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Research article

Metric structure for Riemann-Stieltjes derivable functions on fractals and application

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Abstract: This article presents a metric structure of the space of all Riemann-Stieltjes derivable functions defined over a fractal subset of the real line. Within this framework, we formulate, and analyze a measure of non-compactness tailored to fractal domains. Building upon this foundation, we develop a fixed point theorem in normed linear spaces under a generalized contraction condition, thereby extending Darbo's classical results. To illustrate the applicability of our theoretical findings, we apply this framework to the analysis of a class of fractal α-linear differential equations, particularly focusing on models that exhibit oscillatory behaviors in fractal media settings where traditional methods often fail due to the irregularity of the domain. A numerical simulation using MATLAB supports the theoretical assertions, thus demonstrating that the noncompactness-based conditions imposed on the operator $\mathcal{L}$ are sufficient to ensure convergence within the function space $\mathcal{D}(\phi)$. This approach offers novel insights into solving differential equations in non-Euclidean geometries, thereby emphasizing the interplay between metric structures, the fixed point theory, and mechanical systems in complex media.

Keywords: fractal set; fractal functions; fractal calculus; α-linear differential equations; measure of non-compactness; Riemann-Stieltjes integration; metric space; fixed point; Darbo's theorem
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1. Introduction

Earlier investigations have revealed that fractals can effectively model a range of structures observed in nature, and their geometry has been scrutinized and studied [1–3]. Fractal geometry has been extensively employed to describe and analyze the complex structures of various natural phenomena, including clouds, coastlines, mountains, and blood vessels [1], for which the fractal dimension often exceeds the topological dimension [2]. Traditional calculus has been found insufficient to analyze fractal structures. Consequently, researchers have reformulated several classical concepts for fractal structures, and extended differentiation & integration to accommodate fractal domains. The development of fractal calculus for connected fractal subset of $\mathbb{R}$, and measure-theoretical approaches for disconnected fractal subsets of $\mathbb{R}$ are remarkable points in the analysis. These developments enable the study of traditional calculus with fractal structures, such as Koch curves and Cantor sets [4–6]. The ground of fractal calculus is an adaptation of the Fundamental Theorem of Calculus. The approach of extending classical calculus to fractal calculus is algorithmic, geometric, and physically meaningful. Fractal calculus F^α on fractals $F \subset \mathbb{R}$ is involves derivatives & integrals of order $\alpha \in (0, 1]$, where α is the dimension of F. In developing the framework, we consider both Reimann & Reimann-Stieltjes types of integrals, thereby accommodating the needs of fractal & irregular domains. S_F^α is a generalization of the Cantor staircase function adapted to the fractal set F. For some elementary functions, the F^α-integrals & derivatives can be easily obtained. Sometimes, F^α-differential equations are transformed into ordinary differential equations (ODEs) to find solutions. Some standard results in classical calculus and function spaces have been adapted to F^α-calculus [6, 7].

Golmankhaneh et al. [8] extended the Riemann-Stieltjes calculus to functions supported on fractal sets, thereby introducing fractal derivatives of functions concerning other fractal functions and exploring their properties. To expand the scope of fractals, we are interested in defining a distance function (metric) on the class of all functions defined over a fractal set that is Riemann-Stieltjes derivable concerning a given function. The study of metric spaces and their generalizations has made a significant contribution to the development of modern mathematical analyses. Constructing a metric over such a collection, where traditional Euclidean metrics are inadequate, enables the exploration of their analytical properties, thus providing a foundation for various applications in the mathematical & applied sciences. By developing this framework, we provide a deeper exploration of fractal-based differential equations and the fixed point theory by providing interesting ways into applications in mathematical analyses & mechanics.

The development of fractal calculus has significantly advanced both theoretical & applied aspects of analyses on fractal sets. For example, fractional memristor neural networks have been addressed using adaptive control techniques [9], while numerical schemes that involve the Hausdorff fractal distance have been proposed to solve Poisson equations with Hausdorff derivatives [10]. In particular, Kao et al. [11] investigated projective synchronization in uncertain fractional reaction-diffusion systems that provide a robust framework to control complex fractional-order spatial systems. Moreover, fundamental solutions for Hausdorff derivative partial differential equations (PDEs), and non-Euclidean distance-based formulations have been developed to handle the complexity of fractal media [12, 13]. These contributions collectively underscore the value of fractal-based methods in addressing analytical and computational challenges in modern science and engineering. Two primary directions have emerged: the connected fractal approach, and the measure-theoretical approach for disconnected fractal subsets of $\mathbb{R}$. These frameworks provide tools to analyze classical calculus within fractal structures such as Koch curves and Cantor sets [4, 5]. A foundational concept in this domain is the adaptation of the Fundamental Theorem of Calculus, which underpins the extension from classical to fractal calculus [14].

For certain elementary functions, F^α-integrals and F^α-derivatives can be computed in a closed form.

In some instances, F^α-differential equations can be reduced to ODEs for solution purposes. Moreover, many standard results in classical calculus and function spaces have been extended to the context of F^α-calculus [6, 7]. This framework has found significant applications in physics, including the development of new models for fractal space and time.

In this framework, we examine a powerful tool, the 'measure of non-compactness' for characterizing bounded, and relatively compact fractal subsets within this space. It has a pivotal importance in fixed-point theorems and is applicable to the study of differential, and integral equations. The set-measure of non-compactness was first introduced by Kuratowski [15] in 1930. In a metric space $(\mho, d)$, for a bounded subset $\mathfrak{A}$ of $\mho$, Kuratowski defined the measure of non-compactness as follows:

$$\Delta(\mathfrak{A}) = \inf\{\epsilon > 0 : \mathfrak{A} \subseteq \bigcup_{i=1}^{s} \mathfrak{A}_i,\ \mathfrak{A}_i \subset \mho,\ \operatorname{diam}(\mathfrak{A}_i) < \epsilon,\ i = 1, \ldots, s;\ s \in \mathbb{N}\}, \tag{1.1}$$

where $\operatorname{diam}(\mathfrak{A}_i) = \sup\{d(\xi, y) : \xi, y \in \mathfrak{A}_i\}$.

Later, in 1955, Darbo [16] first used this concept in the fixed point theory and established his famous theorem: if $\mathcal{L}$ be continuous self-mappings on a nonempty, bounded, closed, and convex subset C of a Banach space E, Δ be the Kuratowski measure of non-compactness, defined in (1.1) and there exists $k \in [0, 1)$ such that

$$\Delta(\mathcal{L}(\mathcal{M})) \leq k\,\Delta(\mathcal{M}) \text{ holds for any subset } \mathcal{M} \text{ of } C,$$

then $\mathcal{L}$ has a fixed point.

The Darbo theorem extends the scope of Schauder's theorem [17] by relaxing the strict requirement of compactness that allows the use of non-compact sets as the measure of non-compactness. These make it a powerful tool in functional analyses, especially in contexts where verifying or ensuring compactness is challenging. Notably, the measure of noncompactness has been effectively employed in solving systems of integral equations, and nonlinear fractional equations, as demonstrated in [18,19].

The famous Schauder's fixed point theorem is described below:

Theorem 1.1. *[17] If $\mathcal{L}$ be a compact and continuous self mapping on a closed, convex subset C of a Banach space, then $\mathcal{L}$ has at least one fixed point in C.*

We define the measure of the non-compactness function Δ for a fractal subset of the metric space of Riemann-Stieltjes derivable functions, and study its few properties. Later, we employ Δ to establish Darbo's type fixed point theorem in a Banach space. By developing Darbo's theorem to the distance structure over the space of Riemann-Stieltjes derivable functions, we extend the concept to function spaces with more complex derivatives and non-Euclidean settings, thus offering a new dimension of applicability. In contrast, the obtained results in such spaces defined over fractals introduce non-integer dimensionality and intricate topological properties.

