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*Research article*

# Dynamics of the compact almost automorphic solution for a class of stochastic nonlinear differential equations

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**Abstract:** This paper mainly considers the existence of the compact almost automorphic mild solution for a class of stochastic nonlinear differential equations. More specifically, based on $C_0$-semigroup theory, Hölder inequality, Burkholder–Davis–Gundy inequality and Lebesgue dominated convergence theorem, we obtain that the $K$-mild solution is uniformly continuous and is relatively compact, etc. Combined with the subvariant functional method, we give some sufficient conditions to make sure that there exists at least one minimal $K$-mild solution; further, if the minimal $K$-mild solution is unique, then it is compact and almost automorphic. Moreover, we provide an example to illustrate the main presented results.

**Keywords:** compact almost automorphic solution; $K$-mild solution; existence; stochastic differential equations; subvariant functional method
**Mathematics Subject Classification:** 34C27, 60H10

## 1. Introduction

The concept of almost periodicity proposed by Bohr in [1] has a profound historical background and has been used to explain some curious behavior in astronomy with respect to the sun, planets, and moon. As a generalization, Bochner introduced the concept of almost automorphy when investigating a differential geometry problem, which plays a very important role in the further understanding of almost periodicity [2]. It is essential in the investigation of almost automorphic dynamics of the parabolic, ordinary, and other generalized differential equations [3]. Especially, some dynamics are specific to almost automorphic functions, while periodic functions do not possess these properties [4, 5]. Since then, there have been many researchers who have taken great interest in almost automorphy due to its importance and powerful applications in areas such as biology, physics, and so on; see [6–9] for details.

Although almost automorphic functions have a very wide range of applications, nevertheless, some types of differential equations do not work well; therefore, the concept of compact almost automorphic functions arose naturally. Recently, during the processing of dealing with the delay differential equations and related problems, the almost automorphic functions $f$, $\sigma$: $R \to R$ indicate not that $f(t-\sigma(t))$ is an almost automorphic function, where $R$ stands for the set of all real numbers, but if $f$, $\sigma$: $R \to R$ are compact almost automorphic functions, then $f(t-\sigma(t))$ is also a compact almost automorphic function. As a consequence, the compact almost automorphic functions have their own special charm in the study of the qualitative theory of differential equations. Sometimes, it is necessary to use compact almost automorphy rather than almost automorphy when exploring the differential equations with time-varying delays; see [10–12] for details. However, up to now, the existing results related to the compact almost automorphic solutions of different types of differential equations are still very rare.

In [13], Fink proposed the concept of subvariant functional to show the existence of compact almost automorphic solutions to a class of ordinary differential equation

$$x'(t) = F\left(t, x(t)\right)$$

for $t \in R$, where $F$: $R \times R^n \to R^n$ is a compact almost automorphic function in $t$ uniformly in $x \in K$ for any compact subset $K$ of $R^n$, $n \in \mathbb{N}$, and $\mathbb{N}$ is the set of natural numbers; that is to say, $F$ is continuous, and for any given real sequence $\{\alpha_n\}$, there exists a subsequence $\{\alpha'_n\}$ and a function $G$ such that $\lim\limits_{n\to+\infty} F(t+\alpha'_n, x) = G(t, x)$ and $\lim\limits_{n\to+\infty} G(t-\alpha'_n, x) = F(t, x)$ hold uniformly on $I \times K$ for any compact subset $I$ of $R$. In [14], the authors extended the concept of a compact almost automorphic function in $t$ uniformly associated to the second argument and applied it in Banach space $X$ to prove the existence of the almost automorphic solutions, provided that the differential equation

$$x'(t) = f\left(t, x(t)\right)$$

admits at least a solution with a relatively compact range, where $f$: $R \times X \to X$ is compact and almost automorphic in $t$ uniformly in any compact subset of $X$, i.e., $f$ is continuous, and for the above subsequence $\{\alpha'_n\}$, there exists a function $g$ such that

$$\lim_{n\to+\infty} \sup_{x\in K} \|f(t+\alpha'_n, x) - g(t, x)\| = 0$$

and $\lim\limits_{n\to+\infty} \sup\limits_{x\in K} \|g(t-\alpha'_n, x) - f(t, x)\|$ for any $t \in R$. Based on the operator theory, Cieutat and Ezzinbi in [15] studied the following more general differential equation

$$x'(t) = Ax(t) + f\left(t, x(t)\right),$$

where $A$: $X \to X$ is an infinitesimal generator of a $C_0$-semigroup and $f$: $R \times X \to X$ is compact almost automorphic; moreover, the authors used the subvariant functional method to present that the unique minimal $K$-mild solution is compact almost automorphic. However, there are no studies to explore the existence of the compact almost automorphic mild solutions of stochastic differential equations by utilizing the subvariant functional method.

Inspired by the above discussion, it is significant to propose the concept of $p$-mean compact almost automorphic stochastic processes and apply it to the stochastic differential equation as follows:

$$dZ(t) = AZ(t) + H_1(t, Z(t))dt + H_2(t, Z(t))dW(t) + \int_{|y|_{\mathcal{B}}\geq 1} F(t, Z(t), y)N(dt, dy), \qquad (1.1)$$

where $A$ is an operator; $H_1$, $H_2$, and $F$ are stochastic processes; $W$ is a Brownian motion; $N$ is the Poisson measure of a Lévy process and its compensated Poisson measure denoted by $\tilde{N}$, which will be described concretely in the next section. Due to the fact that we introduce the concept of compact almost automorphic functions into random cases, therefore, it is essential of Burkholder–Davis–Gundy inequality, together with $C_0$-semigroup theory, Hölder inequality, and the Lebesgue dominated convergence theorem, under some suitable assumptions, we obtain that every $K$-mild solution of Eq (1.1) is uniformly continuous with the uniform continuity modulus $\zeta$: $[0, +\infty) \to [0, +\infty)$ satisfying $\lim_{\vartheta\to 0} \zeta(\vartheta) = 0$. Furthermore, based on the concept of compact almost automorphic stochastic processes and the famous Arzela–Ascoli theorem, we prove that the $K$-mild solution $Z(t + \rho'_n)$ is convergent for the real sequence $\{\rho_n\}_{n\in\mathbb{N}}$ uniformly on each compact subset of $R$ and $\{\rho'_n\}_{n\in\mathbb{N}} \subseteq \{\rho_n\}_{n\in\mathbb{N}}$, which lays a solid foundation for the conclusion that there exists a mild solution $Z$ of Eq (1.1) such that

$$\{Z(t) : t \in R\} \subset \overline{\{Z_0(t) : t \geq t_0\}},$$

where $Z_0(t)$ for $t \geq t_0$ is a bounded mild solution of Eq (1.1) and $\{Z_0(t) : t \geq t_0\}$ is relatively compact, obtained by using the Kuratowski measure of noncompactness. Further, according to the subvariant functional method, we give some sufficient conditions to make sure that there exists at least one minimal $K$-mild solution; further, if the minimal $K$-mild solution is unique, then it is a compact almost automorphic mild solution. In addition, we investigate an example to illustrate the main presented results.

An outline of this paper is as follows. In Section 2, we recall some preliminaries and introduce the concept of $p$-mean compact almost automorphic stochastic processes. In Section 3, we establish some properties of the $K$-mild solution of Eq (1.1). In Section 4, based on the subvariant functional method, we prove the existence of the compact almost automorphic mild solution. Finally, we give an suitable example to illustrate our results.

## 2. Preliminaries

Denote by $(\Omega, \mathcal{F}, P)$ a complete probability space endowed with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. Let $\mathcal{H}$, $\mathcal{B}$ be two real separable Hilbert spaces, and let $L^p(\mathcal{H})$ stands for the collection of all $p$-mean integrable $\mathcal{H}$-valued random variables for $p \geq 2$; this is a Banach space under the norm $\|X\|_{L^p} = (E\|X\|^p)^{\frac{1}{p}} < \infty$, where $E$ is the expectation defined on $(\Omega, \mathcal{F}, P)$. Assume $C_b(R, L^p(\mathcal{H}))$ and $C(R, L^p(\mathcal{H}))$ represent the set of all stochastically continuous bounded and stochastically continuous processes, respectively, from $R$ to $L^p(\mathcal{H})$, where $R$ represents the set of real numbers.

Let $\mathbb{K}$ be a separable Hilbert space, and let $q(t)$ in $\mathbb{K}$ be a stationary $\{\mathcal{F}_t\}$-adapted Poisson point process for $t \geq 0$. Define the counting random measure $N_q$ by

$$N_q((0, t], \mathbb{Z}) := \sum_{0\leq u\leq t} \chi_{\mathbb{Z}}(q(u))$$

for any $\mathbb{Z} \in B_\sigma(\mathbb{K})$, where $B_\sigma(\mathbb{K})$ stands for the set of all Borel $\sigma$-algebra of $\mathbb{K}$ and $\chi$ is a characteristic function, which is called the Poisson random measure with respect to the $q$. In addition, denote the compensated Poisson measure by $\tilde{N}(t, d\cdot) = N(t, d\cdot) - t\nu(d\cdot)$, where $\nu$ is a $\sigma$-finite Lévy measure.

Assume that $\mathcal{B} \in B_\sigma(\mathbb{K} - \{0\})$; throughout this paper, for the sake of calculation, denote by

$$b = \int_{|y|_{\mathcal{B}} \geq 1} \nu(dy) < \infty.$$

Next, we give some definitions and lemmas.

