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Research article

New solitary solutions for stochastic Kakutani–Matsuuchi model of internal gravity waves

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Abstract: Here, we consider the stochastic Kakutani–Matsuuchi model (SKMM) perturbed by multiplicative noise in the Itô sense. This model describes the behavior of these waves as they propagate through a stratified fluid medium, such as the Earth's atmosphere or ocean. Internal gravity waves are generated by disturbances in the density or temperature of the fluid and can play a significant role in transporting energy and momentum throughout the system. By applying two different techniques, namely the extended tanh function method and the mapping method, we obtain new periodic soliton, dark soliton, bright soliton, anti-Kink soliton and Kink soliton solutions for SKMM. Because the Kakutani–Matsuuchi model is important in studying internal gravity waves in the atmosphere and oceans, the solutions of the SKMM are beneficial in understanding several fascinating scientific phenomena. Using MATLAB, we exhibit several 2D and 3D graphs that illustrate the impact of the noise on the solutions of SKMM.

Keywords: stochastic PDE; exact solutions; stability by noise; mapping method; extended tanh function method
Mathematics Subject Classification: 60H15, 60H10, 35Q51, 35A20

1. Introduction

Internal gravity waves are waves that propagate within a fluid medium due to buoyancy forces. They play a crucial role in oceanic and atmospheric dynamics, impacting a wide range of phenomena such as ocean circulation, atmospheric convection, and climate variability [1]. The Stochastic Kakutani–Matsuuchi model is a mathematical framework used to study the behavior of internal gravity waves in geophysical flows. The model incorporates stochastic processes to account for the random nature of turbulence and other small-scale processes that can affect the propagation of internal waves.

One key aspect of the stochastic Kakutani–Matsuuchi model is its ability to capture the variability and intermittency of internal gravity waves in the presence of turbulence. By incorporating stochastic terms in the governing equations, the model is able to account for the random fluctuations in the fluid velocity and density fields that can impact the propagation of internal waves. This allows researchers to better understand the complex interactions between internal waves and turbulent eddies, which are essential for accurately predicting the behavior of geophysical flows.

In general, stochastic partial differential equations (SPDEs) are often utilized to study phenomena that are influenced by both deterministic and random factors [2–5]. For instance, in fluid dynamics, SPDEs can be used to model turbulence, which is a complex and chaotic flow regime that is characterized by both deterministic patterns and random fluctuations. By using SPDEs to simulate turbulent flow, engineers can gain insights into its underlying dynamics and develop more accurate models for predicting its effects on structures and devices.

Moreover, obtaining the solutions to PDEs is essential for modeling the dynamics of complex systems and making accurate predictions in diverse scientific and engineering applications. Recently, many useful and efficient techniques have been developed to get the exact solutions for PDEs, such as the sine-Gordon expansion technique [6], the F-expansion method [7], the Riccati equation method [8], the Jacobi elliptic function expansion [9], the (G'/G)-expansion [10, 11], the sine-cosine method [12], the mapping method [13], the $exp(-\phi(\varsigma))$-expansion method [14], the ansatz method [15, 16], the trial function method [17], etc. By obtaining solutions of PDEs, researchers can obtain significant insights into the dynamics of complex systems and make informed decisions about how to manage them effectively.

Here, we consider the stochastic Kakutani–Matsuuchi model (SKMM) in the following form [1,18]:

$$\mathcal{G}_t - \mathcal{G}_{txx} + \mathcal{G}\mathcal{G}_x + \mathcal{G}_x = \rho(\mathcal{G} - \mathcal{G}_{xx})\mathcal{W}_t, \tag{1.1}$$

where $\mathcal{G} = \mathcal{G}(x, t)$ denotes the fluid velocity; $\mathcal{G}_t$ represents the time evolution term; $\mathcal{G}_{txx}$ is the dispersive term; $\mathcal{G}\mathcal{G}_x$ indicates the advective nonlinear steepening of the waves; $\mathcal{G}_x$ stands for the waves' lowest-order dispersive spreading; $\mathcal{W}(t)$ is the standard Wiener process; ρ is the noise strength.

Due to the importance of SKMM (1.1), there are some authors who have obtained their solutions by utilizing different methods, including the auxiliary equation sub-equation method and the Khater II method [19], the finite difference schemes [18], the shifted Jacobi–Gauss–Lobatto collocation [20], and the reliable implicit finite difference method [21]. While the solutions to the stochastic Kakutani–Matsuuchi model have not yet been acquired.

The objective is to determine the exact stochastic solutions of SKMM (1.1) utilizing the expanded tanh function and mapping methods. The SKMM (1.1) has vital applications in studying internal gravity waves in the atmosphere and oceans; therefore, the derived solutions may be utilized to examine many important scientific phenomena. We also give several 2D and 3D graphical representations using the MATLAB application to study how noise impacts the analytical solutions of the SKMM (1.1).

The paper is arranged as follows: The wave equation for the SKMM (1.1) is obtained in Sec. 2 by use of the wave transformation. The stochastic solution of the SKMM (1.1) may be obtained in Sec. 3. In Sec. 4, we discuss how Brownian motion affects the SKMM (1.1) solutions that are produced. The article's findings are finally given.

2. Traveling wave equation for SKMM

The wave transformation is utilized to create the wave equation for SKMM (1.1):

$$\mathcal{G}(x,t) = \mathcal{V}(\zeta)e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}, \quad \zeta = \zeta_1 x + \zeta_2 t, \tag{2.1}$$

where $\mathcal{V}$ is a deterministic real function; ζ_1 and ζ_2 are unknown constants. We see that

$$\begin{aligned} \mathcal{G}_t &= [\zeta_2\mathcal{V}' + \rho\mathcal{V}\mathcal{W}_t + \frac{1}{2}\rho^2\mathcal{V} - \frac{1}{2}\rho^2\mathcal{V}]e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]} \\ &= [\zeta_2\mathcal{V}' + \rho\mathcal{V}\mathcal{W}_t]e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}, \end{aligned} \tag{2.2}$$

with $\mathcal{V}' = \frac{d\mathcal{V}}{d\zeta}$, and

$$\mathcal{G}_{txx} = [\zeta_1^2\zeta_2\mathcal{V}''' + \rho\zeta_1^2\mathcal{V}''\mathcal{W}_t]e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}, \ \mathcal{G}_x = \zeta_1\mathcal{V}'e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{2.3}$$

