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Research article

An upper bound for the semistrong chromatic index of Halin graphs

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Abstract: For a graph G, a semistrong matching is a matching M such that every edge in M contains at least one endpoint of degree one in the induced subgraph $G[V(M)]$. The semistrong chromatic index $\chi'_{ss}(G)$ denotes the minimum number of colors required for a proper edge-coloring where each color class induces a semistrong matching. We study this parameter for Halin graphs, which are planar graphs formed by connecting all leaves of a tree T (with no degree-two vertices) via an outer cycle C. Our main result establishes that for any Halin graph $G = T \cup C$ with maximum degree $\Delta(G)$, the semistrong chromatic index satisfies $\chi'_{ss}(G) \leq \Delta(G)+4$, with equality attained by the wheel graphs W_4 and W_7.

Keywords: semistrong edge-coloring; semistrong chromatic index; Halin graph
Mathematics Subject Classification: 05C15

1. Introduction

For a graph $G = (V,E)$, we denote its order and size by $|G|$ and $\|G\|$, respectively. The neighborhood of a vertex $v \in V$ is denoted by $N(v)$, with $d(v) = |N(v)|$ representing its degree. We write $\Delta(G)$ for the maximum degree, and refer to vertices of degree exactly k (or at least k) as k-vertices (or k^+-vertices). A matching is a set of independent edges. The distance $d(x,y)$ between two vertices $x,y \in V(G)$ equals the minimum number of edges in any path connecting them, while the diameter diam(G) represents the maximum distance between any pair of vertices in G. The distance between the edges e and e' in a graph G is defined as the distance between the vertices corresponding to e and e' in the line graph $L(G)$.

A proper edge-coloring of a graph G is an assignment of colors to edges such that adjacent edges receive distinct colors. The chromatic index $\chi'(G)$ denotes the minimum number of colors required for such a coloring.

A stronger coloring variant, called strong edge-coloring, requires that any two edges within distance two must have different colors. The minimum number of colors needed for such a coloring is the strong chromatic index $\chi_s'(G)$. In 1989, Erdős and Nešetřil [3] conjectured tight upper bounds for $\chi_s'(G)$ in terms of the maximum degree Δ.

Conjecture 1. *[3] For any graph G with maximum degree $\Delta, \chi_s'(G) \leq \frac{5}{4}\Delta^2$ when Δ is even, and $\chi_s'(G) \leq \frac{1}{4}\left(5\Delta^2 - 2\Delta + 1\right)$ when Δ is odd.*

The proposed bounds are known to be tight for certain extremal graphs. For further developments and related results, we recommend the comprehensive survey [2].

A semistrong edge-coloring of a graph G is a proper edge-coloring in which each color class forms a semistrong matching. Specifically, a matching M is called semistrong if for every edge $uv \in M$, it satisfies $d_{G[V(M)]}(u) = 1$ or $d_{G[V(M)]}(v) = 1$. The semistrong chromatic index $\chi_{ss}'(G)$ is the minimum number of colors required for such a coloring. This concept was introduced by Gyárfás and Hubenko [5] in 2005 as a variant of strong edge-coloring.

For any graph G, the following always holds:

$$\chi'(G) \leq \chi_{ss}'(G) \leq \chi_s'(G).$$

The relationship between these chromatic indices has been studied for specific graph classes. Gyárfás and Hubenko [5] showed that $\chi_{ss}'(G) = \chi_s'(G)$ holds for Kneser graphs and subset graphs. This equality was extended to complete graphs and complete bipartite graphs by Lužar, Mockovčiaková, and Soták [10]. In the same paper, the authors proved $\chi_{ss}'(G) \leq 8$ for cubic graphs G, distinct from $K_{3,3}$. Additionally, the general upper bound $\chi_{ss}'(G) \leq \Delta^2$ was established for graphs with maximum degree Δ. The authors further conjectured that this bound could be improved to $\Delta^2 - 1$ for all graphs except $K_{\Delta,\Delta}$, and this conjecture was subsequently verified by Lin and Lin [7]. More recently, these authors [8] obtained refined bounds for planar graphs with large girth, proving that $\chi_{ss}'(G) \leq 2\Delta + 4$ when girth $g(G) \geq 7$, and $\chi_{ss}'(G) \leq 2\Delta + 2$ when $g(G) \geq 8$.

