping and satisfies the demiclosedness principle for each $j = 0, 1, ..., N$ and let $\Im$ be a nonexpansive mapping. If $\{\vartheta_u\}$ is a sequence generated by (4.13), then $\vartheta_u \longrightarrow q$, where q is the UFP of the contraction $P_\Theta G$.*

Proof. Let $p_u = \omega_u - \mu_u \sum_{j=1}^{N} \Im_j^*(I - W_j)\Im_j\omega_u$, and let q be a UFP of the contraction $P_\Theta G$. Then, continuing along the lines of (3.27) in Theorem 3.2, we have

$$\|\theta_u - q\| \leq \|\omega_u - q\|^2 - \epsilon \|(I - W_0) p_u\|^2 - \frac{\mu_u \ell}{2} \sum_{j=1}^{N} \left\|(I - W_j)\Im_j\omega_u\right\|^2.$$

The remaining section of the proof is similar to the proof of Theorem 4.1. We replace

$$-\epsilon\left(\|(I - W_0) p_u\|^2 + \sum_{j=1}^{N} \left\|(I - W_j)\Im_j\omega_u\right\|^2\right)$$

in (4.2) with

$$-\epsilon \|(I - W_0) p_u\|^2 - \frac{\mu_u \ell}{2} \sum_{j=1}^{N} \left\|(I - W_j)\Im_j\omega_u\right\|^2,$$

and use $\mu_u > 0$, $u \geq 1$. The convergence result is established by following the proof of Theorem 4.1 line by line. □

Remark 4.1. *Theorems 3.1, 3.2, 4.1, and 4.2 can be used to approximate the solution of SFPP-multiple output sets (1.3) using nonexpansive mappings in real Hilbert spaces, since every nonexpansive mapping is 0-demicontractive.*

Remark 4.2. *Theorems 3.1, 3.2, 4.1, and 4.2 can be used to approximate the solution of the SFP-multiple output set (1.1), since the projection mapping on a closed convex subset of real Hilbert spaces is nonexpansive and we have $W_0 = P_C$ and $W_j = P_{Q_j}$, $j \in \mathbb{N}$.*

5. Solving split variational inequalities with multiple output sets

Assume that C is a non-empty, closed, and convex subset of a real Hilbert space χ and that$W : C \to C$ is a given mapping. The variational inequality problem (VIP) is described as follows: Find $\vartheta^* \in C$ such that

$$\langle W(\vartheta^*), \mathfrak{R} - \vartheta^* \rangle \geq 0, \text{ for all } \mathfrak{R} \in C.$$

The VIP is a critical problem in nonlinear analysis. Stampacchia [32] introduced this issue to examine problems in elasticity and potential theory. Following that, Lions and Stampacchia [33] demonstrated the existence of a solution for the VIP.The VIP is employed in various fields, like economics, optimization, operations research, and pure/applied mathematics. The VIP can be used to describe many well-known mathematical issues, like convex optimization, complementarity, FPs, and traffic equilibrium. Clearly, the VIP has the solution $\vartheta^* \in C$ if $\vartheta^* = P_C(I - \zeta W)\vartheta^*$. For more details, see [34–36].

To formulate our problem, we need the following definition:

Definition 5.1. *Let $W : C \to \chi$ be a given mapping. If there exists a constant $\hbar > 0$ such that*

$$\langle W\vartheta - W\mathfrak{R}, \vartheta - \mathfrak{R} \rangle \geq \hbar \left\| W\vartheta - W\mathfrak{R} \right\|^2, \textit{ for all } \vartheta, \mathfrak{R} \in C.$$

Then W is called an $\hbar$-inverse strongly monotone ($\hbar$-ISM, for short).

Now, we build the the split variational inequalities (SVIs) with multiple output sets. Let Q_j be a non-empty, closed and convex subset of χ_j for $j = 1, 2, ..., N$. Assume that $T_0 : C \to \chi$ is an $\hbar_0$-ISM and $T_j : C \to \chi$ is an $\hbar_j$-ISM for $j = 1, 2, ..., N$. Then the SVI-multiple output set is described as follows: Find $\vartheta^* \in C$ such that

$$\langle T_0(\vartheta^*), \vartheta - \vartheta^* \rangle \geq 0, \text{ for all } \vartheta \in C,$$

and $\mathfrak{R}^* = \mathfrak{I}_j \vartheta^* \in Q_j$ solves

$$\langle T_j(\mathfrak{R}^*), \mathfrak{R} - \mathfrak{R}^* \rangle \geq 0, \text{ for all } \mathfrak{R} \in Q_j \text{ and } j = 1, 2, ..., N,$$

where $\mathfrak{I}_j$ is a BLO.

The SVI-multiple output set can be treated as an SFP with multiple output sets as follows:

$$\text{Find } \vartheta^* \in Fix(P_C(I - \zeta W_0)) \bigcap \left(\bigcap_{j=1}^{N} \mathfrak{I}_j^{-1} \left(P_{Q_j} \left(I - \zeta W_j \right) \right) \right).$$

It is clear that $P_C(I - \zeta W)$ is nonexpanssive for $\zeta \in [0, 2\hbar]$, provided that $W : \chi \to \chi$ is an $\hbar$-ISM. Because every nonexpansive mapping. is 0-demicontractive, then $P_C\left(I - \zeta W_j\right)$ is 0−demicontractive, for $j = 1, 2, ..., N$. Therefore, Theorems 3.1, 3.2, 4.1, and 4.2 can be applied to solve the supposed SVI-multiple output set of $\hbar_j$-ISMs.

6. Numerical results

In this section, we provide numerical examples to illustrate the convergence of our algorithms and demonstrate the effectiveness of our proposed methodologies. These analyses serve two fundamental objectives: First, they offer valuable insights into the selection of the optimal control settings. Second, they substantiate the enhanced performance of our approaches when compared with previously published methodologies. It is important to note that the error term, denoted as TOL, remains consistent across all methods and computations outlined in this section. The comparison is as follows:

Algo. (8) in [37] (shortly, Alg1) and Algo. (6) (shortly, NewAlg1);

Algo. (33) in [37] (shortly, Alg2) and Algo. (30) (shortly, NewAlg2);

Algo. (45) in [37] (shortly, Alg3) and Algo. (42) (shortly, NewAlg3);

Algo. (58) in [37] (shortly, Alg4)and Algo. (54) (shortly, NewAlg4).