We subsequently apply Darbo's theorem to find a solution of the fractal α-linear ODE via its associated fractal Volterra integral equation-an area not traditionally addressed in the context of Darbo-type fixed point theorems. The fractal oscillator model serves as a mathematical instrument that builds upon conventional oscillator models by integrating fractal dimensions, thereby facilitating a more precise depiction of real-world systems characterized by complex, irregular, or self-similar structures [20]. This approach is physically relevant to phenomena such as vibrations in porous materials, wave propagation in fractal antennas, and energy dissipation in irregular structures. Notably, a fractal microelectromechanical system (MEMS) oscillator can overcome the pull-in instability completely in a fractal space [21]. Additionally, fractal oscillations exhibit behaviors distinct from traditional oscillations. Wang and colleagues uncovered the intriguing periodic characteristics of the fractional Sasa-Satsuma equation [22].

In this context, we consider the model in fractal media:

$$D_G^{2\alpha} y + \frac{w^2}{q} y = 0, \tag{1.2}$$

where w, q are constant. This fractal ordinary differential equation (FODE) shows an oscillatory behavior, thereby introducing an important physical application beyond purely mathematical generalizations. The solutions to these fractal differential equations are not differentiable in the sense of ordinary calculus. The inadequacy of traditional approaches underscores the growing relevance of fractal structures in the study of differential, and integral equations. The extended form of Darbo's theorem demonstrates an advanced understanding of fractal concepts into the fixed point theory.

The article is organized as follows. Section 2 consists of some preliminary results. In Section 3, we construct a metric space $(\mathcal{D}(\phi), d)$ over the Riemann-Stieltjes derivable function defined over a fractal set, and establish its completeness. Additionally, we define a measure of non-compactness of the subsets of that metric space. In Section 4, we define a Banach space over the linear set $\mathcal{D}(\phi)$, whose induced metric is the metric defined in Section 3. Next, we introduce a fixed point theorem using the measure of non-compactness function, which generalizes Darbo's fixed point theorem over $(\mathcal{D}(\phi), \|\cdot\|)$. Section 5 contains a significant application that demonstrates the practical utility of the developed concepts. Here, we show how the Ψ-measure reductive contraction theorem can be used in finding solutions to fractal differential equations that exhibit a non-classical sense of ordinary calculus properties. We provide a simulation example that supports the existence result, and also illustrates the practical efficiency and numerical stability of the iterative scheme under the assumptions provided in Section 5.

2. Preliminaries

The primary focus of this study is the formulation of a metric over the collection of all functions defined over a fractal set that are Riemann-Stieltjes derivable with respect to a function, and to measure the non-compactness of the subsets of that metric space and its application on the fixed point theory. To conduct this study, we need to provide the following definitions and results.

Definition 2.1. *[23] A metric space is a pair $(\mho, d)$ of a non-empty set $\mho$ equipped with a function d, which satisfies the following conditions:*

(d1) Non-negativity: $d(a, u) \geq 0$ and $d(a, u) = 0$ if and only if $a = u$;

(d2) Symmetry: $d(a, u) = d(u, a)$; and

(d3) Triangle inequality: $d(a, u) \leq d(a, \omega) + d(\omega, u)$
for all $a, u, \omega \in \mho$.

The following are probably well known.

A subset $\mathfrak{G}$ of the metric space $(\mho, d)$ is called an open set if for any $y \in \mathfrak{G}$, there exists $r > 0$ such that the open ball $B_r(y) \subset \mathfrak{G}$; it is called closed if $\mho \setminus \mathfrak{G}$ is an open set. $u \in \mho$ is called a limit point of $\mathfrak{G}$ if each open ball $B_r(y)$ is centered on u, $B_r(y) \cap (\mathfrak{G} \setminus \{u\}) \neq \Phi$. The diameter of $\mathfrak{G}$ is defined by $diam(\mathfrak{G}) = \sup\{d(a, \omega) : a, \Omega \in \mathfrak{G}\}$. $\mathfrak{G}$ is called bounded if $diam(\mathfrak{G})$ is finite. A sequence $\{r_n\}$ in $(\mho, d)$ is said to converge to $a \in \mho$ if, for any given number $E > 0$, there is $N_0 \in \mathbb{N}$ such that $d(r_n, a) < E$ for all $n \geq N_0$. $\{r_n\}$ is called a Cauchy sequence if, for any given number $E > 0$, there exists $N_0 \in \mathbb{N}$ such that $d(r_n, r_m) < E$ for all $n, m \geq N_0$. If every Cauchy sequence in $(\mho, d)$ converges to a point in $\mho$, then $(\mho, d)$ is called complete. $(\mho, d)$ is said to be compact if every open cover of $\mho$ has a finite sub-cover. Every compact subset of $(\mho, d)$ is closed, and bounded. $(\mho, d)$ is said to be sequentially compact if every sequence in $\mho$ has a convergent subsequence in $\mho$.

Definition 2.2. *[24] Let $(\mho, d)$ be a metric space, and $\mathcal{H}(\mho)$ is the set of the non-empty compact subsets of $\mho$. For subsets P, Q of $\mathcal{H}(\mho)$, the Hausdorff distance $d_H(P, Q)$ is defined as follows:*

$$d_H(P, Q) = \max\left\{\sup_{\eth \in P} d(\omega, Q), \sup_{\vartheta \in Q} d(P, \vartheta)\right\},$$

where $d(\eth, Q) = \inf_{\vartheta \in Q} d(\eth, \vartheta)$, and $d(P, \vartheta) = \inf_{\eth \in P} d(\eth, \vartheta)$.

Next, we recall some properties of the measure of the non-compactness function Δ defined on subsets of a metric space.

Theorem 2.3. *[25] Let $A, \mathfrak{A}$ be bounded subsets of a complete metric space $(\mho, d)$. Then,*

(i) $\Delta(\mathfrak{A}) = 0 \iff \overline{\mathfrak{A}}$ *is compact;*

(ii) $\Delta(\mathfrak{A}) = \Delta(\overline{\mathfrak{A}})$*;*

(iii) If $A \subseteq \mathfrak{A}$ *then* $\Delta(A) \leq \Delta(\mathfrak{A})$*;*

(iv) $\Delta(A \cup \mathfrak{A}) = \max\{\Delta(A), \Delta(\mathfrak{A})\}$*;*

(v) $\Delta(A \cap \mathfrak{A}) \leq \min\{\Delta(A), \Delta(\mathfrak{A})\}$*;*

(vi) $|\Delta(A) - \Delta(\mathfrak{A})| \leq 2d_H(A, \mathfrak{A})$.

Throughout this paper, we assume that $(E, \|\cdot\|)$ is a normed linear space. For $\mathfrak{X} \subset E$, $\overline{\mathfrak{X}}$, and $conv(\mathfrak{X})$ denote the closure, and closed convex hull of $\mathfrak{X}$, respectively. The set of all nonempty bounded subsets of E is denoted by $\mathcal{M}_E$, and $\mathcal{N}_E$ stands for the subfamily consisting of all compact subsets of E.

Proposition 2.4. *[16] Let $\Delta : \mathcal{M}_E \to [0, \infty]$ be the measure of non-compactness function in E; then, it satisfies the following conditions:*

(i) The kernel of Δ, $\mathrm{Ker}(\Delta) = \{\mathfrak{X} \in \mathcal{M}_E : \Delta(\mathfrak{X}) = 0\}$ *is nonempty, and* $\mathrm{Ker}(\Delta) \subseteq \mathcal{N}_E$*;*

(ii) $\mathfrak{X} \subseteq \mathcal{Y} \Rightarrow \Delta(\mathfrak{X}) \leq \Delta(\mathcal{Y})$*;*

(iii) $\Delta(\overline{\mathfrak{X}}) = \Delta(\mathrm{Conv}(\mathfrak{X})) = \Delta(\mathfrak{X})$*;*

(iv) For any $\lambda \in [0, 1]$, $\Delta(\lambda\mathfrak{X} + (1 - \lambda)\mathcal{Y}) \leq \lambda\Delta(\mathfrak{X}) + (1 - \lambda)\Delta(\mathcal{Y})$*; and*

(v) If $\{\mathfrak{X}_n\} \subseteq \mathcal{M}_E$ *is a sequence of closed sets such that* $\mathfrak{X}_{n+1} \subseteq \mathfrak{X}_n$ *for all* $n \geq 1$, *and* $\lim_{n\to\infty} \Delta(\mathfrak{X}_n) = 0$, *then* $\mathfrak{X}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{X}_n \neq \Phi$.