A Halin graph is a planar graph G constructed as follows. Let T be a tree with $|T| \geq 4$, called the characteristic tree, where every vertex is either a 1-vertex (called a leaf) or 3^+-vertex. Embedding T in the plane and adding a cycle C, called the adjoint cycle, that connects all leaves of T in cyclic order yields $G = T \cup C$, with C bounding outer face. A subclass of Halin graphs is the family of wheel graphs $W_n (n \geq 3)$ where the characteristic tree is a star $K_{1,n}$ and the adjoint cycle links all n leaves. The necklace is a particular Halin graph whose characteristic tree is a caterpillar, denoted N_{e_h} ($h \geq 1$). The characteristic tree T consists of a path $v_0 v_1 \cdots v_h v_{h+1}$ along with additional leaves $v_1', v_2', \ldots, v_h'$, where each v_i' is adjacent only to v_i for $1 \leq i \leq h$. The adjoint cycle C_{h+2} is formed by connecting $v_0, v_1', v_2', \ldots, v_h', v_{h+1}$ in sequence. Examples of N_{e_2} and N_{e_4} are illustrated in Figure 1.

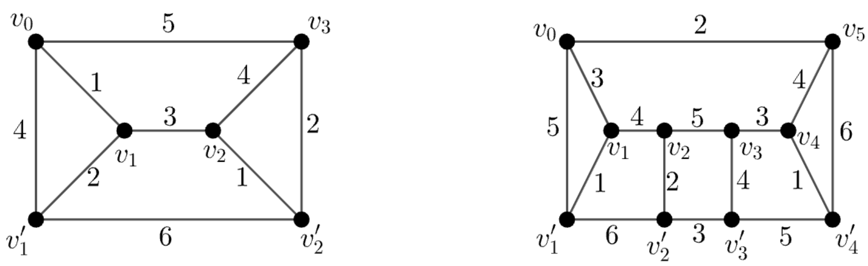

Figure 1. N_{e_2} and N_{e_4}.

By definition, for any graph G, we have $\sigma(G) \leq \chi'_s(G)$, where

$$\sigma(G) := \max\{d_G(u) + d_G(v) - 1 \mid uv \in E(G)\}.$$

For trees, we can establish a stronger result.

Lemma 1. *[4] For any tree T, $\chi'_s(T) = \sigma(T) \leq 2\Delta(T) - 1$.*

For cubic Halin graphs, significant progress has been made in determining the bounds of $\chi'_s(G)$. Shiu, Lam, and Tam [12] and Lih and Liu [11] established initial upper bounds. Moreover, exact values were determined for specific families by Shiu and Tam [13] and Chang and Liu [1]. The investigation of general Halin graphs has yielded important developments, Lai, Lih, and Tsai first proved that for $G = T \cup C \notin \{Ne_2, W_n\}$ where $n \not\equiv 0(\bmod 3)$, $\chi'_s(G) \leqslant \chi'_s(T) + 3$. Subsequently, Yang and Wu [14] improved this bound, which was shown to be sharp for an infinite family of graphs.

Theorem 1. *[14] For any Halin graph $G = T \cup C \notin \{W_n, N_{e_2}, N_{e_4}\}$, $\chi'_s(G) \leq \chi'_s(T) + 2$.*

For wheel graphs W_n, Lai, Lih, and Tsai [9] completely determined the strong chromatic index: $\chi'_s(W_n) = n + 3$ if $n \equiv 0(\bmod 3)$, $\chi'_s(W_n) = n + 5$ if $n = 5$, and $\chi'_s(W_n) = n + 4$ otherwise. For necklace graphs N_{e_h}, Shiu, Lam, and Tam [12] established the following exact values: $\chi'_s(N_{e_2}) = 9$; $\chi'_s(N_{e_4}) = 8$; $\chi'_s(N_{e_h}) = 7$ if $h \geq 6$ and is even; and $\chi'_s(N_{e_h}) = 6$ if h is odd. Building upon Lemma 1 and Theorem 1, we obtain the following result, which was independently proved by Hu, Lih, and Liu [6].

Theorem 2. *[6] For any Halin graph $G \notin \{N_{e_2}, N_{e_4}\}$, $\chi'_s(G) \leqslant 2\Delta(G) + 1$.*

This paper investigates the semistrong edge-coloring problem for Halin graphs. Our main result establishes a tight upper bound on the semistrong chromatic index in terms of the maximum degree.

