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Research article

Equivalent curves in $\mathbb{E}^n$

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Abstract: In this paper, we first define an equivalence relation for curves in $\mathbb{E}^n$. Based on this equivalence relation, we investigate the relationships between the Frenet frame and curvatures of equivalent curves. Next, we introduce the concept of linearly dependent curvatures in $\mathbb{E}^n$ and examine its implications for equivalent curves. Building on this concept and the proposed equivalence relation, we present a method to construct (1,3)-Bertrand curves in $\mathbb{E}^4$. Additionally, we derive the relationships between the harmonic curvatures of equivalent curves and use these relationships to establish several properties of equivalent helical curves. These results enable systematic construction of curves with prescribed geometric properties.

Keywords: combescure transformation; curve theory; equivalent curves; helix; Bertrand curve
Mathematics Subject Classification: 53A04, 53C21

1. Introduction

Obtaining a curve from an arbitrary curve plays an important role in geometry, which is frequently interested in the theory of curves. For instance, various studies have focused on deriving Bertrand curves from spherical curves [1], PH-curves from planar curves [2, 3], and Bertrand curves from Salkowski curves [4]. In such investigations, the Frenet frame is commonly analyzed, as there may exist relationships between the tangent, normal, and binormal vectors of one curve and those of another. Similarly, the curvature functions of these curve pairs may be interrelated [5].

The curves whose tangents at corresponding points are parallel are associated with a Combescure transformation [6]. In this case, the Frenet frames of the curves are identical. That is, the tangent, normal, and binormal vectors of such curves coincide. Therefore, it is possible to investigate the properties of a β curve obtained by Combescure transformation from a curve α and the relationships between these curves by using the Frenet frame and curvature functions of the curves. For instance, Bhat and Baskar investigated the necessary conditions for a curve β, obtained through a Combescure transformation from a curve α in $\mathbb{E}^3$, to be a Mannheim curve [7]. Additionally, the properties and characterizations of Bertrand and (1,3)-Bertrand curves are given in [8–11]. Moreover, Camcı et al. investigated the necessary conditions for obtaining helix, spherical helix, Bertrand, Mannheim, Salkowsky, and anti-Salkowsky curves from a curve by Combecsure transformation [12].

The concept of equivalence relations holds significant importance in the theory of curves, similar to its role in algebra and analysis. The following expression defines an equivalence relation between such curves $\alpha,\beta : I \to \mathbb{E}^3$ $\alpha \equiv \beta$ if and only if $\beta(s) = \varphi'(s)\alpha(s) - \int \varphi''(s)\alpha(s)ds$ where $\varphi : I \to \mathbb{R}$ is a non-constant differentiable function [12]. Furthermore, let α and β be curves characterized by the Frenet apparatus $\{T, N, B, \kappa, \tau, s\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$, respectively. If these curves are equivalent according to the given equivalence relation, then the following properties hold [12]:

$$T = T^*, \quad N = N^*, \quad B = B^*. \tag{1.1}$$

$$\varphi'\kappa^* = \kappa, \quad \varphi'\tau^* = \tau, \quad \varphi' ds = ds^*. \tag{1.2}$$

The theory of curves also finds widespread applications in engineering and physical sciences, particularly in areas involving fluid dynamics, thermal management, and electromagnetic field analysis [13–15]. Consequently, the equivalence relation developed in this work provides a versatile mathematical framework that can be extended to these diverse application domains.

2. Equivalent curves in $\mathbb{E}^n$

While the equivalence in $\mathbb{E}^3$ is well-studied [2, 12, 16], modern applications in n-dimensional optimization and string theory motivate this generalization. Therefore, we extend the equivalence relation from $\mathbb{E}^3$ to $\mathbb{E}^n$ with the following definition.

Definition 2.1. *Let $\alpha : I \to \mathbb{E}^n$ be a regular curve, and let $\varphi' : I \to \mathbb{R}$ be a non-zero differentiable function. The curve β is then defined as*

$$\beta : I \to \mathbb{E}^n, \quad \beta(s) = \varphi'(s)\alpha(s) - \int \varphi''(s)\alpha(s)ds. \tag{2.1}$$

This equivalence relation generalizes the Combescure transformation naturally. Geometrically, it represents a curve scaled by φ' and adjusted by the offset term $\int \varphi''(s)\alpha(s)ds$. For instance, choosing $\varphi(s) = s$ yields $\beta = \alpha$.

