AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 16045–16067.
DOI: 10.3934/math.2025719
Received: 02 April 2025
Revised: 22 June 2025
Accepted: 08 July 2025
Published: 16 July 2025

Research article

On generalized α-ψ-Geraghty contractive mappings with integral type and fixed point theorems in quasi-partial metric spaces

He Liu and Hongyan Guan*

School of Mathematics and Systems Science, Shenyang Normal University, Shenyang 110034, China

* **Correspondence:** Email: guanhy8010@163.com; Tel: +8602486505824.

Abstract: In this paper, we mainly studied the existence and uniqueness of fixed points of α-ψ-Geraghty contractive mappings of integral type in quasi-partial metric spaces. We also gave examples to support our results.

Keywords: common fixed point, α-admissibility, quasi-partial metric, generalized α-ψ-Geraghty contractive mapping
Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

Fixed point theory, as a core branch of modern mathematics, has widely permeated into numerous fields such as functional analysis, topology, and differential equations. The Banach fixed point theorem [1], popularly known as the Banach contraction mapping principle, is a rewarding result in fixed point theory. The idea and method of constructing a convergent sequence and proving that its only limit is the only fixed point of a mapping is widely used. Subsequently, more and more mathematical scholars began to study fixed point theory, and extended the result in different spaces. Among them, some mathematicians defined the concept of quasi-metric spaces. In 1994, Matthews [2] defined partial metric spaces. By combined quasi-metric space and partial metric space, Karapinar [3] et al. introduced the concept of quasi-partial metric spaces. Fixed point theory has been widely applied in various fields. As a result, more and more mathematicians have devoted themselves to its research.

With the development of fixed point theory, the contraction form in metric spaces has been expanded. In 2012, Samet [4] et al. introduced a new class of α-ψ-contractive mappings in metric spaces, and then Karapinar [5] et al. introduced α-(ψ, ϕ) contractive mappings in quasi-partial metric spaces. The authors also proved the existence and uniqueness of fixed points of α-(ψ, ϕ) contractive mappings in these spaces.

In this paper, we introduce a new type of α-ψ-contractive mapping of integral type and prove the existence and uniqueness of fixed points and common fixed points of the contractive mappings in the setting of quasi-partial metric spaces.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{N}$ denotes the set of all positive integers.

The famous Banach contraction principle is as follows.

Theorem 1.1. [1] Let f be a mapping from a complete metric space (X, d) into itself satisfying

$$d(fx, fy) \le cd(x, y),$$

where $c \in (0, 1)$ is a constant. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

In 2002, Branciari [6] gave an integral version of the Banach contraction principle and showed the following fixed point theorem.

Theorem 1.2. [6] Let f be a mapping from a complete metric space (X, d) into itself satisfying

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X,$$

where $c \in (0, 1)$ is a constant and $\varphi \in \Phi$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

2. Preliminaries

In this section, some basic definitions and lemmas are introduced which take on an important role for this article.

Let

$\Phi = \{\varphi$: φ: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$, and $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0\}$,

$\Phi_1 = \{\psi : \psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and nondecreasing , and $\psi(x + y) \le \psi(x) + \psi(y), \forall x, y > 0\}$.

Example 2.1. Let $f(x) = \frac{x}{x+1}, x \in \mathbb{R}^+$. It is obvious that $f \in \Phi_1$.

First, we recall the concept of quasi-metric spaces.

Definition 2.2. [7] Let $X \neq \emptyset$. A quasi-metric is a function $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, z) \le d(x, y) + d(y, z)$ *for* $x, y, z \in X$.

The pair (X, d) is called a quasi-metric space.

Definition 2.3. [8] Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X, $x \in X$. A sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Definition 2.4. [8] Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to be left Cauchy if for every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for any $n \ge m > N$, $d(x_n, x_m) < \varepsilon$.

(ii) $\{x_n\}$ is said to be right Cauchy if for every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for any $n \geq m > N$, $d(x_m, x_n) < \varepsilon$.

(iii) $\{x_n\}$ is said to be Cauchy if and only if $\{x_n\}$ is both left Cauchy and right Cauchy.

Definition 2.5. [8] Let (X, d) be a quasi-metric space.

(i) (X, d) is left complete if and only if every left Cauchy in X converges.

(ii) (X, d) is right complete if and only if every right Cauchy in X converges.

(iii) (X, d) is complete if and only if every Cauchy in X converges.

Then, we outline the definition of partial metric spaces.

Definition 2.6. [3] Let $X \neq \emptyset$. A partial metric is a function $p: X \times X \to \mathbb{R}^+$ that satisfies

(i) $p(x, y) = p(y, x)$,

(ii) if $0 \leq q(x, x) = q(x, y) = q(y, y)$, then $x = y$,

(iii) $q(x, x) \leq q(x, y)$,

(iv) $q(x, z) + q(y, y) \leq q(x, y) + q(y, z)$,

for all $x, y, z \in X$. The pair (X, p) is called a partial metric space.

Proposition 2.7. [3] Let (X, p) be a partial metric space. Set $q(x, y) = p(x, y) - p(x, x)$ for $x, y \in X$. Then (X, q) is a quasi-partial metric space and their topologies coincide.

Definition 2.8. [3] Let (X, p) be a partial metric space. Then,

(i) a sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if

$$p(x, x) = \lim_{n \to \infty} p(x, x_n).$$

(ii) A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if

$$\lim_{n,m \to \infty} p(x_n, x_m) \text{ exists (and is finite).}$$

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m).$$

Next, we introduce the definition of quasi-partial metric spaces.

Definiton 2.9. [3] Let $X \neq \emptyset$. A quasi-partial metric is a function $q: X \times X \to \mathbb{R}^+$ that satisfies

(i) if $0 \leq q(x, x) = q(x, y) = q(y, y)$, then $x = y$,

(ii) $q(x, x) \leq q(x, y)$,

(iii) $q(x, x) \leq q(y, x)$,

(iv) $q(x, z) + q(y, y) \leq q(x, y) + q(y, z)$,

for all $x, y, z \in X$. The pair (X, q) is called a quasi-partial metric space.

Note that for a quasi-partial metric q on X, the function $d_q : X \times X \to \mathbb{R}^+$ defined by

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y)$$

is a metric on X (see [3]).

Example 2.10. Let $X = [-2, 2]$. We define $q : X \times X \to \mathbb{R}^+$ as $q(x, y) = |x - y| + |x|$. Then (X, q) is a quasi-partial metric space.

Proof. Here $q(x, x) = q(x, y) = q(y, y) \Rightarrow |x| = |x - y| + |x| = |y| \Rightarrow x = y$.

Let $x, y \in X$. It follows from $|x| \leq |x-y| + |x|$ that $q(x,x) \leq q(x,y)$.

In addition, $q(y,x) = |y-x| + |y| = |x-y| + |y| \geq |x-y+y| = |x| = q(x,x)$.

Note that

$$|x-z| + |x| + |y| \leq |x-y| + |x| + |y-z| + |y|.$$

So, $q(x,z) + q(y,y) \leq q(x,y) + q(y,z)$.

Therefore, (X,q) is a quasi-partial metric space.

Definition 2.11. [3] Let (X,q) be a quasi-partial metric space. Then,

(i) a sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if

$$q(x,x) = \lim_{n\to\infty} q(x,x_n) = \lim_{n\to\infty} q(x_n,x).$$

(ii) A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if

$$\lim_{n,m\to\infty} q(x_m,x_n) \text{ and } \lim_{n,m\to\infty} q(x_n,x_m) \text{ exist (and are finite).}$$

(iii) A quasi-partial metric space (X,q) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_q, to a point $x \in X$ such that

$$q(x,x) = \lim_{n,m\to\infty} q(x_m,x_n) = \lim_{n,m\to\infty} q(x_n,x_m).$$

Lemma 2.12. [3] Let (X,q) be a quasi-partial metric space, (X,p) be a partial metric space, and (X,d_q) be the corresponding metric space. The following statements are equivalent:

(i) (X,q) is complete.

(ii) (X,p) is complete.

(iii) (X,d_q) is complete.