Plugging Eq (2.1) into Eq (1.1) and utilizing (2.2-2.3), we obtain

$$-\zeta_1^2\zeta_2\mathcal{V}''' + (\zeta_2 + \zeta_1)\mathcal{V}' + \zeta_1\mathcal{V}\mathcal{V}'e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]} = 0. \tag{2.4}$$

Taking the expectations, we have

$$-\zeta_1^2\zeta_2\mathcal{V}''' + (\zeta_2 + \zeta_1)\mathcal{V}' + \zeta_1\mathcal{V}\mathcal{V}'e^{-\frac{1}{2}\rho^2 t}\mathbb{E}e^{[\rho\mathcal{W}(t)]} = 0. \tag{2.5}$$

Since $\mathcal{W}(t)$ has a normal distribution, then $\mathbb{E}(e^{\rho\mathcal{W}(t)}) = e^{\frac{1}{2}\rho^2 t}$ (see for more detail [22]). Hence, Eq (2.5) tends to

$$-\zeta_1^2\zeta_2\mathcal{V}''' + (\zeta_2 + \zeta_1)\mathcal{V}' + \zeta_1\mathcal{V}\mathcal{V}' = 0. \tag{2.6}$$

After integrating Eq (2.6) once and neglecting the constant of integration for simplicity, we have

$$\mathcal{V}'' + \mathcal{H}_1\mathcal{V} + \mathcal{H}_2\mathcal{V}^2 = 0, \tag{2.7}$$

where

$$\mathcal{A}_1 = \frac{-\zeta_2 - \zeta_1}{\zeta_1^2\zeta_2} \text{ and } \mathcal{A}_2 = \frac{-1}{2\zeta_1\zeta_2}.$$

3. Exact solutions of SKMM

We solve the wave equation (2.7) by using the extended tanh function method [23] and mapping method [24, 25]. The SKMM solutions (1.1) may then be derived by applying the wave transformation (2.1).

3.1. Extended tanh function method

Suppose the solution $\mathcal{V}$ to Eq (2.7) is

$$\mathcal{V}(\zeta) = \ell_0 + \sum_{k=1}^{M}\ell_k\mathcal{X}^k, \tag{3.1}$$

where $\mathcal{X}$ solves

$$\mathcal{X}' = \mathcal{X}^2 + \varpi. \tag{3.2}$$

It is necessary to balance $\mathcal{V}^2$ with $\mathcal{V}''$ in Eq (2.7) in order to calculate M as follows:

$$2M = M + 2 \ \Rightarrow \ M = 2.$$

Rewriting Eq (3.1) with $M = 2$

$$\mathcal{V}(\zeta) = \ell_0 + \ell_1\mathcal{X}+\ell_2\mathcal{X}^2. \tag{3.3}$$

The solutions to Eq (3.2) are presented as follows:

$$\mathcal{X}(\zeta) = \sqrt{\varpi}\tan(\sqrt{\varpi}\zeta) \ \text{ or } \mathcal{X}(\zeta) = -\sqrt{\varpi}\cot(\sqrt{\varpi}\zeta), \tag{3.4}$$

if $\varpi > 0$, or

$$\mathcal{X}(\zeta) = -\sqrt{-\varpi}\tanh(\sqrt{-\varpi}\zeta) \ \text{ or } \mathcal{X}(\zeta) = -\sqrt{-\varpi}\coth(\sqrt{-\varpi}\zeta), \tag{3.5}$$

if $\varpi < 0$, or

$$\mathcal{X}(\zeta) = \frac{-1}{\zeta}, \tag{3.6}$$

if $\varpi = 0$.

Now, setting Eq (3.3) into Eq (2.7), we obtain

$$\begin{aligned}&(6\ell_2 + \mathcal{A}_2\ell_2^2)\mathcal{X}^4 + (2\ell_1 + 2\mathcal{A}_2\ell_1\ell_2)\mathcal{X}^3 + (8\varpi\ell_2 + 2\ell_0\ell_2\mathcal{A}_2 + \ell_1^2\mathcal{A}_2 + \mathcal{A}_1\ell_2)\mathcal{X}^2\\ &(2\varpi\ell_1 + \mathcal{A}_1\ell_1 + 2\mathcal{A}_2\ell_0\ell_1)\mathcal{X} + (2\varpi^2\ell_2 + \mathcal{A}_1\ell_0 + \mathcal{A}_2\ell_0^2) = 0.\end{aligned}$$

Assigning zero to the coefficients of every power of $\mathcal{X}$, yields:

$$6\ell_2 + \mathcal{A}_2\ell_2^2 = 0,$$

$$2\ell_1 + 2\mathcal{A}_2\ell_1\ell_2 = 0,$$

$$8\varpi\ell_2 + 2\ell_0\ell_2\mathcal{A}_2 + \ell_1^2\mathcal{A}_2 + \mathcal{A}_1\ell_2 = 0,$$

$$2\varpi\ell_1 + \mathcal{A}_1\ell_1 + 2\mathcal{A}_2\ell_0\ell_1 = 0,$$

and

$$2\varpi^2\ell_2 + \mathcal{A}_1\ell_0 + \mathcal{A}_2\ell_0^2 = 0.$$

These equations can be solved to obtain the next two sets of solutions:

Set I:

$$\ell_0 = \frac{-3\mathcal{A}_1}{2\mathcal{A}_2}, \ \ell_1 = 0, \ \ell_2 = \frac{-6}{\mathcal{A}_2}, \ \varpi = \frac{\mathcal{A}_1}{4}. \tag{3.7}$$

Set II:

$$\ell_0 = \frac{\mathcal{A}_1}{2\mathcal{A}_2}, \ \ell_1 = 0, \ \ell_2 = \frac{-6}{\mathcal{A}_2}, \ \varpi = \frac{-\mathcal{A}_1}{4}. \tag{3.8}$$

For the Set I: Equation (2.7) has the next solution:

$$\mathcal{V}(\zeta) = \frac{-3\mathcal{A}_1}{2\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\mathcal{X}^2(\zeta).$$

There are three different cases for $\mathcal{X}(\zeta)$:

Case 1: If $\varpi = \frac{\mathcal{A}_1}{4} > 0$, then with (3.4) we have

$$\mathcal{V}(\zeta) = \frac{-6\varpi}{\mathcal{A}_2} - \frac{6\varpi}{\mathcal{A}_2}\tan^2(\sqrt{\varpi}\zeta) = -\frac{6\varpi}{\mathcal{A}_2}\sec^2\left(\sqrt{\varpi}\zeta\right),$$

and

$$\mathcal{V}(\zeta) = \frac{-6\varpi}{\mathcal{A}_2} - \frac{6\varpi}{\mathcal{A}_2}\cot^2(\sqrt{\varpi}\zeta) = \frac{-6\varpi}{\mathcal{A}_2}\csc^2\left(\sqrt{\varpi}\zeta\right).$$

Therefore, the SKMM solution (1.1) is

$$\mathcal{G}(x,t) = -\frac{6\varpi}{\mathcal{A}_2}\sec^2\left(\sqrt{\varpi}(\zeta_1 x + \zeta_2 t)\right)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{3.9}$$

and

$$\mathcal{G}(x,t) = \frac{-6\varpi}{\mathcal{A}_2}\csc^2\left(\sqrt{\varpi}(\zeta_1 x + \zeta_2 t)\right)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{3.10}$$

Case 2: If $\varpi = \frac{\mathcal{A}_1}{4} < 0$, then by using (3.5), we obtain

$$\mathcal{V}(\zeta) = \frac{-6\varpi}{\mathcal{A}_2} + \frac{6\varpi}{\mathcal{A}_2}\tanh^2(\sqrt{-\varpi}\zeta) = \frac{-6\varpi}{\mathcal{A}_2}\text{sech}^2\left(\sqrt{-\varpi}\zeta\right),$$

and

$$\mathcal{V}(\zeta) = \frac{-6\varpi}{\mathcal{A}_2} + \frac{6\varpi}{\mathcal{A}_2}\coth^2\left(\sqrt{-\varpi}\zeta\right) = \frac{6\varpi}{\mathcal{A}_2}\text{csch}^2\left(\sqrt{-\varpi}\zeta\right).$$

Thus, the SKMM solution (1.1) is

$$\mathcal{G}(x,t) = \frac{-6\varpi}{\mathcal{A}_2}\text{sech}^2\left(\sqrt{-\varpi}(\zeta_1 x + \zeta_2 t)\right)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{3.11}$$

and

$$\mathcal{G}(x,t) = \frac{6\varpi}{\mathcal{A}_2}\text{csch}^2\left(\sqrt{-\varpi}(\zeta_1 x + \zeta_2 t)\right)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{3.12}$$

Case 3: If $\varpi = \mathcal{A}_1 = 0$, then by using (3.6) we obtain

$$\mathcal{V}(\zeta) = \frac{6}{\mathcal{A}_2\zeta^2}.$$

Therefore, the SKMM solution (1.1) is

$$\mathcal{G}(x,t) = \left(\frac{-6}{\mathcal{A}_2(\zeta_1 x + \zeta_2 t)^2}\right)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{3.13}$$

For the Set II: Equation (2.7) has the solution

$$\mathcal{V}(\zeta) = \frac{-2\varpi}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\mathcal{X}^2(\zeta).$$

There are three distinct cases for $\mathcal{X}(\zeta)$:

Case 1: If $\varpi > 0$, then by using (3.4), we obtain

$$\mathcal{V}(\zeta) = \frac{-2\varpi}{\mathcal{A}_2} - \frac{6\varpi}{\mathcal{A}_2}\tan^2\left(\sqrt{\varpi}\zeta\right),$$

and

$$\mathcal{V}(\zeta) = \frac{-2\varpi}{\mathcal{A}_2} - \frac{6\varpi}{\mathcal{A}_2}\cot^2\left(\sqrt{\varpi}\zeta\right).$$

Thus, the SKMM solution (1.1) is

$$\mathcal{G}(x,t) = \Big(\frac{-2\varpi}{\mathcal{A}_2} - \frac{6\varpi}{\mathcal{A}_2}\tan^2(\sqrt{\varpi}(\zeta_1 x + \zeta_2 t)\Big)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{3.14}$$

and

$$\mathcal{G}(x,t) = \Big(\frac{-2\varpi}{\mathcal{A}_2} - \frac{6\varpi}{\mathcal{A}_2}\cot^2(\sqrt{\varpi}(\zeta_1 x + \zeta_2 t))\Big)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{3.15}$$

Case 2: If $\varpi < 0$, then by using (3.5) we obtain

$$\mathcal{V}(\zeta) = \frac{-2\varpi}{\mathcal{A}_2} + \frac{6\varpi}{\mathcal{A}_2}\tanh^2\left(\sqrt{-\varpi}\zeta\right),$$

and

$$\mathcal{V}(\zeta) = \frac{-2\varpi}{\mathcal{A}_2} + \frac{6\varpi}{\mathcal{A}_2}\coth^2\left(\sqrt{-\varpi}\zeta\right).$$

Therefore, the SKMM solution (1.1) is

$$\mathcal{G}(x,t) = \Big(\frac{-2\varpi}{\mathcal{A}_2} + \frac{6\varpi}{\mathcal{A}_2}\tanh^2(\sqrt{-\varpi}(\zeta_1 x + \zeta_2 t))\Big)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}, \tag{3.16}$$

and

$$\mathcal{G}(x,t) = \Big(\frac{-2\varpi}{\mathcal{A}_2} + \frac{6\varpi}{\mathcal{A}_2}\coth^2(\sqrt{-\varpi}(\zeta_1 x + \zeta_2 t))\Big)e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{3.17}$$

Case 3: If $\varpi = 0$, then by using (3.6) we obtain

$$\mathcal{V}(\zeta) = \frac{6}{\mathcal{A}_2}\frac{1}{\zeta^2}.$$

Thus, the SKMM solution (1.1) is

$$\mathcal{G}(x,t) = \frac{6}{\mathcal{A}_2}\frac{1}{(\zeta_1 x + \zeta_2 t)^2}e^{(\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t)}. \tag{3.18}$$