Theorem 3. *For any Halin graph G, $\chi'_{ss}(G) \leq \Delta(G) + 4$, with equality attained by the wheel graphs W_4 and W_7.*

2. Proof of Theorem 3

For a partial semistrong edge-coloring Ψ of graph G, we define the color set $C_\Psi(v)$ at the vertex $v \in V(G)$ to be $C_\Psi(v) := \{\Psi(e) \mid e \in E(G) \text{ is incident to } v\}$.

Lemma 2. *For the cycle C_n, we have*

$$\chi'_{ss}(C_n) = \begin{cases} 4 & \text{if } n = 4 \text{ or } n = 7, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let us consider the cycle $C_n = (v_0, v_1, \ldots, v_{n-1}, v_0)$ with edges $e_i = v_{i-1}v_i$ for $0 \leq i \leq n-1$, where indices are taken modulo n. For the 4-cycle C_4, it is straightforward to verify that $\chi'_{ss}(C_4) = 4$. For the 7-cycle C_7, we observe that any maximal semistrong matching contains at most 2 edges, while the graph has 7 edges in total. This implies the lower bound $\chi'_{ss}(C_7) \geq \lceil 7/2 \rceil = 4$. To establish the upper bound, consider the edge-coloring Ψ defined by $\Psi(e_1) = \Psi(e_4) = 1$, $\Psi(e_2) = \Psi(e_5) = 2$, $\Psi(e_3) = \Psi(e_6) = 3$, and $\Psi(e_0) = 4$. This coloring partitions the edges into four semistrong matchings, proving that $\chi'_{ss}(C_7) \leq 4$. Combining both bounds yields $\chi'_{ss}(C_7) = 4$.

Otherwise, $n \neq 4, 7$. Note that $\chi'_{ss}(C_n) \geq 3$. We construct explicit semistrong 3-edge-colorings depending on the value of n mod 3.

Case 1. $n \equiv 0 \pmod 3$.

Define periodic coloring Ψ:

$$\begin{aligned}\Psi(e_0) = \Psi(e_3) = \cdots = \Psi(e_{n-3}) = 1,\\ \Psi(e_1) = \Psi(e_4) = \cdots = \Psi(e_{n-2}) = 2,\\ \Psi(e_2) = \Psi(e_5) = \cdots = \Psi(e_{n-1}) = 3.\end{aligned}$$

Case 2. $n \equiv 1 \pmod 3$.

Adjusted terminal coloring Ψ:

$$\begin{aligned}&\Psi(e_0) = \Psi(e_3) = \cdots = \Psi(e_{n-7}) = \Psi(e_{n-3}) = 1,\\ &\Psi(e_1) = \Psi(e_4) = \cdots = \Psi(e_{n-6}) = \Psi(e_{n-4}) = \Psi(e_{n-1}) = 2,\\ &\Psi(e_2) = \Psi(e_5) = \cdots = \Psi(e_{n-5}) = \Psi(e_{n-2}) = 3.\end{aligned}$$

Case 3. $n \equiv 2 \pmod 3$.

Shifted coloring Ψ:

$$\begin{aligned}\Psi(e_0) = \Psi(e_3) = \cdots = \Psi(e_{n-2}) = 1,\\ \Psi(e_1) = \Psi(e_4) = \cdots = \Psi(e_{n-1}) = 2,\\ \Psi(e_2) = \Psi(e_5) = \cdots = \Psi(e_{n-3}) = 3.\end{aligned}$$

In all cases, Ψ is a semistrong 3-edge-coloring. Hence, $\chi'_{ss}(C_n) = 3$ for $n \neq 4, 7$. □

Lemma 3. *For the wheel $W_n = K_{1,n} \cup C_n$, we have*

$$\chi'_{ss}(W_n) = \begin{cases} n+4 & \text{if } n = 4 \text{ or } n = 7,\\ n+3 & \text{otherwise.}\end{cases}$$

Proof. Any edge in $K_{1,n}$ and any edge in C_n are either adjacent or do not form a semistrong matching. Hence, $\chi'_{ss}(W_n) = \|K_n\| + \chi'_{ss}(C_n)$. □