Example 6.1. *Let* $\mho = \mathbb{R}^3$*. For* $j = 0, 1, 2, 3$*, we define the mappings* $\mathcal{W}_j : \mathbb{R}^3 \to \mathbb{R}^3$ *as follows:*

$$\mathcal{W}_0(\vartheta, w, \varpi) = (0, w, \varpi),$$

$$\mathcal{W}_1(\vartheta, w, \varpi) = \begin{cases} \frac{1}{2}(\vartheta, w, \varpi), & \textit{if } \|(\vartheta, w, \varpi)\| > 1, \\ -3(\vartheta, w, \varpi), & \textit{if } \|(\vartheta, w, \varpi)\| \le 1. \end{cases}$$

$$\mathcal{W}_2(\vartheta, w, \varpi) = \frac{1}{4}(\vartheta, w, \varpi),$$

and

$$\mathcal{W}_3(\vartheta, w, \varpi) = \frac{1}{6}(\vartheta, w, \varpi), \quad \forall (\vartheta, w, \varpi) \in \mathbb{R}^3.$$

Here, $\mathcal{W}_1$ *is a* $\frac{1}{2}$*-demicontractive mapping, while* $\mathcal{W}_0$*,* $\mathcal{W}_2$*, and* $\mathcal{W}_3$ *are* 0*-demicontractive mappings. Now, for* $i = 1, 2, 3$*, let*

$$\mathfrak{I}_j = \frac{1}{j}\begin{pmatrix} 7 & -3 & -5 \\ -2 & 4 & -2 \\ -5 & -2 & 7 \end{pmatrix}.$$

We seek $\vartheta^* \in \mho$ *such that*

$$\vartheta^* \in Fix(\mathcal{W}_0) \cap \Big(\bigcap_{j=1}^{3} \mathfrak{I}_j^{-1}(Fix(\mathcal{W}_j))\Big).$$

It is evident that the solution to the specified problem is $\vartheta^* = (0, 0, 0)$*. The experimental control parameters are selected as follows:*

(i) For the weak convergence results with fixed and variable step sizes, we set

$$\alpha_u = 0.75, \kappa_u = 0.64, \mu = 0.45, \varrho_u = 0.65, \lambda = 0.45.$$

(ii) For the strong convergence results with fixed and variable step sizes, we choose

$$\alpha_u = 0.75, \kappa_u = 0.64, \mu = 0.45, \varrho_u = 0.65, \lambda = 0.45, \zeta_u = \frac{1}{u+1}.$$

We perform a comparative analysis between the proposed iterative methods and those introduced by Majee et al. [37], focusing on the number of iterations across various initial points. The experiment's stopping criterion is defined as $TOL < \epsilon$*, where* $TOL = \|\vartheta_{u+1} - \vartheta_u\|$ *and* $\epsilon > 0$ *is a predetermined value. These comparisons are presented in Figures 1–8 and Tables 1–4.*

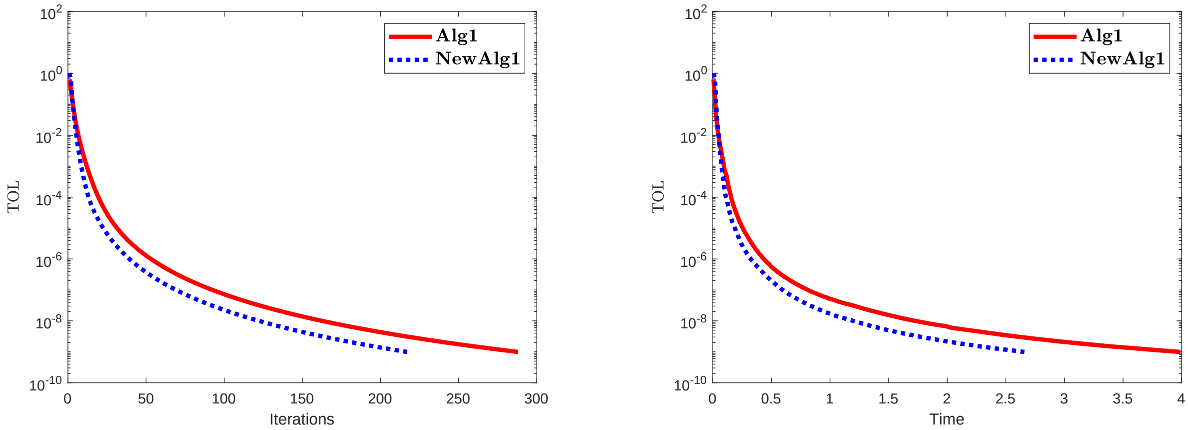


Figure 1. A numerical comparison has been conducted by contrasting Algorithm (8) outlined in [37] with our introduced Algorithm (6). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyzes with $\vartheta_0 = (1, 1, 1)^T$.

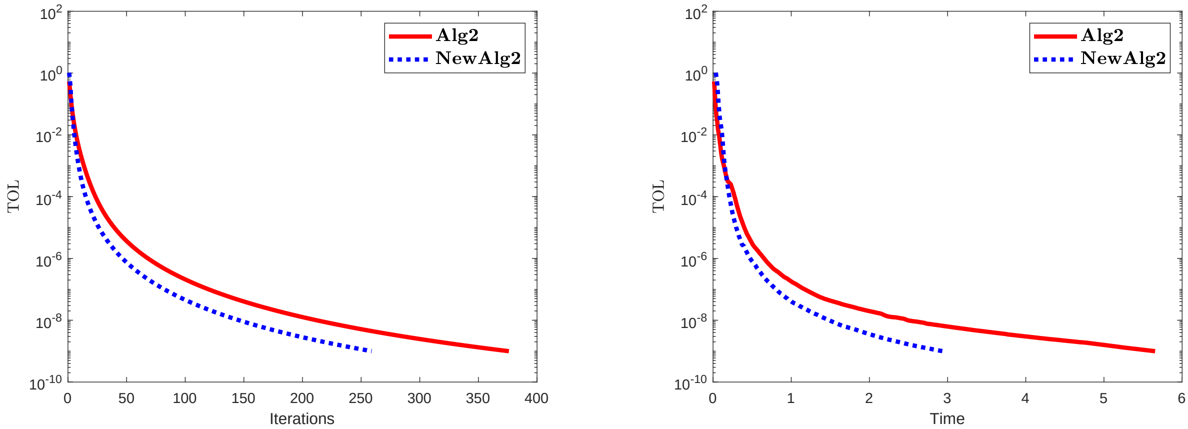


Figure 2. A numerical comparison has been conducted by contrasting Algorithm (33) outlined in [37] with our introduced Algorithm (30). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 1, 1)^T$.