Remark 2.1. *From now on, for the equivalent curves α and β in $\mathbb{E}^n$, let $\{V_1, V_2, ..., V_n\}$ and $\left\{V_1^*, V_2^*, ..., V_n^*\right\}$ denote the Frenet frame $\{k_1, k_2, ..., k_{n-1}\}$, and $\left\{k_1^*, k_2^*, ..., k_{n-1}^*\right\}$ denote the curvatures of the curves α and β, respectively. The following theorem generalizes Eqs (1.1) and (1.2).*

Theorem 2.1. *Let $\alpha : I \to \mathbb{E}^n$ be a regular curve and $\beta : I \to \mathbb{E}^n$ be an equivalent of the curve α with the equivalence relation given in (2.1). Then the followings hold for all $s \in I$:*

$$i.\ V_{j+1}^* \;=\; V_{j+1}\,,\ 0 \le j \le n-1, \tag{2.2}$$

$$ii.\ \frac{k_j}{k_j^*} = \varphi^{'},\ 1 \leq j \leq n-1. \tag{2.3}$$

Proof. Without loss of generality, α can be chosen as a unit speed curve with arc-length parameter s. Because $\beta^{'}(s) = \varphi^{'}(s)\alpha'(s) = \varphi^{'}(s)V_1$, we have

$$V_1^* = \frac{\beta'(s)}{\|\beta'(s)\|} = V_1.$$

Since $V_1^* = V_1$, taking the derivative of both sides yields $\varphi' k_1^* V_2^* = k_1 V_2$. From this equation, we deduce that

$$\varphi' k_1^* = k_1 \quad \text{and} \quad V_2^* = V_2.$$

That means (2.2) and (2.3) are satisfied for $j = 1$. Continuing the proof by induction method, let (2.2) and (2.3) hold for any $j = 1, 2, ..., m$. So we have

$$V_{m+1}^* = V_{m+1} \tag{2.4}$$

$$\varphi^{'} k_m^* = k_m \tag{2.5}$$

$$V_m^* = V_m. \tag{2.6}$$

Differentiating both sides of $V_{m+1}^* = V_{m+1}$ yields

$$-\varphi'(s)k_m^* V_m^* + \varphi'(s)k_{m+1}^* V_{m+2}^* = -k_m V_m + k_{m+1} V_{m+2}. \tag{2.7}$$

Applying Eqs (2.5) and (2.6) in (2.7), we obtain

$$\varphi'(s)k_{m+1}^* V_{m+2}^* = k_{m+1} V_{m+2}. \tag{2.8}$$

Thus we get

$$\varphi'(s)k_{m+1}^* = k_{m+1} \quad \text{and} \quad V_{m+2}^* = V_{m+2}.$$

Hence Eqs (2.2) and (2.3) hold true for $j = m + 1$. This completes the proof. □

Example 2.1. *Let $\alpha : I \to \mathbb{E}^4$ be a smooth curve parameterized by arc-length s, defined as*

$$\alpha(s) = \left(\cos\left(\frac{2s}{\sqrt{13}}\right), \sin\left(\frac{2s}{\sqrt{13}}s\right), 3\cos\left(\frac{s}{\sqrt{13}}\right), 3\sin\left(\frac{s}{\sqrt{13}}\right)\right).$$

Frenet vectors and curvatures of this curve are:

$$V_1 = \left(-\frac{2}{\sqrt{13}}\sin\left(\frac{2s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}\cos\left(\frac{2s}{\sqrt{13}}\right), -\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right), \frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right)\right)$$

$$V_2 = \left(-\frac{4}{5}\cos\left(\frac{2s}{\sqrt{13}}\right), -\frac{4}{5}\sin\left(\frac{2s}{\sqrt{13}}\right), -\frac{3}{5}\cos\left(\frac{s}{\sqrt{13}}\right), -\frac{3}{5}\sin\left(\frac{s}{\sqrt{13}}\right)\right)$$