Lemma 4. *Let T be a tree. Then,*

$$\Delta(T) \leqslant \chi'_{\rm ss}(T) \leqslant \Delta(T) + 1.$$

Furthermore, if T has $diam(T) \leq 4$ or contains exactly one maximum degree vertex, then

$$\chi'_{\rm ss}(T) = \Delta(T).$$

For $\Delta(T) \geq 2$, if there exist two maximum degree vertices at distance exactly three, then

$$\chi'_{\rm ss}(T) = \Delta(T) + 1.$$

Proof. The chromatic index clearly provides a lower bound for the semistrong chromatic index, establishing $\Delta(T) \leq \chi'_{\rm ss}(T)$. If $T \cong K_{1,n}$, then $\chi'_{ss}(T) = \chi'(T) = \Delta(T)$. If T is a non-star tree, then $\|T\| \geq 3$, and we prove the upper bound by induction on $\|T\|$. When $\|T\| = 3$, then $T \cong P_4$ and $\chi'_{\rm ss}(T) = \Delta(T) = 2$. Assume the result holds for all trees with fewer edges than $\|T\|$. Let $P := v_0v_1 \ldots v_{k-1}v_k$ be a longest path in T with $d(v_{k-1}) \geq d(v_1)$ and consider $T' := T - v_0$. By the induction hypothesis, $\chi'_{ss}(T') \leqslant \Delta(T')+1 = \Delta(T)+1$. Let Ψ be a semistrong edge-coloring of T' using $\Delta(T)+1$ colors. For the edge v_0v_1, the forbidden colors are $C_\Psi(v_1) \cup \{\Psi(v_2v_3)\}$ with $|C_\Psi(v_1)| \leq d(v_1)-1$. Thus, the number of available colors is at least $\Delta(T)+1-d(v_1) \geq 1$, establishing $\chi'_{ss}(T) \leqslant \Delta(T)+1$.

Given a tree T with diameter at most 4, if T is a star $K_{1,n}$, then $\operatorname{diam}(T) \leq 4$ and $\chi'_{ss}(T) = \chi'(T) = \Delta(T)$. For non-star trees, proceed by induction on $\|T\|$. When $\|T\| = 3$, then $T \cong P_4$ and $\chi'_{ss}(T) = \Delta(T) = 2$. Let $P := v_0v_1 \ldots v_{k-1}v_k$ be a longest path with $d(v_{k-1}) \geq d(v_1)$, where $k \leq 4$. For $T' := T - v_0$, the induction gives $\chi'_{ss}(T') = \Delta(T') = \Delta(T)$. Given a semistrong $\Delta(T)$-edge-coloring Ψ of T', the forbidden colors for v_0v_1 is $C_\Psi(v_1)$ with $|C_\Psi(v_1)| \leq \Delta(T)-1$, leaving at least one available color.

Let T be a tree with exactly one maximum degree vertex. Similarly, if T is a star, then $\chi'_{ss}(T) = \chi'(T) = \Delta(T)$. For non-star trees, by induction on $\|T\|$, $\|T\| = 3$, then $T \cong P_4$ and $\chi'_{ss}(T) = \Delta(T) = 2$. Let $P := v_0v_1 \ldots v_{k-1}v_k$ be a longest path in T with $d(v_{k-1}) \geq d(v_1)$. It follows that $d(v_1) \leq \Delta(T)-1$. For $T' := T - v_0$, we have $\chi'_{ss}(T') = \Delta(T') = \Delta(T)$ by the inductive hypothesis. Let Ψ be a semistrong $\Delta(T)$-edge-coloring of T'. The forbidden colors for v_0v_1 are $C_\Psi(v_1) \cup \{\Psi(v_2v_3)\}$ with $|C_\Psi(v_1)| \leq \Delta(T)-2$, leaving at least one available color.