Now, we collect key definitions from fractal calculus and Riemann-Stieltjes derivable functions to set our work of metric structures using this framework.

For two fractal sets $\mathfrak{F}, \mathfrak{G} \subset [p, q] \subset \mathbb{R}$, $\mathfrak{F} \times \mathfrak{G}$ denotes the Cartesian product of fractal sets, and is defined by $\mathfrak{F} \times \mathfrak{G} = \{(p, q) : p \in \mathfrak{F}, q \in \mathfrak{G}\}$.

Definition 2.5. *[6] Let $J = [p, q] \subset \mathbb{R}$. The indicator function I of $\mathfrak{F}$ is defined by the following:*

$$I(\mathfrak{F}, J) = \begin{cases} 1, & \text{if } \mathfrak{F} \cap J \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 2.6. *[6] The coarse-grained measure of $\mathfrak{F} \cap [p, q]$ is defined as follows:*

$$\Delta_\delta^\alpha(\mathfrak{F}, p, q) = \inf_{|\mathfrak{P}| \leq \delta} \sum_{\iota=0}^{n-1} \Gamma(\alpha + 1)(\xi_{\iota+1} - \xi_\iota)^\alpha I(\mathfrak{F}, [\xi_\iota, \xi_{\iota+1}]),$$

where $|\mathfrak{P}| = \max_{0 \leq \iota \leq n-1} (\xi_{\iota+1} - \xi_\iota)$, where $[p, q] = \{\xi_0 = p, \xi_1, \ldots, \xi_n = q\}$, and $0 < \alpha \leq 1$.

Definition 2.7. *[6] The measure function of $\mathfrak{F}$ is defined as follows:*

$$\Delta^\alpha(\mathfrak{F}, p, q) = \lim_{\delta \to 0} \Delta_\delta^\alpha(\mathfrak{F}, p, q).$$

Definition 2.8. *[6] Let $a_0 \in \mathbb{R}$ be a fixed number. Then, the integral staircase function of $\mathfrak{F}$ is defined as follows:*

$$S_{\mathfrak{F}}^\alpha(\xi) = \begin{cases} \Delta^\alpha(\mathfrak{F}, a_0, \xi), & \text{if } \xi \geq a_0, \\ -\Delta^\alpha(\mathfrak{F}, \xi, a_0), & \text{otherwise}. \end{cases}$$

Definition 2.9. *[6] Let $0 < \alpha \leq 1$. If $\gamma^\alpha(\mathfrak{F}, p, q)$ is finite, then for all $\xi, \vartheta \in (p, q)$ such that $\xi < \vartheta$, the following conditions hold:*

(i) $S_{\mathfrak{F}}^\alpha(\xi)$ is increasing in ξ;

(ii) If $\mathfrak{F} \cap (\xi, \vartheta) = \phi$, then $S_{\mathfrak{F}}^\alpha$ is a constant in $[\xi, \vartheta]$;

(iii) $S_{\mathfrak{F}}^\alpha(\vartheta) - S_{\mathfrak{F}}^\alpha(\xi) = \gamma^\alpha(\mathfrak{F}, \xi, \vartheta)$; and

(iv) $S_{\mathfrak{F}}^\alpha$ is continuous on (p, q).

Definition 2.10. *[6] The fractal dimension of $\mathfrak{F} \cap [p, q]$ is defined as follows:*

$$\dim_\gamma(\mathfrak{F} \cap [p, q]) = \inf\{\alpha : \Delta^\alpha(\mathfrak{F}, p, q) = 0\} = \sup\{\alpha : \Delta^\alpha(\mathfrak{F}, p, q) = \infty\}.$$

Definition 2.11. *[7] The $\mathfrak{F}$-limit of a function $\Xi : \mathfrak{F} \to \mathbb{R}$ is defined as follows: for given any $\varkappa > 0$, there exists a positive E such that $\vartheta \in \mathfrak{F}$ and $|\vartheta - \xi| < E \implies |\Xi(\vartheta) - \Xi(\xi)| < \varkappa$.*

Definition 2.12. *[7] A function $\Xi : \mathfrak{F} \to \mathbb{R}$ is said to be $\mathfrak{F}$-continuous at $\xi \in \mathfrak{F}$ if $\Xi(\xi) = \mathfrak{F}\text{-}\lim_{\vartheta \to \xi} \Xi(\vartheta)$ whenever the $\mathfrak{F}\text{-}\lim$ exists.*

The concept of fractal differentiability naturally extends from the classical idea of the derivative when viewed through the lens of Riemann-Stieltjes integration. As discussed by Castillo and Chapinz [14], the Fundamental Theorem of Calculus for the Riemann-Stieltjes integral offers a versatile tool to generalize differentiation with respect to a non-constant integrator. This insight motivates the definition of a ϕ-fractal derivative.

Definition 2.13. *[8] Suppose $\xi_0 \in \mathfrak{F}$ and $\Xi, \phi : \mathfrak{F} \to \mathbb{R}$ are two functions such that ϕ is $\mathfrak{F}$-continuous, and monotonically increasing. Then, Ξ is said to be ϕ-fractal differentiable at ξ_0 if*

$$\mathfrak{F}\text{-}\lim_{\xi \to \xi_0} \frac{\Xi(\xi) - \Xi(\xi_0)}{\phi(\xi) - \phi(\xi_0)} \tag{2.1}$$

exists. If this limit exists, then its value is denoted by $D^{\alpha}_{\mathfrak{F},\phi}\Xi(\xi_0)$, which is called the $\mathfrak{F}^{\alpha}_{\phi}(\xi)$-derivative of Ξ at ξ_0.

It is important to note that the Riemann-Stieltjes derivative to the fractal set is different from the Riemann-Stieltjes derivative; here, $\phi(\xi)$ is monotonically increasing but not differentiable in the conventional sense of ordinary calculus. Of course, if $\phi(\xi) = S^{\alpha}_{\Xi}(\xi)$, then the $\mathfrak{F}^{\alpha}_{\phi(\xi)}$-derivative of Ξ is a $\mathfrak{F}^{\alpha}$-derivative of Ξ at ξ_0.

Since we use fractal Volterra integral equations to find a solution of periodic ODEs in our main result, we recall its definition from [26] by A. K. Golmankhaneh et al.

Definition 2.14. *(i) [26] A first kind fractal Volterra integral equation is defined as follows:*

$$\Xi(\xi) = \int_a^{\xi} \mathfrak{K}(\xi, t)\, \phi(t)\, d^{\alpha}_{\mathfrak{F}}t. \tag{2.2}$$

(ii) [26] A second kind fractal Volterra integral equation is defined as follows:

$$\phi(\xi) = \Xi(\xi) + \lambda \int_a^{\xi} \mathfrak{K}(\xi, t)\, \phi(t)\, d^{\alpha}_{\mathfrak{F}}t, \tag{2.3}$$

where $\Xi : \mathfrak{F} \to \mathbb{R}$ is a known function, $\lambda \in \mathbb{R}$ is a parameter, $\mathfrak{K} : \mathfrak{F} \times \mathfrak{F} \to \mathbb{R}$ is the kernel, and ϕ is the unknown function we have to solve for.