$$V_3 = \left(\frac{3}{\sqrt{13}}\sin\left(\frac{2s}{\sqrt{13}}\right), -\frac{3}{\sqrt{13}}\cos\left(\frac{2s}{\sqrt{13}}\right), -\frac{2}{\sqrt{13}}\sin\left(\frac{2s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}\cos\left(\frac{2s}{\sqrt{13}}\right)\right)$$

$$V_4=\left(\frac{3}{5}\cos\left(\frac{2s}{\sqrt{13}}\right),\frac{3}{5}\sin\left(\frac{2s}{\sqrt{13}}\right),-\frac{4}{5}\cos\left(\frac{s}{\sqrt{13}}\right),-\frac{4}{5}\sin\left(\frac{s}{\sqrt{13}}\right)\right)$$

and $k_1=5/13$*,* $k_2=18/65$*, and* $k_3=2/5$*. Choosing* $\varphi'(s)=2s$*, we find the equivalent of this curve* $\beta(s)$ *as follows:*

$$\beta(s)=\begin{pmatrix} 2s\cos\left(\frac{2s}{\sqrt{13}}\right)-\sqrt{13}\sin\left(\frac{2s}{\sqrt{13}}\right) \\ 2s\sin\left(\frac{2s}{\sqrt{13}}\right)+\sqrt{13}\cos\left(\frac{2s}{\sqrt{13}}\right) \\ 6s\cos\left(\frac{s}{\sqrt{13}}\right)-6\sqrt{13}\sin\left(\frac{s}{\sqrt{13}}\right) \\ 6s\sin\left(\frac{s}{\sqrt{13}}\right)+6\sqrt{13}\cos\left(\frac{s}{\sqrt{13}}\right) \end{pmatrix}.$$

Frenet vectors and curvatures of this curve are:

$$V_1^*=\left(-\frac{2}{\sqrt{13}}\sin\left(\frac{2s}{\sqrt{13}}\right),\frac{2}{\sqrt{13}}\cos\left(\frac{2s}{\sqrt{13}}\right),-\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right),\frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right)\right)$$

$$V_2^*=\left(-\frac{4}{5}\cos\left(\frac{2s}{\sqrt{13}}\right),-\frac{4}{5}\sin\left(\frac{2s}{\sqrt{13}}\right),-\frac{3}{5}\cos\left(\frac{s}{\sqrt{13}}\right),-\frac{3}{5}\sin\left(\frac{s}{\sqrt{13}}\right)\right)$$

$$V_3^*=\left(\frac{3}{\sqrt{13}}\sin\left(\frac{2s}{\sqrt{13}}\right),-\frac{3}{\sqrt{13}}\cos\left(\frac{2s}{\sqrt{13}}\right),-\frac{2}{\sqrt{13}}\sin\left(\frac{2s}{\sqrt{13}}\right),\frac{2}{\sqrt{13}}\cos\left(\frac{2s}{\sqrt{13}}\right)\right)$$

$$V_4^*=\left(\frac{3}{5}\cos\left(\frac{2s}{\sqrt{13}}\right),\frac{3}{5}\sin\left(\frac{2s}{\sqrt{13}}\right),-\frac{4}{5}\cos\left(\frac{s}{\sqrt{13}}\right),-\frac{4}{5}\sin\left(\frac{s}{\sqrt{13}}\right)\right)$$

and $k_1^*=5/26s$*,* $k_2=9/65s$ *and* $k_3^*=1/5s$*.*

Definition 2.2. *Let* $\alpha: I\to\mathbb{E}^n$ *be a regular curve with curvatures* $k_1,k_2,...,k_n$*. If there exist constants* $c_{1,}c_2,...,c_{n-1}\in\mathbb{R}$*, not all zero, such that*

$$c_1k_1+c_2k_2+...+c_{n-1}k_{n-1}=0 \quad \textit{for all } s\in I \tag{2.9}$$

then the curve α *is said to have linearly dependent curvatures.*

Corollary 2.1. *Non-degenerate helix curves have linearly dependent curvatures in* $\mathbb{E}^3$*.*

Corollary 2.2. *A curve is called an W-curve, if it has constant Frenet curvatures. So, every W-curve in* $\mathbb{E}^n$ *has linearly dependent curvatures.*