3.2. Mapping method

Supposing that the solution of Eq (2.7) takes the following, with $M = 2$, form:

$$\mathcal{V}(\zeta) = \ell_0 + \ell_1\mathcal{M} + \ell_2\mathcal{M}^2, \tag{3.19}$$

where $\mathcal{M}$ solves

$$\mathcal{M}' = \sqrt{\gamma_1\mathcal{M}^4 + \gamma_2\mathcal{M}^2 + \gamma_3}, \tag{3.20}$$

where $\gamma_1,\ \gamma_2$, and γ_3 are real parameters. Differentiating Eq (3.19) twice and utilizing (3.20), we get

$$\mathcal{V}'' = \ell_1(\gamma_2\mathcal{M} + 2\gamma_1\mathcal{M}^3) + 2\ell_2(\gamma_3 + 2\gamma_2\mathcal{M}^2 + 3\gamma_1\mathcal{M}^4). \tag{3.21}$$

Setting Eq (3.19) and Eq (3.21) into Eq (2.7), we have

$$(6\ell_2\gamma_1 + \mathcal{A}_2\ell_2^2)\mathcal{M}^4 + (2\gamma_1\ell_1 + 2\ell_1\ell_2\mathcal{A}_2)\mathcal{M}^3 + (4\ell_2\gamma_2 + 2\mathcal{A}_2\ell_0\ell_2 + \ell_1^2$$
$$+\ell_2\mathcal{A}_1)\mathcal{M}^2 + (\ell_1\gamma_2 + 2\mathcal{A}_2\ell_0\ell_1 + \mathcal{A}_1\ell_1)\mathcal{M}+(2\gamma_3\ell_2 + \mathcal{A}_1\ell_0 + \mathcal{A}_2\ell_0^2) = 0.$$

If we set all of the coefficients of $\mathcal{M}^k$ to zero, we have an algebraic system of equations. When we solve this system for $\gamma_2^2 - 3\gamma_3\gamma_1 > 0$, we obtain these two sets:
Set I:

$$\ell_0 = (\frac{-2\gamma_2 - 2\sqrt{(\gamma_2^2 - 3\gamma_3\gamma_1)}}{\mathcal{A}_2}),\ \ \ell_1 = 0,\ \ell_2 = \frac{-6\gamma_1}{\mathcal{A}_2},\ \ \mathcal{A}_1 = 4\sqrt{(\gamma_2^2 - 3\gamma_3\gamma_1)}.$$

Set II:

$$\ell_0 = (\frac{-2\gamma_2 + 2\sqrt{(\gamma_2^2 - 3\gamma_3\gamma_1)}}{\mathcal{A}_2}),\ \ \ell_1 = 0,\ \ell_2 = \frac{-6\gamma_1}{\mathcal{A}_2},\ \ \mathcal{A}_1 = -4\sqrt{(\gamma_2^2 - 3\gamma_3\gamma_1)}.$$

Set I: The solution of Eq (2.7) is

$$\mathcal{V}(\zeta) = \frac{-2\gamma_2 - 2\sqrt{(\gamma_2^2 - 3\gamma_3\gamma_1)}}{\mathcal{A}_2} - \frac{6\gamma_1}{\mathcal{A}_2}\mathcal{M}^2(\zeta). \tag{3.22}$$

There are several cases relying on the value of $\gamma_2,\ \gamma_3$, and γ_1 such that $\gamma_2^2 - 3\gamma_3\gamma_1 > 0$ as follows:

Case I-1: If $\gamma_2 = -(1 + \omega^2),\ \gamma_3 = 1$, and $\gamma_1 = \omega^2$, then $\mathcal{M}(\zeta) = sn(\zeta)$. Therefore, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\begin{aligned}\mathcal{G}(x,t) &= \Big(\frac{2 + 2\omega^2 - 2\sqrt{\omega^4 - \omega^2 + 1}}{\mathcal{A}_2} \\ &\quad -\frac{6\omega^2}{\mathcal{A}_2}sn^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}.\end{aligned} \tag{3.23}$$

If $\omega \to 1$, then Eq (3.23) transfers to

$$\mathcal{G}(x,t) = \Big(\frac{2}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\tanh^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.24}$$

Case I-2: If $\gamma_2 = 2\omega^2 - 1,\ \gamma_3 = -\omega^2(1-\omega^2)$, and $\gamma_1 = 1$, then $\mathcal{M}(\zeta) = ds(\zeta)$. Therefore, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\begin{aligned}\mathcal{G}(x,t) &= \Big(\frac{2 - 4\omega^2 - 2\sqrt{\omega^4 - \omega^2 + 1}}{\mathcal{A}_2} \\ &\quad -\frac{6}{\mathcal{A}_2}ds^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}.\end{aligned} \tag{3.25}$$

If $\omega \to 1$, then Eq (3.25) transfers to

$$\mathcal{G}(x,t) = \Big(\frac{-4}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\operatorname{csch}^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \tag{3.26}$$

When $\omega \to 0$, then Eq (3.25) transfers to

$$\mathcal{G}(x,t) = -\frac{6}{\mathcal{A}_2}\csc^2(\zeta_1 x + \zeta_2 t)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \tag{3.27}$$

Case I-3: If $\gamma_2 = 2 - \omega^2$, $\gamma_3 = (1 - \omega^2)$, and $\gamma_1 = 1$, then $\mathcal{M}(\zeta) = cs(\zeta)$. Therefore, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\begin{aligned} \mathcal{G}(x,t) &= \Big(\frac{2\omega^2 - 4 - 2\sqrt{\omega^4 + \omega^2 + 1}}{\mathcal{A}_2} \\ &\quad -\frac{6}{\mathcal{A}_2}cs^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \end{aligned} \tag{3.28}$$

If $\omega \to 1$, then Eq (3.28) transfers to

$$\mathcal{G}(x,t) = \Big(\frac{-4}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\operatorname{csch}^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \tag{3.29}$$

When $\omega \to 0$, then Eq (3.28) transfers to

$$\begin{aligned} \mathcal{G}(x,t) &= [\frac{-6}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\cot^2(\zeta_1 x + \zeta_2 t)]e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]} \\ &= -\frac{6}{\mathcal{A}_2}\csc^2(\zeta_1 x + \zeta_2 t)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \end{aligned} \tag{3.30}$$