**Definition 2.1.** *A stochastically continuous stochastic process $g : R \to L^p(\mathcal{H})$ is said to be p-mean almost automorphic provided that for every real sequence $\{t_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{t'_n\}_{n\in\mathbb{N}}$ and a stochastic process $g_1 : R \to L^p(\mathcal{H})$ such that*

$$\lim_{n\to+\infty} E\|g(t+t'_n) - g_1(t)\|^p = 0, \tag{2.1}$$

$$\lim_{n\to+\infty} E\|g_1(t-t'_n) - g(t)\|^p = 0 \tag{2.2}$$

*hold for each $t \in R$. Denote by $AA(R, L^p(\mathcal{H}))$ the family of all such stochastic processes, which is a Banach space equipped with the norm $\|g\|_\infty = \sup_{t\in R} \|g(t)\|_{L^p}$. If the limits in (2.1)-(2.2) hold uniformly on any compact subset of $L^p(\mathcal{H})$, then g is called p-mean compact almost automorphic, and we denote by $AA^c(R, L^p(\mathcal{H}))$ the collection of all such stochastic processes.*

**Remark 2.1.** *From the definition of a compact almost automorphic stochastic process, it follows that*

$$AA^c(R, L^p(\mathcal{H})) \subset AA(R, L^p(\mathcal{H})) \subset C_b(R, L^p(\mathcal{H})).$$

**Definition 2.2.** *The stochastic process $g : R \times L^p(\mathcal{H}) \to L^p(\mathcal{H})$ is called p-mean almost automorphic in t uniformly associated to the second variable if*
*(i) $g \in C(R \times L^p(\mathcal{H}), L^p(\mathcal{H}))$;*
*(ii) for all compact subsets K of $L^p(\mathcal{H})$ and for every real sequence $\{t_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{t'_n\}_{n\in\mathbb{N}}$ and a stochastic process $g_1 : R \times L^p(\mathcal{H}) \to L^p(\mathcal{H})$ such that*

$$\lim_{n\to+\infty} \sup_{z\in K} E\|g(t+t'_n, z) - g_1(t, z)\|^p = 0,$$

$$\lim_{n\to+\infty} \sup_{z\in K} E\|g_1(t-t'_n, z) - g(t, z)\|^p = 0$$

*for any $t \in R$. Denote by $AA^c(R \times L^p(\mathcal{H}), L^p(\mathcal{H}))$ the family of all such stochastic processes.*

**Remark 2.2.** *From Definitions 2.1 and 2.2, it follows that $g \in AA^c(R \times L^p(\mathcal{H}), L^p(\mathcal{H}))$ if and only if for any $z \in K$, $g(\cdot, z) \in AA^c(R, L^p(\mathcal{H}))$ and for any $\varepsilon > 0$, there exists $\delta > 0$, and for any $z_1, z_2 \in K$, if $E\|z_1 - z_2\|^p < \delta$, then*

$$\sup_{t\in R} E\|g(t, z_1) - g(t, z_2)\|^p < \varepsilon.$$

**Definition 2.3.** *The stochastic process $f$: $R \times L^p(\mathcal{H}) \times \mathcal{B} \to L^p(\mathcal{H})$ is called Poisson p-mean almost automorphic in $t \in R$ uniformly associated to the second variable if*
*(i) $f$ is stochastically continuous;*
*(ii) for all compact subsets K of $L^p(\mathcal{H})$ and for every real sequence $\{t_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{t'_n\}_{n\in\mathbb{N}}$ and a stochastic process $f_1 : R \times L^p(\mathcal{H}) \times \mathcal{B} \to L^p(\mathcal{H})$ such that*

$$\lim_{n\to+\infty} \sup_{z\in K} \int_{|y|_{\mathcal{B}} \geq 1} E\|f(t+t'_n, z, y) - f_1(t, z, y)\|^p \nu(dy) = 0,$$

$$\lim_{n\to+\infty}\sup_{z\in K}\int_{|y|_{\mathcal{B}}\geq 1} E\|f_1(t-t_n',z,y)-f(t,z,y)\|^p\nu(dy)=0$$

*for any $t\in R$. Denote the set of all such functions by $PAA^c(R\times L^p(\mathcal{H})\times\mathcal{B},L^p(\mathcal{H}))$.*

**Remark 2.3.** *If $f\in PAA^c(R\times L^p(\mathcal{H})\times\mathcal{B},L^p(\mathcal{H}))$, it follows that for any $\varepsilon>0$, there exists $\delta>0$, for any $z_1,z_2\in K$, if $E\|z_1-z_2\|^p<\delta$, then*

$$\sup_{t\in R}\int_{|y|_{\mathcal{B}}\geq 1} E\|f(t,z_1,y)-f(t,z_2,y)\|^p\nu(dy)<\varepsilon.$$

## 3. Main results

In this section, let $I_0=[t_0,+\infty)$ or $I_0=R$. The following hypotheses will be required:

(I) $A$ is the infinitesimal generator of a $C_0$-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$.

(II) $\{\mathcal{T}(t)\}_{t\geq 0}$ is compact.

(III)$H_i\in AA^c(R\times L^p(\mathcal{H}),L^p(\mathcal{H}))$ and $F\in PAA^c(R\times L^p(\mathcal{H})\times\mathcal{B},L^p(\mathcal{H}))$.

Next, we will further consider the Eq (1.1), where $H_1$: $R\times L^p(\mathcal{H})\to L^p(\mathcal{H})$, $H_2$: $R\times L^p(\mathcal{H})\to L^p(\mathcal{H})$ and $F$: $R\times L^p(\mathcal{H})\times\mathcal{B}\to L^p(\mathcal{H})$ are $\mathcal{F}_t$-measurable. Assume that

$$\mathcal{K}_{H_i}=\sup_{t\in I_0}\sup_{Z\in K}E\|H_i(t,Z(t))\|^p<+\infty,\ \text{for } i=1,2, \tag{3.1}$$

$$\mathcal{K}_F=\sup_{t\in I_0}\sup_{Z\in K}\int_{|y|_{\mathcal{B}}\geq 1}E\|F(t,Z(t),y)\|^p\nu(dy)<+\infty. \tag{3.2}$$

If $Z\in C(I_0,L^p(\mathcal{H}))$ satisfies $Z(t)\in K$ for any $t\in I_0$ and

$$\begin{aligned}Z(t)=&\mathcal{T}(t-\tau)Z(\tau)+\int_\tau^t\mathcal{T}(t-m)H_1(m,Z(m))dm+\int_\tau^t\mathcal{T}(t-m)H_2(m,Z(m))dW(m)\\&+\int_\tau^t\mathcal{T}(t-m)\int_{|y|_{\mathcal{B}}\geq 1}F(m,Z(m),y)N(dm,dy),\quad\text{for } t\geq\tau,\end{aligned} \tag{3.3}$$

where $K$ is a compact subset of $L^p(\mathcal{H})$, then $Z$ is called a $K$-mild solution on $I_0$ of Eq (1.1).

**Remark 3.1.** *In the following statement, $p\geq 2$ is satisfied unless otherwise stated.*

**Lemma 3.1.** *Assume that (I), (3.1), and (3.2) hold; then the $K$-mild solution $Z$: $I_0\to L^p(\mathcal{H})$ of Eq (1.1) satisfies*

$$E\|Z(t)-Z(\tau)\|^p\leq 4^{p-1}\zeta(|t-\tau|), \tag{3.4}$$

*where $\zeta$: $[0,+\infty)\to[0,+\infty)$ and $\lim_{\vartheta\to 0}\zeta(\vartheta)=0$.*

*Proof.* Since $Z$ is a $K$-mild solution of Eq (1.1), therefore

$$\begin{aligned}\frac{E\|Z(t)-Z(\tau)\|^p}{4^{p-1}}\leq& E\|\mathcal{T}(t-\tau)Z(\tau)-Z(\tau)\|^p+E\left\|\int_\tau^t\mathcal{T}(t-m)H_1(m,Z(m))dm\right\|^p\\&+E\left\|\int_\tau^t\mathcal{T}(t-m)H_2(m,Z(m))dW(m)\right\|^p\end{aligned}$$

$$+ E\left\|\int_{\tau}^{t}\mathcal{T}(t-m)\int_{|y|_{\mathcal{B}}\geq 1}F(m,Z(m),y)N(dm,dy)\right\|^{p}$$
$$=\Phi_{t,\tau}(1)+\Phi_{t,\tau}(2)+\Phi_{t,\tau}(3)+\Phi_{t,\tau}(4).$$

From (I), it follows that there exist constants $\omega \geq 0$ and $M \geq 1$ satisfying

$$\|\mathcal{T}(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{3.5}$$

Define $\zeta$: $[0,+\infty) \to [0,+\infty)$ and

$$\zeta(\vartheta)=\begin{cases}\sup\limits_{u\in L^{p}(\mathcal{H})}E\|\mathcal{T}(\vartheta)u-u\|^{p}+\dfrac{M^{p}}{p\omega}\Big\{\Big[\mathcal{K}_{H_1}+(2b)^{p-1}\mathcal{K}_{F}\Big]\vartheta^{p-1} \\ \quad +C_{p}\Big(\mathcal{K}_{H_2}+2^{p-1}b^{\frac{p-2}{2}}\mathcal{K}_{F}\Big)\vartheta^{\frac{p-2}{2}}+2^{p-1}C_{p}\mathcal{K}_{F}\Big\}\Big(e^{p\omega\vartheta}-1\Big), & \text{for } p>2,\\ \sup\limits_{u\in L^{2}(\mathcal{H})}E\|\mathcal{T}(\vartheta)u-u\|^{2}+\dfrac{M^{2}}{2\omega}\left[(\mathcal{K}_{H_1}+2b\mathcal{K}_{F})\vartheta+\mathcal{K}_{H_2}+2\mathcal{K}_{F}\right]\Big(e^{2\omega\vartheta}-1\Big), & \text{for } p=2,\end{cases}$$

where $C_p > 0$ is constant related to $p$.