Consider a tree T containing two maximum degree vertices at distance exactly three. Let $P = v_1v_2v_3v_4$ be a path in T with $d(v_1) = d(v_4) = \Delta(T) \geq 2$. Assume, for contradiction, that $\chi'_{ss}(T) = \Delta(T)$ and let Ψ be a semistrong $\Delta(T)$-edge-coloring of T. Let $\Psi(v_2v_3) = \alpha \in \{1, 2, \ldots, \Delta(T)\}$. Since v_1 and v_4 are maximum degree vertices, there exists an edge v_0v_1 (or v_4v_5) incident to v_1 (or v_4) with $\Psi(v_0v_1) = \alpha$ (or $\Psi(v_4v_5) = \alpha$). However, the matching v_0v_1, v_2v_3, v_4v_5 is not a semistrong matching, a contradiction. Therefore, we must have $\chi'_{ss}(T) = \Delta(T)+1$. □

Let $G = T \cup C$ be a Halin graph. In the tree T, an edge is called pendant if it incident to a leaf. Then, there exists a bijection between the set of leaves of T and the set of pendant edges in T. Let $v_1, v_2, \ldots, v_\ell$ denote the leaves of T ordered clockwise along C. And, let $e_1, e_2, \ldots, e_\ell$ be their corresponding pendant edges. Two pendant edges e_i and e_j of T are consecutive if their leaves v_i and v_j are adjacent in C. For example, the edges $v_1v'_1$ and $v_2v'_2$ are consecutive, as are $v_1v'_1$ and v_1v_0, as shown in Figure 1.

Lemma 5. *Let T be a tree without 2-vertices and maximum degree $\Delta(T) \geq 4$. Then, there exists a semistrong $(\Delta(T)+1)$-edge-coloring of T in which any two consecutive pendant edges are assigned distinct colors.*

Proof. If T is a star $K_{1,n}$, then any proper $\Delta(T)$-edge-coloring of T is automatically a semistrong $\Delta(T)$-edge-coloring. In addition, any two consecutive pendant edges receive distinct colors, as they are adjacent in T. For non-star trees T, we proceed by induction on $\|T\|$. For the base case $\|T\| = 6$, the statement holds by the coloring of Figure 2. Now assume $\|T\| \geq 7$ and let $P := v_0v_1 \ldots v_{k-1}v_k$ be a longest path in T with $d(v_{k-1}) \geq d(v_1)$, where $k \geq 3$. We consider two cases:

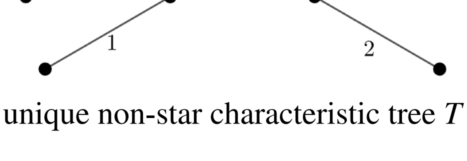

Figure 2. The unique non-star characteristic tree T with $\|T\| = 6$.

Case 1. $d(v_1) \geq 4$.

Let $u_0, u_1, \ldots, u_{d(v_1)-2}$ be the neighbors of v_1, excluding v_2, in clockwise order, where $u_1 = v_0$. Define $T' := T - v_0$. Since T' is a tree without 2-vertices and $\Delta(T') = \Delta(T)$, the induction hypothesis yields a semistrong $(\Delta(T)+1)$-edge-coloring Ψ for T' with distinct colors on consecutive pendant edges.

For the edge v_0v_1, the forbidden colors are $C_\Psi(v_1) \cup \{\Psi(v_2v_3)\}$. Noting that $|C_\Psi(v_1)| = d(v_1) - 1$, the number of forbidden colors is at most $d(v_1)$. Therefore, the number of available colors is at least

$$(\Delta(T)+1) - d(v_1) \geq 1.$$

Choosing any available color preserves the required properties.

Case 2. $d(v_1) = 3$.

Let v_0, u_0 be the neighbors of v_1, excluding v_2, in clockwise order. Define

$$\begin{aligned} e' &:= \text{first pendant edge counterclockwise from } v_0v_1; \\ e'' &:= \text{first pendant edge clockwise from } u_0v_1. \end{aligned}$$

Construct $T^{'} = T - \{v_0, u_0\}$. Since T' is a tree containing no 2-vertices and $\Delta(T') = \Delta(T)$, by the induction hypothesis, T' admits a semistrong $(\Delta(T)+1)$-edge-coloring Ψ where any two consecutive pendant edges receive distinct colors. Hence, we have $\Psi(v_1v_2) \neq \Psi(e')$ and $\Psi(v_1v_2) \neq \Psi(e'')$. This coloring can be extended to a semistrong $(\Delta(T)+1)$-edge-coloring of T such that consecutive pendant edges remain distinctly colored. If $\Psi(e') \neq \Psi(e'')$, we assign the color $\Psi(e'')$ to v_0v_1 and a color different from $\{\Psi(v_1v_2), \Psi(v_2v_3), \Psi(e'')\}$ to u_0v_1. If $\Psi(e') = \Psi(e'')$, then we assign a color different from $\{\Psi(v_1v_2), \Psi(v_2v_3), \Psi(e')\}$ to v_0v_1, say color α, and a color different from $\{\Psi(v_1v_2), \Psi(v_2v_3), \Psi(e'), \alpha\}$ to u_0v_1. Since $(\Delta(T)+1) - 4 = \Delta(T) - 3 \geq 1$, such color assignments are always possible. This completes the induction step. □