Moreover,

$$\begin{aligned}\lim_{n\to\infty} d_q(x,x_n) = 0 \Leftrightarrow p(x,x) &= \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m)\\ \Leftrightarrow q(x,x) &= \lim_{n\to\infty} q(x,x_n) = \lim_{n,m\to\infty} q(x_n,x_m)\\ &= \lim_{n\to\infty} q(x_n,x) = \lim_{n,m\to\infty} q(x_m,x_n).\end{aligned}$$

Definition 2.13. [4] Let X be a nonempty set and $\alpha : X \times X \to [0,\infty)$ be a given mapping. A mapping $f : X \to X$ is said to be an α−admissible mapping if the following condition holds:

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(fx,fy) \geq 1, \forall x,y \in X.$$

Definition 2.14. [9] Let X be a nonempty set and $\alpha : X \times X \to [0,\infty)$, $f, g : X \to X$. A mapping $f : X \to X$ is called a g-α-admissible mapping if for all $x,y \in X$, $\alpha(gx,gy) \geq 1$ means $\alpha(fx,fy) \geq 1$.

Definition 2.15. [10] Let X be a nonempty set. If a pair of self-mappings f and g satisfy that for some $x \in X$ with $v = fx = gx$, then x is the coincidence point of f and g, and v is the point of coincidence of f and g. The set of all coincidence points of f and g is denoted as $C(f,g)$.

Definition 2.16. [10] Let X be a nonempty set. If a pair of self-mappings f and g satisfy that for every $x \in C(f,g)$, $fgx = gfx$, then f and g are said to be weakly compatible.

Definition 2.17. [11] One says that $\varphi \in \Phi$ is an integral subadditive if for each $a, b > 0$, there is

$$\int_0^{a+b} \varphi(t)dt \le \int_0^a \varphi(t)dt + \int_0^b \varphi(t)dt.$$

Let Φ_s denotes the class of all integral subadditive functions $\varphi \in \Phi$.

Lemma 2.18. [6] Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 2.19. [12] Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. If $\limsup_{n\to\infty} r_n = a$, then

$$\int_0^a \varphi(t)dt \le \limsup_{n\to\infty} \int_0^{r_n} \varphi(t)dt.$$

If $\liminf_{n\to\infty} r_n = a$, then

$$\liminf_{n\to\infty} \int_0^{r_n} \varphi(t)dt \le \int_0^a \varphi(t)dt.$$

3. Main results

In this section, we mainly study the existence and uniqueness of fixed points and common fixed points of contraction mappings of integral type in a complete quasi-partial metric space.

Definition 3.1. Let (X, q) be a quasi-partial metric space. A mapping $f : X \to X$ is called a generalized α-ψ-contractive mapping of integral type if for all $x, y \in X$,

$$\alpha(x,y) \int_0^{\psi(q(fx,fy)-q(fx,fx))} \varphi(t)dt \le L(q(x,y)) \int_0^{\psi(h\cdot M(x,y))} \varphi(t)dt, \tag{3.1}$$

where $\varphi \in \Phi_s$, $\psi \in \Phi_1$, $L : \mathbb{R}^+ \to [0, \frac{1}{2})$, $h \in (0,1)$, and

$$\begin{aligned} M(x,y) = \max\{&q(fy,fx) - q(fy,fy), q(x,y) - q(x,x), q(fx,fy) - q(fx,fx), q(y,x) - q(y,y) \\ &\frac{q(x,fy)[q(fy,x) - q(fy,fx)]}{1 + q(x,y) + q(fx,fy)}, \frac{q(y,fx)[q(y,fy) - q(y,fx)]}{1 + q(y,fy) + q(x,fx)}\}. \end{aligned} \tag{3.2}$$

Theorem 3.2. Let (X, q) be a complete quasi-partial metric space and $f : X \to X$ be a generalized α-ψ-contractive mapping of integral type. If the following conditions are satisfied:

(i) f is an α-admissible mapping,

(ii) $\exists x_0 \in X$, such that $\alpha(x_0, fx_0) \ge 1$ and $\alpha(fx_0, x_0) \ge 1$,

(iii) for every sequence$\{x_n\} \subset X$, if $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x) \ge 1$ and $\alpha(x, x_n) \ge 1$,

(iv) if $Fix(f) \ne \emptyset$, then for $x, y \in Fix(f)$, there is $\alpha(x,y) \ge 1$ and $\alpha(y,x) \ge 1$,

then f has a unique fixed point in X.

Proof. We define the Picard iteration sequence $\{x_n\}$ by

$$x_{n+1} = fx_n$$

for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then we have $x_{n_0} \in Fix(f)$ and hence the conclusion holds. Now we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. It follows that

$$q(x_n, x_{n+1}) > 0,\ q(x_{n+1}, x_n) > 0.$$

According to the conditions (i) and (ii), we have

$$\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1,$$

$$\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geq 1,$$

$$\alpha(x_2, x_3) = \alpha(fx_1, fx_2) \geq 1,$$

$$\cdots .$$

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \alpha(x_{n+1}, x_n) \geq 1.$$

It follows that f is a generalized α-ψ-contraction mapping and

$$\begin{aligned}\int_0^{\psi(q(fx_{n-1},fx_n)-q(fx_{n-1},fx_{n-1}))}\varphi(t)dt &\leq \alpha(x_{n-1}, x_n)\int_0^{\psi(q(fx_{n-1},fx_n)-q(fx_{n-1},fx_{n-1}))}\varphi(t)dt \\ &\leq L(q(x_{n-1}, x_n))\int_0^{\psi(h\cdot M(x_{n-1},x_n))}\varphi(t)dt \\ &< \frac{1}{2}\int_0^{\psi(h\cdot M(x_{n-1},x_n))}\varphi(t)dt.\end{aligned} \tag{3.3}$$

Similarly, one can get

$$\int_0^{\psi(q(fx_n,fx_{n-1})-q(fx_n,fx_n))}\varphi(t)dt < \frac{1}{2}\int_0^{\psi(h\cdot M(x_n,x_{n-1}))}\varphi(t)dt. \tag{3.4}$$

By the integral subadditivity, we obtain

$$\int_0^{\psi(q(x,y)+q(y,x))}\varphi(t)dt \leq \int_0^{\psi(q(x,y))+\psi(q(y,x))}\varphi(t)dt \leq \int_0^{\psi(q(x,y))}\varphi(t)dt + \int_0^{\psi(q(y,x))}\varphi(t)dt.$$

So using (3.3) and (3.4), we gain

$$\begin{aligned}\int_0^{\psi(d_q(fx_{n-1},fx_n))}\varphi(t)dt &= \int_0^{\psi(q(fx_{n-1},fx_n)-q(fx_{n-1},fx_{n-1})+q(fx_n,fx_{n-1})-q(fx_n,fx_n))}\varphi(t)dt \\ &\leq \int_0^{\psi(q(fx_{n-1},fx_n)-q(fx_{n-1},fx_{n-1}))}\varphi(t)dt + \int_0^{\psi(q(fx_n,fx_{n-1})-q(fx_n,fx_n))}\varphi(t)dt \\ &< \frac{1}{2}\int_0^{\psi(h\cdot M(x_{n-1},x_n))}\varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot M(x_n,x_{n-1}))}\varphi(t)dt.\end{aligned} \tag{3.5}$$

Note that for each $n \in \mathbb{N}$,

$$\begin{aligned} M(x_{n-1}, x_n) = \max\{& q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}), q(x_{n-1}, x_n) - q(x_{n-1}, x_{n-1}), q(x_n, x_{n+1}) - q(x_n, x_n), \\ & q(x_n, x_{n-1}) - q(x_n, x_n), \frac{q(x_{n-1}, x_{n+1})[q(x_{n+1}, x_{n-1}) - q(x_{n+1}, x_n)]}{1 + q(x_{n-1}, x_n) + q(x_n, x_{n+1})}, \\ & \frac{q(x_n, x_n)[q(x_n, x_{n+1}) - q(x_n, x_n)]}{1 + q(x_n, x_{n+1}) + q(x_{n-1}, x_n)}\}. \end{aligned}$$

Since q is a quasi-partial metric, we have

$$q(x_n, x_n) \leq q(x_n, x_{n+1}),$$

$$q(x_{n+1}, x_{n-1}) \leq q(x_{n+1}, x_n) + q(x_n, x_{n-1}) - q(x_n, x_n),$$

$$q(x_{n-1}, x_{n+1}) \leq q(x_{n-1}, x_n) + q(x_n, x_{n+1}) - q(x_n, x_n).$$

Therefore,

$$\frac{q(x_{n-1}, x_{n+1})[q(x_{n+1}, x_{n-1}) - q(x_{n+1}, x_n)]}{1 + q(x_{n-1}, x_n) + q(x_n, x_{n+1})} \leq q(x_n, x_{n-1}) - q(x_n, x_n),$$

$$\frac{q(x_n, x_n)[q(x_n, x_{n+1}) - q(x_n, x_n)]}{1 + q(x_n, x_{n+1}) + q(x_{n-1}, x_n)} \leq q(x_n, x_{n+1}) - q(x_n, x_n).$$