3. Metric space over the collection of Riemann-Stieltjes derivable functions, and measure of non-compactness function

In this section, we introduce a metric function over the collection of Riemann-Stieltjes derivable functions defined over a fractal set $G \subset \mathbb{R}$ that presents an analytical foundation to study the applications of fractal-based differential equations and the fixed point theory in mathematical analyses and mechanics.

Suppose $\phi : G \to \mathbb{R}$ is a G-continuous, and monotone increasing function. In this article, $\mathcal{D}(\phi)$ denotes the set of all functions $g : G \to \mathbb{R}$ that are Riemann-Stieltjes derivable with respect to the function ϕ. Next, we define a function $d : \mathcal{D}(\phi) \times \mathcal{D}(\phi) \to \mathbb{R}$ by the following:

$$d(f, \Xi) = \sup_{\xi \in G} \left| f(\xi) - \Xi(\xi) \right| + \sup_{\xi \in G} \left| D^{\alpha}_{G,\phi(\xi)} f(\xi) - D^{\alpha}_{G,\phi(\xi)} \Xi(\xi) \right|, \tag{3.1}$$

where $D^{\alpha}_{G,\phi} f(\xi)$ is the Riemann-Stieltjes derivative of f with respect to the function ϕ over the fractal set G. Then, the function d is a metric on $\mathcal{D}(\phi)$.

Remark 3.1. *The expression in Eq (3.1) involves the application of a Riemann-Stieltjes derivative in the context of fractal calculus. More precisely, the derivative $D^{\alpha}_{G,\phi}(.)$ is the Ξ-derivative given in*

Definition 2.13, where Ξ is considered $\mathfrak{F}$-differentiable with respect to a $\mathfrak{F}$-continuous, and increasing function ϕ. The operator defined in (2.1) defines the derivative $D^{\alpha}_{G,\phi}(.)$, which we use to build the metric structure in this section. Therefore, the derivative operator in Eq (3.1) should be interpreted as a generalized Riemann-Stieltjes derivative adapted to the fractal set, where the involved functions are $\mathfrak{F}$-differentiable. For clarity, we will denote it consistently as $D^{\alpha}_{G,\phi}(.)$ throughout the article.

Theorem 3.2. *Every subset $\mathcal{M}$ of $\mathcal{D}(\phi)$ is compact if every sequence $\{f_l\}$ in $\mathcal{M}$ has a convergent subsequence in $\mathcal{M}$.*

Proof. With the given assumption, to prove $\mathcal{M} \subseteq \mathcal{D}(\phi)$ is compact, it is enough to prove that $\mathcal{M}$ is closed, and bounded.

First, we show that $\mathcal{M} \subset \mathcal{D}(\phi)$ is bounded. For the contradiction, assume that $\mathcal{M}$ is unbounded. This means that for some $f_0 \in \mathcal{D}(\phi)$, and for every $l \in \mathbb{N}$, there exists $f_l \in M$ such that

$$d(f_l, f_0) > l.$$

Therefore, we obtain a sequence $\{f_l\} \subset \mathcal{M}$ such that

$$\begin{aligned} &d(f_l, f_0) \to \infty \text{ as } l \to \infty \\ or\ &\sup_{\xi\in G} |f_l(\xi) - f_0(\xi)| + \sup_{\xi\in G} \left|D^{\alpha}_{G,\phi(\xi)} f_l(\xi) - D^{\alpha}_{G,\phi(\xi)} f_0(\xi)\right| \to \infty \quad \text{as } l \to \infty. \end{aligned}$$

This implies that either $\sup_{\xi\in G} \left|f_l(\xi) - f_0(\xi)\right|$ or $\sup_{\xi\in G} \left|D^{\alpha}_{G,\phi(\xi)} f_l(\xi) - D^{\alpha}_{G,\phi(\xi)} f_0(\xi)\right|$ escapes to infinity.

In particular, this implies that either or both of them are unbounded, and cannot contain any convergent subsequence, because convergent sequences in metric spaces are bounded, and any subsequence of $\{f_l\}$ would also diverge in the metric from f_0. Hence, the subsequence also cannot converge. However, this contradicts the assumed condition of the stated theorem. Therefore, $\mathcal{M}$ must be bounded.

Let $\{f_r\}$ be a sequence in $\mathcal{M}$ that converges to some $f \in \mathcal{D}(\phi)$. By assumption, there exists a subsequence $\{f_{r_k}\}$ of $\{f_l\}$ that converges to some element in M. However, $\{f_r\}$ converges to f, and since limits of a subsequence is same as the mother sequence in metric spaces, so any convergent subsequence must also converge to f. Therefore, the subsequence $\{f_{l_k}\}$ converges to f, and the assumption of the theorem guarantees that the limit f must be in $\mathcal{M}$. Therefore, $\mathcal{M}$ is closed. Hence, the proof is complete. □

In our next theorem, we prove the completeness of $(\mathcal{D}(\phi), d)$ followed by the next lemma.

Lemma 3.3. *Let $\{f_l(\xi)\}$ be a sequence in $\mathcal{D}(\phi)$ which converges uniformly to $f : G \to \mathbb{R}$. Then, the limit function $f \in \mathcal{D}(\phi)$, and $\lim_{l\to\infty} D^{\alpha}_{G,\phi(\xi)} f_l(\xi) = D^{\alpha}_{G,\phi(\xi)} f(\xi)$, $\xi \in G$.*

Proof. Since each $f_l \in \mathcal{D}(\phi)$, $l \in \mathbb{N}$, so the limit $D^{\alpha}_{G,\phi(\xi)} f_l(\xi)$ exists for each $\xi \in G$ where $D^{\alpha}_{G,\phi(\xi_0)} f_l(\xi_0) = G\text{-}\lim_{\xi\to\xi_0} \frac{f_n(\xi)-f_n(\xi_0)}{\phi(\xi)-\phi(\xi_0)}$. Again, $f_l \to f$ uniformly implies $f_l(c) \to f(c)$ as $l \to \infty$ for each $c \in G$. Then, $f_l(\xi)$ and $f_l(\xi_0)$ converge to $f(\xi)$ and $f(\xi_0)$ respectively. Again, since ϕ is monotonic increasing and continuous in any nbd of ξ_0, the term $\{\phi(\xi) - \phi(\xi_0)\}$ is bounded and does not admit zero value at any case. Hence, the limit $G\text{-}\lim\limits_{\xi\to\xi_0}\frac{f(\xi)-f(\xi_0)}{\phi(\xi)-\phi(\xi_0)}$ exists finitely. Therefore, $f \in \mathcal{D}(\phi)$. Now, interchanging the limit, we obtain the following:

$$\lim_{n\to\infty} G\text{-}\lim_{\xi\to\xi_0}\frac{f_n(\xi)-f_n(\xi_0)}{\phi(\xi)-\phi(\xi_0)} = G\text{-}\lim_{\xi\to\xi_0}\frac{f(\xi)-f(\xi_0)}{\phi(\xi)-\phi(\xi_0)}$$
$$or\ \lim_{n\to\infty} D^{\alpha}_{G,\phi(\xi_0)} f_n(\xi_0) = G\text{-}\lim_{\xi\to\xi_0}\frac{f(\xi)-f(\xi_0)}{\phi(\xi)-\phi(\xi_0)}.$$

This yields $\lim\limits_{n\to\infty} D^{\alpha}_{G,\phi(\xi_0)}f_n(\xi_0) = D^{\alpha}_{G,\phi(\xi_0)}f(\xi_0)$. □