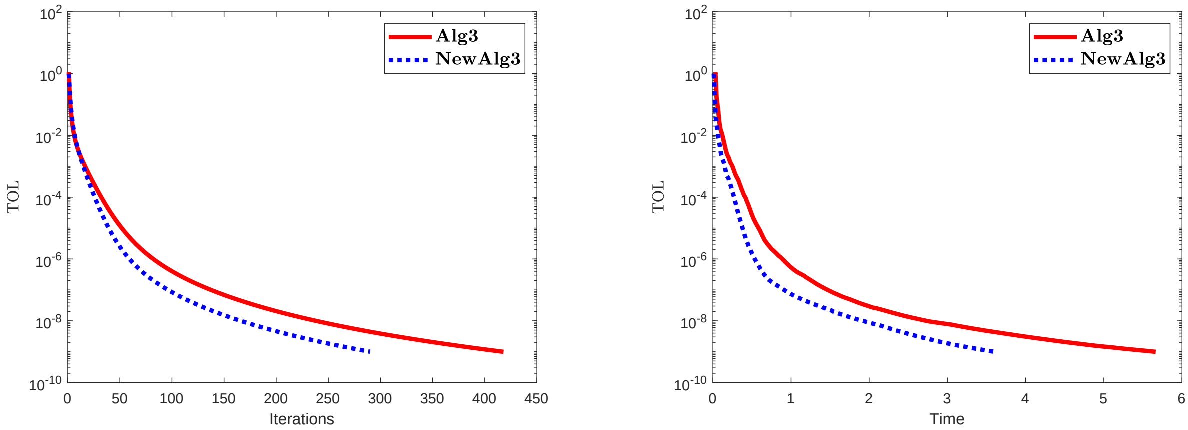


Figure 3. A numerical comparison has been conducted by contrasting Algorithm (45) outlined in [37] with our introduced Algorithm (42). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 1, 1)^T$.

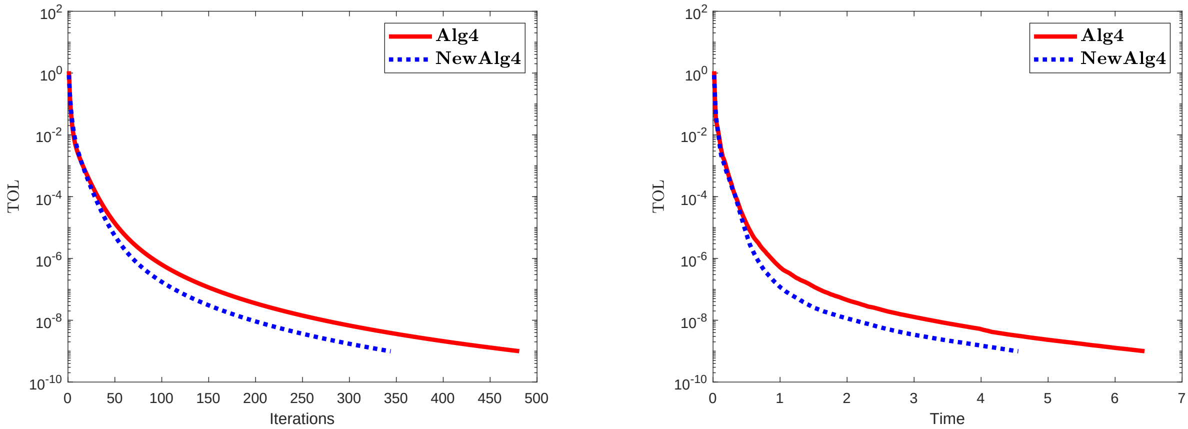


Figure 4. A numerical comparison has been conducted by contrasting Algorithm (58) outlined in [37] with our introduced Algorithm (54). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 1, 1)^T$.

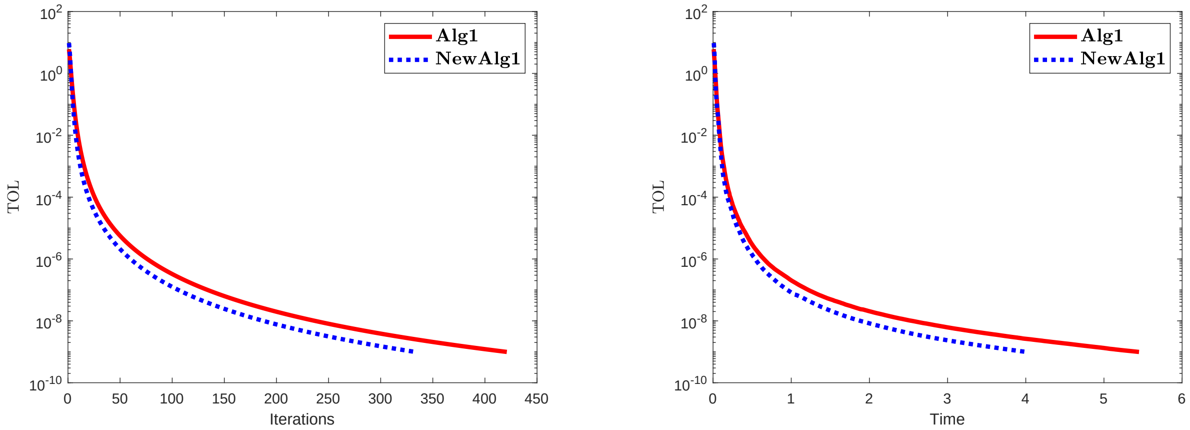


Figure 5. A numerical comparison has been conducted by contrasting Algorithm (8) outlined in [37] with our introduced Algorithm (6). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 2, 3)^T$.