Because of the strong continuity of the $C_0$-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, the $\mathcal{T}(\cdot)u : [0,+\infty) \to L^p(\mathcal{H})$ is continuous for each $u \in L^p(\mathcal{H})$, then $\lim\limits_{\vartheta\to 0}\mathcal{T}(\vartheta)u = u$, for $u \in K$. Combined with Banach–Steinhaus' theorem, one deduces

$$\sup_{u\in K}E\|\mathcal{T}(\vartheta)u-u\|^{p} \to 0 \text{ as } \vartheta \to 0.$$

In view of $\mathcal{K}_F < +\infty$ and $\mathcal{K}_{H_i} < +\infty$ for $i = 1, 2$, therefore, $\zeta(\vartheta) \to 0$ as $\vartheta \to 0$. Next, it remains to prove (3.4) holds for $\vartheta = |t-\tau|$.

Based on (3.1), (3.2), and (3.5), Hölder inequality, and Burkholder–Davis–Gundy inequality, it yields for $p > 2$ that

$$\begin{aligned}\Phi_{t,\tau}(2)+\Phi_{t,\tau}(3) \leq & M^{p}E\left(\int_{\tau}^{t}e^{\omega(t-m)}\|H_1(m,Z(m))\|dm\right)^{p}\\ & + M^{p}C_{p}E\left(\int_{\tau}^{t}e^{2\omega(t-m)}\|H_2(m,Z(m))\|^{2}dm\right)^{\frac{p}{2}}\\ \leq & M^{p}\left(\int_{\tau}^{t}e^{\omega(t-m)}dm\right)^{p-1}\int_{\tau}^{t}e^{\omega(t-m)}E\|H_1(m,Z(m))\|^{p}dm\\ & + M^{p}C_{p}\left(\int_{\tau}^{t}e^{2\omega(t-m)}dm\right)^{\frac{p-2}{2}}\int_{\tau}^{t}e^{2\omega(t-m)}E\|H_2(m,Z(m))\|^{p}dm\\ \leq & M^{p}\mathcal{K}_{H_1}\left(\int_{\tau}^{t}e^{\omega(t-m)}dm\right)^{p}+M^{p}C_{p}\mathcal{K}_{H_2}\left(\int_{\tau}^{t}e^{2\omega(t-m)}dm\right)^{\frac{p}{2}}\\ \leq & M^{p}\left[\mathcal{K}_{H_1}(t-\tau)^{p-1}+C_{p}\mathcal{K}_{H_2}(t-\tau)^{\frac{p-2}{2}}\right]\int_{\tau}^{t}e^{p\omega(t-m)}dm\end{aligned} \tag{3.6}$$

and

$$
\begin{aligned}
\Phi_{t,\tau}(4) \leq & 2^{p-1}E\left\|\int_\tau^t \mathcal{T}(t-m)\int_{|y|_\mathcal{B}\geq 1} F(m,Z(m),y)\tilde{N}(dm,dy)\right\|^p \\
& + 2^{p-1}E\left\|\int_\tau^t \mathcal{T}(t-m)\int_{|y|_\mathcal{B}\geq 1} F(m,Z(m),y)\nu(dy)dm\right\|^p \\
\leq & 2^{p-1}M^pC_pE\left(\int_\tau^t e^{2\omega(t-m)}\int_{|y|_\mathcal{B}\geq 1}\|F(m,Z(m),y)\|^2\nu(dy)dm\right)^{\frac{p}{2}} \\
& + 2^{p-1}M^pC_p\int_\tau^t e^{p\omega(t-m)}\int_{|y|_\mathcal{B}\geq 1}E\|F(m,Z(m),y)\|^p\nu(dy)dm \\
& + 2^{p-1}M^p\left(\int_\tau^t e^{\omega(t-m)}dm\right)^{p-1}\int_\tau^t e^{\omega(t-m)}E\left(\int_{|y|_\mathcal{B}\geq 1}\|F(m,Z(m),y)\|\nu(dy)\right)^p dm \\
\leq & 2^{p-1}b^{\frac{p-2}{2}}M^pC_p\left(\int_\tau^t e^{2\omega(t-m)}dm\right)^{\frac{p-2}{2}}\int_\tau^t e^{2\omega(t-m)}\int_{|y|_\mathcal{B}\geq 1}E\|F(m,Z(m),y)\|^p\nu(dy)dm \\
& + 2^{p-1}M^pC_p\int_\tau^t e^{p\omega(t-m)}\int_{|y|_\mathcal{B}\geq 1}E\|F(m,Z(m),y)\|^p\nu(dy)dm \\
& + (2b)^{p-1}M^p\left(\int_\tau^t e^{\omega(t-m)}dm\right)^{p-1}\int_\tau^t e^{\omega(t-m)}\int_{|y|_\mathcal{B}\geq 1}E\|F(m,Z(m),y)\|^p\nu(dy)dm \\
\leq & 2^{p-1}M^p\mathcal{K}_F\left[b^{\frac{p-2}{2}}C_p\left(\int_\tau^t e^{2\omega(t-m)}dm\right)^{\frac{p}{2}} + C_p\int_\tau^t e^{p\omega(t-m)}dm + b^{p-1}\left(\int_\tau^t e^{\omega(t-m)}dm\right)^p\right] \\
\leq & 2^{p-1}M^p\mathcal{K}_F\left[b^{\frac{p-2}{2}}C_p(t-\tau)^{\frac{p-2}{2}} + C_p + b^{p-1}(t-\tau)^{p-1}\right]\int_\tau^t e^{p\omega(t-m)}dm. \qquad (3.7)
\end{aligned}
$$

Similarly, we calculate for $p = 2$ that

$$
\begin{aligned}
& \Phi_{t,\tau}(2) + \Phi_{t,\tau}(3) \\
\leq & M^2\int_\tau^t e^{\omega(t-m)}dm\int_\tau^t e^{\omega(t-m)}E\|H_1(m,Z(m))\|^2dm \\
& + M^2\int_\tau^t e^{2\omega(t-m)}E\|H_2(m,Z(m))\|^2dm \\
\leq & M^2\left[\mathcal{K}_{H_1}(t-\tau) + \mathcal{K}_{H_2}\right]\int_\tau^t e^{2\omega(t-m)}dm \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_{t,\tau}(4) \leq & 2M^2\int_\tau^t e^{2\omega(t-m)}\int_{|y|_\mathcal{B}\geq 1}E\|F(m,Z(m),y)\|^2\nu(dy)dm \\
& + 2M^2\int_\tau^t e^{\omega(t-m)}dm\int_\tau^t e^{\omega(t-m)}E\left(\int_{|y|_\mathcal{B}\geq 1}\|F(m,Z(m),y)\|\nu(dy)\right)^2dm \\
\leq & 2M^2\mathcal{K}_F\left[1 + b(t-\tau)\right]\int_\tau^t e^{2\omega(t-m)}dm. \qquad (3.9)
\end{aligned}
$$

According to (3.6)–(3.9) and the definition of $\zeta$, we obtain

$$\begin{aligned}
&\Phi_{t,\tau}(1) + \Phi_{t,\tau}(2) + \Phi_{t,\tau}(3) + \Phi_{t,\tau}(4)\\
\le& \sup_{u\in K} E\|\mathcal{T}(t-\tau)u - u\|^p + M^p\left\{\left[\mathcal{K}_{H_1}(t-\tau)^{p-1} + C_p\mathcal{K}_{H_2}(t-\tau)^{\frac{p-2}{2}}\right]\right.\\
&\left. + 2^{p-1}\mathcal{K}_F\left[b^{\frac{p-2}{2}}C_p(t-\tau)^{\frac{p-2}{2}} + C_p + b^{p-1}(t-\tau)^{p-1}\right]\right\}\int_\tau^t e^{p\omega(t-m)}dm\\
\le& \sup_{u\in K} E\|\mathcal{T}(|t-\tau|)u - u\|^p + \frac{M^p}{p\omega}\left\{\left[\mathcal{K}_{H_1} + (2b)^{p-1}\mathcal{K}_F\right]|t-\tau|^{p-1}\right.\\
&\left. + C_p\left(\mathcal{K}_{H_2} + 2^{p-1}b^{\frac{p-2}{2}}\mathcal{K}_F\right)|t-\tau|^{\frac{p-2}{2}} + 2^{p-1}C_p\mathcal{K}_F\right\}\left[e^{p\omega|t-\tau|} - 1\right]\\
=&\zeta(|t-\tau|), \text{ for } p > 2
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_{t,\tau}(1) + \Phi_{t,\tau}(2) + \Phi_{t,\tau}(3) + \Phi_{t,\tau}(4)\\
\le& \sup_{u\in K} E\|\mathcal{T}(|t-\tau|)u - u\|^2 + \frac{M^2}{2\omega}\left[(\mathcal{K}_{H_1} + 2b\mathcal{K}_F)|t-\tau| + \mathcal{K}_{H_2} + 2\mathcal{K}_F\right]\left(e^{2\omega|t-\tau|} - 1\right)\\
=&\zeta(|t-\tau|), \text{ for } p = 2,
\end{aligned}$$

which indicates (3.4) holds. □

**Lemma 3.2.** *Assume that (I), (3.1), and (3.2) hold. Let $Z$ be a $K$-mild solution of Eq (1.1) and stochastic processes $H_i(t,\cdot)$ and $F(t,\cdot,y)$ belong to $C(K, L^p(\mathcal{H}))$ for $i = 1,2$. If for any real sequence $\{\rho_n\}_{n\in\mathbb{N}}$, there exists $\{\rho'_n\}_{n\in\mathbb{N}} \subseteq \{\rho_n\}_{n\in\mathbb{N}}$ and stochastic processes $\tilde{G}_i$: $R\times K \to L^p(\mathcal{H})$, $\tilde{H}$: $R\times K\times\mathcal{B} \to L^p(\mathcal{H})$ satisfying*

$$\lim_{n\to+\infty}\sup_{Z\in K} E\|H_i(t+\rho'_n, Z) - \tilde{G}_i(t,Z)\|^p = 0, \tag{3.10}$$

$$\lim_{n\to+\infty}\sup_{Z\in K}\int_{|y|_{\mathcal{B}}\ge 1} E\|F(t+\rho'_n, Z, y) - \tilde{H}(t,Z,y)\|^p \nu(dy) = 0 \tag{3.11}$$