Case I-4: If $\gamma_2 = -\omega^2,\ \gamma_3 = 2\omega^2 - 1$,and $\gamma_1 = 1 - \omega^2$, then $\mathcal{M}(\zeta) = cn(\zeta)$. Therefore, the SKMM solution (1.1) is

$$\begin{aligned} \mathcal{G}(x,t) &= \Big(\frac{2\omega^2 - 2\sqrt{(7\omega^4 - 9\omega^2 + 3)}}{\mathcal{A}_2} \\ &\quad -\frac{6(1-\omega^2)}{\mathcal{A}_2}cn^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \end{aligned} \tag{3.31}$$

If $\omega \to 0$, then Eq (3.31) becomes

$$\mathcal{G}(x,t) = \Big(\frac{-2\sqrt{3}}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\cos^2(\zeta_1 x + \zeta_2 t)\Big)e^{[\rho \mathcal{W}(t) - \frac{1}{2}\rho^2 t]}. \tag{3.32}$$

Case I-5: If $\gamma_2 = 2\omega^2 - 1,\ \gamma_3 = (1 - \omega^2)$, and $\gamma_1 = -\omega^2$, then $\mathcal{M}(\zeta) = ds(\zeta)$. Therefore, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\mathcal{G}(x,t) = \Big(\frac{2-4\omega^2-2\sqrt{\omega^4-\omega^2+1}}{\mathcal{A}_2} + \frac{6\omega^2}{\mathcal{A}_2} ds^2(\zeta_1 x+\zeta_2 t)\Big) e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.33}$$

If $\omega \to 1$, then Eq (3.33) transfers to

$$\mathcal{G}(x,t) = \Big(\frac{-4}{\mathcal{A}_2} + \frac{6}{\mathcal{A}_2}\operatorname{sech}^2(\zeta_1 x+\zeta_2 t)\Big) e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.34}$$

Case I-6: If $\gamma_2 = 2-\omega^2$, $\gamma_3 = (\omega^2-1)$, and $\gamma_1 = -1$, then $\mathcal{M}(\zeta) = dn(\zeta)$. Thus, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\mathcal{G}(x,t) = \Big(\frac{2\omega^2-4-2\sqrt{\omega^4-\omega^2+1}}{\mathcal{A}_2} + \frac{6}{\mathcal{A}_2} dn^2(\zeta_1 x+\zeta_2 t)\Big) e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.35}$$

If $\omega \to 1$, then Eq (3.35) transfers to Eq (3.34).

Case I-7: If $\gamma_2 = \frac{(\omega^2-2)}{2}$, $\gamma_3 = \frac{1}{4}$, and $\gamma_1 = \frac{\omega^2}{4}$, then $\mathcal{M}(\zeta) = \frac{sn(\zeta)}{1\pm dn(\zeta)}$. Therefore, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\mathcal{G}(x,t) = \Big(\frac{4-2\omega^2-\sqrt{4\omega^4-19\omega^2+16}}{2\mathcal{A}_2} - \frac{3\omega^2}{2\mathcal{A}_2}\frac{sn^2(\zeta_1 x+\zeta_2 t)}{(1\pm dn(\zeta_1 x+\zeta_2 t))^2}\Big) e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.36}$$

If $\omega \to 1$, then Eq (3.36) changes into

$$\mathcal{G}(x,t) = e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}\Big(\frac{1}{2\mathcal{A}_2} - \frac{3}{2\mathcal{A}_2}\frac{\tanh^2(\zeta_1 x+\zeta_2 t)}{(1\pm \operatorname{sech}(\zeta_1 x+\zeta_2 t))^2}\Big). \tag{3.37}$$

We can write Eq (3.37) as

$$\mathcal{G}(x,t) = e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}\Big(\frac{1}{2\mathcal{A}_2} - \frac{3}{2\mathcal{A}_2}(\coth(\zeta_1 x+\zeta_2 t)\mp \operatorname{csch}(\zeta_1 x+\zeta_2 t))^2\Big). \tag{3.38}$$

Case I-8: If $\gamma_1 = \frac{-1}{4}$, $\gamma_2 = \frac{(\omega^2+1)}{2}$, and $\gamma_3 = \frac{-(1-\omega^2)^2}{4}$, then $\mathcal{M}(\zeta) = \omega cn(\zeta)\pm dn(\zeta)$. Thus, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\mathcal{G}(x,t) = \Big(\frac{-2\omega^2-2-\sqrt{\omega^4+14\omega^2+1}}{2\mathcal{A}_2} + \frac{3}{2\mathcal{A}_2}(\omega cn(\zeta_1 x+\zeta_2 t)\pm dn(\zeta_1 x+\zeta_2 t))^2\Big) e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.39}$$

If $\omega \to 1$, then Eq (3.39) turns to

$$\mathcal{G}(x,t) = \Big(\frac{-4}{\mathcal{A}_2} - \frac{6}{\mathcal{A}_2}\operatorname{sech}^2(\zeta_1 x+\zeta_2 t)\Big) e^{[\rho \mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.40}$$

Case I-9: If $\gamma_1 = \frac{\omega^2-1}{4}$, $\gamma_2 = \frac{(\omega^2+1)}{2}$, and $\gamma_3 = \frac{\omega^2-1}{4}$, then $\mathcal{M}(\zeta) = \frac{dn(\zeta)}{1\pm sn(\zeta)}$. Thus, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\begin{aligned}\mathcal{G}(x,t) &= e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}\Big(\frac{-2\omega^2-2-\sqrt{\omega^4+14\omega^2+1}}{2\mathcal{A}_2} \\ &\quad -\frac{3(\omega^2-1)}{2\mathcal{A}_2}\frac{dn^2(\zeta_1 x+\zeta_2 t)}{\big(1\pm sn(\zeta_1 x+\zeta_2 t)\big)^2}\Big).\end{aligned} \tag{3.41}$$

When $\omega \to 0$, then Eq (3.41) becomes

$$\mathcal{G}(x,t) = \frac{-3}{2\mathcal{A}_2}\Big(1-\frac{1}{\big(1\pm\sin(\zeta_1 x+\zeta_2 t)\big)^2}\Big)e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.42}$$