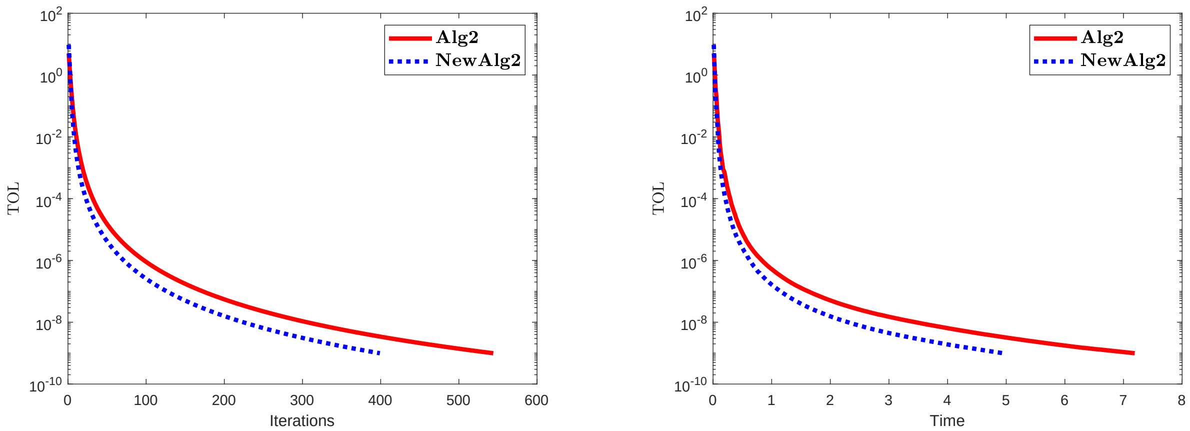


Figure 6. A numerical comparison has been conducted by contrasting Algorithm (33) outlined in [37] with our introduced Algorithm (30). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 2, 3)^T$.

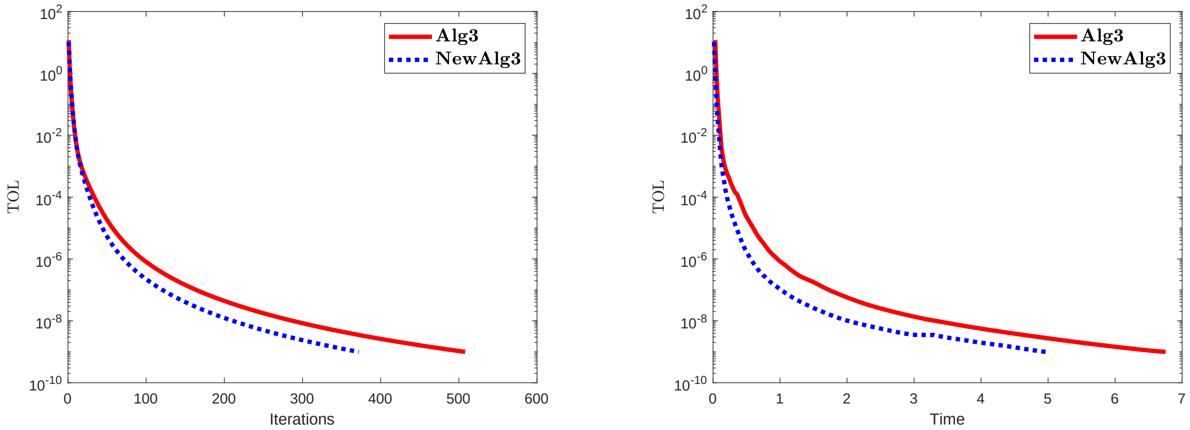


Figure 7. A numerical comparison has been conducted by contrasting Algorithm (45) outlined in [37] with our introduced Algorithm (42). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 2, 3)^T$.

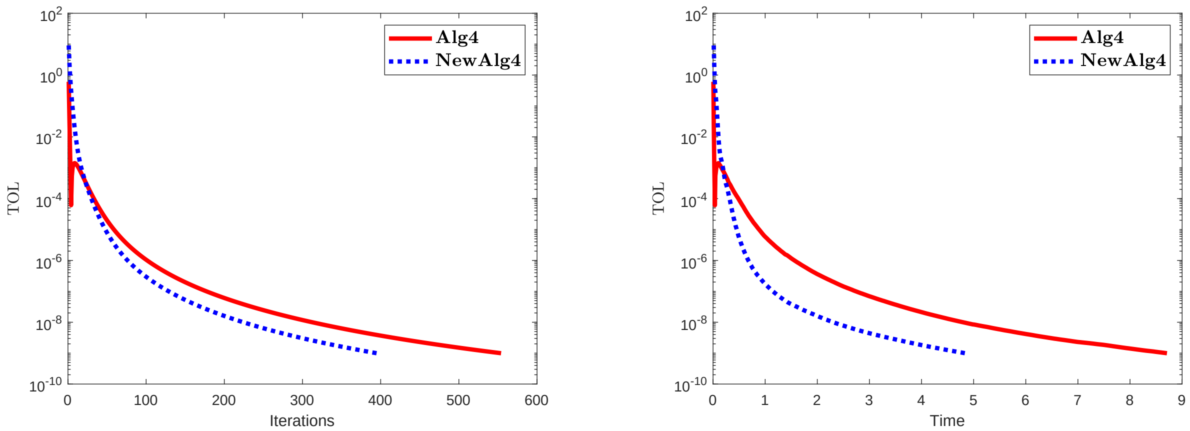


Figure 8. A numerical comparison has been conducted by contrasting Algorithm (58) outlined in [37] with our introduced Algorithm (54). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.1 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = (1, 2, 3)^T$.