*for $t\in R$, then*

$$\lim_{n\to+\infty} E\|Z(t+\rho'_n) - Z^*(t)\|^p = 0 \tag{3.12}$$

*uniformly on each compact subset of $R$, where $Z^*$ is a $K$-mild solution of*

$$dZ^*(t) = AZ^*(t) + \tilde{G}_1(t, Z^*(t))dt + \tilde{G}_2(t, Z^*(t))dW(t) + \int_{|y|_{\mathcal{B}}\ge 1}\tilde{H}(t, Z^*(t), y)N(dt, dy). \tag{3.13}$$

*Proof.* Since $Z$ is a $K$-mild solution of Eq (1.1), then $Z\in C(R, L^p(\mathcal{H}))$ and $Z(t)\in K$ for $t\in R$. Based on Lemma 3.1, it follows that

$$E\|Z(t) - Z(\tau)\|^p \le 4^{p-1}\zeta(|t-\tau|) \text{ for } t,\tau\in R,$$

where the function $\zeta$ satisfies $\zeta(\vartheta)\to 0$ as $\vartheta\to 0$. This implies that the $K$-mild solution $Z$ is uniformly continuous. For the above real sequence $\{\rho'_n\}_{n\in\mathbb{N}}$, it is not difficult to obtain that $\{Z(t+\rho'_n)\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $R$. By utilizing Arzela–Ascoli theorem, it has $\{Z(t+\rho_n')\}_{n\in\mathbb{N}}$ as relatively compact; therefore, there exists a stochastic process $Z^*$ such that (3.12) holds.

Next, we will show $Z^*$ is a $K$-mild solution of (3.13). Based on the fact that $Z$ is a $K$-mild solution of Eq (1.1), obviously, $Z(t+\rho_n') \in K$, together with (3.12), it obtains $Z^* \in K$. According to the expression of $Z$ in Eq (3.3), then

$$\begin{aligned}
Z(t+\rho_n') =& \mathcal{T}(t-\kappa)Z(\kappa+\rho_n') + \int_{\kappa+\rho_n'}^{t+\rho_n'} \mathcal{T}(t+\rho_n'-m)H_1(m, Z(m))dm \\
&+ \int_{\kappa+\rho_n'}^{t+\rho_n'} \mathcal{T}(t+\rho_n'-m)H_2(m, Z(m))dW(m) \\
&+ \int_{\kappa+\rho_n'}^{t+\rho_n'} \mathcal{T}(t+\rho_n'-m) \int_{|y|_{\mathcal{B}}\geq 1} F(m, Z(m), y)N(dm, dy) \\
=& \mathcal{T}(t-\kappa)Z(\kappa+\rho_n') + \int_{\kappa}^{t} \mathcal{T}(t-r)H_1(r+\rho_n', Z(r+\rho_n'))dr \\
&+ \int_{\kappa}^{t} \mathcal{T}(t-r)H_2(r+\rho_n', Z(r+\rho_n'))dW(r+\rho_n') \\
&+ \int_{\kappa}^{t} \mathcal{T}(t-r) \int_{|y|_{\mathcal{B}}\geq 1} F(r+\rho_n', Z(r+\rho_n'), y)N(d(r+\rho_n'), dy)
\end{aligned}$$

for $t \geq \kappa$ and $t, \kappa \in R$.

Define $\tilde{Z}$: $R \to L^p(\mathcal{H})$ such that

$$\begin{aligned}
\tilde{Z}(t) =& \mathcal{T}(t-\kappa)Z^*(\kappa) + \int_{\kappa}^{t} \mathcal{T}(t-r)\tilde{G}_1(r, Z^*(r))dr \\
&+ \int_{\kappa}^{t} \mathcal{T}(t-r)\tilde{G}_2(r, Z^*(r))dW(r) \\
&+ \int_{\kappa}^{t} \mathcal{T}(t-r) \int_{|y|_{\mathcal{B}}\geq 1} \tilde{H}(r, Z^*(r), y)N(dr, dy), \quad \text{for } t \geq \kappa.
\end{aligned}$$

From the expressions of stochastic processes $Z(t+\rho_n')$ and $\tilde{Z}(t)$, it yields

$$E\|Z(t+\rho_n') - \tilde{Z}(t)\|^p \leq 4^{p-1}\left\{E\left\|\mathcal{T}(t-\kappa)[Z(\kappa+\rho_n') - Z^*(\kappa)]\right\|^p + \Phi_{H_1,\tilde{G}_1}(t) + \Phi_{H_2,\tilde{G}_2}(t) + \Phi_{F,\tilde{H}}(t)\right\},$$

where

$$\begin{aligned}
\Phi_{H_1,\tilde{G}_1}(t) =& E\left\|\int_{\kappa}^{t} \mathcal{T}(t-r)\left[H_1(r+\rho_n', Z(r+\rho_n')) - \tilde{G}_1(r, Z^*(r))\right]dr\right\|^p, \\
\Phi_{H_2,\tilde{G}_2}(t) =& E\left\|\int_{\kappa}^{t} \mathcal{T}(t-r)\left[H_2(r+\rho_n', Z(r+\rho_n')) - \tilde{G}_2(r, Z^*(r))\right]dW_0(r)\right\|^p, \\
\Phi_{F,\tilde{H}}(t) =& E\left\|\int_{\kappa}^{t} \mathcal{T}(t-r)\int_{|y|_{\mathcal{B}}\geq 1}\left[F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\right]N_0(dr, dy)\right\|^p,
\end{aligned}$$

where $W_0(r) = W(r+\rho_n') - W(\rho_n')$ and $N_0(r, y) = N(r+\rho_n', y) - N(\rho_n', y)$, then $W_0$ and $N_0$ have the same distribution as $W$ and $N$, respectively. By using (I), it follows that (3.5) holds. Taking an analogous method to the calculation of (3.6)-(3.7), it follows for $p > 2$ that

$$\begin{aligned}
\Phi_{H_1,\tilde{G}_1}(t) \leq& \Delta_{1,p} \int_{\kappa}^{t} e^{\omega(t-r)} E\|H_1(r+\rho_n', Z(r+\rho_n')) - \tilde{G}_1(r, Z^*(r))\|^p dr,\\
\Phi_{H_2,\tilde{G}_2}(t) \leq& \Delta_{2,p} \int_{\kappa}^{t} e^{2\omega(t-r)} E\|H_2(r+\rho_n', Z(r+\rho_n')) - \tilde{G}_2(r, Z^*(r))\|^p dr,\\
\Phi_{F,\tilde{H}}(t) \leq& \Delta_{3,p} \int_{\kappa}^{t} e^{2\omega(t-r)} \int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^p \nu(dy)dr\\
&+ \Delta_{4,p} \int_{\kappa}^{t} e^{p\omega(t-r)} \int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^p \nu(dy)dr\\
&+ \Delta_{5,p} \int_{\kappa}^{t} e^{\omega(t-r)} \int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^p \nu(dy)dr,
\end{aligned}$$

where

$$\Delta_{1,p} = M^p \Re_{1,p}, \quad \Delta_{2,p} = M^p C_p \Re_{2,p}, \quad \Delta_{3,p} = 2^{p-1} b^{\frac{p-2}{2}} M^p C_p \Re_{2,p}, \quad \Delta_{4,p} = 2^{p-1} M^p C_p,$$
$$\Delta_{5,p} = (2b)^{p-1} M^p \Re_{1,p}, \quad \Re_{1,p} = \left[\frac{e^{\omega(t-\kappa)}-1}{\omega}\right]^{p-1}, \quad \Re_{2,p} = \left[\frac{e^{2\omega(t-\kappa)}-1}{2\omega}\right]^{\frac{p-2}{2}}.$$