Proof of Theorem 3. Assume $\Delta(G) = 3$ and $G \notin \{N_{e_2}, N_{e_4}\}$. By Theorem 2, we have

$$\chi'_{ss}(G) \leq \chi'_s(G) \leq 7 = \Delta(G) + 4.$$

If $G \in \{N_{e_2}, N_{e_4}\}$, then the coloring in Figure 1 yields $\chi'_{ss}(G) \leq 6 < 7 = \Delta(G) + 4$. Thus, we may assume $\Delta(G) \geq 4$ in what follows.

Observe that $\Delta(T) = \Delta(G)$. Let n_d denote the number of d-vertices in the characteristic tree T. Note that the adjoint cycle C has length n_1. Applying Lemma 5, we obtain a semistrong $(\Delta(G)+1)$-edge-coloring Ψ_1 of T where consecutive pendant edges are assigned distinct colors from $\{1, 2, \ldots, \Delta(G) + 1\}$. For cases where $n_1 \neq \{4, 7\}$, Lemma 2 guarantees the existence of a semistrong 3-edge-coloring Ψ_2 of C using the color set $\{\Delta(G)+2, \Delta(G)+3, \Delta(G)+4\}$. We then construct a global semistrong edge-coloring Ψ of G by combining these colorings:

$$\Psi(e) = \begin{cases} \Psi_1(e) & \text{if } e \in T, \\ \Psi_2(e) & \text{if } e \in C. \end{cases}$$

Thus, $\chi'_{ss}(G) \leq \Delta(G) + 4$. We may therefore restrict our attention to the cases where $n_1 = 4$ or $n_1 = 7$.

By the handshake,

$$\sum_{d=1}^{\Delta(G)} dn_d = 2(n-1) = 2\left(\sum_{d=1}^{\Delta(G)} n_d - 1\right).$$

which simplifies to

$$n_1 = \sum_{d=3}^{\Delta(G)} (d-2)n_d + 2. \tag{2.1}$$

Case 1. $n_1 = 4$.

Since $\Delta(G) \geq 4$, Equation (2.1) implies $n_4 = 1$, and thus $G \cong W_4$. By Lemma 3, we obtain $\chi'_{ss}(G) = 8 = \Delta(G) + 4$.

Case 2. $n_1 = 7$.

Given $\Delta(G) \geq 4$, the equation $n_3 + 2n_4 + 3n_5 + 4n_6 + 5n_7 = 5$ admits exactly six solutions: $n_7 = 1$; $n_3 = n_6 = 1$; $n_4 = n_5 = 1$; $n_3 = 2$ and $n_5 = 1$; $n_3 = 1$ and $n_4 = 2$; $n_3 = 3$ and $n_4 = 1$.

Case 2.1. $n_7 = 1$.

Here, $G \cong W_7$, and Lemma 3 yields $\chi'_{ss}(G) = 11 = \Delta(G) + 4$.

Case 2.2. $n_3 = n_6 = 1$.

Figure 3(1) provides an explicit semistrong edge-coloring using 8 colors, proving $\chi'_{ss}(G) \leq 8 < 10 = \Delta(G) + 4$.

Case 2.3. $n_4 = n_5 = 1$.

The coloring in Figure 3(2) establishes $\chi'_{ss}(G) \leq 8 < 9 = \Delta(G) + 4$.

Case 2.4. $n_3 = 2$ and $n_5 = 1$.

There exist three non-isomorphic graphs, as shown in Figure 3(3–5), each satisfying $\chi'_{ss}(G) \leq 7 < 9 = \Delta(G) + 4$.

Case 2.5. $n_3 = 1$ and $n_4 = 2$.

For the three non-isomorphic graphs depicted in Figure 3(6–8), we have $\chi'_{ss}(G) \leq 7 < 8 = \Delta(G) + 4$.