Table 1. The data-set presented here corresponds to the numerical values associated with Figures 1–8, specifically in terms of the number of iterations.

ϑ_0	Alg1	Alg2	Alg3	Alg4
$(1, 1, 1)^T$	288	376	418	481
$(1, 2, 3)^T$	421	544	508	544

Table 2. The data-set presented here corresponds to the numerical values associated with Figures 1–8, specifically in terms of execution time measured in seconds.

ϑ_0	Alg1	Alg2	Alg3	Alg4
$(1,1,1)^T$	3.98910080000000	5.65416570000000	5.66388190000000	6.43818780000000
$(1,2,3)^T$	5.44849080000000	7.19120470000000	6.75355650000000	8.70876290000000

Table 3. The data-set presented here corresponds to the numerical values associated with Figures 1–8, specifically in terms of the number of iterations.

ϑ_0	NewAlg1	NewAlg2	NewAlg3	NewAlg4
$(1,1,1)^T$	217	259	290	344
$(1,2,3)^T$	333	399	372	395

Table 4. The data-set presented here corresponds to the numerical values associated with Figures 1–8, specifically in terms of execution time measured in seconds.

ϑ_0	NewAlg1	NewAlg2	NewAlg3	NewAlg4
$(1,1,1)^T$	2.65832610000000	2.94047930000000	3.59740310000000	4.55457900000000
$(1,2,3)^T$	4.00560800000000	4.95051310000000	4.96351510000000	4.83952040000000

Example 6.2. *Let us consider the Hilbert space* $\mho = L^2([0,1])$ *with the inner product defined as*

$$\langle \vartheta, w \rangle = \int_0^1 \vartheta(\tau) w(\tau)\, d\tau, \ \forall \vartheta, w \in \mho,$$

and the induced norm given by

$$\|\vartheta\| = \sqrt{\int_0^1 |\vartheta(\tau)|^2 d\tau}.$$

Now, let C *and* Q_j*, where* $j = 1, 2, 3$*, be the closed and convex subsets of* $L^2([0,1])$ *described as*

$$C = \left\{ \vartheta \in L^2[0,1] : \langle \vartheta(\tau), 3\tau^2 \rangle = 0 \right\},$$

$$Q_j = \{ \vartheta \in L^2[0,1] : \langle \vartheta(\tau), \frac{\tau}{2+j} \rangle \geq -j \}.$$

For $j = 1, 2, 3$*, let* $\Im_j : L^2[0,1] \to L^2([0,1])$ *be a BLO defined by*

$$\Im_j(\vartheta(\tau)) = \vartheta(\tau).$$

The projection formulas onto C *and* Q_j *are given by*

$$P_C(\vartheta(\tau)) = \begin{cases} \vartheta(\tau) - \frac{\langle \vartheta(\tau), 3\tau^2 \rangle 3\tau^2}{\|3\tau^2\|^2}, & if\, \langle \vartheta(\tau), 3\tau^2 \rangle \neq 0, \\ \vartheta(\tau), & if\, \langle \vartheta(\tau), 3\tau^2 \rangle = 0, \end{cases}$$

and

$$P_{Q_j}(\vartheta(\tau)) = \begin{cases} \vartheta(\tau) - \frac{\langle \vartheta(\tau), \frac{-\tau}{2+j} \rangle - j}{\| \frac{-\tau}{2+j} \|^2} \left(\frac{-\tau}{2+j} \right), & \text{if } \langle \vartheta(\tau), \frac{\tau}{2+j} \rangle < -j, \\ \vartheta(\tau), & \text{if } \langle \vartheta(\tau), \frac{\tau}{2+j} \rangle \geq -j. \end{cases}$$

The numerical comparisons are presented in Figures 9–12, and Tables 5 and 6. The control parameters for the experiment are chosen as follows:

(i) For the weak convergence results with fixed and variable step sizes, we set

$$\alpha_u = 0.655, \kappa_u = 0.554, \mu = 0.545, \varrho_u = 0.465, \lambda = 0.545.$$

(ii) For the strong convergence results with fixed and variable step sizes, we choose

$$\alpha_u = 0.655, \kappa_u = 0.554, \mu = 0.545, \varrho_u = 0.465, \lambda = 0.545, \zeta_u = \frac{1}{2u+2}.$$

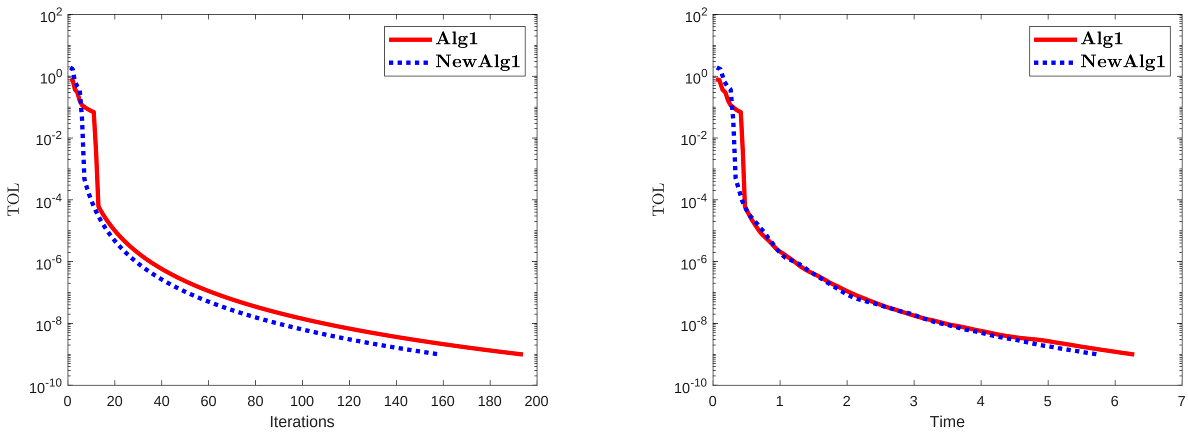


Figure 9. A numerical comparison has been conducted by contrasting Algorithm (8) outlined in [37] with our introduced Algorithm (6). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.2 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = 2\tau^2$.