Theorem 3.4. *The pair $(\mathcal{D}(\phi), d)$ is a complete metric space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{D}(\phi)$. Then, for a given positive number ϵ, there exists $\mathfrak{N} \in \mathbb{N}$ such that

$$\begin{aligned}
&d(f_l, f_r) < \epsilon \quad \forall l, r \geq \mathfrak{N}\\
\implies &\sup_{\xi\in G}|f_l(\xi)-f_r(\xi)| + \sup_{\xi\in G}|D^{\alpha}_{G,\phi(\xi)}f_l(\xi) - D^{\alpha}_{G,\phi(\xi)}f_r(\xi)| < \epsilon \quad \forall l, r \geq \mathfrak{N}\\
\implies &\sup_{\xi\in G}|f_l(\xi)-f_r(\xi)| < \epsilon\ \ \forall l, r \geq N\ \ \&\ \ \sup_{\xi\in G}|D^{\alpha}_{G,\phi(\xi)}f_l(\xi) - D^{\alpha}_{G,\phi(\xi)}f_r(\xi)| < \epsilon\ \ \forall l, r \geq \mathfrak{N}\\
\implies &|f_l(\xi)-f_r(\xi)| < \epsilon\ \ \forall l, r \geq \mathfrak{N}\ \ \&\ \ |D^{\alpha}_{G,\phi(\xi)}f_l(\xi) - D^{\alpha}_{G,\phi(\xi)}f_r(\xi)| < \epsilon\ \ \forall l, r \geq \mathfrak{N}\ \ \&\ \ \forall \xi \in G.
\end{aligned}$$

This expounds that for each $c \in G$, both $\{f_n(c)\}$ and $\{D^{\alpha}_{G,\phi(c)}f_n(c)\}$ are Cauchy sequences of real numbers. Thus, there exists a function $f : G \to \mathbb{R}$ such that for each $c \in G$, $f_n(c) \to f(c)$ pointwise as $n \to \infty$. Similarly, for each $c \in G$, $D^{\alpha}_{G,\phi(c)}f_n(c)$ converges to some limit $g(c)$ where $g : G \to \mathbb{R}$. We claim that $f \in \mathcal{D}(\phi)$, and $g(c) = D^{\alpha}_{G,\phi(c)}f(c),\ c \in G$.

Since $d(f_l, f_r) \to 0$ as $l, r \to \infty$ implies the sequences $\{f_l\}$ and $\{D^{\alpha}_{G,\phi}f_l\}$ converge uniformly to f and g, respectively. Again, each $f_l \in \mathcal{D}(\phi)$ and $\{f_l\}$ converge uniformly to f; therefore, using Lemma 3.3 we conclude that $f \in \mathcal{D}(\phi)$ and $\lim\limits_{l\to\infty} D^{\alpha}_{G,\phi(c)}f_l(c) = D^{\alpha}_{G,\phi(c)}f(c)$, $c \in G$. Hence, the uniqueness of the limit point implies $g(c) = D^{\alpha}_{F,\phi(c)}f(c)$ for each $c \in F$. Thus, the limit point of $\{f_l\}$ lies in $\mathcal{D}(\phi)$, thus concluding the proof. □

3.1. Measure of non-compactness over fractal set of $\mathcal{D}(\phi)$

Let $\mathcal{M}$, $\mathfrak{S}$ be subsets of the metric space $(\mathcal{D}(\phi), d)$, and $\mathfrak{E} > 0$; then, $\mathfrak{S}$ is called $\mathfrak{E}$-net of $\mathcal{M}$ if there exists $g \in \mathfrak{S}$ such that $d(f, g) < \mathfrak{E}$. If the set $\mathfrak{S}$ is finite, then the $\mathfrak{E}$-net of $\mathcal{M}$ is called finite $\mathfrak{E}$-net. Then, we can define the measure of non-compactness over a fractal set G on $\mathcal{D}(\phi)$ as

$$\inf\ \{\mathfrak{E} > 0 : G \subseteq \bigcup_{i=1}^{\mathfrak{N}} \mathfrak{A}_i \text{ where } \mathfrak{A}_i \subseteq \mathcal{D}(\phi),\ i = 1, 2, \cdots, \mathfrak{N} \text{ with } diam(\mathfrak{A}_i) \leq \mathfrak{E}\},$$

and denote is $\Delta(F)$, where $diam(\mathfrak{A}_i)$ denotes the diameter of the set $\mathfrak{A}_i$ defined by $diam(\mathfrak{A}_i) = \sup\{d(f, g) : f, g \in \mathfrak{A}_i\}$, and d stands for the metric in $\mathcal{D}(\phi)$ defined in (3.1).

Remark 3.5. *The functional Δ adapts Kuratowski's classical measure of noncompactness to the fractal setting by computing diameters with respect to a fractal metric space $(\mathcal{D}(\phi), d)$, in which the traditional*

Euclidean notions are replaced by fractal-modified norms, and fractional orders. Unlike classical measures based on Euclidean balls, here Δ reflects the underlying fractal geometry, and measure (e.g., involving α-order differences), and can capture both the irregularity and non-integer dimensionality, which is crucial to analyze operators on fractal function spaces.

The basic properties are held by Δ, and can be proven in a similar logic to Theorem 2.3.

Theorem 3.6. *Let $\mathscr{O}$, $\mathscr{A}$ be subsets of $\mathcal{D}(\phi)$:*

(i) $\mathscr{O}$ *is bounded if and only if* $\Delta(\mathscr{O}) < \infty$*;*

(ii) If $\mathscr{O}$ *is compact, then* $\Delta(\mathscr{O}) = 0$*; conversely, if* $\Delta(\mathscr{O}) = 0$ *and* $\mathscr{O}$ *is complete, then* $\mathscr{O}_1$ *is compact;*

(iii) If $\mathscr{O} \subset \mathscr{A}$*, then* $\Delta(\mathscr{O}) \leq \Delta(\mathscr{A})$*;*

(iv) $\Delta(\mathscr{O} \cup \mathscr{A}) = \max\{\Delta(\mathscr{O}), \Delta(\mathscr{A})\}$*;*

(v) $\Delta(\mathscr{O} \cap \mathscr{A}) \leq \min\{\Delta(\mathscr{O}), \Delta(\mathscr{A})\}$*; the equality does not hold in general; and*

(vi) Continuity with respect to the Hausdorff Distance: $|\Delta(\mathscr{O}) - \Delta(\mathscr{A})| \leq 2d_H(\mathscr{O}, \mathscr{A})$*.*

4. Fixed point theorem on normed linear space

In this section, we establish of Darbo's type fixed point theorem. Before that, recall the set $\mathcal{D}(\phi)$ of Section 3. Then, observe that $\mathcal{D}(\phi)$ is a real vector space with usual operations of pointwise addition and scalar multiplication. Henceforth, using the existing metric d of Section 3, we can define a norm $\|\cdot\|$ on $\mathcal{D}(\phi)$ as follows:

$$\|f\| = d(f, 0) = \sup_{\xi \in G} |\, f(\xi)\,| + \sup_{\xi \in G} |\, D^{\alpha}_{G,\phi(\xi)} f(\xi)\,|, \ \forall f \in \mathcal{D}(\phi). \tag{4.1}$$

Since $(\mathcal{D}(\phi), d)$ is a complete metric space and the defined norm $\|\cdot\|$ induced the metric d, the space $(\mathcal{D}(\phi), \|\cdot\|)$ forms a Banach space, which enables us to apply various fixed point theorems.

Now, we establish a measure of non-compactness in $(\mathcal{D}(\phi), \|\cdot\|)$. For a subset $\mathcal{M}$ of $\mathcal{D}(\phi)$, we recall the definition of the diameter of $\mathcal{M}$, $diam(\mathcal{M}) = \sup\{\,\|a - \vartheta\| : a, \vartheta \in \mathcal{M}\,\}$.