Similar to the estimate of (3.8)-(3.9), it obtains for $p = 2$ that

$$\begin{aligned}
\Phi_{H_1,\tilde{G}_1}(t) \leq& M^2 \left[\frac{e^{\omega(t-\kappa)}-1}{\omega}\right] \int_{\kappa}^{t} e^{\omega(t-r)} E\|H_1(r+\rho_n', Z(r+\rho_n')) - \tilde{G}_1(r, Z^*(r))\|^2 dr,\\
\Phi_{H_2,\tilde{G}_2}(t) \leq& M^2 \int_{\kappa}^{t} e^{2\omega(t-r)} E\|H_2(r+\rho_n', Z(r+\rho_n')) - \tilde{G}_2(r, Z^*(r))\|^2 dr,\\
\Phi_{F,\tilde{H}}(t) \leq& 2M^2 \int_{\kappa}^{t} e^{2\omega(t-r)} \int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^2 \nu(dy)dr\\
&+ 2bM^2 \left[\frac{e^{\omega(t-\kappa)}-1}{\omega}\right] \int_{\kappa}^{t} e^{\omega(t-r)} \int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^2 \nu(dy)dr.
\end{aligned}$$

To complete the proof, we only need to prove

$$E\|Z(t+\rho_n') - \tilde{Z}(t)\|^p \to 0 \text{ as } n \to +\infty. \tag{3.14}$$

Since (3.12) holds, therefore

$$E\left\|\mathcal{T}(t-\kappa)[Z(\kappa+\rho_n') - Z^*(\kappa)]\right\|^p \to 0 \text{ as } n \to +\infty. \tag{3.15}$$

Considering $F(t,\cdot,y)$ belongs to $C(K, L^p(\mathcal{H}))$, then $\tilde{H}(t,\cdot,y)$ also belongs to $C(K, L^p(\mathcal{H}))$ logically. Choosing

$$\Delta_{max} = \max\{\Delta_{3,p}, \Delta_{4,p}, \Delta_{5,p}, 2M^2, 2bM^2\},$$

based on (3.11)-(3.12) and the Lebesgue dominated convergence theorem, it calculates from the valuation of $\Phi_{F,\tilde{H}}(t)$ for the case of $p > 2$ and $p = 2$ that

$$\begin{aligned}\Phi_{F,\tilde{H}}(t) \leq & 3\Delta_{max}\int_{\kappa}^{t} e^{p\omega(t-r)}\int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^p \nu(dy)dr \\ \leq & 3\cdot 2^{p-1}\Delta_{max}\left\{\int_{\kappa}^{t} e^{p\omega(t-r)}\sup_{Z\in K}\int_{|y|_{\mathcal{B}}\geq 1} E\|F(r+\rho_n', Z, y) - \tilde{H}(r, Z, y)\|^p \nu(dy)dr \right. \\ & \left. + \int_{\kappa}^{t} e^{p\omega(t-r)}\int_{|y|_{\mathcal{B}}\geq 1} E\|\tilde{H}(r, Z(r+\rho_n'), y) - \tilde{H}(r, Z^*(r), y)\|^p \nu(dy)dr\right\} \\ \rightarrow & 0, \text{ as } n\rightarrow +\infty. \end{aligned} \tag{3.16}$$

Similarly, it has $\Phi_{H_1,\tilde{G}_1}(t) + \Phi_{H_2,\tilde{G}_2}(t) \rightarrow 0$ as $n \rightarrow +\infty$. Together with (3.15)-(3.16), it follows that (3.14) holds. Further, (3.12), (3.14), and the uniqueness of the limits yield that $Z^*(t) = \tilde{Z}(t)$ for $t \in R$, hence,

$$\begin{aligned}Z^*(t) = & \mathcal{T}(t-\kappa)Z^*(\kappa) + \int_{\kappa}^{t}\mathcal{T}(t-r)\tilde{G}_1(r, Z^*(r))dr + \int_{\kappa}^{t}\mathcal{T}(t-r)\tilde{G}_2(r, Z^*(r))dW(r) \\ & + \int_{\kappa}^{t}\mathcal{T}(t-r)\int_{|y|_{\mathcal{B}}\geq 1}\tilde{H}(r, Z^*(r), y)N(dr, dy), \quad \text{for } t\geq\kappa,\end{aligned}$$

which implies that $Z^*(t)$ is a $K$-mild solution of (3.13) based on $Z^* \in K$. □

**Lemma 3.3.** *Assume that (I), (II), (3.1), and (3.2) hold. Let $Z_0(t)$ for $t \geq t_0$ be a bounded mild solution of Eq (1.1); then the set $\{Z_0(t) : t \geq t_0\} \subset L^p(\mathcal{H})$ is relatively compact.*

*Proof.* For a sufficiently small $\varsigma \in (0, 0.1)$, it follows for $t \geq t_0 + 0.1$ that

$$\begin{aligned}Z_0(t) = & \mathcal{T}(\varsigma)Z_0(t-\varsigma) + \int_{t-\varsigma}^{t}\mathcal{T}(t-\tau)H_1(\tau, Z_0(\tau))d\tau + \int_{t-\varsigma}^{t}\mathcal{T}(t-\tau)H_2(\tau, Z_0(\tau))dW(\tau) \\ & + \int_{t-\varsigma}^{t}\mathcal{T}(t-\tau)\int_{|y|_{\mathcal{B}}\geq 1}F(\tau, Z_0(\tau), y)N(d\tau, dy) \\ = & \Lambda_{Z_0}(t,\varsigma) + \Lambda_{H_1,Z_0}(t,\varsigma) + \Lambda_{H_2,Z_0}(t,\varsigma) + \Lambda_{F,Z_0}(t,\varsigma).\end{aligned}$$

Based on (II), we obtain $\Lambda_{Z_0}(t,\varsigma) \in \mathcal{T}(\varsigma)\bar{D}(0, \|Z_0\|_\infty)$, and $\mathcal{T}(\varsigma)\bar{D}(0, \|Z_0\|_\infty)$ is a relatively compact set of $L^p(\mathcal{H})$, where

$$\|Z_0\|_\infty = \sup_{t\in R} E\|Z_0(t)\|^p < +\infty$$

and $\bar{D}(0, \|Z_0\|_\infty)$ is the closed ball with the center 0 and radius $\|Z_0\|_\infty$.

From (I), it follows (3.5) holds. Based on (3.2), Hölder and Burkholder–Davis–Gundy inequality, it yields

$$\begin{aligned}& E\|\Lambda_{F,Z_0}(t,\varsigma)\|^p \\ \leq & 2^{p-1}M^pC_pE\left(\int_{t-\varsigma}^{t} e^{2\omega(t-\tau)}\int_{|y|_{\mathcal{B}}\geq 1}\|F(\tau, Z_0(\tau), y)\|^2\nu(dy)d\tau\right)^{\frac{p}{2}} \\ & + 2^{p-1}M^pC_p\int_{t-\varsigma}^{t} e^{p\omega(t-\tau)}\int_{|y|_{\mathcal{B}}\geq 1}E\|F(\tau, Z_0(\tau), y)\|^p\nu(dy)d\tau\end{aligned}$$

$$
\begin{aligned}
&+2^{p-1}M^p\left(\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2(p-1)}(t-\tau)}d\tau\right)^{p-1}\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2}(t-\tau)}E\left(\int_{|y|_{\mathcal{B}}\geq 1}\|F(\tau,Z_0(\tau),y)\|\nu(dy)\right)^{p}d\tau\\
\leq&2^{p-1}b^{\frac{p-2}{2}}M^pC_p\left(\int_{t-\varsigma}^{t}e^{\frac{p\omega}{p-2}(t-\tau)}d\tau\right)^{\frac{p-2}{2}}\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2}(t-\tau)}\int_{|y|_{\mathcal{B}}\geq 1}E\|F(\tau,Z_0(\tau),y)\|^p\nu(dy)d\tau\\
&+2^{p-1}M^pC_p\int_{t-\varsigma}^{t}e^{p\omega(t-\tau)}\int_{|y|_{\mathcal{B}}\geq 1}E\|F(\tau,Z_0(\tau),y)\|^p\nu(dy)d\tau\\
&+(2b)^{p-1}M^p\left(\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2(p-1)}(t-\tau)}d\tau\right)^{p-1}\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2}(t-\tau)}\int_{|y|_{\mathcal{B}}\geq 1}E\|F(\tau,Z_0(\tau),y)\|^p\nu(dy)d\tau\\
\leq&\frac{2^pM^p\mathcal{K}_F}{p\omega}\left\{C_p\left[\frac{b(p-2)}{p\omega}\left(e^{\frac{p\omega\varsigma}{p-2}}-1\right)\right]^{\frac{p-2}{2}}\left(e^{\frac{p\omega\varsigma}{2}}-1\right)+\frac{C_p}{2}\left(e^{p\omega\varsigma}-1\right)\right.\\
&\left.+\left[\frac{2b(p-1)}{p\omega}\left(e^{\frac{p\omega\varsigma}{2(p-1)}}-1\right)\right]^{p-1}\left(e^{\frac{p\omega\varsigma}{2}}-1\right)\right\},\ \text{for } p>2. \qquad (3.17)
\end{aligned}
$$