As a result,

$$\begin{aligned} M(x_{n-1}, x_n) \leq \max\{& q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}), q(x_{n-1}, x_n) - q(x_{n-1}, x_{n-1}), q(x_n, x_{n+1}) - q(x_n, x_n), \\ & q(x_n, x_{n-1}) - q(x_n, x_n)\}. \end{aligned}$$

In the same way, we know

$$\begin{aligned} M(x_n, x_{n-1}) \leq \max\{& q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}), q(x_{n-1}, x_n) - q(x_{n-1}, x_{n-1}), q(x_n, x_{n+1}) - q(x_n, x_n), \\ & q(x_n, x_{n-1}) - q(x_n, x_n)\}. \end{aligned}$$

According to the above inequality, we obtain

$$\begin{aligned} \max\{M(x_{n-1}, x_n), M(x_n, x_{n-1})\} \leq \max\{& q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}), q(x_n, x_{n+1}) - q(x_n, x_n), \\ & q(x_{n-1}, x_n) - q(x_{n-1}, x_{n-1}), q(x_n, x_{n-1}) - q(x_n, x_n)\}. \end{aligned}$$

Since $\psi \in \Phi_1$, we get

$$\begin{aligned} \psi(h \cdot \max\{M(x_{n-1}, x_n), M(x_n, x_{n-1})\}) \leq \psi(h \cdot \max\{& q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}), q(x_n, x_{n+1}) - q(x_n, x_n), \\ & q(x_{n-1}, x_n) - q(x_{n-1}, x_{n-1}), q(x_n, x_{n-1}) - q(x_n, x_n)\}). \end{aligned}$$

We assume

$$\begin{aligned} & \max\{q(x_{n-1}, x_n) - q(x_{n-1}, x_{n-1}), q(x_n, x_{n-1}) - q(x_n, x_n)\} \\ & \leq \max\{q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}), q(x_n, x_{n+1}) - q(x_n, x_n)\} \\ & \leq q(x_{n+1}, x_n) - q(x_{n+1}, x_{n+1}) + q(x_n, x_{n+1}) - q(x_n, x_n). \end{aligned}$$

By using (3.5), we derive

$$\begin{aligned}
\int_0^{\psi(d_q(fx_{n-1},fx_n))}\varphi(t)dt &< \frac{1}{2}\int_0^{\psi(h\cdot M(x_{n-1},x_n))}\varphi(t)dt+\frac{1}{2}\int_0^{\psi(h\cdot M(x_n,x_{n-1}))}\varphi(t)dt\\
&< \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(x_{n-1},x_n),M(x_n,x_{n-1})\})}\varphi(t)dt+\frac{1}{2}\int_0^{\psi(h\cdot\max\{M(x_{n-1},x_n),M(x_n,x_{n-1})\})}\varphi(t)dt\\
&< \int_0^{\psi(h\cdot\max\{M(x_{n-1},x_n),M(x_n,x_{n-1})\})}\varphi(t)dt\\
&\le \int_0^{\psi(h\cdot(q(x_{n+1},x_n)-q(x_{n+1},x_{n+1})+q(x_n,x_{n+1})-q(x_n,x_n))}\varphi(t)dt\\
&= \int_0^{\psi(h\cdot d_q(fx_{n-1},fx_n))}\varphi(t)dt\\
&< \int_0^{\psi(d_q(fx_{n-1},fx_n))}\varphi(t)dt,
\end{aligned}$$

which is a contradiction. This implies

$$\max\{M(x_{n-1},x_n),M(x_n,x_{n-1})\}\le\max\{q(x_{n-1},x_n)-q(x_{n-1},x_{n-1}),q(x_n,x_{n-1})-q(x_n,x_n)\}.$$

At the same time, we obtain

$$\int_0^{\psi(d_q(x_n,x_{n+1}))}\varphi(t)dt\le\int_0^{\psi(h\cdot d_q(x_{n-1},x_n))}\varphi(t)dt. \tag{3.6}$$

Because ψ is nondecreasing, we have

$$d_q(x_n,x_{n+1})\le hd_q(x_{n-1},x_n). \tag{3.7}$$

According to (3.7), it can be concluded that $\{d_q(x_n,x_{n+1})\}$ is a decreasing sequence in $\mathbb{R}^+$. It follows that there exists $r\ge 0$ such that $\lim_{n\to\infty}d_q(x_n,x_{n+1})=r$. Assume $r>0$ and take the upper limit as $n\to\infty$ in (3.6), and one can deduce

$$0<\int_0^{\psi(r)}\varphi(t)dt=\limsup_{n\to\infty}\int_0^{\psi(d_q(x_n,x_{n+1}))}\varphi(t)dt\le\limsup_{n\to\infty}\int_0^{\psi(h\cdot d_q(x_{n-1},x_n))}\varphi(t)dt<\int_0^{\psi(r)}\varphi(t)dt.$$

This conflicts with the facts. Consequently, $r=0$. So we have $\lim_{n\to\infty}d_q(x_n,x_{n+1})=0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence in (X,d_q). In view of (3.7), we obtain

$$d_q(x_n,x_{n+1})\le hd_q(x_{n-1},x_n)\le h^2d_q(x_{n-2},x_{n-1})\le\cdots\le h^nd_q(x_0,x_1).$$

Thus, for all $m\in\mathbb{R}^+$,

$$\begin{aligned}
d_q(x_n,x_{n+m}) &\le d_q(x_n,x_{n+1})+d_q(x_{n+1},x_{n+2})+\cdots+d_q(x_{n+m-1},x_{n+m})\\
&\le (h^n+h^{n+1}+\cdots+h^{n+m-1})d_q(x_0,x_1)\\
&\le (h^n+h^{n+1}+\cdots+h^{n+m-1}+\cdots)d_q(x_0,x_1)\\
&\le \frac{h^n}{1-h}d_q(x_0,x_1)\quad\to 0\quad\text{as } n\to\infty.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, d_q), which means $\{x_n\}$ is a Cauchy sequence in (X, q). By the completeness of (X, q), we arrive at the fact that (X, d_q) also is a complete space. It follows that there exists $u \in X$ such that $\lim_{n\to\infty} d_q(x_n, u) = 0$. From that we conclude

$$q(u,u) = \lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = \lim_{n,m\to\infty} q(x_n, x_m) = \lim_{m,n\to\infty} q(x_m, x_n).$$

Next we prove $fu = u$. Assume to the contrary that $fu \neq u$. It follows that

$$\begin{aligned} M(x_{n-1}, u) = \max\{&q(fu, x_n) - q(fu, fu), q(x_{n-1}, u) - q(x_{n-1}, x_{n-1}), q(x_n, fu) - q(x_n, x_n), \\ &q(u, x_{n-1}) - q(u,u), \frac{q(x_{n-1}, fu)[q(fu, x_{n-1}) - q(fu, x_n)]}{1 + q(x_n, fu) + q(x_{n-1}, u)}, \\ &\frac{q(u, x_n)[q(u, fu) - q(u, x_n)]}{1 + q(x_{n-1}, x_n) + q(u, fu)}\}. \end{aligned} \tag{3.8}$$

Taking the inferior limit as $n \to \infty$ in (3.8), one can obtain

$$\begin{aligned} \liminf_{n\to\infty} M(x_{n-1}, u) \leq &\max\{q(fu, u) - q(fu, fu), q(u,u) - q(u,u), q(u, fu) - q(u,u), q(u,u) - q(u,u), \\ &\liminf_{n\to\infty} \frac{q(x_{n-1}, fu)}{1 + q(x_n, fu) + q(x_{n-1}, u)} \cdot \limsup_{n\to\infty}[q(fu, x_{n-1}) - q(fu, x_n)], \\ &\liminf_{n\to\infty} \frac{q(u, x_n)}{1 + q(x_{n-1}, x_n) + q(u, fu)} \cdot \limsup_{n\to\infty}[q(u, fu) - q(u, x_n)]\} \\ \leq &\max\{q(fu, u) - q(fu, fu), 0, q(u, fu) - q(u,u), 0, 0, q(u, fu) - q(u,u)\} \\ \leq &\max\{q(fu, u) - q(fu, fu), q(u, fu) - q(u,u)\} \\ \leq &\, q(fu, u) - q(fu, fu) + q(u, fu) - q(u,u) \\ = &\, d_q(u, fu). \end{aligned}$$