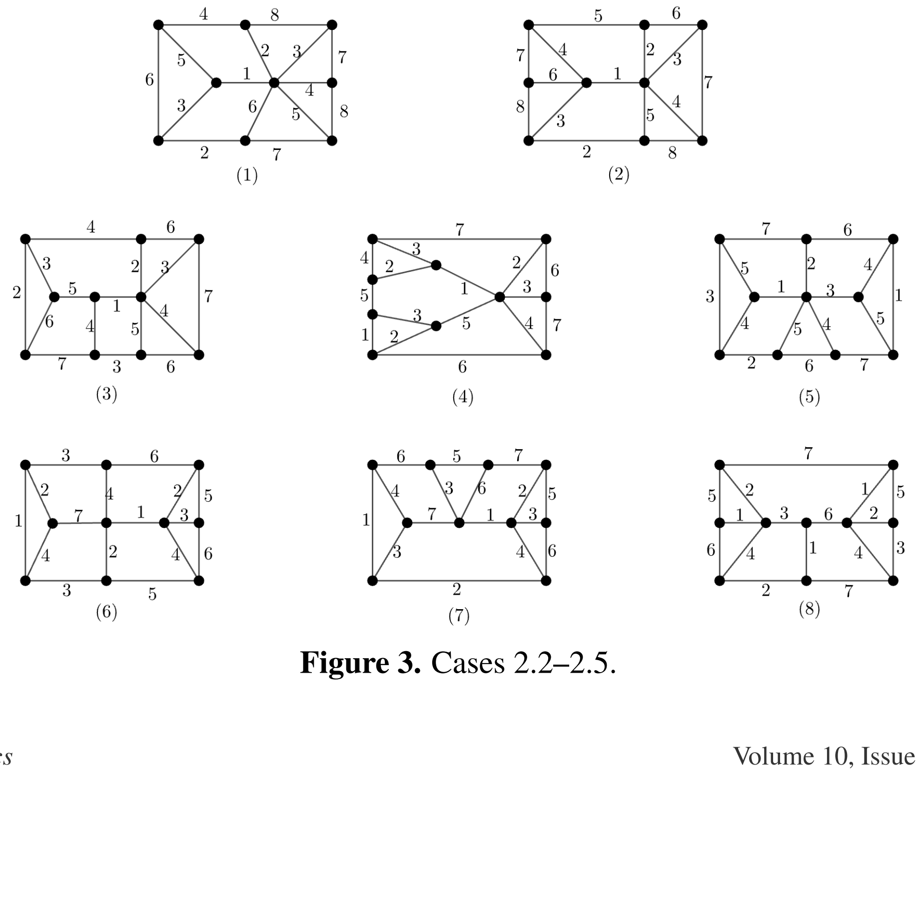

Figure 3. Cases 2.2–2.5.

Case 2.6. $n_3 = 3$ and $n_4 = 1$.

The seven non-isomorphic graphs, as shown in Figure 4, admit explicit colorings showing $\chi'_{ss}(G) \leq 6 < 8 = \Delta(G) + 4$.