Definition 4.1. *Let U denote a closed unitary ball in $\mathcal{D}(\phi)$. We define the Hausdorff measure of non-compactness of a bounded subset $B \subset \mathcal{D}(\phi)$ by the following:*

$$\Delta(B) = \inf\{\varepsilon > 0 : \textit{there exists a finite fractal set } \mathfrak{F} \subset \mathcal{D}(\phi) \textit{ such that } B \subset \mathfrak{F} + \varepsilon\, U\}.$$

The basic properties of Δ hold in $(\mathcal{D}(\phi), \|\cdot\|)$.

Definition 4.2. *Let $\mathfrak{M}, \mathcal{N}$ be subsets of $\mathcal{D}(\phi)$. Then, following properties are true:*

(i) $\Delta(a\mathfrak{M}) = a\Delta(\mathfrak{M})$ *for any scalar a;*

(ii) $\Delta(\mathfrak{M}) = \Delta(\overline{\mathfrak{M}}) = \Delta(conv(\mathfrak{M}))$*, where* $\overline{\mathfrak{M}}$ *and* $conv(\mathfrak{M})$ *denote the closure, and convex hull of* $\mathfrak{M}$*; and*

(iii) $\Delta(\mathfrak{M} + \mathcal{N}) \leq \Delta(\mathfrak{M}) + \Delta(\mathcal{N})$.

The proofs are similar to Proposition 2.4.

In this connection, we define the following contraction mapping and establish the criterion of existence of a fixed point.

Definition 4.3. *Let $(\mho, d)$ be a metric space, and $\mathcal{L} : \mho \to \mho$ be a bounded continuous mapping. If for some $k \in (0,1)$, T satisfies*

$$\Delta(\mathcal{L}(\mathfrak{Q})) \leq \Delta(\mathfrak{Q}) - \Psi(\Delta(\mathfrak{Q})), \tag{4.2}$$

for all bounded subsets $\mathfrak{Q}$ of $\mho$, where Δ is the measure of non-compactness function on $\mho$, and $\Psi : [0,\infty) \to [0,\infty)$ is monotone increasing function with the property $\Psi(t) = 0 \iff t = 0$, then $\mathcal{L}$ is said to be a $\Psi -$ measure reductive contraction.

Theorem 4.4. *Let $\mathfrak{C}$ be a nonempty closed bounded subset of a Banach space $(E, \|\cdot\|)$, and $\mathcal{L} : \mathfrak{C} \to \mathfrak{C}$ be a Ψ-measure reductive contraction; then, $\mathcal{L}$ has a fixed point in E.*

Proof. We define a sequence $\{\mathfrak{C}_n\}$ by letting $\mathfrak{C}_0 = \mathfrak{C}$, $\mathfrak{C}_n = conv(\mathcal{L}\mathfrak{C}_{n-1})$, $n \geq 1$. Then, we have the following:

$$\mathcal{L}\mathfrak{C}_0 = \mathcal{L}\mathfrak{C} \subseteq \mathfrak{C} = \mathfrak{C}_0,\ \mathfrak{C}_1 = conv(\mathcal{L}\mathfrak{C}_0) \subseteq conv(\mathfrak{C}_0) = \mathfrak{C}_0,\ \mathfrak{C}_2 = conv(\mathcal{L}\mathfrak{C}_1) \subseteq conv(\mathfrak{C}_1) = \mathfrak{C}_1.$$

Continuing this process, we obtain the following:

$$\mathfrak{C}_0 \supseteq \mathfrak{C}_1 \supseteq \mathfrak{C}_2 \supseteq \cdots.$$

Now, if there is a $K \in \mathbb{N}$ such that $\Delta(\mathfrak{C}_K) = 0$, then $\mathfrak{C}_K$ is compact since $\mathcal{L}\mathfrak{C}_K \subset conv(\mathcal{L}\mathfrak{C}_K) = \mathfrak{C}_{K+1} \subset \mathfrak{C}_K$. In this case, Theorem 1.1 ensures the existence of a fixed point for $\mathcal{L}$.

Thus, we can assume that, $\Delta(\mathfrak{C}_l) \neq 0$ for all $l \in \mathbb{N} \cup \{0\}$. Again, for any $n \in \mathbb{N}$,

$$\begin{aligned}\Delta(\mathfrak{C}_{n+1}) =& \Delta(conv(\mathcal{L}\mathfrak{C}_n))\\ =& \Delta(\mathcal{L}\mathfrak{C}_n)\\ \leq& \Delta(\mathfrak{C}_n) - \Psi(\Delta(\mathfrak{C}_n)) \text{ (by condition (4.2))}\\ <& \Delta(\mathfrak{C}_n).\end{aligned}$$

This shows that $\{\Delta(\mathfrak{C}_n)\}$ is a monotone decreasing sequence bounded below by zero. Thus, there exists $r \in [0, \infty)$ such that $\lim_{n\to\infty} \Delta(\mathfrak{C}_n) = r$. We claim that $r = 0$. On the contradictory, assume that $r \neq 0$. Then,

$$\begin{aligned}&\Delta(\mathfrak{C}_{n+1}) \leq \Delta(\mathfrak{C}_n) - \Psi(\Delta(\mathfrak{C}_n))\\ \Longrightarrow& \lim_{n\to\infty} \Delta(\mathfrak{C}_{n+1}) \leq \lim_{n\to\infty} \Delta(\mathfrak{C}_n) - \lim_{n\to\infty} \Psi(\Delta(\mathfrak{C}_n))\\ \Longrightarrow& \lim_{n\to\infty} \Psi(\Delta(\mathfrak{C}_n)) \leq 0\\ \Longrightarrow& \lim_{n\to\infty} \Psi(\Delta(\mathfrak{C}_n)) = 0\\ \Longrightarrow& \lim_{n\to\infty} \Delta(\mathfrak{C}_n) = 0.\end{aligned}$$

Hence, we arrived at a contradiction. Thus, we get $r = 0$.

Since $\mathfrak{C}_{n+1} \subseteq \mathfrak{C}_n$, and $\mathcal{L}\mathfrak{C}_n \subseteq \mathfrak{C}_n$, $\forall n \geq 1$, then, from the property (vi) of Proposition 2.4, we obtain that $\mathfrak{C}_\infty = \cap_{n=1}^{\infty} \mathfrak{C}_n$ is a nonempty convex closed bounded set that is invariant under $\mathcal{L}$ and belongs to $Ker(\Delta)$. Hence, Theorem 1.1 gives the desired result. □

The following example illustrates the applicability of our theorem, which demonstrates the existence of a fixed point for a self-mapping defined on a closed, bounded, and convex subset of $\mathcal{D}(\phi)$ over the Cantor set.

Example 4.5. *Let G be the Cantor set, and $\phi : G \to \mathbb{R}$ be defined by $\phi(\xi) = \xi^2$, $\forall \xi \in G$, which is continuous and monotone increasing on G. Consider the closed bounded convex subset $\mathfrak{C} = \{f \in \mathcal{D}(\phi) : \|f\| \leq 1\}$ of $\mathcal{D}(\phi)$, and define a mapping $\mathcal{L} : \mathfrak{C} \to \mathfrak{C}$ by $\mathcal{L}(f) = \frac{f}{2}$, for all $f \in \mathfrak{C}$. Then,*

$$\| f \| = \sup_{\xi \in G} \left| f(\xi) \right| + \sup_{\xi \in G} \left| D^{\alpha}_{G,\phi(\xi)} f(\xi) \right|$$

implies

$$\begin{aligned} \|\mathcal{L}f\| = \sup_{\xi\in G}\left|\mathcal{L}f(\xi)\right| + \sup_{\xi\in G}\left|D^{\alpha}_{G,\phi(\xi)}\mathcal{L}f(\xi)\right| &= \sup_{\xi\in G}\left|\frac{f(\xi)}{2}\right| + \sup_{\xi\in G}\left|D^{\alpha}_{G,\phi(\xi)}\frac{f(\xi)}{2}\right| \\ &= \frac{1}{2}\|f\| \leq \frac{1}{2} < 1. \end{aligned}$$

Thus, $\mathcal{L}(\mathfrak{C}) \subset \mathfrak{C}$. Therefore, $\mathcal{L}$ is a bounded and continuous mapping.