Similarly, it has

$$
\begin{aligned}
E\|\Lambda_{H_2,Z_0}(t,\varsigma)\|^p\leq&M^pC_pE\left(\int_{t-\varsigma}^{t}e^{2\omega(t-\tau)}\|H_2(\tau,Z_0(\tau))\|^2d\tau\right)^{\frac{p}{2}}\\
\leq&M^pC_p\left(\int_{t-\varsigma}^{t}e^{\frac{p\omega}{p-2}(t-\tau)}d\tau\right)^{\frac{p-2}{2}}\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2}(t-\tau)}E\|H_2(\tau,Z_0(\tau))\|^pd\tau\\
\leq&\frac{2M^pC_p\mathcal{K}_{H_2}}{p\omega}\left[\frac{p-2}{p\omega}\left(e^{\frac{p\omega\varsigma}{p-2}}-1\right)\right]^{\frac{p-2}{2}}\left(e^{\frac{p\omega\varsigma}{2}}-1\right),\ \text{for } p>2 \qquad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
E\|\Lambda_{H_1,Z_0}(t,\varsigma)\|^p\leq&M^p\left(\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2(p-1)}(t-\tau)}d\tau\right)^{p-1}\int_{t-\varsigma}^{t}e^{\frac{p\omega}{2}(t-\tau)}E\|H_1(\tau,Z_0(\tau))\|^pd\tau\\
\leq&\frac{2M^p\mathcal{K}_{H_1}}{p\omega}\left[\frac{2(p-1)}{p\omega}\left(e^{\frac{p\omega\varsigma}{2(p-1)}}-1\right)\right]^{p-1}\left(e^{\frac{p\omega\varsigma}{2}}-1\right),\ \text{for } p>2. \qquad (3.19)
\end{aligned}
$$

For the case of $p=2$, it follows that

$$
E\|\Lambda_{H_1,Z_0}(t,\varsigma)\|^2\leq\frac{M^2\mathcal{K}_{H_1}}{\omega^2}\left(e^{\omega\varsigma}-1\right)^2, \qquad (3.20)
$$

$$
E\|\Lambda_{H_2,Z_0}(t,\varsigma)\|^2\leq\frac{M^2\mathcal{K}_{H_2}}{2\omega}\left(e^{2\omega\varsigma}-1\right), \qquad (3.21)
$$

$$
E\|\Lambda_{F,Z_0}(t,\varsigma)\|^2\leq\frac{M^2\mathcal{K}_F}{\omega}\left(e^{2\omega\varsigma}-1\right)+\frac{2bM^2\mathcal{K}_F}{\omega^2}\left(e^{\omega\varsigma}-1\right)^2. \qquad (3.22)
$$

Define

$$
d_{H_1}(\varsigma)=\max\left\{\frac{2M^p\mathcal{K}_{H_1}}{p\omega}\left[\frac{2(p-1)}{p\omega}\left(e^{\frac{p\omega\varsigma}{2(p-1)}}-1\right)\right]^{p-1}\left(e^{\frac{p\omega\varsigma}{2}}-1\right),\ \frac{M^2\mathcal{K}_{H_1}}{\omega^2}\left(e^{\omega\varsigma}-1\right)^2\right\},
$$

$$d_{H_2}(\varsigma) = \max\left\{\frac{2M^pC_p\mathcal{K}_{H_2}}{p\omega}\left[\frac{p-2}{p\omega}\left(e^{\frac{p\omega\varsigma}{p-2}}-1\right)\right]^{\frac{p-2}{2}}\left(e^{\frac{p\omega\varsigma}{2}}-1\right),\ \frac{M^2\mathcal{K}_{H_2}}{2\omega}\left(e^{2\omega\varsigma}-1\right)\right\},$$

$$d_F(\varsigma) = \max\left\{\frac{2^pM^p\mathcal{K}_F}{p\omega}\left[C_p\left[\frac{b(p-2)}{p\omega}\left(e^{\frac{p\omega\varsigma}{p-2}}-1\right)\right]^{\frac{p-2}{2}}\left(e^{\frac{p\omega\varsigma}{2}}-1\right)+\frac{C_p}{2}\left(e^{p\omega\varsigma}-1\right)\right.\right.$$
$$\left.\left.+\left[\frac{2b(p-1)}{p\omega}\left(e^{\frac{p\omega\varsigma}{2(p-1)}}-1\right)\right]^{p-1}\left(e^{\frac{p\omega\varsigma}{2}}-1\right)\right],\ \frac{M^2\mathcal{K}_F}{\omega}\left(e^{2\omega\varsigma}-1\right)+\frac{2bM^2\mathcal{K}_F}{\omega^2}\left(e^{\omega\varsigma}-1\right)^2\right\},$$

from (3.17)–(3.22), it yields that $\Lambda_{H_1,Z_0}(t,\varsigma) \in \bar{D}(0, d_{H_1}(\varsigma))$, $\Lambda_{H_2,Z_0}(t,\varsigma) \in \bar{D}(0, d_{H_2}(\varsigma))$ and $\Lambda_{F,Z_0}(t,\varsigma) \in \bar{D}(0, d_F(\varsigma))$. In addition, it obtains

$$\begin{aligned}&\{Z_0(t) : t \geq t_0\}\\ \subset&\{Z_0(t) : t_0 \leq t \leq t_0 + 0.1\} \cup \{Z_0(t) : t \geq t_0 + 0.1\}\\ \subset&\{Z_0(t) : t_0 \leq t \leq t_0 + 0.1\} \cup \mathcal{T}(\varsigma)\bar{D}(0, \|Z_0\|_\infty) \cup \bar{D}(0, d_{H_1}(\varsigma)) \cup \bar{D}(0, d_{H_2}(\varsigma)) \cup \bar{D}(0, d_F(\varsigma)).\end{aligned}$$

By using the Kuratowski measure of noncompactness $\gamma(\cdot)$ for any bounded set in $L^p(\mathcal{H})$, then

$$\begin{aligned}\gamma(\{Z_0(t) : t \geq t_0\}) \leq&\gamma(\{Z_0(t) : t_0 \leq t \leq t_0 + 0.1\}) + \gamma\left(\mathcal{T}(\varsigma)\bar{D}(0, \|Z_0\|_\infty)\right)\\ &+ \gamma\left(\bar{D}(0, d_{H_1}(\varsigma))\right) + \gamma\left(\bar{D}(0, d_{H_2}(\varsigma))\right) + \gamma\left(\bar{D}(0, d_F(\varsigma))\right).\end{aligned}$$

In view of the relative compactness of $\{Z_0(t) : t_0 \leq t \leq t_0 + 0.1\}$ and $\mathcal{T}(\varsigma)\bar{D}(0, \|Z_0\|_\infty)$, therefore,

$$\gamma(\{Z_0(t) : t_0 \leq t \leq t_0 + 0.1\}) = \gamma\left(\mathcal{T}(\varsigma)\bar{D}(0, \|Z_0\|_\infty)\right) = 0$$

and

$$\begin{aligned}\gamma(\{Z_0(t) : t \geq t_0\}) \leq&\gamma\left(\bar{D}(0, d_{H_1}(\varsigma))\right) + \gamma\left(\bar{D}(0, d_{H_2}(\varsigma))\right) + \gamma\left(\bar{D}(0, d_F(\varsigma))\right)\\ \leq&2\left[d_{H_1}(\varsigma) + d_{H_2}(\varsigma) + d_F(\varsigma)\right].\end{aligned}$$

From (3.1)-(3.2), then $\mathcal{K}_{H_i} < +\infty$ and $\mathcal{K}_F < +\infty$ for $i = 1, 2$; therefore, $d_{H_1}(\varsigma) \to 0$, $d_{H_2}(\varsigma) \to 0$, and $d_F(\varsigma) \to 0$ as $\varsigma \to 0$. Further, it follows

$$\gamma(\{Z_0(t) : t \geq t_0\}) = 0,$$

which indicates that $\{Z_0(t) : t \geq t_0\}$ is relatively compact by utilizing the properties of the Kuratowski measure of noncompactness. □

**Lemma 3.4.** *Assume that (I)–(III) hold. Let $Z_0(t)$ for $t \geq t_0$ be a bounded mild solution of Eq (1.1); then there exists a mild solution $Z$ of Eq (1.1) such that $\{Z(t) : t \in R\} \subset \overline{\{Z_0(t) : t \geq t_0\}}$.*

*Proof.* From (III), it follows that (3.1)-(3.2) are satisfied. According to the conclusion of Lemma 3.3, then $\{Z_0(t) : t \geq t_0\}$ is relatively compact and $K = \overline{\{Z_0(t) : t \geq t_0\}}$ is a compact set in $L^p(\mathcal{H})$. Obviously, $Z_0(t) \in K$ for all $t \geq t_0$; furthermore, $Z_0$ is a $K$-mild solution of Eq (1.1). Since $H_i \in AA^c(R \times L^p(\mathcal{H}), L^p(\mathcal{H}))$ and $F \in PAA^c(R \times L^p(\mathcal{H}) \times \mathcal{B}, L^p(\mathcal{H}))$ for $i = 1, 2$, then for any real sequence $\{\mu_n\}$, there exists a subsequence $\{\mu_n'\}$ and stochastic processes $\tilde{R}_i$: $R \times L^p(\mathcal{H}) \to L^p(\mathcal{H})$ and $\tilde{J}$: $R \times L^p(\mathcal{H}) \times \mathcal{B} \to L^p(\mathcal{H})$ such that

$$\lim_{n\to+\infty} \sup_{Z\in K} E\|H_i(t+\mu_n', Z) - \tilde{R}_i(t, Z)\|^p = 0, \tag{3.23}$$

$$\lim_{n\to+\infty} \sup_{Z\in K} \int_{|y|_{\mathcal{B}}\geq 1} E\|F(t+\mu_n', Z, y) - \tilde{J}(t, Z, y)\|^p \nu(dy) = 0 \tag{3.24}$$

and

$$\lim_{n\to+\infty} \sup_{Z\in K} E\|\tilde{R}_i(t-\mu_n', Z) - H_i(t, Z)\|^p = 0, \tag{3.25}$$

$$\lim_{n\to+\infty} \sup_{Z\in K} \int_{|y|_{\mathcal{B}}\geq 1} E\|\tilde{J}(t-\mu_n', Z, y) - F(t, Z, y)\|^p \nu(dy) = 0. \tag{3.26}$$