By the same token, we have

$$\begin{aligned} \liminf_{n\to\infty} M(u, x_{n-1}) &\leq \max\{q(fu, u) - q(fu, fu), q(u, fu) - q(u,u)\} \\ &\leq q(fu, u) - q(fu, fu) + q(u, fu) - q(u,u) \\ &= d_q(u, fu). \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} \max\{M(x_{n-1}, u), M(u, x_{n-1})\} \leq d_q(u, fu).$$

By (iii), we get $\alpha(u, x_n) \geq 1$ and $\alpha(x_n, u) \geq 1$. As a result, we derive

$$\begin{aligned}
\int_0^{\psi(d_q(fx_{n-1},fu))}\varphi(t)dt &= \int_0^{\psi(q(fx_{n-1},fu)-q(fx_{n-1},fx_{n-1})+q(fu,fx_{n-1})-q(fu,fu))}\varphi(t)dt \\
&\leq \int_0^{\psi(q(fx_{n-1},fu)-q(fx_{n-1},fx_{n-1}))}\varphi(t)dt + \int_0^{\psi(q(fu,fx_{n-1})-q(fu,fu))}\varphi(t)dt \\
&\leq \alpha(x_{n-1},u)\int_0^{\psi(q(fx_{n-1},fu)-q(fx_{n-1},fx_{n-1}))}\varphi(t)dt + \alpha(x_{n-1},u)\int_0^{\psi(q(fu,fx_{n-1})-q(fu,fu))}\varphi(t)dt \\
&\leq L(d(x_{n-1},u))\int_0^{\psi(h\cdot M(x_{n-1},u))}\varphi(t)dt + L(d(u,x_{n-1}))\int_0^{\psi(h\cdot M(u,x_{n-1}))}\varphi(t)dt \\
&< \frac{1}{2}\int_0^{\psi(h\cdot M(x_{n-1},u))}\varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot M(u,x_{n-1}))}\varphi(t)dt \\
&< \frac{1}{2}\int_0^{\psi(h\cdot \max\{M(x_{n-1},u),M(u,x_{n-1})\})}\varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot \max\{M(x_{n-1},u),M(u,x_{n-1})\})}\varphi(t)dt \\
&< \int_0^{\psi(h\cdot \max\{M(x_{n-1},u),M(u,x_{n-1})\})}\varphi(t)dt.
\end{aligned} \tag{3.9}$$

Taking the inferior limit as $n \to \infty$ in (3.9), one can deduce

$$\begin{aligned}
0 &< \int_0^{\psi(d_q(u,fu))}\varphi(t)dt \\
&= \liminf_{n\to\infty}\int_0^{\psi(d_q(fx_{n-1},fu))}\varphi(t)dt \\
&\leq \liminf_{n\to\infty}\int_0^{\psi(h\cdot \max\{M(x_{n-1},u),M(u,x_{n-1})\})}\varphi(t)dt \\
&\leq \int_0^{\psi(hd_q(u,fu))}\varphi(t)dt \\
&< \int_0^{\psi(d_q(u,fu))}\varphi(t)dt,
\end{aligned}$$

which is a contradiction. Hence we get $fu = u$.

Finally, we prove that f has a unique fixed point in X. Suppose that f has another fixed point $v \in X \setminus \{u\}$. It is easy to get that

$$\begin{aligned}
M(u,v) &= \max\{q(fv,fu)-q(fv,fv), q(u,v)-q(u,u), q(fu,fv)-q(fu,fu), q(v,u)-q(v,v) \\
&\quad \frac{q(u,fv)[q(fv,u)-q(fv,fu)]}{1+q(u,v)+q(fu,fv)}, \frac{q(v,fu)[q(v,fv)-q(v,fu)]}{1+q(v,fv)+q(u,fu)}\} \\
&\leq \max\{q(u,v)-q(u,u), q(v,u)-q(v,v)\} \\
&\leq q(u,v)-q(u,u)+q(v,u)-q(v,v) \\
&= d_q(u,v).
\end{aligned}$$

In the same way, we gain

$$M(v,u) \leq \max\{q(u,v)-q(u,u), q(v,u)-q(v,v)\} \leq q(u,v)-q(u,u)+q(v,u)-q(v,v) = d_q(u,v).$$

Therefore, $\max\{M(v,u), M(v,u)\} \leq d_q(u,v)$. By the condition (iv), we have $\alpha(u,v) > 1$ and $\alpha(v,u) > 1$. Consequently,

$$\begin{aligned}
\int_0^{\psi(d_q(fu,fv))} \varphi(t)dt &= \int_0^{\psi(q(fu,fv)-q(fu,fu)+q(fv,fu)-q(fv,fv))} \varphi(t)dt \\
&\leq \int_0^{\psi(q(fu,fv)-q(fu,fu))} \varphi(t)dt + \int_0^{\psi(q(fv,fu)-q(fv,fv))} \varphi(t)dt \\
&\leq \alpha(u,v)\int_0^{\psi(q(fu,fu)-q(fu,fu))} \varphi(t)dt + \alpha(v,u)\int_0^{\psi(q(fv,fu)-q(fv,fv))} \varphi(t)dt \\
&\leq L(d(u,v))\int_0^{\psi(h\cdot M(u,v))} \varphi(t)dt + L(d(v,u))\int_0^{\psi(h\cdot M(v,u))} \varphi(t)dt \\
&< \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(u,v),M(v,u)\})} \varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(u,v),M(v,u)\})} \varphi(t)dt \\
&< \int_0^{\psi(h\cdot\max\{M(u,v),M(v,u)\})} \varphi(t)dt \\
&< \int_0^{\psi(d_q(u,v))} \varphi(t)dt.
\end{aligned}$$

This is impossible. To sum up, we derive $u = v$. The proof is complete.

Example 3.3. Let $X = [0, \frac{\pi}{2}]$. Define $q(x,y) = |x-y| + |x|$ and $f(x) = \frac{\sin x}{2}$. For $\alpha(x,y) = 1$, $\varphi(t) = 2t$, $\psi(t) = t$, $L(x) = \frac{4}{9}$, $h = \frac{3}{4}$, we conclude that f has a unique fixed point in X.

In fact, according to this statement, we get

$$q(fx,fy) - q(fx,fx) = |\frac{\sin x}{2} - \frac{\sin y}{2}| + |\frac{\sin x}{2}| - |\frac{\sin x}{2}| = \frac{1}{2}|\sin x - \sin y|$$

and

$$q(x,y) - q(x,x) = |x-y| + |x| - |x| = |x-y|.$$

So, we derive

$$h \cdot M(x,y) \geq \frac{3}{4} \cdot |x-y| = \frac{3}{4}|x-y|.$$

From the above inequality, it can be concluded that

$$\int_0^{\psi(q(fx,fy)-q(fx,fx))} \varphi(t)dt = (\frac{1}{2}|\sin x - \sin y|)^2 = \frac{1}{4}|\sin x - \sin y|^2.$$

Similarly, we have

$$\int_0^{\psi(h\cdot M(x,y))} \varphi(t)dt \geq \int_0^{\psi(\frac{3}{4}|x-y|)} \varphi(t)dt = (\frac{3}{4}|x-y|)^2 = \frac{9}{16}|x-y|^2.$$

Hence, we obtain

$$L(q(x,y))\int_0^{\psi(h\cdot M(x,y))} \varphi(t)dt \geq \frac{4}{9} \cdot \frac{9}{16}|x-y|^2 = \frac{1}{4}|x-y|^2.$$

Since $X = [0, \frac{\pi}{2}]$,

$$\frac{1}{4}|\sin x - \sin y|^2 \leq \frac{1}{4}|x - y|^2.$$

As a result, we conclude that f satisfies the conditions of Theorem 3.2, which means f has a unique fixed point in X. In this example, 0 is the unique fixed point of f.

Theorem 3.4. Let (X, p) be a complete partial metric space and $f : X \to X$ be a self-mapping. If for all $x, y \in X$,

$$\int_0^{\psi(p(fx,fy)-p(fx,fx))} \varphi(t)dt \leq \int_0^{\psi(h\cdot(p(x,y)-p(x,x))} \varphi(t)dt,$$

where $\varphi \in \Phi_s$, $\psi \in \Phi_1$, and $h \in (0, 1)$, then f has a unique fixed point in X.