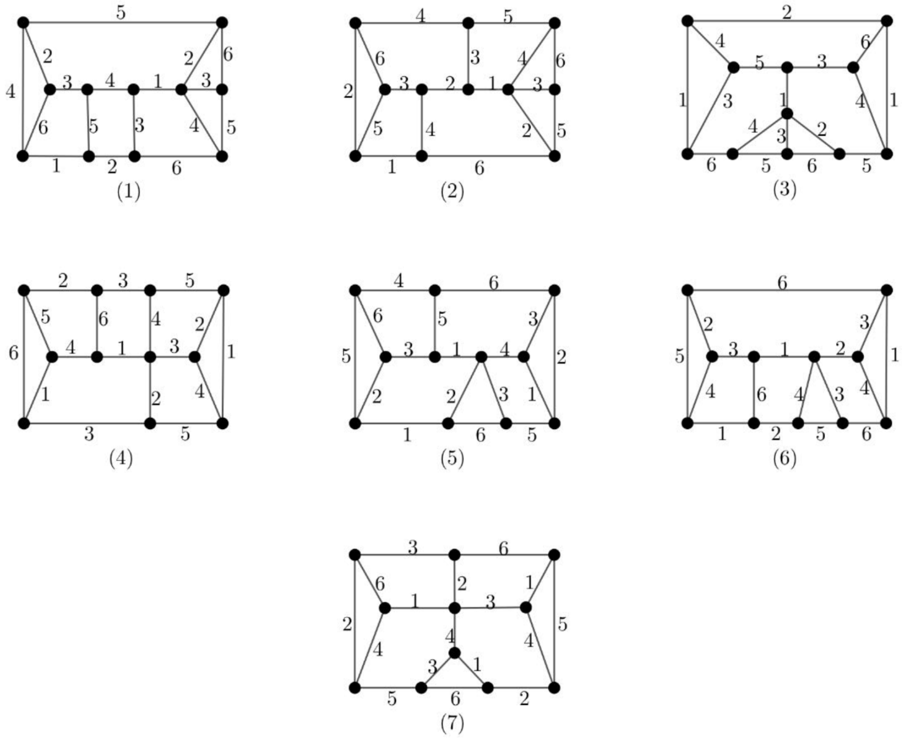

Figure 4. Case 2.6.

3. Conclusions

We believe that the bounds presented in this paper, although tight, can be improved. In particular, we are not aware of any family of Halin graphs G, other than W_4 and W_7, that attains the bound $\chi'_{ss}(G) = \Delta(G) + 4$. Therefore, we propose the following.

Conjecture 2. *For any Halin graph $G \notin \{W_4, W_7\}$, $\chi'_{ss}(G) \leq \Delta(G) + 3$.*

An affirmative resolution would demonstrate sharpness, as infinite families (e.g., the graph in Figure 5) attain this bound. The construction involves a characteristic tree T with diam$(T) \leq 4$ or unique maximum-degree vertex v ($\Delta(G) \geq 7$). In such cases, the $\Delta(G)$ edges incident to v require distinct colors $\{1, 2, \ldots, \Delta(G)\}$. Additional edges $e_1, e_2, \ldots, e_{\Delta(G)-1}$ need colors outside $\{1, 2, \ldots, \Delta(G)\}$. For $\Delta(G) \geq 7$, the edges e_1 and e_2 must receive different colors. This implies that the five edges e_1, e_2, e_3, e_4, e_5 shown in Figure 5 only have two admissible colors, from the set $\{\Delta(G)+1, \Delta(G)+2\}$. This forces edges e_1, e_3, e_5 to share a color, violating the semistrong coloring condition. We conclude that $\chi'_{ss}(G) \geq \Delta(G) + 3$. This construction can be extended to create infinite families of Halin graphs attaining the bound, proving its sharpness.

We have recently confirmed Conjecture 2 for cubic Halin graphs ($\Delta(G) = 3$), obtaining the following result: any cubic Halin graphs G, $\chi'_{ss}(G) \leq 6$. Moreover, for complete cubic Halin graphs $G \neq W_3$, $\chi'_{ss}(G) = 5$; for necklace graphs G, $\chi'_{ss}(G) = 6$.

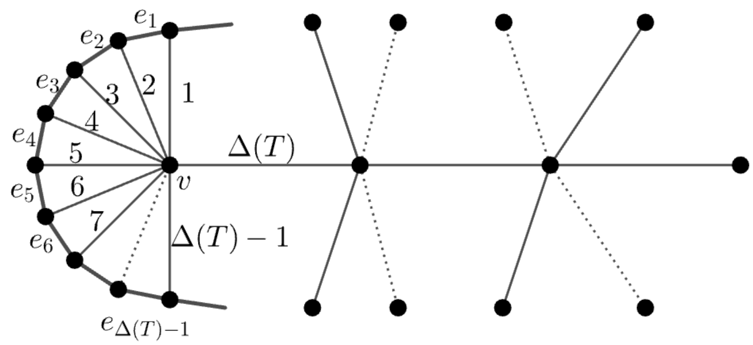

Figure 5. An example showing sharp bound of Conjecture 2.

Conjecture 3. *If $G = T \cup C$ is a Halin graph other than W_4 or W_7, then $\chi'_{ss}(G) \leq \chi'_{ss}(T) + 3$.*

This would generalize Theorem 3 via Lemma 1, remaining sharp for infinite families (e.g., Figure 5), where $\chi'_{ss}(T) = \Delta(T)$ holds.

Author contributions

Jianxin Luo: Investigation, Visualization, Writing-original draft; Jiangxu Kong: Methodology, Supervision, Writing-review & editing, Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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