Let $\Psi : [0, \infty) \to [0, \infty)$ by $\Psi(\tau) = \frac{\tau}{3}$, $\forall \tau \in [0, \infty)$, which is a monotone increasing function that satisfies $\Psi(\tau) = 0 \iff \tau = 0$.

Since $\mathcal{L}$ scales each function of $\mathfrak{C}$ by $\frac{1}{2}$, for $Q \subset \mathfrak{C}$, we have the following:

$$\begin{aligned} \Delta(\mathcal{L}(Q)) &= \frac{1}{2}\Delta(Q) \\ &< \frac{2}{3}\Delta(Q) = \Delta(Q) - \Psi(\Delta(Q)). \end{aligned}$$

Thus, $\mathcal{L}$ meets the conditions of Theorem 4.4, and henceforth, $\mathcal{L}$ has a fixed point in $\mathfrak{C}$, which is the null mapping.

5. Application by solving fractal α-linear differential equation in a mechanic perspective

In this section, we analyze the fractal α-linear differential equation and establish the existence of its solutions by transforming it into its equivalent fractal Volterra integral equation form. This approach ensures the solvability of the fractal differential equation in the metric framework of Riemann-Stieltjes derivable functions over a fractal set as an application of Theorem 4.4.

For this, we consider the Banach space $(\mathcal{D}(\phi), \|\cdot\|)$ of Riemann-Stieltjes derivable functions over a fractal set $G \subseteq [p, q] \subset \mathbb{R}$ equipped with norm defined on (4.1).

Next, recall the integral staircase function $S^{\alpha}_{G}(\xi)$ defined for the bounded fractal set G with the fractal dimension α, and $d^{\alpha}_{G}\tau$, the fractal measure. Since we consider S^{α}_{G} over the bounded fractal set G, so we can find the non-negative constant M such that

$$| S^{\alpha}_{G}(\xi) - S^{\alpha}_{G}(\tau) | \leq M, \ \forall \xi, \tau \in G.$$

Again, there exists $A > 0$ such that $\left|\int_a^{\xi} d^{\alpha}_{G}\tau\right| \leq A$. Additionally, let $M' = \sup_{\xi,\tau\in G}\left|D^{\alpha}_{G,\phi(\xi)}\left(S^{\alpha}_{G}(\xi) - S^{\alpha}_{G}(\tau)\right)\right|$.

Now, we consider the following fractal linear oscillator equation:

$$D_G^{2\alpha} y + \frac{w^2}{R} y = 0 \text{ with initial conditions } y(a) = 0 \ \& \ D_G^{\alpha} y(a) = c, \tag{5.1}$$

where w, R are constant with $R = (M + M')A + 1$.

This equation plays a crucial role in demonstrating the practical significance of the developed metric structure and the fixed point framework. It involves the generalized fractional derivative D_G^{α} that models dynamic systems where fractal-like behaviors and memory effects are present, making it highly relevant in mechanics and complex systems analyses. By solving this equation within our framework, we demonstrate how the proposed approach effectively addresses noncompactness in function spaces, thus offering a novel perspective on fractal differential equations, and their applications. Such kinds of equations are specially used in vibrational mechanics, wave propagation, or oscillatory systems in fractal media. The presence of the fractional derivative D_G^{α} indicates that the system exhibits memory effects and anomalous dynamics, which are common in fracture mechanics and nonlinear dynamical systems.

Using Eqs (4.1) and (4.9) of [26], we can write the fractal Volterra integral equation that corresponds to the FODE equation (5.1) as follows:

$$y(\xi) = c\, S_G^{\alpha}(\xi) + \frac{w^2}{R} \int_a^{\xi} \Big(S_G^{\alpha}(\tau) - S_G^{\alpha}(\xi) \Big) y(\tau)\, d_G^{\alpha}\tau, \ \ \xi \in G, \tag{5.2}$$

where $S_G^{\alpha}(\xi)$ is the integral staircase function defined for the fractal set $\mathfrak{F}$ with the fractal dimension α, $d_G^{\alpha} t$ is the fractal measure, and $(S_G^{\alpha}(\xi) - S_G^{\alpha}(\tau))$ is the kernel, which is written as $K(\xi, \tau)$.

To examine the existence of the solution of the FODE (5.1), we find a solution of the equivalent second kind fractal Volterra integral equation (5.2). Thus, we define the following operator over $\mathcal{D}(\phi)$:

$$\mathcal{L}y(\xi) = c\, S_G^{\alpha}(\xi) - \frac{w^2}{R} \int_a^{\xi} K(\xi, \tau)\, y(\tau)\, d_G^{\alpha}\tau, \ \ \xi \in G. \tag{5.3}$$

Then, we observe the following:

(i) $\mathcal{L}$ is linear;

(ii) $\mathcal{L}$ is continuous since it involves continuous functions $S_G^{\alpha}(\xi)$ & $y(\xi)$, $\xi \in G$;

(iii) $K(\xi, \tau)$ is continuous on $G \times G$; and

(iv) the boundedness of $S_G^{\alpha}(\xi)$ ensures that for any bounded set $B \subset \mathcal{D}(\phi)$, $\mathcal{L}(B)$ is bounded.

Next, consider a bounded subset Q of $\mathcal{D}(\phi)$. Then, for any $\Xi_1, \Xi_2 \in Q$,

$$\mathcal{L}(\Xi_1(\xi)) - \mathcal{L}(\Xi_2(\xi)) = \frac{w^2}{R} \int_a^{\xi} K(\xi, t)\, (\Xi_1(t) - \Xi_2(t))\, d_G^{\alpha} t, \ \ \xi \in G,$$

and hence

$$\begin{aligned}
&\|\mathcal{L}\Xi_1 - \mathcal{L}\Xi_2\| \\
&= \sup_{\xi \in G} \ |\mathcal{L}\Xi_1(\xi) - \mathcal{L}\Xi_2(\xi)| + \sup_{\xi \in G} \left| D^{\alpha}_{G,\phi(\xi)} (\mathcal{L}\Xi_1(\xi) - \mathcal{L}\Xi_2(\xi)) \right| \\
&= \sup_{\xi \in G} \ \left| \frac{w^2}{R} \int_a^{\xi} K(\xi, t)(\Xi_1(t) - \Xi_2(t))\, d_G^{\alpha} t \right| + \sup_{\xi \in G} \ \left| D^{\alpha}_{G,\phi(\xi)} \left(\frac{w^2}{R} \int_a^{\xi} K(\xi, t)(\Xi_1(t) - \Xi_2(t))\, d_G^{\alpha} t \right) \right|.
\end{aligned}$$

In this stage, we can write

$$
\begin{aligned}
&\sup_{\xi\in G}\left|\frac{w^2}{R}\int_a^{\xi} K(\xi,t)\,(\Xi_1(t)-\Xi_2(t))\,d_G^{\alpha}t\right| \\
&\leq \frac{w^2}{R}\,\|K(\xi,t)\|\left|\int_a^{\xi} d_G^{\alpha}t\right|\ \sup_{t\in G}|\Xi_1(t)-\Xi_2(t)| \\
&\leq \frac{w^2}{R}\,MA\ \sup_{t\in G}|\Xi_1(t)-\Xi_2(t)|
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{\xi\in G}\left|D^{\alpha}_{G,\phi(\xi)}\left(\frac{w^2}{R}\int_a^{\xi} K(\xi,t)(\Xi_1(t)-\Xi_2(t))\,d_G^{\alpha}t\right)\right| \\
\leq&\frac{w^2}{R}\left|\int_a^{\xi} d_G^{\alpha}t\right|\left\{\sup_{\xi\in G}\left|K(\xi,t)D^{\alpha}_{G,\phi(\xi)}(\Xi_1(t)-\Xi_2(t))\right|+\sup_{\xi\in G}\ \left|(\Xi_1(t)-\Xi_2(t))D^{\alpha}_{G,\phi(\xi)}\,K(\xi,t)\right|\right\} \\
\leq&\frac{w^2}{R}A\left\{M\sup_{t\in G}\ \left|D^{\alpha}_{G,\phi(\xi)}\,(\Xi_1(t)-\Xi_2(t))\right|+M'\sup_{t\in G}\ |\Xi_1(t)-\Xi_2(t)|\right\}.
\end{aligned}
$$