Define the interval $(\lambda, +\infty)$ that satisfies $\lambda + \mu_n' \geq t_0$ for sufficiently large $n \in \mathbb{N}$; then $t \to Z_0(\cdot + \mu_n')$ is defined on $(\lambda, +\infty)$ and $Z_0(t + \mu_n') \in K$ for any $t \geq \lambda$. Based on $H_i(t, \cdot) \in C(K, L^p(\mathcal{H}))$, $F(t, \cdot, y) \in C(K, L^p(\mathcal{H}))$, and Eqs (3.23) and (3.24) that are equivalent to replacing $\tilde{G}_i$, $\tilde{H}$ and $\rho_n''$ in (3.10) and (3.11) with $\tilde{R}_i$, $\tilde{J}$ and $\mu_n'$, respectively, it follows from Lemma 3.2 that there exists $Z_0^*(t)$ such that

$$\begin{aligned} Z_0(t+\mu_n') =& \mathcal{T}(t-s)Z_0(s+\mu_n') + \int_s^t \mathcal{T}(t-\varrho)H_1(\varrho+\mu_n', Z_0(\varrho+\mu_n'))d\varrho \\ &+ \int_s^t \mathcal{T}(t-\varrho)H_2(\varrho+\mu_n', Z_0(\varrho+\mu_n'))dW(\varrho+\mu_n') \\ &+ \int_s^t \mathcal{T}(t-\varrho)\int_{|y|_{\mathcal{B}}\geq 1} F(\varrho+\mu_n', Z_0(\varrho+\mu_n'), y)N(d(\varrho+\mu_n'), dy) \end{aligned}$$

uniformly converges to

$$\begin{aligned} Z_0^*(t) =& \mathcal{T}(t-s)Z_0^*(s) + \int_s^t \mathcal{T}(t-\varrho)\tilde{R}_1(\varrho, Z_0^*(\varrho))d\varrho + \int_s^t \mathcal{T}(t-\varrho)\tilde{R}_2(\varrho, Z_0^*(\varrho))dW(\varrho) \\ &+ \int_s^t \mathcal{T}(t-\varrho)\int_{|y|_{\mathcal{B}}\geq 1} \tilde{J}(\varrho, Z_0^*(r), y)N(d\varrho, dy), \quad \text{for } t \geq s \end{aligned}$$

on each compact subset of $R$ as $n \to +\infty$, where $Z_0^*$ is a $K$-mild solution of

$$dZ_0^*(t) = AZ_0^*(t) + \tilde{R}_1(t, Z_0^*(t))dt + \tilde{R}_2(t, Z_0^*(t))dW(t) + \int_{|y|_{\mathcal{B}}\geq 1} \tilde{J}(t, Z_0^*(t), y)N(dt, dy).$$

Since (III) holds, then

$$\tilde{R}_i(t, \cdot) \in C(K, L^p(\mathcal{H})), \quad \sup_{t\in I_0} \sup_{Z\in K} E\|\tilde{R}_i(t, Z)\|^p < +\infty, \text{ for } i = 1, 2, \tag{3.27}$$

and

$$\tilde{J}(t, \cdot, y) \in C(K, L^p(\mathcal{H})), \quad \sup_{t\in I_0} \sup_{Z\in K} \int_{|y|_{\mathcal{B}}\geq 1} E\|\tilde{J}(t, Z(t), y)\|^p \nu(dy) < +\infty. \tag{3.28}$$

Based on Eqs (3.25) and (3.26), which are equivalent to the equalities obtained by replacing $H_i$, $F$, $\tilde{G}_i$, $\tilde{H}$ and $\rho_n''$ in (3.10) and (3.11) with $\tilde{R}_i$, $\tilde{J}$, $H_i$, $F$, and $-\mu_n'$, respectively. Together with (3.27) and (3.28), it follows from Lemma 3.2 that there exists a $K$-mild solution $Z(t)$ of Eq (1.1), therefore, $Z(t) \in K$, in other words, $\{Z(t) : t \in R\} \subset \overline{\{Z_0(t) : t \geq t_0\}}$. □

**Remark 3.2.** *Lemmas 3.1–3.3 still hold by replacing (3.1)-(3.2) with (III).*

In order to introduce the main conclusions in this section, we require the description of the subvariant functional method that was first proposed by Fink in [13]. Later, Cieutat and Ezzinbi in [15] modified this concept. Denote by $C_K(R, L^p(\mathcal{H})) = \{Z \in C(R, L^p(\mathcal{H})) : Z(t) \in K \text{ for } t \in R\}$, the mapping $\xi_K$: $C_K(R, L^p(\mathcal{H})) \to R$ is called a subvariant functional with respect to the compact set $K$ if $\xi_K$ satisfies:

(i) $\xi_K$ is translation invariant; that is, $\xi_K(z_\tau) = \xi_K(z)$ for each $\tau \in R$, where $z_\tau(\cdot) = z(\tau + \cdot)$;

(ii) if $\lim_{n\to+\infty} z_n = x$ uniformly on each compact subset of $L^p(\mathcal{H})$, then $\xi_K(x) \le \liminf_{n\to+\infty} \xi_K(z_n)$.

Besides, let $\mathcal{A}_K$ stand for the family of all $K$-mild solutions on $R$ of Eq (1.1); if $Z_* \in \mathcal{A}_K$ and $\xi_K(z_*) = \inf_{z\in\mathcal{A}_K} \xi_K(z)$, then $z_*$ is called a minimal $K$-mild solution of Eq (1.1).

**Remark 3.3.** *The conditions (i) and (ii) imply that if $\lim_{n\to+\infty} z_{\tau_n} = x$ uniformly on each compact subset of $L^p(\mathcal{H})$, then $\xi_K(x) \le \xi_K(z)$.*

**Theorem 3.1.** *Let (I)–(III) hold. Assume that Eq (1.1) admits at least a bounded mild solution $Z_0$: $[t_0, +\infty) \to R$ and $\xi_K$ is a subvariant functional with respect to the compact set K; then Eq (1.1) has at least a minimal K-mild solution.*

*Proof.* From Lemma 3.3, it follows that $\{Z_0(t) : t \ge t_0\}$ is relatively compact; then $K = \overline{\{Z_0(t) : t \ge t_0\}}$ is a compact set. Define $\eta = \inf_{Z\in\mathcal{A}_K} \xi_K(Z)$; according to Lemma 3.4, it obtains that there exists a $K$-mild solution $Z(t)$ of Eq (1.1) satisfying $\{Z(t) : t \in R\} \subset \overline{\{Z_0(t) : t \ge t_0\}}$, which indicates that $\mathcal{A}_K$ is a nonempty subset of $L^p(\mathcal{H})$ and $\eta$ is well defined. Further, there exists a sequence $\{Z_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_K$ satisfying

$$\lim_{n\to+\infty} \xi_K(Z_n) = \eta. \tag{3.29}$$

Let $\mathcal{E}_n = \{Z_n(t) : t \in R\}$, where $Z_n \in C(R, L^p(\mathcal{H}))$; obviously, $\mathcal{E}_n \subset K$ for each $t \in R$. Based on Lemma 3.1, it follows that there exists a function $\zeta$: $[0, +\infty) \to [0, +\infty)$ such that $\zeta(\vartheta) \to 0$ as $\vartheta \to 0$ and

$$E\|Z_n(t) - Z_n(\tau)\|^p \le 4^{p-1}\zeta(|t - \tau|),$$

which implies that $\mathcal{E}_n$ is equicontinuous. By utilizing the well-known Arzela–Ascoli theorem, it derives that the set $\mathcal{E}_n$ is relatively compact; thus, for the above real sequence $\{Z_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{Z'_n\}_{n\in\mathbb{N}} \subset \{Z_n\}_{n\in\mathbb{N}}$ and a stochastic process $\hat{Z}$: $R \to L^p(\mathcal{H})$ satisfying

$$\lim_{n\to+\infty} E\|Z'_n(t) - \hat{Z}(t)\|^p = 0 \tag{3.30}$$

uniformly on each compact subset of $R$. In addition, (III) yields that stochastic processes $H_1$, $H_2$, and $F$ are uniformly continuous with respect to the second variable. Therefore, it yields

$$\lim_{n\to+\infty} E\|H_i(t, Z'_n(t)) - H_i(t, \hat{Z}(t))\|^p = 0 \text{ for } i = 1, 2, \tag{3.31}$$

and

$$\lim_{n\to+\infty} \int_{|y|_{\mathcal{B}}\ge1} E\|F(t, Z'_n(t), y) - F(t, \hat{Z}(t), y)\|^p \nu(dy) = 0. \tag{3.32}$$