Proof. We define the Picard iteration sequence $\{x_n\}$ by

$$x_{n+1} = fx_n$$

for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then we have $x_{n_0} \in Fix(f)$ and hence the conclusion holds. Now we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From the contraction conditions, it can be obtained that

$$p(x_n, x_{n+1}) - p(x_n, x_n) = p(fx_{n-1}, fx_n) - p(fx_{n-1}, fx_{n-1}) \leq h[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})].$$

Since $q(x, y) = p(x, y) - p(x, x)$, we have

$$q(fx_{n-1}, fx_n) \leq hq(x_{n-1}, x_n).$$

Therefore,

$$q(x_n, x_{n+1}) \leq hq(x_{n-1}, x_n) \leq h^2 q(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n q(x_0, x_1).$$

For all $m \in \mathbb{R}^+$, we get

$$\begin{aligned}
q(x_n, x_{n+m}) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{n+m-1}, x_{n+m}) \\
&\leq (h^n + h^{n+1} + \cdots + h^{n+m-1})q(x_0, x_1) \\
&\leq (h^n + h^{n+1} + \cdots + h^{n+m-1} + \cdots)q(x_0, x_1) \\
&\leq \frac{h^n}{1-h} q(x_0, x_1) \quad \to 0 \ (n \to \infty).
\end{aligned}$$

By the same token, one can deduce

$$\lim_{n\to\infty} q(x_{n+m}, x_n) = 0.$$

As a result, $\{x_n\}$ is a Cauchy sequence in (X, q). Since (X, p) is complete, then (X, q) is complete. There exists $u \in X$ such that $\lim_{n\to\infty} q(x_n, x_m) = \lim_{n\to\infty} q(x_n, u) = q(u, u)$.

Next, we prove $fu = u$. Since

$$\int_0^{\psi(q(fx_{n-1},fu))} \varphi(t)dt \leq \int_0^{\psi(h(q(x_{n-1},u)))} \varphi(t)dt,$$

taking the upper limit as $n \to \infty$, we conclude

$$\begin{aligned}\int_0^{\psi(q(u,fu))} \varphi(t)dt &\le \limsup_{n\to\infty} \int_0^{\psi(q(fx_{n-1},fu))} \varphi(t)dt \\ &\le \limsup_{n\to\infty} \int_0^{\psi(h(q(x_{n-1},u)))} \varphi(t)dt \\ &= \int_0^{\psi(h(q(u,u)))} \varphi(t)dt.\end{aligned}$$

Then,

$$q(u, fu) \le hq(u, u),$$

which is a contradiction. So we obtain $fu = u$.

Finally, we prove the uniqueness. Suppose that f has another fixed point $v \in X \setminus \{u\}$. There exists

$$\int_0^{\psi(q(fu,fv))} \varphi(t)dt \le \int_0^{\psi(h(q(u,v)))} \varphi(t)dt.$$

Hence,

$$q(u, v) = q(fu, fv) \le hq(u, v).$$

This is contradictory. To sum up, we derive $u = v$. This completes the proof.

Definition 3.5. Let (X, q) be a quasi-partial metric space and $f, g : X \to X$ be given mappings with $f(X) \subseteq g(X)$. Suppose f, g are generalized α-ψ-contractive mappings of integral type if for all $x, y \in X$,

$$\alpha(gx, gy) \int_0^{\psi(q(fx,fy)-q(fx,fx))} \varphi(t)dt \le L(q(gx, gy)) \int_0^{\psi(h\cdot M(x,y))} \varphi(t)dt, \tag{3.10}$$

where $\varphi \in \Phi_s,\ \psi \in \Phi_1, L : \mathbb{R}^+ \to [0, \frac{1}{2}), h \in (0, 1)$, and

$$\begin{aligned}M(x, y) = \max\{&q(gy, fy) - q(gy, gy), q(fy, gy) - q(fy, fy), q(gx, fx) - q(gx, gx), \\ &q(fx, gx) - q(fx, fx), \frac{q(fx, gy)[q(fy, fx) - q(fy, fy)]}{1 + q(fx, fy) + q(gx, gy)}, \\ &\frac{q(gy, fx)[q(gx, gy) - q(gx, gx)]}{1 + q(gx, fx) + q(gy, gx)}\}.\end{aligned} \tag{3.11}$$

Theorem 3.6. Let (X, q) be a complete quasi-partial metric space. Assume that f, g are generalized α-ψ-contractive mappings of integral type. If the following conditions are satisfied:

(i) f is a $g - \alpha-$admissible mapping,

(ii) $\exists x_0 \in X$, such that $\alpha(gx_0, fx_0) \ge 1$, and $\alpha(fx_0, gx_0) \ge 1$,

(iii) for a sequence $\{y_n\} \subset X$, if $\lim_{n\to\infty} gy_n = ga$, then $\alpha(gy_n, ga) \ge 1$ and $\alpha(ga, gy_n) \ge 1$,

(iv) if $e, f \in C(f, g)$, then $\alpha(ge, gf) \ge 1$ and $\alpha(gf, ge) \ge 1$,

(v) if $Fix(f) \ne \emptyset$, then if $s, t \in Fix(f)$, there are $\alpha(gs, gt) \ge 1$ and $\alpha(gt, gs) \ge 1$,

(vi) f and g are weakly compatible,

(vii) $g(X)$ is closed,

then f and g have a unique fixed point in X.

Proof. By condition (ii), there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$. According to $f(X) \subseteq g(X)$, we define sequence $\{y_n\}$ and $\{x_n\}$ such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. If there is some n such that $y_n = y_{n+1}$, then $y_{n+1} = y_n = fx_{n+1} = gx_{n+1}$, and hence f and g have coincidence points. Now we assume $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$, and from the condition (i), we have

$$\begin{gathered}\alpha(gx_0, fx_0) \geq 1, \alpha(gx_0, gx_1) \geq 1,\\ \alpha(fx_0, fx_1) \geq 1, \alpha(gx_1, gx_2) \geq 1,\\ \alpha(fx_1, fx_2) \geq 1, \alpha(gx_2, gx_3) \geq 1,\\ \cdots .\end{gathered}$$

Therefore, we obtain $\alpha(gx_n, gx_{n+1}) \geq 1,\ \alpha(y_{n-1}, y_n) \geq 1$.

In the same way, we have $\alpha(gx_{n+1}, gx_n) \geq 1, \alpha(y_n, y_{n-1}) \geq 1$. It follows that

$$\begin{aligned}\int_0^{\psi(q(y_n,y_{n+1}))} \varphi(t)dt &= \int_0^{\psi(q(fx_n,fx_{n+1}))} \varphi(t)dt\\ &\leq \alpha(gx_n, gx_{n+1}) \int_0^{\psi(q(fx_n,fx_{n+1}))} \varphi(t)dt\\ &\leq L(q(gx_n, gx_{n+1})) \int_0^{\psi(h\cdot M(x_n,x_{n+1}))} \varphi(t)dt\\ &< \frac{1}{2} \int_0^{\psi(h\cdot M(x_n,x_{n+1}))} \varphi(t)dt.\end{aligned}$$

Similarly, we get

$$\int_0^{\psi(q(y_{n+1},y_n))} \varphi(t)dt = \int_0^{\psi(q(fx_{n+1},fx_n))} \varphi(t)dt < \frac{1}{2} \int_0^{\psi(h\cdot M(x_{n+1},x_n))} \varphi(t)dt.$$

Note that for each $n \in \mathbb{N}$,

$$\begin{aligned}M(x_n, x_{n+1}) = \max\{&q(gx_{n+1}, fx_{n+1}) - q(gx_{n+1}, gx_{n+1}), q(gx_n, fx_n) - q(gx_n, gx_n),\\ &q(fx_{n+1}, gx_{n+1}) - q(fx_{n+1}, fx_{n+1}), q(fx_n, gx_n) - q(fx_n, fx_n),\\ &\frac{q(fx_n, gx_{n+1})[q(fx_{n+1}, fx_n) - q(fx_{n+1}, fx_{n+1})]}{1 + q(fx_n, fx_{n+1}) + q(gx_n, gx_{n+1})},\\ &\frac{q(gx_{n+1}, fx_n)[q(gx_n, gx_{n+1}) - q(gx_n, gx_n)]}{1 + q(gx_n, fx_n) + q(gx_{n+1}, gx_n)}\}\\ = \max\{&q(y_n, y_{n+1}) - q(y_n, y_n), q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}),\\ &q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1}), q(y_n, y_{n-1}) - q(y_n, y_n),\\ &\frac{q(y_n, y_n)[q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1})]}{1 + q(y_n, y_{n+1}) + q(y_{n-1}, y_n)},\\ &\frac{q(y_n, y_n)[q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1})]}{1 + q(y_{n-1}, y_n) + q(y_n, y_{n-1})}\}.\end{aligned}$$