Remark 2.2. *W-curves in odd-dimensional Euclidean space* $\mathbb{E}^{2n+1}$ *can be regarded as generalized helices; however, this characterization does not extend to the even-dimensional case* $\mathbb{E}^{2n}$ *[17]. Consequently, the curve* $\alpha(s)$ *given in Example (2.1) is not a generalized helix but an W-curve. Hence, it provides a substantive application of Corollary 2.2.*

Definition 2.3. *The equivalence class of a curve* $\alpha: I\to\mathbb{E}^n$ *is defined as*

$$[a]=\left\{\beta: I\to\mathbb{E}^n \mid \beta(s)=\varphi^{'}(s)\alpha(s)-\int\varphi^{''}(s)\alpha(s)ds\right\}$$

where $\varphi^{'}: I\to\mathbb{R}$ *is a non-zero differentiable function.*

Corollary 2.3. *For a non-planar curve $\alpha : I \to \mathbb{E}^3$, the following proposition holds:*

$$\alpha \text{ has linearly dependent curvatures if and only if } \alpha \text{ is a helical curve.}$$

Theorem 2.2. *Let $\alpha : I \to \mathbb{E}^n$ has linearly dependent curvatures. Then for all $\beta \in [\alpha]$, has linearly dependent curvatures.*

Proof. By means of Theorem 2.1 it is calculated that

$$\varphi' k_i^* = k_i \Rightarrow \varphi' = \frac{k_i}{k_i^*} \Rightarrow k_i = \varphi' k_i^* \ , 1 \leq i < n-1.$$

Because $\alpha : I \to \mathbb{E}^n$ has linearly dependent curvatures, there exist constants $c_{1,}c_2, ..., c_{n-1} \in \mathbb{R}$ such that

$$c_1 k_1 + c_2 k_2 + ... + c_{n-1} k_{n-1} = 0 \tag{2.10}$$

holds. For $1 \leq i \leq n-1$, substituting k_i with $\varphi' k_i^*$ in (2.4) we obtain

$$c_1 k_1^* + c_2 k_2^* + ... + c_{n-1} k_{n-1}^* = 0.$$

That means β has linearly dependent curvatures. □

Theorem 2.3. *[11] Let C be a C^∞-special Frenet curve in $\mathbb{E}^4$ with curvature functions k_1, k_2, and k_3. Then C is a $(1,3)$-Bertrand curve if and only if there exist constant real numbers c_1, c_2, c_3, and c_4 satisfying*

$$c_1 k_2(s) - c_2 k_3(s) \neq 0 \tag{i}$$

$$c_1 k_1(s) + c_3 (c_1 k_2(s) - c_2 k_3(s)) = 1 \tag{ii}$$

$$c_3 k_1(s) - k_2(s) = c_4 k_3(s) \tag{iii}$$

$$(c_3^2 - 1) k_1(s) k_2(s) + c_3 \left\{ k_1^2(s) - k_2^2(s) - k_3^2(s) \right\} \neq 0 \tag{iv}$$

for all $s \in I$.

Corollary 2.4. *According to (iii) of Theorem 2.3, every (1,3)-Bertrand curve has linearly dependent curvatures.*

The following theorem provides a constructive method for obtaining a (1,3)-Bertrand curve and serves as an application of the linear dependence among the curvatures of a curve.

Theorem 2.4. *Let $\alpha : I \to \mathbb{E}^4$ be a regular curve whose curvatures $k_1, k_2,$ and k_3 satisfy the relation*

$$c_3 k_1(s) - k_2(s) = c_4 k_3(s) \tag{2.11}$$

where c_3 and c_4 are real constants. If the function $\varphi' : I \to \mathbb{R}$ is defined by

$$\varphi'(s) = c_1 k_1(s) + c_3 (c_1 k_2(s) - c_2 k_3(s)) \tag{2.12}$$

then, the equivalent curve of α is a (1,3)-Bertrand curve.