Case I-10: If $\gamma_1 = \frac{1-\omega^2}{4}$, $\gamma_2 = \frac{(1-\omega^2)}{2}$, and $\gamma_3 = \frac{1-\omega^2}{4}$, then $\mathcal{M}(\zeta) = \frac{cn(\zeta)}{1\pm sn(\zeta)}$. Therefore, the SKMM solution (1.1), utilizing Eqs (2.1) and (3.22), is

$$\begin{aligned}\mathcal{G}(x,t) &= e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}\Big(\frac{2\omega^2-2-\sqrt{\omega^4-2\omega^2+1}}{2\mathcal{A}_2} \\ &\quad -\frac{3(1-\omega^2)}{2\mathcal{A}_2}\frac{cn^2(\zeta_1 x+\zeta_2 t)}{\big(1\pm sn(\zeta_1 x+\zeta_2 t)\big)^2}\Big).\end{aligned} \tag{3.43}$$

When $\omega \to 0$, then Eq (3.43) turns to

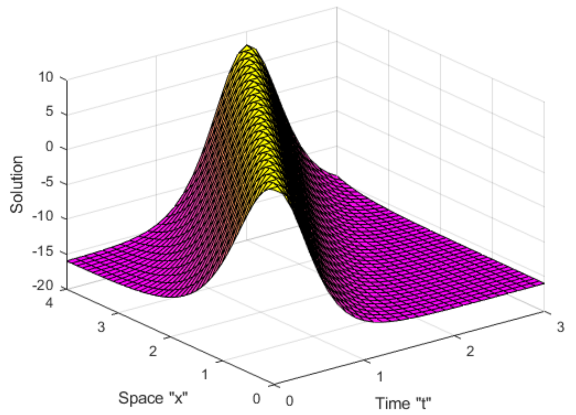
$$\mathcal{G}(x,t) = \frac{-3}{2\mathcal{A}_2}\Big(1+\frac{\cos^2(\zeta_1 x+\zeta_2 t)}{\big(1\pm\sin(\zeta_1 x+\zeta_2 t)\big)^2}\Big)e^{[\rho\mathcal{W}(t)-\frac{1}{2}\rho^2 t]}. \tag{3.44}$$

Set II: The same solutions with different coefficients may be obtained by using the same procedures as with the first set.

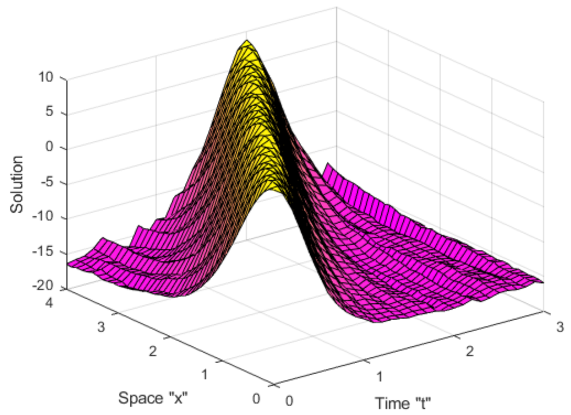
4. Impacts of noise

Now, let us demonstrate the influence of noise on the achieved solution of the SKMM (1.1). Many figures, including Eqs (3.16), (3.23), and (3.40), are supplied to demonstrate the behavior of some of the obtained solutions as follows.

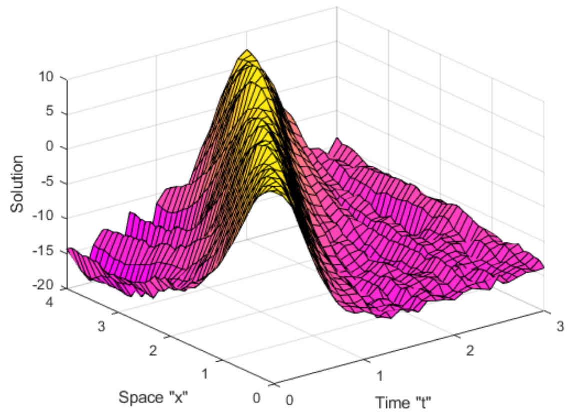
Figures 1–3 indicate that when noise is eliminated (i.e., when $\rho = 0$), there are several solutions, such as periodic soliton, bright soliton, dark soliton, among others. Introducing noise with $\rho = 0.1, 0.3, 1, 2$ causes the surface to become significantly flatter, following minor transit patterns, as seen by the 2D graph. This indicates that when noise is present, the solutions of SKMM (1.1) tend to converge toward zero.