Because q is a quasi-partial metric, we have

$$q(y_n, y_n) \leq q(y_n, y_{n+1}),$$

$$q(y_n, y_n) \leq q(y_{n-1}, y_n),$$

and then

$$\frac{q(y_n, y_n)[q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1})]}{1 + q(y_n, y_{n+1}) + q(y_{n-1}, y_n)} \leq q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1}),$$

$$\frac{q(y_n, y_n)[q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1})]}{1 + q(y_{n-1}, y_n) + q(y_n, y_{n-1})} \leq q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}).$$

Hence,

$$\begin{aligned} M(x_n, x_{n+1}) \leq \max\{&q(y_n, y_{n+1}) - q(y_n, y_n), q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}), q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1}),\\ &q(y_n, y_{n-1}) - q(y_n, y_n)\}. \end{aligned}$$

By the same token, we gain

$$\begin{aligned} M(x_{n+1}, x_n) \leq \max\{&q(y_n, y_{n+1}) - q(y_n, y_n), q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}), q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1}),\\ &q(y_n, y_{n-1}) - q(y_n, y_n)\}. \end{aligned}$$

Thus, one can deduce

$$\begin{aligned} \max\{M(x_{n+1}, x_n), M(x_n, x_{n+1})\} \leq \max\{&q(y_n, y_{n+1}) - q(y_n, y_n), q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}),\\ &q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1}), q(y_n, y_{n-1}) - q(y_n, y_n)\}. \end{aligned}$$

Since $\psi \in \Phi_1$, we get

$$\begin{aligned} \psi(\max\{M(x_{n+1}, x_n), M(x_n, x_{n+1})\}) \leq \psi(\max\{&q(y_n, y_{n+1}) - q(y_n, y_n), q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}),\\ &q(y_{n+1}, y_n) - q(y_{n+1}, y_{n+1}), q(y_n, y_{n-1}) - q(y_n, y_n)\}). \end{aligned}$$

By the subadditivity of the integral, one can derive

$$\int_0^{\psi(q(x,y)+q(y,x))} \varphi(t)dt \leq \int_0^{\psi(q(x,y))+\psi(q(y,x))} \varphi(t)dt \leq \int_0^{\psi(q(x,y))} \varphi(t)dt + \int_0^{\psi(q(y,x))} \varphi(t)dt,$$

which implies

$$\begin{aligned} \int_0^{\psi(d_q(y_n,y_{n+1}))} \varphi(t)dt &= \int_0^{\psi(d_q(fx_n,fx_{n+1}))} \varphi(t)dt\\ &\leq \int_0^{\psi(q(fx_n,fx_{n+1}))} \varphi(t)dt + \int_0^{\psi(q(fx_{n+1},fx_n))} \varphi(t)dt\\ &< \frac{1}{2}\int_0^{\psi(h\cdot M(x_n,x_{n+1}))} \varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot M(x_{n+1},x_n))} \varphi(t)dt\\ &< \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(x_{n+1},x_n),M(x_n,x_{n+1})\})} \varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(x_{n+1},x_n),M(x_n,x_{n+1})\})} \varphi(t)dt\\ &< \int_0^{\psi(h\cdot\max\{M(x_{n+1},x_n),M(x_n,x_{n+1})\})} \varphi(t)dt. \end{aligned}$$

Similar to the proof of Theorem 3.2, we can conclude

$$\lim_{n\to\infty} d_q(y_n, y_{n+1}) = 0,$$

and $\{y_n\}$ is a Cauchy sequence in (X, d_q). Hence $\{y_n\}$ is a Cauchy sequence in (X, q).

Since $g(X)$ is closed, there exists $a \in g(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = a.$$

As a result, we have

$$\begin{aligned}\lim_{n\to\infty} d_q(y_n, a) = 0 \Leftrightarrow q(a,a) &= \lim_{n\to\infty} q(y_n, a) = \lim_{n,m\to\infty} q(y_n, y_m)\\ &= \lim_{n\to\infty} q(a, y_n) = \lim_{n,m\to\infty} q(y_m, y_n).\end{aligned}$$

Next, we prove $gu = fu$. We choose $u \in g(X)$ that satisfies $gu = a$. By condition (iii), we get $\alpha(gx_n, gu) \geq 1$ and $\alpha(gu, gx_n) \geq 1$. If $gu \neq fu$, then $fu \neq a$. It follows that

$$\begin{aligned}M(x_n, u) = \max\{&q(gu, fu) - q(gu, gu), q(gx_n, fx_n) - q(gx_n, gx_n), q(fu, gu) - q(fu, fu),\\ &q(fx_n, gx_n) - q(fx_n, fx_n), \frac{q(fx_n, gu)[q(fu, fx_n) - q(fu, fu)]}{1 + q(fx_n, fu) + q(gx_n, gu)},\\ &\frac{q(gu, fx_n)[q(gx_n, gu) - q(gx_n, gx_n)]}{1 + q(gx_n, fx_n) + q(gu, gx_n)}\}\\ = \max\{&q(a, fu) - q(a,a), q(y_{n-1}, y_n) - q(y_{n-1}, y_{n-1}), q(fu, a) - q(fu, fu),\\ &q(y_n, y_{n-1}) - q(y_n, y_n), \frac{q(y_n, a)[q(fu, y_n) - q(fu, fu)]}{1 + q(y_n, fu) + q(y_{n-1}, a)},\\ &\frac{q(a, y_n)[q(y_{n-1}, a) - q(y_{n-1}, y_{n-1})]}{1 + q(y_{n-1}, y_n) + q(a, y_{n-1})}\}.\end{aligned} \tag{3.12}$$

Taking the inferior limit as $n \to \infty$ in (3.12), we derive

$$\begin{aligned}\liminf_{n\to\infty} M(x_n, u) \leq &\max\{q(a, fu) - q(a,a), q(a,a) - q(a,a), q(fu, a) - q(fu, fu), q(a,a) - q(a,a),\\ &\liminf_{n\to\infty} \frac{q(y_n, a)}{1 + q(y_n, fu) + q(y_{n-1}, a)} \cdot \limsup_{n\to\infty} [q(fu, y_n) - q(fu, fu)],\\ &\liminf_{n\to\infty} \frac{q(a, y_n)}{1 + q(y_{n-1}, y_n) + q(a, y_{n-1})} \cdot \limsup_{n\to\infty} [q(y_{n-1}, a) - q(y_{n-1}, y_{n-1})]\}\\ \leq &\max\{q(a, fu) - q(a,a), 0, q(fu, a) - q(fu, fu), 0, q(fu, a) - q(fu, fu), 0\}\\ \leq &\max\{q(a, fu) - q(a,a), q(fu, a) - q(fu, fu)\}.\end{aligned}$$

In the same way, we obtain

$$\liminf_{n\to\infty} M(u, x_n) \leq \max\{q(a, fu) - q(a,a), q(fu, a) - q(fu, fu)\}.$$

Therefore, it can be concluded that

$$\liminf_{n\to\infty} \max\{M(x_n, u), M(u, x_n)\} \leq \max\{q(a, fu) - q(a,a), q(fu, a) - q(fu, fu)\}.$$

From the above inequality, one can deduce

$$\begin{aligned}
\int_0^{\psi(d_q(fx_n,fu))}\varphi(t)dt &\le \int_0^{\psi(q(fx_n,fu))}\varphi(t)dt+\int_0^{\psi(q(fu,fx_n))}\varphi(t)dt\\
&\le \alpha(gx_n,gu)\int_0^{\psi(q(fx_n,fu))}\varphi(t)dt+\alpha(gu,gx_n)\int_0^{\psi(q(fu,fx_n))}\varphi(t)dt\\
&\le L(q(gx_n,gu))\int_0^{\psi(h\cdot M(x_n,u))}\varphi(t)dt+L(q(gu,gx_n))\int_0^{\psi(h\cdot M(u,x_n)}\varphi(t)dt \qquad (3.13)\\
&< \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(x_n,u),M(u,x_n)\})}\varphi(t)dt+\frac{1}{2}\int_0^{\psi(h\cdot\max\{M(u,x_n),M(x_n,u)\})}\varphi(t)dt)\\
&< \int_0^{\psi(h\cdot\max\{M(x_n,u),M(u,x_n)\})}\varphi(t)dt.
\end{aligned}$$

Letting $n\to\infty$ in (3.13), we gain

$$\begin{aligned}
\int_0^{\psi(d_q(a,fu))}\varphi(t)dt &= \liminf_{n\to\infty}\int_0^{\psi(d_q(fx_n,fu))}\varphi(t)dt\\
&\le \liminf_{n\to\infty}\int_0^{\psi(h\cdot\max\{M(x_n,u),M(u,x_n)\})}\varphi(t)dt\\
&\le \int_0^{\psi(h\cdot\max\{q(a,fu)-q(a,a),q(fu,a)-q(fu,fu)\})}\varphi(t)dt\\
&\le \int_0^{\psi(h(q(a,fu)-q(a,a)+q(fu,a)-q(fu,fu)))}\varphi(t)dt\\
&= \int_0^{\psi(hd_q(a,fu))}\varphi(t)dt\\
&< \int_0^{\psi(d_q(a,fu))}\varphi(t)dt,
\end{aligned}$$

which is a contradiction. Consequently, we derive $fu=gu=a$ and u is a coincidence point of f and g.