Combining the above two relations, we obtain the following:

$$
\begin{aligned}
&\|\mathcal{L}\Xi_1-\mathcal{L}\Xi_2\| \\
=&\frac{w^2}{R}\,MA\ \sup_{t\in G}|\Xi_1(t)-\Xi_2(t)|+\frac{w^2}{R}A\left\{M\sup_{t\in G}\left|D^{\alpha}_{G,\phi(\xi)}(\Xi_1(t)-\Xi_2(t))\right|+M'\sup_{t\in G}|\Xi_1(t)-\Xi_2(t)|\right\} \\
<&\frac{w^2}{R}\,A(M+M')\ \sup_{t\in G}\ |\Xi_1(t)-\Xi_2(t)|+\frac{w^2}{R}\,A(M+M')\ \sup_{t\in G}\ \left|D^{\alpha}_{G,\phi(\xi)}(\Xi_1(t)-\Xi_2(t))\right| \\
=&\frac{w^2}{R}\,A(M+M')\ \left(\sup_{t\in G}|\Xi_1(t)-\Xi_2(t)|+\sup_{t\in G}\left|D^{\alpha}_{G,\phi(\xi)}(\Xi_1(t)-\Xi_2(t))\right|\right) \\
=&\frac{w^2}{R}A(M+M')\|\Xi_1-\Xi_2\|,
\end{aligned}
$$

which yields $diam(\mathcal{L}Q)\leq diam(Q)$. Thus, the measure of non-compactness satisfies $\Delta(\mathcal{L}Q)\leq\Delta(Q)$.

Next, consider a function $\Psi:[0,\infty)\rightarrow[0,\infty)$ by

$$\Psi(t)=\ t\ \left(1-\frac{A(M+M')w^2}{R}\right),\ \forall t\in[0,\infty).$$

Then, Ψ is monotone a increasing function, and $\Psi(t)=0$ iff $t=0$. Moreover, the measure of non-compactness Δ satisfies the following:

$$\Delta(\mathcal{L}Q)\leq\Delta(Q)-\Psi(\Delta(Q)).$$

Hence, $\mathcal{L}$ satisfies all the conditions of Theorem 4.4, which implies that $\mathcal{L}$ has a fixed point in $\mathcal{D}(\phi)$ that corresponds to the solution of the fractal integral equation (5.3). Consequently, this solution also serves as a solution to the FODE (5.1).

Numerical example

To verify the validity of the theoretical result established in Theorem 4.4, we perform a simulation by choosing the following:
(i) a numerical approximation of $G \subset [a, b] \subset \mathbb{R}$ as a bounded set;
(ii) define $S_G^\alpha(\tau) \approx \tau^\alpha$;
(iii) approximate the fractal integral using a weighted Reimann sum, $d_G^\alpha(\tau) \approx (\tau_{i+1}^\alpha - \tau_i^\alpha)$;
and consider the fractal oscillator equation (5.1) with parameters $a = 0$, $b = 1$, $\alpha = 0.8$ (fractal dimension), $c = w = M = M' = A = 1.0$. Therefore, $R = 3$.

Let $\{\xi_i\}_{i=0}^N$ be an equally spaced grid in $[0, 1]$ and simulates $S_G^\alpha(\xi_i) = \xi_i^\alpha$.

Initialize, $y_0(\xi_i) = c \cdot \xi_i^\alpha$, and iterate

$$y_{n+1}(\xi_i) = c \cdot \xi_i^\alpha + \frac{w^2}{R} \sum_{j=0}^{i} \left(\xi_j^\alpha - \xi_i^\alpha\right) y_n(\xi_j) \cdot (\xi_{j+1} - \xi_j)^\alpha,$$

using successive approximations until it converges.

Using MATLAB, we obtained Figure 1, which illustrates the convergence behavior of the iterative scheme used to numerically solve the fractal Volterra integral equation associated with the fractal oscillator model. The vertical axis represents the logarithmic scale of the supremum norm error $\|y_{n+1} - y_n\|_\infty$ between two successive iterates, while the horizontal axis denotes the iteration number. After a few iterations, the sequence $\{y_n(\xi)\}$ uniformly converges, thus confirming the contractive nature of the associated integral operator $\mathcal{L}$, and validating the theoretical result established via the Darbo-type fixed point theorem.

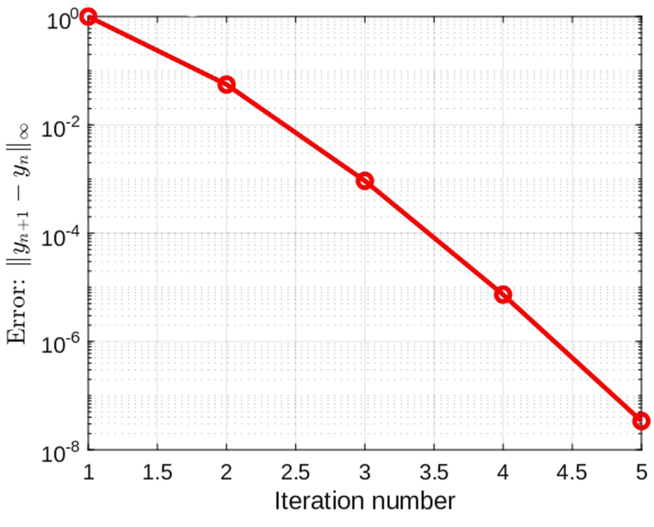

Figure 1. Convergence of $y_n(\xi)$ for the fractal oscillator equation.

6. Conclusions

In this article, we constructed a complete metric space over the Riemann-Stieltjes derivable function defined over a fractal set. Then, a measure of the non-compactness function Δ was introduced for a fractal subset of this metric space, and several of its key properties were explored. We established a fixed point theorem in a metric space of Riemann-Stieltjes derivable functions over fractals, thus extending Darbo's theorem through a novel contraction condition involving the measure of non-compactness. To illustrate the applicability of our results, we applied the framework to a fractal α-linear differential equation that modeled oscillatory systems in fractal media, where classical methods may be insufficient. We demonstrated the existence of solutions, thus offering a new perspective on solving differential equations in complex, non-Euclidean settings. A supporting simulation verified the existing result, the efficiency and stability of the proposed iterative scheme. The observed convergence behavior confirmed that the conditions that involved the measure of noncompactness, imposed on the integral operator $\mathcal{L}$ were sufficient to guarantee the existence of a fixed point within the function space $\mathcal{D}(\phi)$. This convergence validated the applicability of the abstract results to fractal differential equations, and affirmed the reliability of the method to solve such problems in practice.

These findings open new directions to investigate mechanical systems with fractal structures, and memory effects, further bridging the gap between the abstract fixed point theory and real-world applications across scientific and engineering domains, where concepts such as fractal functions, fractal differentials, and integral equations can be explored.

Author contributions

Mohammad Sajid: Funding acquisition, writing – review & editing; Abhishikta Das: Conceptualization, formal analysis, validation, writing – original draft; Hemanta Kalita: Conceptualization, validation, supervision, writing – review & editing.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors affirm that there are no conflicts of interest to disclose.

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