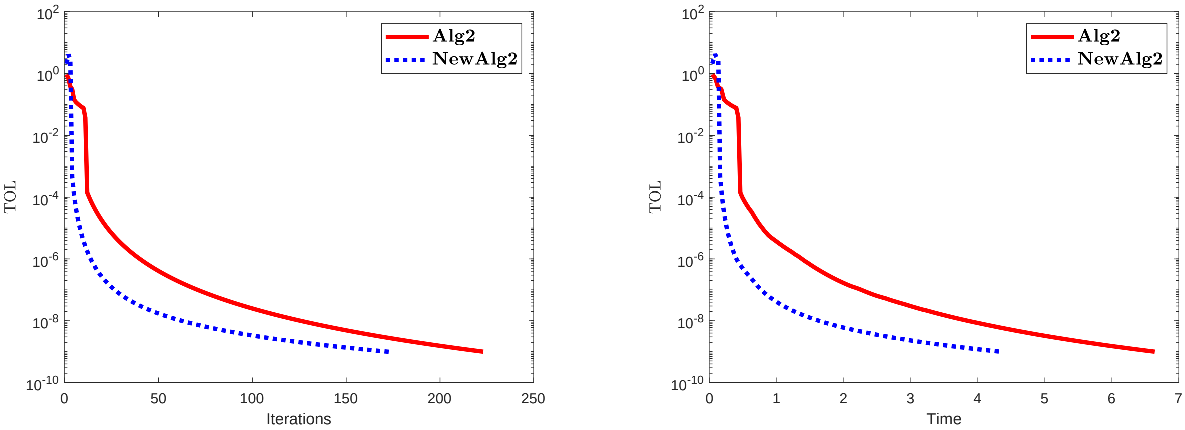


Figure 10. A numerical comparison has been conducted by contrasting Algorithm (33) outlined in [37] with our introduced Algorithm (30). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.2 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = 2\tau^2$.

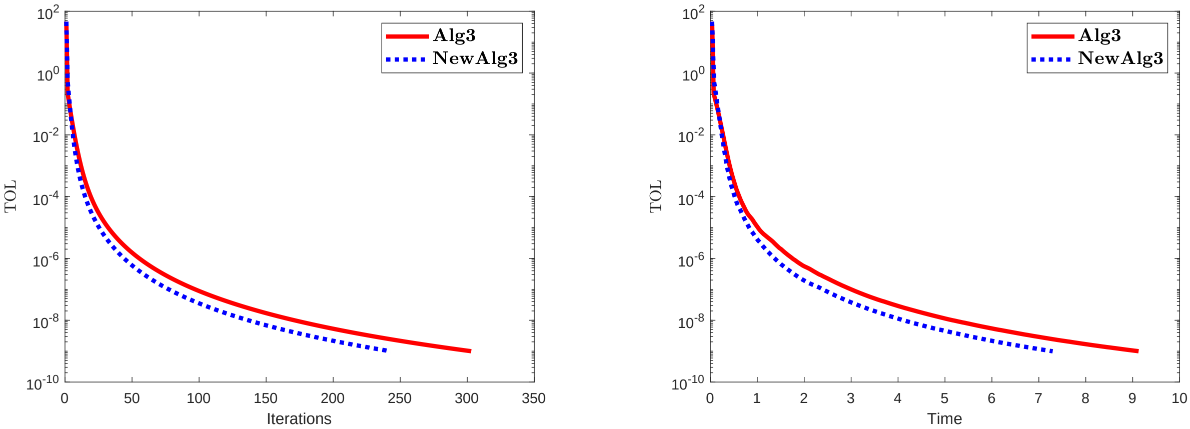
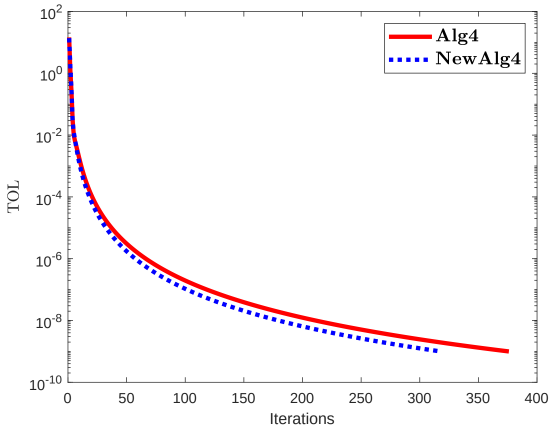


Figure 11. A numerical comparison has been conducted by contrasting Algorithm (45) outlined in [37] with our introduced Algorithm (42). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.2 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = 2\tau^2$.



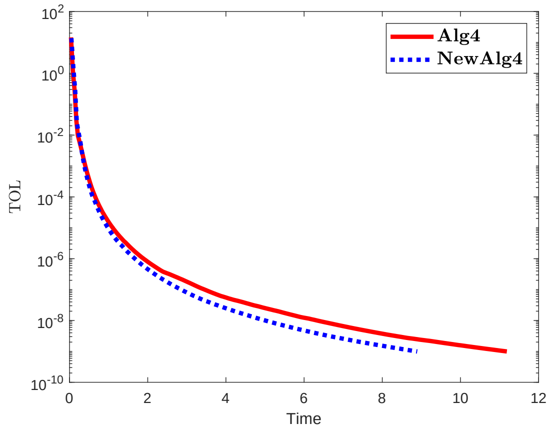


Figure 12. A numerical comparison has been conducted by contrasting Algorithm (58) outlined in [37] with our introduced Algorithm (54). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.2 serves as an illustrative demonstration of these comparative analyses with $\vartheta_0 = 2\tau^2$.

Table 5. The data-set presented here corresponds to the numerical values associated with Figures 9–12, specifically in terms of the number of iterations.

	Alg1	Alg2	Alg3	Alg4
Iterations	194	223	303	376
Time	6.28606670000000	6.63806650000000	9.12589670000000	11.1870605000000

Table 6. The data-set presented here corresponds to the numerical values associated with Figures 9–12, specifically in terms of the number of iterations.