Because f and g are weakly compatible, we conclude $fa=fgu=gfu=ga$. Now we assume $fa\ne a$, namely, $fa\ne fu$. We get $\alpha(ga,gu)\ge 1$ and $\alpha(gu,ga)\ge 1$ with $a,u\in C(f,g)$, and then

$$\begin{aligned}
M(a,u) &= \max\{q(gu,fu)-q(gu,gu),q(fu,gu)-q(fu,fu),q(ga,fa)-q(ga,ga),\\
&\qquad q(fa,ga)-q(fa,fa),\frac{q(fa,gu)[q(fu,fa)-q(fu,fu)]}{1+q(fa,fu)+q(ga,gu)},\\
&\qquad \frac{q(gu,fa)[q(ga,gu)-q(ga,ga)]}{1+q(ga,fa)+q(gu,ga)}\}\\
&\le \max\{0,0,0,0,q(a,fa)-q(a,a),q(fa,a)-q(fa,fa)\}\\
&= \max\{q(a,fa)-q(a,a),q(fa,a)-q(fa,fa)\}.
\end{aligned}$$

By the same token, we have $M(u,a)\le\max\{q(a,fa)-q(a,a),q(fa,a)-q(fa,fa)\}$.

Further,

$$
\begin{aligned}
0 &< \int_0^{\psi(d_q(fa,a))} \varphi(t)dt \\
&\leq \int_0^{\psi(q(fa,fu))} \varphi(t)dt + \int_0^{\psi(q(fu,fa))} \varphi(t)dt \\
&\leq \alpha(ga,gu) \int_0^{\psi(q(fa,fu))} \varphi(t)dt + \alpha(gu,ga) \int_0^{\psi(q(fu,fa))} \varphi(t)dt \\
&\leq L(q(ga,gu)) \int_0^{\psi(h\cdot M(a,u))} \varphi(t)dt + L(q(gu,ga)) \int_0^{\psi(h\cdot M(u,a))} \varphi(t)dt \\
&< \frac{1}{2} \int_0^{\psi(h\cdot \max\{M(a,u),M(u,a)\})} \varphi(t)dt + \frac{1}{2} \int_0^{\psi(h\cdot \max\{M(a,u),M(u,a)\})} \varphi(t)dt \\
&< \int_0^{\psi(h(q(a,fa)-q(a,a)+q(fa,a)-q(fa,fa)))} \varphi(t)dt \\
&= \int_0^{\psi(hd_q(a,fa))} \varphi(t)dt \\
&< \int_0^{\psi(d_q(a,fu))} \varphi(t)dt.
\end{aligned}
$$

This is a contraction. It implies $fa = ga = a$. Therefore a is a fixed point of f and g.

Finally, we prove the uniqueness. Let a and c be two different common fixed points of f and g, and specifically $a \neq c$. Since $a, c \in Fix(f)$, we have $\alpha(ga, gc) \geq 1$ and $\alpha(gc, ga) \geq 1$. After that, one can obtain

$$
\begin{aligned}
M(a,c) = \max\{&q(gc,fc) - q(gc,gc), q(fc,gc) - q(fc,fc), q(ga,fa) - q(ga,ga), \\
&q(fa,ga) - q(fa,fa), \frac{q(fa,gc)[q(fc,fa) - q(fc,fc)]}{1 + q(fa,fc) + q(ga,gc)}, \\
&\frac{q(gc,fa)[q(ga,gc) - q(ga,ga)]}{1 + q(ga,fa) + q(gc,ga)}\} \\
\leq \max\{&0,0,0,0, q(c,a) - q(c,c), q(a,c) - q(a,a)\} \\
= \max\{&q(c,a) - q(c,c), q(a,c) - q(a,a)\}.
\end{aligned}
$$

In the same way, one can deduce $M(c,a) \leq \max\{q(c,a)-q(c,c),\ q(a,c)-q(a,a)\}$. It follows that

$$\begin{aligned}
0 &< \int_0^{\psi(d_q(a,c))} \varphi(t)dt \\
&\leq \int_0^{\psi(q(fa,fc))} \varphi(t)dt + \int_0^{\psi(q(fc,fa))} \varphi(t)dt \\
&\leq \alpha(ga,gc)\int_0^{\psi(q(fa,fc))} \varphi(t)dt + \alpha(gc,ga)\int_0^{\psi(q(fc,fa))} \varphi(t)dt \\
&\leq L(q(ga,gc))\int_0^{\psi(h\cdot M(a,c))} \varphi(t)dt + L(q(gc,ga))\int_0^{\psi(h\cdot M(c,a))} \varphi(t)dt \\
&< \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(a,c),M(c,a)\})} \varphi(t)dt + \frac{1}{2}\int_0^{\psi(h\cdot\max\{M(a,c),M(c,a)\})} \varphi(t)dt \\
&< \int_0^{\psi(h(q(a,c)-q(a,a)+q(c,a)-q(c,c)))} \varphi(t)dt \\
&= \int_0^{\psi(hd_q(a,c))} \varphi(t)dt \\
&< \int_0^{\psi(d_q(a,c))} \varphi(t)dt.
\end{aligned}$$

This is contradictory, so $a \neq c$. Consequently, a is the unique common fixed point of f and g. This completes the proof.

Example 3.7. Let $X = [0, \frac{\pi}{4}]$. Define $q(x,y) = |x-y| + |x|$ and $f(x) = \frac{\sin x}{2}$, $g(x) = x - \frac{\sin x}{2}$. For $\alpha(x,y) = 1$, $\varphi(t) = 2t$, $\psi(t) = t$, $L(x) = \frac{25}{64}$, and $h = \frac{4}{5}$, we conclude that f and g have a unique fixed point in X.

According to the statement, we get

$$q(fx,fy) - q(fx,fx) = |\frac{\sin x}{2} - \frac{\sin y}{2}| + |\frac{\sin x}{2}| - |\frac{\sin x}{2}| = \frac{1}{2}|\sin x - \sin y|,$$

and then

$$\int_0^{\psi(q(fx,fy)-q(fx,fx)} \varphi(t)dt = (\frac{1}{2}|\sin x - \sin y|)^2 = \frac{1}{4}|\sin x - \sin y|^2.$$

We consider two cases as follows.

Case 1. $x \geqslant y$. We arrive at

$$q(gx,fx) - q(gx,gx) = |x - \frac{\sin x}{2} - \frac{\sin x}{2}| + |x - \frac{\sin x}{2}| - |x - \frac{\sin x}{2}| = |x - \sin x|.$$

So, we derive

$$h \cdot M(x,y) \geq \frac{4}{5} \cdot |x - \sin x| = \frac{4}{5}|x - \sin x|.$$

This implies

$$\int_0^{\psi(h\cdot M(x,y))} \varphi(t)dt \geq \int_0^{\psi(\frac{4}{5}|x-\sin x|)} \varphi(t)dt = (\frac{4}{5}|x-\sin x|)^2 = \frac{16}{25}|x-\sin x|^2.$$

Thus, we obtain

$$L(q(x,y))\int_0^{\psi(h\cdot M(x,y))}\varphi(t)dt \geq \frac{25}{64}\cdot\frac{16}{25}|x-\sin x|^2 = \frac{1}{4}|x-\sin x|^2.$$

Since $X = [0, \frac{\pi}{4}]$,

$$\frac{1}{4}|\sin x - \sin y|^2 \leqslant \frac{1}{4}|x-\sin x|^2.$$

Case 2. $x < y$. In this case, we have

$$q(gy, fy) - q(gy, gy) = |y - \frac{\sin y}{2} - \frac{\sin y}{2}| + |y - \frac{\sin y}{2}| - |y - \frac{\sin y}{2}| = |y - \sin y|.$$

So, one can deduce

$$h \cdot M(x,y) \geq \frac{4}{5}\cdot|y-\sin y| = \frac{4}{5}|y-\sin y|.$$

It can be concluded that

$$\int_0^{\psi(h\cdot M(x,y))}\varphi(t)dt \geq \int_0^{\psi(\frac{4}{5}|y-\sin y|)}\varphi(t)dt = (\frac{4}{5}|y-\sin y|)^2 = \frac{16}{25}|y-\sin y|^2.$$

As a result, we gain

$$L(q(x,y))\int_0^{\psi(h\cdot M(x,y))}\varphi(t)dt \geq \frac{25}{64}\cdot\frac{16}{25}|y-\sin y|^2 = \frac{1}{4}|y-\sin y|^2.$$

Due to $X = [0, \frac{\pi}{4}]$, we obtain

$$\frac{1}{4}|\sin x - \sin y|^2 \leqslant \frac{1}{4}|y-\sin y|^2.$$

In summary, we can conclude that f and g satisfy the conditions of the Theorem 3.5, which means f and g have a unique fixed point in X. In this example, 0 is the unique fixed point of f and g.