Replacing the Eqs (3.10) and (3.11) with (3.31) and (3.32) and taking a similar discussion to the proof of (3.14) in Lemma 3.2, it follows that

$$Z'_n(t) = \mathcal{T}(t-\epsilon)Z'_n(\epsilon) + \int_\epsilon^t \mathcal{T}(t-\sigma)H_1(\sigma, Z'_n(\sigma))d\sigma + \int_\epsilon^t \mathcal{T}(t-\sigma)H_2(\sigma, Z'_n(\sigma))dW(\sigma) + \int_\epsilon^t \mathcal{T}(t-\sigma)\int_{|y|_{\mathcal{B}}\geq 1} F(\sigma, Z'_n(\sigma), y)N(d\sigma, dy)$$

converges to

$$\hat{Z}(t) = \mathcal{T}(t-\epsilon)\hat{Z}(\epsilon) + \int_\epsilon^t \mathcal{T}(t-\sigma)H_1(\sigma, \hat{Z}(\sigma))d\sigma + \int_\epsilon^t \mathcal{T}(t-\sigma)H_2(\sigma, \hat{Z}(\sigma))dW(\sigma) + \int_\epsilon^t \mathcal{T}(t-\sigma)\int_{|y|_{\mathcal{B}}\geq 1} F(\sigma, \hat{Z}(\sigma), y)N(d\sigma, dy),$$

where $\hat{Z}$ is a $K$-mild solution of Eq (1.1), that is, $\hat{Z} \in \mathcal{A}_K$. Since $\eta = \inf_{Z\in\mathcal{A}_K} \xi_K(Z)$, then $\eta \leq \xi_K(\hat{Z})$. Due to the fact that $\xi_K$ is a subvariant functional with respect to the compact set $K$, then (3.30) yields

$$\xi_K(\hat{Z}) \leq \liminf_{n\to+\infty} \xi_K(Z'_n) = \eta.$$

Further, $\eta = \xi_K(\hat{Z})$, that is, Eq (1.1) has at least a minimal $K$-mild solution. □

**Theorem 3.2.** *Let (I)–(III) hold. Assume that Eq (1.1) admits at least a bounded mild solution* $Z_0$*:* $[t_0, +\infty) \to R$ *and* $\xi_K$ *is a subvariant functional with respect to the compact set K; if Eq (1.1) has a unique minimal K-mild solution, then it is compact almost automorphic.*

*Proof.* From Theorem 3.1, it follows that Eq (1.1) has at least a minimal $K$-mild solution $\hat{Z}$. Without loss of generality, let $\hat{Z}$ be a unique minimal $K$-mild solution; it remains to prove $\hat{Z}$ is compact and almost automorphic. From (III), then for any real sequence $\{\mu_n\}$, there exists a subsequence $\{\mu'_n\}$ and stochastic processes $\tilde{R}_i$: $R \times L^p(\mathcal{H}) \to L^p(\mathcal{H})$ and $\tilde{J}$: $R \times L^p(\mathcal{H}) \times \mathcal{B} \to L^p(\mathcal{H})$ such that (3.23)–(3.26) hold. By using $H_i(t, \cdot) \in C(K, L^p(\mathcal{H}))$, $F(t, \cdot, y) \in C(K, L^p(\mathcal{H}))$, and Eqs (3.23) and (3.24) that are equivalent to replacing $\tilde{G}_i$, $\tilde{H}$ and $\rho''_n$ in (3.10)-(3.11) with $\tilde{R}_i$, $\tilde{J}$, and $\mu'_n$, respectively, it follows from Lemma 3.2 that there exists a stochastic process $\hat{Z}^*$ satisfying

$$\lim_{n\to+\infty} E\|\hat{Z}(t+\mu'_n) - \hat{Z}^*(t)\|^p = 0 \tag{3.33}$$

uniformly on each subset in $L^p(\mathcal{H})$, where $\hat{Z}^*$ is a $K$-mild solution of

$$d\hat{Z}^*(t) = A\hat{Z}^*(t) + \tilde{R}_1(t, \hat{Z}^*(t))dt + \tilde{R}_2(t, \hat{Z}^*(t))dW(t) + \int_{|y|_{\mathcal{B}}\geq 1} \tilde{J}(t, \hat{Z}^*(t), y)N(dt, dy).$$

Since (III) implies (3.27)-(3.28) hold, therefore, based on Eqs (3.25) and (3.26), it is equivalent to replacing $H_i$, $F$, $\tilde{G}_i$, $\tilde{H}$, and $\rho''_n$ in (3.10)-(3.11) with $\tilde{R}_i$, $\tilde{J}$, $H_i$, $F$, and $-\mu'_n$, respectively. By reusing Lemma 3.2, then, there exists a stochastic process $\hat{Z}^{**}$ satisfying

$$\lim_{n\to+\infty} E\|\hat{Z}^*(t-\mu'_n) - \hat{Z}^{**}(t)\|^p = 0 \tag{3.34}$$

uniformly on each subset in $L^p(\mathcal{H})$, where $\hat{Z}^{**}$ is a $K$-mild solution of Eq (1.1). Based on the properties of subvariant functional $\xi_K$ and (3.33)-(3.34), it obtains that $\xi_K(\hat{Z}^{**}) \leq \xi_K(\hat{Z}^{*}) \leq \xi_K(\hat{Z})$; therefore, $\xi_K(\hat{Z}^{**}) \leq \xi_K(\hat{Z})$. Since $\hat{Z}$ is a unique minimal $K$-mild solution, then $\hat{Z}^{**} = \hat{Z}$; moreover, it deduces

$$\lim_{n\to+\infty} E\|\hat{Z}^*(t-\mu_n') - \hat{Z}(t)\|^p = 0. \tag{3.35}$$

Combining (3.33) and (3.35), it follows that $\hat{Z}$ is compact and almost automorphic. □

**Example 3.1.** *Consider the stochastic heat equations with Dirichlet boundary conditions as follows:*

$$\begin{cases} du(t,\xi) = \frac{\partial^2 u(t,\xi)}{\partial \xi^2}dt + h_1(t,u(t,\xi))dt + h_2(t,u(t,\xi))dW(t,\xi) \\ \qquad\qquad + f(t,u(t,\xi))dY(t,\xi),\ (t,\xi) \in R\times(0,1), \\ u(t,0) = u(t,1) = 0, \quad t\in R. \end{cases} \tag{3.36}$$

*Define*

$$D(A) = \left\{\xi \in C^1[0,1] \| \xi' \text{ is absolutely continuous on } [0,1], \xi'' \in L^2[0,1], \xi(0) = \xi(1) = 0\right\},$$

*then $A$ generates a compact $C_0$-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $L^2[0,1]$ such that $\|\mathcal{T}(t)\| \leq e^{-\pi^2 t}$ for $t \geq 0$; therefore; (I) and (II) are satisfied. Assume that*

$$h_i(t,u) = \eta_0 \sin\frac{1}{2+\cos t + \cos\sqrt{3}t} + \tilde{h}(\eta),\ i = 1,2,$$
$$f(t,u) = \eta_0(\sin t + \sin\sqrt{2}t) + \tilde{f}(\eta),$$

*where $\eta_0 \in L^p[0,1]$, $\tilde{h}$, $\tilde{f}$ are locally Lipschitz such that*

$$\limsup_{|\lambda|\to+\infty}\frac{\tilde{h}(\lambda)}{\lambda} = \limsup_{|\lambda|\to+\infty}\frac{\tilde{f}(\lambda)}{\lambda} < \pi^2$$

*and $\tilde{h}(0) = \tilde{f}(0) = 0$. In addition, denote by $A = \frac{\partial}{\partial\xi^2}$, $Z(t) = u(t,\xi)$, $H_i(t,Z(t)) = h_i(t,u)$ and $F(t,Z(t)) = f(t,u)$; then (3.36) can be expressed in the form (1.1) and (III) holds. Because (III) indicates (3.1) and (3.2) hold; therefore, the results obtained in Section 3 are valid.*

## 4. Conclusions

This paper mainly introduced the concept of $p$-mean compact almost automorphic stochastic processes and applied it to the stochastic differential equation (1.1), which has not been investigated so far. Based on $C_0$-semigroup theory, Hölder inequality, Burkholder–Davis–Gundy inequality, and the Lebesgue dominated convergence theorem, under assumptions (I) and (3.1)-(3.2), we obtain that every $K$-mild solution of Eq (1.1) is uniformly continuous with the uniform continuity modulus $\zeta$: $[0,+\infty) \to [0,+\infty)$ such that $\zeta(\vartheta) \to 0$ as $\vartheta \to 0$. This result is the so-called Lemma 3.1, which is used to prove Lemmas 3.2–3.4. Under the assumptions in Lemma 3.1, if $H_i(t,\cdot)$ and $F(t,\cdot,y)$ belong to $C(K, L^p(\mathcal{H}))$ for $i = 1,2$ and (3.10)-(3.11) hold, from the concept of compact almost automorphic stochastic processes and the famous Arzela-Ascoli theorem, for any real sequence $\{\rho_n\}_{n\in\mathbb{N}}$, it obtains that the $K$-mild solution $Z(t + \rho'_n)$ is convergent uniformly on each compact subset of $R$, which lays a solid foundation for the conclusion that there exists a mild solution $Z$ of Eq (1.1) such that $\{Z(t) : t \in R\} \subset \overline{\{Z_0(t) : t \geq t_0\}}$, where $Z_0(t)$ for $t \geq t_0$ is a bounded mild solution of Eq (1.1) and $\{Z_0(t) : t \geq t_0\}$ is relatively compact. Further, by utilizing the subvariant functional method, we give some sufficient conditions to make sure that there exists at least one minimal $K$-mild solution; further, if the minimal $K$-mild solution is unique, then it is a compact almost automorphic mild solution. The exploration of the compact almost automorphic stochastic processes in this paper is challenging and meaningful; one is the difficulty in finding some reasonable analytical and computational skills to prove the desired result, and another is the value of the subvariant functional method applied to stochastic differential equations.

## Use of Generative-AI tools declaration

The author declares that he/she has not used Artiffcial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares no conflict of interest in this paper.

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