	NewAlg1	NewAlg2	NewAlg3	NewAlg4
Iterations	159	173	243	318
Time	5.75115640000000	4.35383970000000	7.29541340000000	8.89380250000000

Example 6.3. *Let C and Q_j, $j = 1, 2, 3$, be the closed and convex subsets of $\mathbb{R}^N$, $\mathbb{R}^{2N}$, $\mathbb{R}^{3N}$, and $\mathbb{R}^{4N}$, respectively. Define C and Q_j as follows:*

$$C = \{\vartheta \in \mathbb{R}^N : \|\vartheta - \eta^1\| \leq c^1\},$$

and

$$Q_j = \{\vartheta \in \mathbb{R}^{N(j+1)} : \|\vartheta - \eta_j^1\| \leq c_j^1\},$$

Here, η^1 and η_j^1 are vectors with coordinates randomly generated in $[-1, 1]$, c^1 is a real number randomly generated in $[N, 2N]$, and c_j^1 is a real number randomly generated in $[(j+1)N, 2(j+1)N]$ for all $j = 1, 2, 3$.

Let $\mathfrak{I}_j : \mathbb{R}^N \to \mathbb{R}^{(i+1)N}$ be a BLO with elements randomly generated in the closed interval $[-5, 5]$. It is straightforward to verify that

$$C \cap (\cap_{j=1}^{3} \mathfrak{I}_j^{-1}(Q_j)) \neq \emptyset,$$

since the zero element of $\mathbb{R}^N$ belongs to the set.

The projection formulas on the closed and convex balls C and Q_j are given by

$$P_C(\vartheta) = \eta^1 + \frac{c^1(\vartheta - \eta^1)}{\max\{\|\vartheta - \eta^1\|, c^1\}},$$

and

$$P_{Q_j}(\vartheta) = \eta_j^1 + \frac{c_j^1(\vartheta - \eta_j^1)}{\max\{\|\vartheta - \eta_j^1\|, c_j^1\}}.$$

The stopping criterion for the experiment is set as $TOL < \epsilon$, where

$$TOL = \|(I - P_{Q_1})\mathfrak{I}_1\vartheta_u\| + \|(I - P_{Q_2})\mathfrak{I}_2\vartheta_u\| + \|(I - P_{Q_3})\mathfrak{I}_3\vartheta_u\|.$$

The results of the experiment are reported in Figures 13–16, and Tables 7 and 8. The control parameters for the experiment are chosen as follows:

(i) For the weak convergence results with fixed and variable step sizes, we set

$$\alpha_u = 0.655, \kappa_u = 0.554, \mu = 0.545, \varrho_u = 0.465, \lambda = 0.545.$$

(ii) For the strong convergence results with fixed and variable step sizes, we choose

$$\alpha_u = 0.655, \kappa_u = 0.554, \mu = 0.545, \varrho_u = 0.465, \lambda = 0.545, \zeta_u = \frac{1}{2u+2}.$$

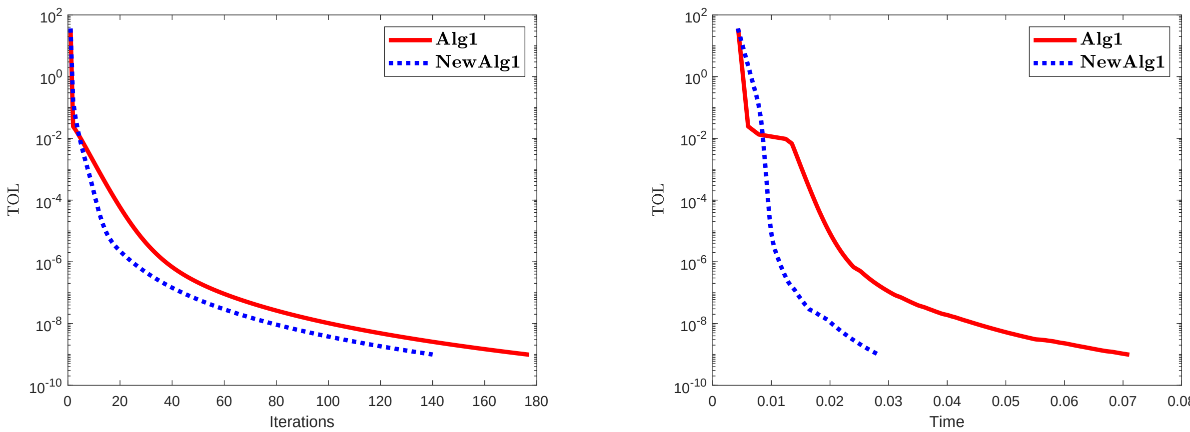


Figure 13. A numerical comparison has been conducted by contrasting Algorithm (8) outlined in [37] with our introduced Algorithm (6). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.3 serves as an illustrative demonstration of these comparative analyses.

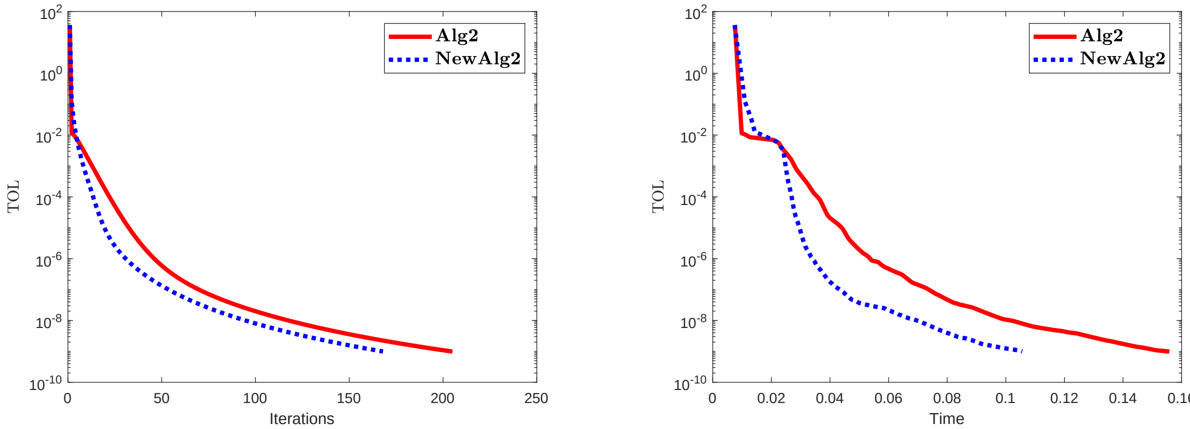


Figure 14. A numerical comparison has been conducted by contrasting Algorithm (33) outlined in [37] with our introduced Algorithm (30). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.3 serves as an illustrative demonstration of these comparative analyses.