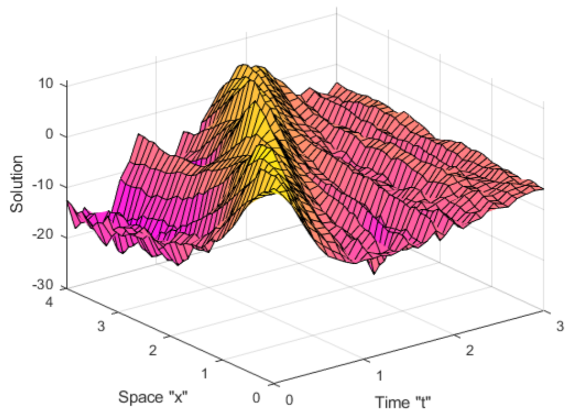
(a) $\rho = 0$



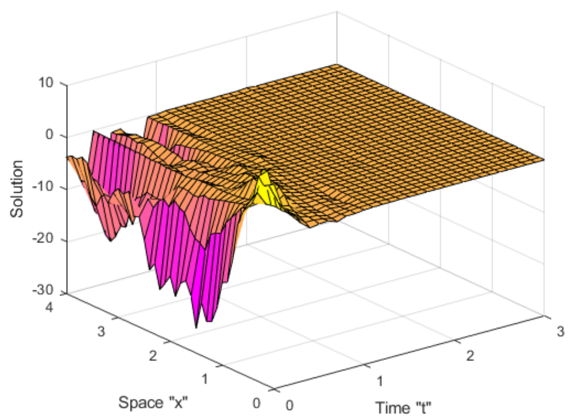
(b) $\rho = 0.1$



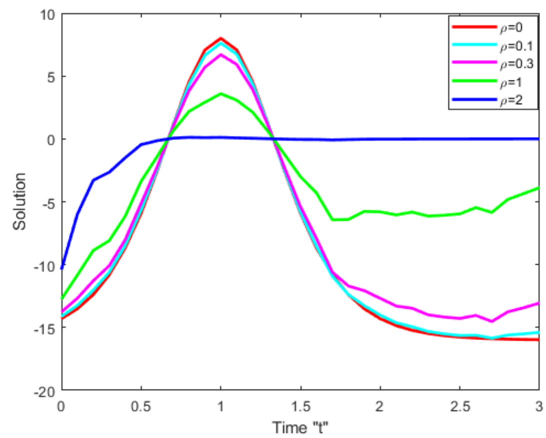
(c) $\rho = 0.3$



(d) $\rho = 1$

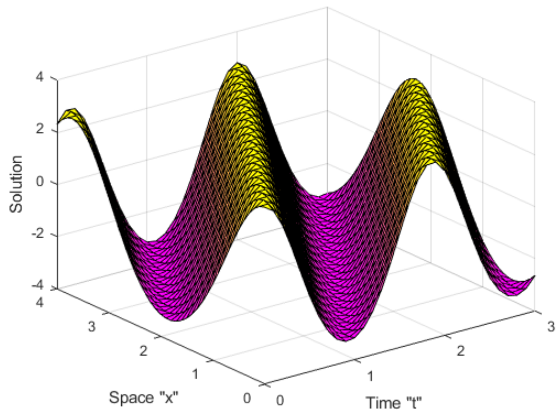


(e) $\rho = 2$

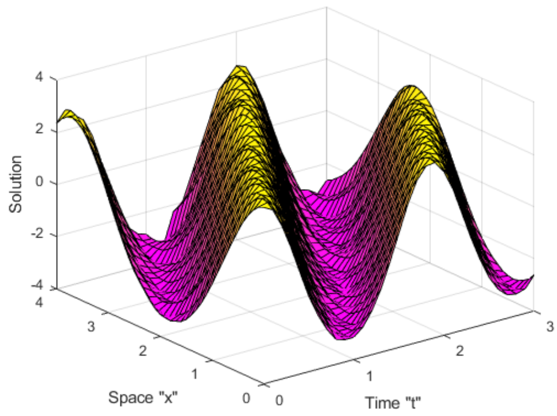


(f) $\rho = 0,\ 0.1,\ 0.3,\ 1,\ 2$

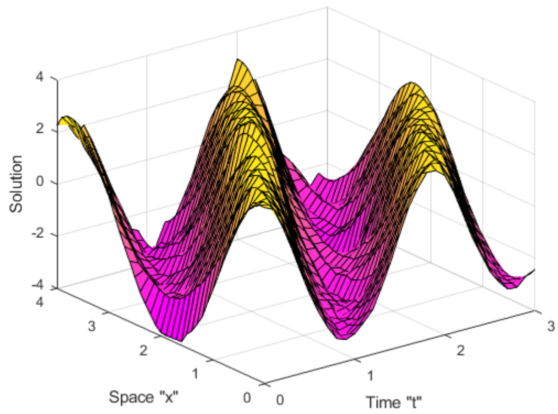
Figure 1. (a–e) present 3D-profile of solution $\mathcal{G}(x,t)$ in Eq (3.16) with $\zeta_1 = 1,\ \zeta_2 = -2,\ t \in [0,3],\ \ x \in [0,4]$, and with distinct ρ, (f) presents 2D-profile of Eq (3.16) with various ρ.