4. Application

In this section, we will derive an existence solution based on Theorem 3.4, thereby solving the following integral equation:

$$u(t) = \int_0^T K(t, r, u(r))dr, \tag{4.1}$$

where $t \in [0, T]$, $T > 0$, $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$, and $u : [0, T] \to \mathbb{R}$.

Let $X = C[0, T]$. Define $p : X \times X \to \mathbb{R}^+$ by

$$p(x,y) = \sup_{t\in[0,T]}|x(t)-y(t)| + \max\{\sup_{t\in[0,T]}|x(t)|, \sup_{t\in[0,T]}|y(t)|\}.$$

It is obvious that (X, p) is a $p-$complete partial metric space.

Define f by

$$fu(t) = \int_0^T K(t, r, u(r))dr.$$

Theorem 4.1. Let (X, p) be a complete partial metric space and $f : X \to X$ be a self-mapping. If the following conditions are satisfied:

(i) the mapping $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous,

(ii) there exists a continuous function $\eta : [0, T] \times [0, T] \to \mathbb{R}^+$ such that $\int_0^T \eta(t, r)dr = 1$,

(iii) for each $x, y \in X$, $t, r \in [0, T]$, and $h \in (0, 1)$, we have

$$|K(t, r, x(r)) - K(t, r, y(r))| \leq \frac{h}{1 + h}\eta(t, r)|x(t) - y(t)|,$$

$$\max_{t,r\in[0,T]} \{|K(t, r, x(r))|, |K(t, r, y(r))|\} \leq \frac{h}{1 + h}\eta(t, r) \max_{t,r\in[0,T]} \{|x(t)|, |y(t)|\},$$

$$|K(t, r, x(r))| \geq \frac{h}{1 + h}\eta(t, r)|x(t)|,$$

then the integral equation (4.1) has a unique solution $\xi \in C[0, T]$.

Proof. For $x, y \in X$, by virtue of conditions (i)–(iii), we have

$$p(fx, fy) - p(fx, fx) = \sup_{t\in[0,T]} |fx(t) - fy(t)| + \max\{\sup_{t\in[0,T]} |fx(t)|, \sup_{t\in[0,T]} |fy(t)|\} - \sup_{t\in[0,T]} |fx(t)|.$$

So that

$$\begin{aligned}
\sup_{t\in[0,T]} (|fx(t) - fy(t)|) &= \sup_{t\in[0,T]} (|\int_0^T K(t, r, x(r))dr - \int_0^T K(t, r, y(r))dr|) \\
&\leq \sup_{t\in[0,T]} (\int_0^T |K(t, r, x(r)) - K(t, r, y(r))|dr) \\
&\leq \sup_{t\in[0,T]} (\int_0^T \frac{h}{1 + h}\eta(t, r)|x(t) - y(t)|dr) \\
&\leq \sup_{t\in[0,T]} (\frac{h}{1 + h}|x(t) - y(t)|) \\
&= \frac{h}{1 + h} \sup_{t\in[0,T]} |x(t) - y(t)|.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\max\{\sup_{t\in[0,T]} |fx(t)|, \sup_{t\in[0,T]} |fy(t)|\} - \sup_{t\in[0,T]} |fx(t)| \\
&= \max\{\sup_{t\in[0,T]} |\int_0^T K(t, r, x(r))dr|, \sup_{t\in[0,T]} |\int_0^T K(t, r, y(r))dr|\} - \sup_{t\in[0,T]} |\int_0^T K(t, r, x(r))dr| \\
&\leq \sup_{t\in[0,T]} (\int_0^T \max\{|K(t, r, x(r))|, |K(t, r, y(r))|\}dr) - \sup_{t\in[0,T]} \int_0^T \frac{h}{1 + h}\eta(t, r)|x(t)|dr \\
&\leq \sup_{t\in[0,T]} (\int_0^T \frac{h}{1 + h}\eta(t, r) \max\{|x(t)|, |y(t)|\}dr) - \sup_{t\in[0,T]} \int_0^T \frac{h}{1 + h}\eta(t, r)|x(t)|dr \\
&\leq \frac{h}{1 + h} \max\{\sup_{t\in[0,T]} |x(t)|, \sup_{t\in[0,T]} |y(t)|\} - \frac{h}{1 + h} \sup_{t\in[0,T]} |x(t)| \\
&= \frac{h}{1 + h}(\max\{\sup_{t\in[0,T]} |x(t)|, \sup_{t\in[0,T]} |y(t)|\} - \sup_{t\in[0,T]} |x(t)|).
\end{aligned}$$

We suppose

$$H = \sup_{t\in[0,T]} |x(t) - y(t)| + \max\{ \sup_{t\in[0,T]} |x(t)|, \sup_{t\in[0,T]} |y(t)|\} - \sup_{t\in[0,T]} |x(t)|,$$

and by the above equation, we obtain

$$\int_0^{\psi(p(fx,fy)-p(fx,fx))} \varphi(t)dt \leq \int_0^{\psi(\frac{h}{1+h}H)} \varphi(t)dt \leq \int_0^{\psi((hH)} \varphi(t)dt = \int_0^{\psi h((p(x,y)-p(x,x)))} \varphi(t)dt.$$

Therefore, all the conditions of Theorem 3.4 hold. As a result, the mapping f has a unique point $\xi \in C[0,T]$, which is a solution of the integral equation (4.1).

5. Conclusions

In this manuscript, we introduced a new class of generalized α-ψ-contractive mappings and proved the existence and uniqueness of fixed points and common fixed points in the framework of quasi-partial metric spaces. Furthermore, two examples were given to support our results.

Author contributions

All authors contributed equally and significantly in writing this article. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was financially supported by the Natural Science Foundation of Liaoning Province (No:2024-MS-108).

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

References

1. S. Banach, Surles operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
2. S. G. Matthews, Partial metric topology, *Ann. NY Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x
3. E. Karapinar, I. M. Erhan, A. Ozturk, Fixed point theorems on quasi-partial metric spaces, *Math. Comput. Model.,* **57** (2013), 2442–2448. https://doi.org/10.1016/j.mcm.2012.06.036

4. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.,* **75** (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014
5. E. Karapinar, L. Gholizadeh, H. H. Alsulami, M. Noorwali, α-(ψ, ϕ) Contractive mappings on quasi-partial metric spaces, *Fixed Point Theory A.,* **105** (2015), 1–20. https://doi.org/10.1186/s13663-015-0352-z
6. A. Branciari, A fixed point theorem for mapings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.,* **29** (2002), 531–536. https://doi.org/10.1155/S0161171202007524
7. D. Doitchinov, On completeness in quasi-metric spaces, *Topol. Appl.,* **30** (1988), 127–148. https://doi.org/10.1016/0166-8641(88)90012-0
8. M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, *Fixed Point Theory A.,* **2012** (2012), 1–7. https://doi.org/10.1186/1687-1812-2012-210
9. Y. Hao, H. Guan, On some common fixed point results for weakly contraction mappings with application, *J. Funct. Space.,* **2021** (2021), 5573983. https://doi.org/10.1155/2021/5573983
10. G. Jungck, Compatible mapings and common fixed points, *Int. J. Math. Math. Sci.,* **9** (1986), 771–779. https://doi.org/10.1155/S0161171286000935
11. M. U. Ali, T. Kamran, E. Karapinar, An approach to existence of fixed points of generalized contractive multivalued mapings of integral type via admissible mapping, *Abstr. Appl. Anal.,* **2014** (2014), 141489. https://doi.org/10.1155/2014/141489
12. H. Guan, J. Gou, Common fixed point theorems for contractive mappings of integral type in b-metric spaces, *Math. Notes,* **115** (2024), 538–554. https://doi.org/10.1134/S0001434624030258