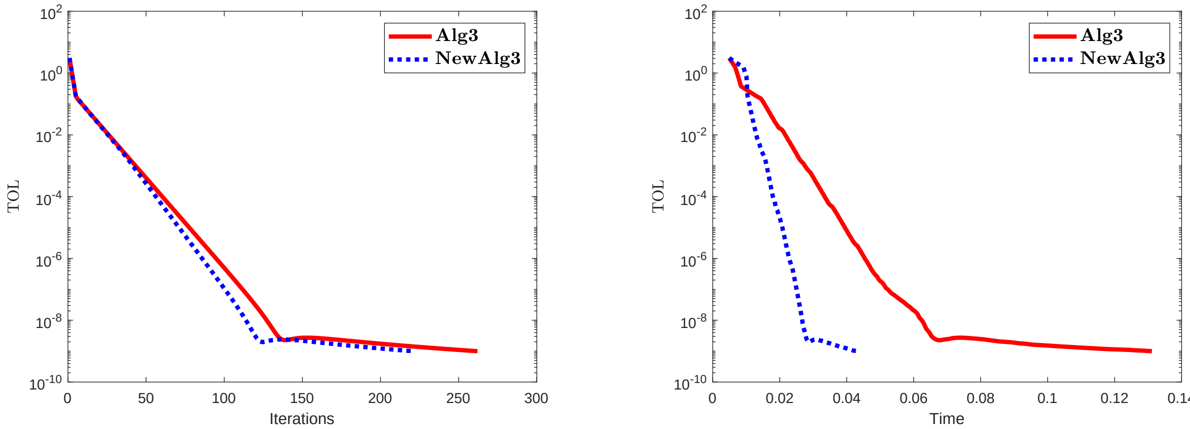


Figure 15. A numerical comparison has been conducted by contrasting Algorithm (45) outlined in [37] with our introduced Algorithm (42). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.3 serves as an illustrative demonstration of these comparative analyses.

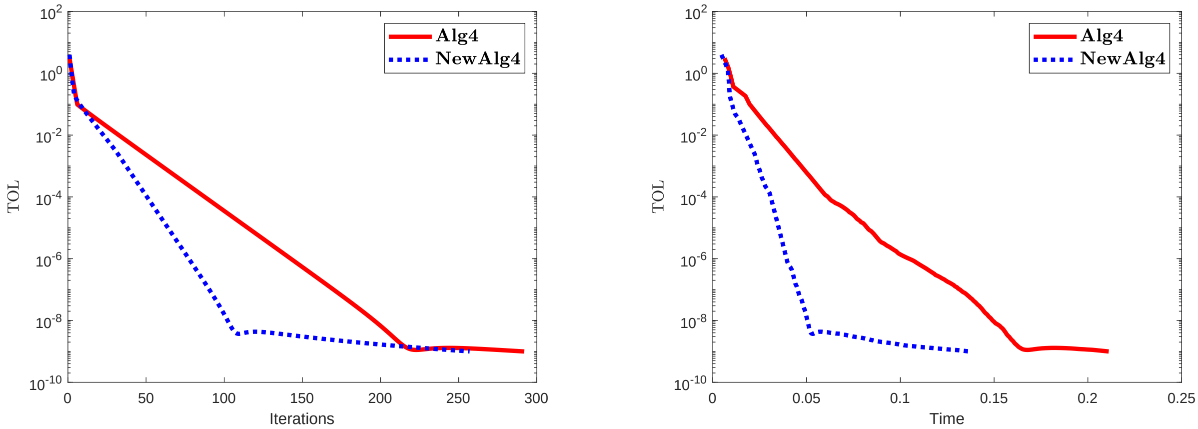


Figure 16. A numerical comparison has been conducted by contrasting Algorithm (58) outlined in [37] with our introduced Algorithm (54). This assessment encompasses evaluations of both the iteration count and the execution time, and Example 6.3 serves as an illustrative demonstration of these comparative analyses.

Table 7. The dataset presented here corresponds to the numerical values associated with Figures 13–16, specifically in terms of the number of iterations.

	Alg1	Alg2	Alg3	Alg4
Iterations	177	205	262	292
Time	0.0710745000000000	0.155804500000000	0.131109200000000	0.211114800000000

Table 8. The dataset presented here corresponds to the numerical values associated with Figures 13–16, specifically in terms of the number of iterations.

	NewAlg1	NewAlg2	NewAlg3	NewAlg4
Iterations	140	168	221	257
Time	0.0282480000000000	0.105587200000000	0.0435061000000000	0.136429400000000

7. Conclusions and future work

This study develops an inertial Mann-type algorithm to find approximate solutions for SFPPs involving demicontractive mappings in real Hilbert spaces. We rigorously prove the algorithm's weak and strong convergence under mild parameter constraints, using two-step iterative schemes with both fixed and variable step sizes. Numerical experiments validate the algorithm's effectiveness. Future work includes extending these methods to broader split and bilevel problems in Banach spaces, incorporating various step size strategies and inertial terms, and exploring their applicability to Hadamard manifolds to comprehensively assess the two-step inertial methods' performance across diverse conditions.

Author contributions

H. A. Hammad: writing-original draft, writing-review and editing, validation; H. ur Rehman: writing-original draft, writing-review and editing, resources, methodology; M. De la Sen: conceptualization, funding acquisition, writing-review and editing, visualization. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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