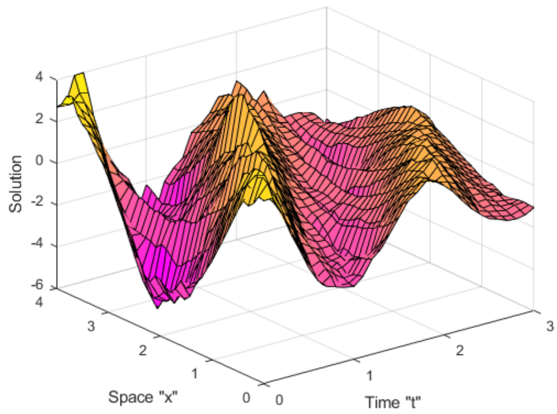
(a) $\rho = 0$



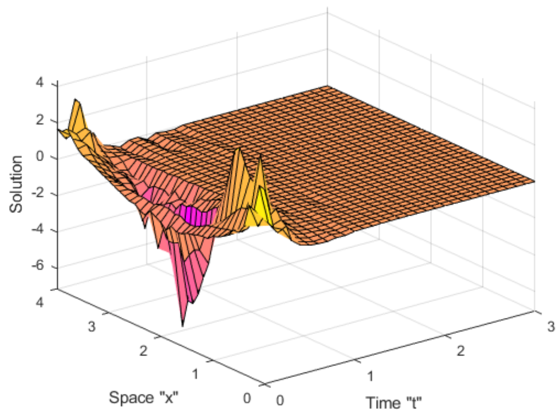
(b) $\rho = 0.1$



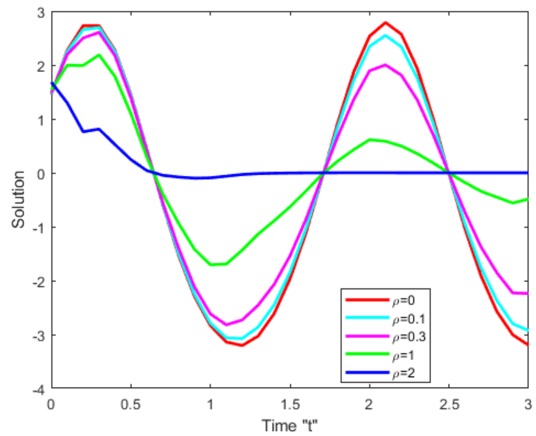
(c) $\rho = 0.3$



(d) $\rho = 1$

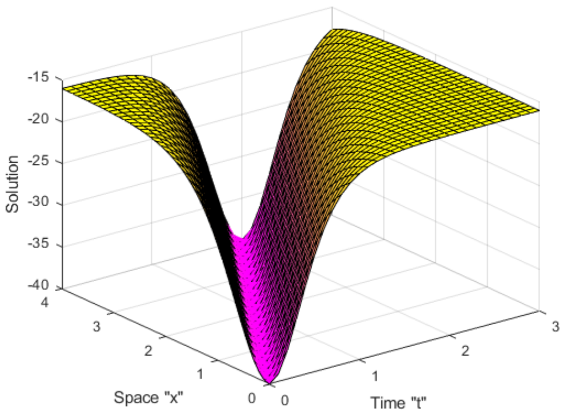


(e) $\rho = 2$

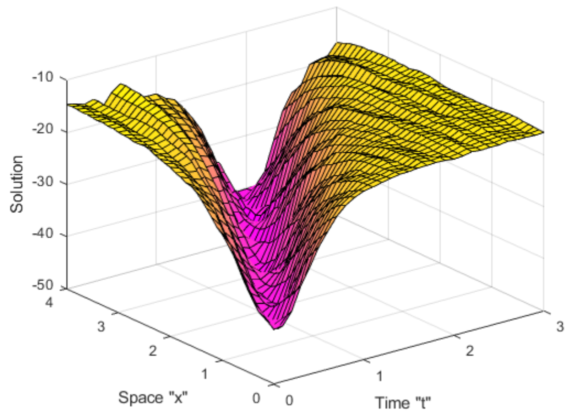


(f) $\rho = 0,\ 0.1,\ 0.3,\ 1,\ 2$

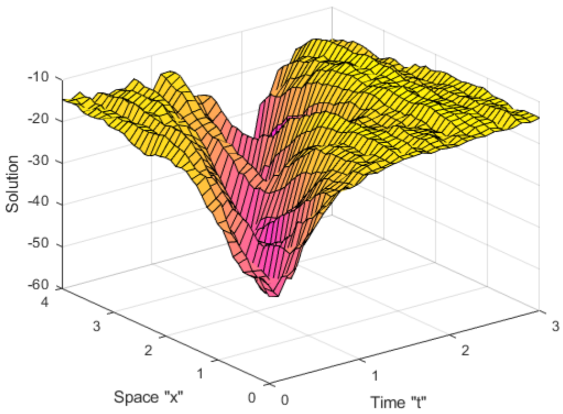
Figure 2. (a–e) present 3D-profile of solution $\mathcal{G}(x, t)$ in Eq (3.23) with $\check{n} = 0.5$, $\zeta_1 = 1$, $\zeta_2 = -2$, $t \in [0, 3]$, $x \in [0, 4]$, and with distinct ρ, (f) presents 2D-profile of Eq (3.23) with different ρ.



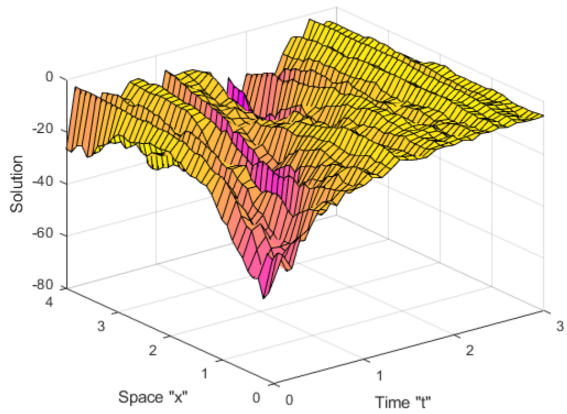
(a) $\rho = 0$



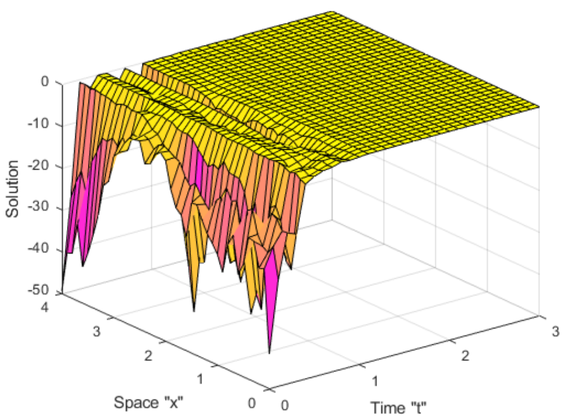
(b) $\rho = 0.1$



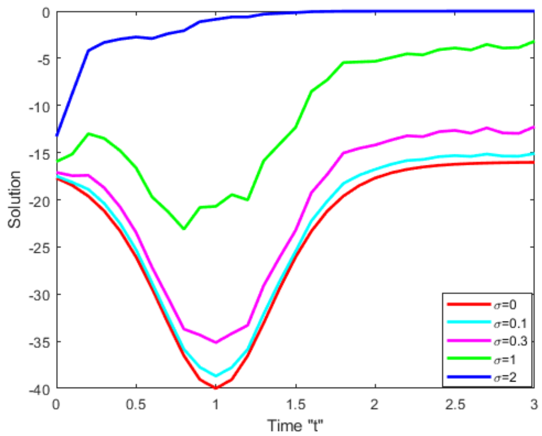
(c) $\rho = 0.3$



(d) $\rho = 1$



(e) $\rho = 2$



(f) $\rho = 0,\ 0.1,\ 0.3,\ 1,\ 2$

Figure 3. (a–e) present 3D-profile of solution $\mathcal{G}(x,t)$ in Eq (3.40) with $\zeta_1 = 1,\ \zeta_2 = -2,\ t \in [0,3],\ x \in [0,4]$, and with distinct ρ, (f) presents 2D-profile of Eq (3.40) with various ρ.

5. Conclusions

In this study, we investigated the stochastic Kakutani–Matsuuchi model (SKMM) perturbed in the Itô sense by multiplicative noise. We obtained new bright, dark, periodic, kink, and anti-kink solitons solutions for SKMM by applying two different techniques, namely the extended tanh function method and the mapping method. Because the Kakutani–Matsuuchi model is important in studying internal gravity waves in the atmosphere and oceans, the solutions of the stochastic Kakutani–Matsuuchi model are useful in understanding several fascinating scientific phenomena. The influence of the Wiener process on the analytical solutions of SKMM is demonstrated by a number of 2D and 3D graphs, which we present using MATLAB. We deduced that the white noise kept the solutions around zero. In future work, we can acquire the exact solutions for the Kakutani–Matsuuchi model with additive noise.

Author contributions

A. H. Alblowy: methodology, validation, formal analysis, funding acquisition, writing original draft preparation, writing—review and editing; W.W. Mohammed: methodology, software, validation, formal analysis, writing original draft preparation, writing—review and editing; E. M. Elsayed: methodology, software, validation, formal analysis, writing original draft preparation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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