Proof. Let $\beta : I \to \mathbb{E}^4$ denote the equivalent of α defining $\varphi^{'}(s)$ as in (2.12). Upon substituting $k_i(s)$ with $\varphi'(s)k_i^*(s)$ for $1 \leq i \leq 3$ in (2.11), it becomes straightforward to verify that

$$c_3k_1^*(s) - k_2^*(s) = c_4k_3^*(s). \tag{2.13}$$

From 2.12, we obtain

$$\begin{aligned} c_1k_1^*(s) + c_3\left(c_1k_2^*(s) - c_2k_3^*(s)\right) &= \frac{1}{\varphi^{'}}(c_1k_1(s) + c_3\left(c_1k_2(s) - c_2k_3(s)\right)) \\ &= 1. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) β is a $(1,3)$-Bertrand curve. □

Example 2.2. *Let $\alpha : I \to \mathbb{E}^4$ be a smooth curve parameterized by arc-length s, defined as*

$$\alpha(s) = \left(\cos\left(\frac{2s}{\sqrt{13}}\right), \sin\left(\frac{2s}{\sqrt{13}}s\right), 3\cos\left(\frac{s}{\sqrt{13}}\right), 3\sin\left(\frac{s}{\sqrt{13}}\right)\right).$$

It is shown that the curve α has curvatures $k_1 = 5/13$, $k_2 = 18/65$ and $k_3 = 2/5$. A straightforward verification shows that the constants $c_3 = -11/10$ and $c_4 = -7/4$ satisfy Eq (2.11). Selecting the parameters $c_1 = 33/20$ and $c_2 = 11/5$ reduces (2.12) to $\varphi(s) = \frac{11}{10}$. Hence, the equivalent of α can be calculated as

$$\beta(s) = \left(\frac{11}{10}\cos\left(\frac{2s}{\sqrt{13}}\right), \frac{11}{10}\sin\left(\frac{2s}{\sqrt{13}}s\right), \frac{33}{10}\cos\left(\frac{s}{\sqrt{13}}\right), \frac{33}{10}\sin\left(\frac{s}{\sqrt{13}}\right)\right).$$

Thus $\beta(s)$ is a (1,3)-Bertand curve.

Definition 2.4. *[18] Let $\alpha : I \to \mathbb{E}^n$ be a regular curve with curvatures $k_1, k_2, ..., k_{n-1}$ and arc-length parameter s. Then its harmonic curvatures are defined as follows:*

$$\begin{aligned} H_0 &= 0, \\ H_1 &= \frac{k_1}{k_2}, \\ H_i &= \left(\frac{d}{ds}H_{i-1} + H_{i-2}k_i\right)\frac{1}{k_{i+1}}, i = 2, 3, ..., n-2. \end{aligned}$$

Theorem 2.5. *Let $\alpha : I \to \mathbb{E}^n$ be a regular curve with harmonic curvatures $H_0, H_1, ..., H_{n-2}$. For $\beta \in [\alpha]$, let $H_0^*, H_1^*, H_2^*, ..., H_{n-2}^*$ be its harmonic curvatures. Then*

$$H_i^* = H_i, \quad 0 \leq i \leq n-2. \tag{2.15}$$

Proof. Without loss of generality, α can be chosen as a unit speed curve with arclength parameter s and curvatures $k_1, k_2, ..., k_{n-1}$. Let $k_1^*, k_2^*, ..., k_{n-1}^*$ be the curvatures of the curve $\beta \in [\alpha]$ with arc-length parameter s^*. For $i = 0$ and 1, it's clear that $H_0 = H_0^*$ and $H_1 = H_1^*$. If we calculate ds^*, we find

$$s^* = \int \|\beta'(s)\|\, ds = \int \varphi^{'}(s)\, \|a'(s)\|\, ds = \int \varphi^{'}(s) ds \Rightarrow ds^* = \varphi^{'}(s) ds.$$

Substituting $ds^* = \varphi^{'}(s)ds$, $k_3^* = \varphi^{'}k_3$ and $H_1^* = H_1$, for $i = 2$, it is calculated that

$$\begin{aligned} H_2 &= \left(\frac{d}{ds}H_1 + H_0 k_2\right)\frac{1}{k_3} \\ &= \frac{dH_1^*}{ds^*}\frac{ds^*}{\varphi' ds}\frac{\varphi^{'}}{k_3^*} \\ &= H_2^*. \end{aligned}$$

Continuing by the induction method, let (2.15) hold for any $i = 1, 2, 3, ..., m$. Thus

$$\begin{aligned} H_{m+1} &= \left(\frac{d}{ds}H_{m-1} + H_{m-2}k_m\right)\frac{1}{k_{m+1}} \\ &= \left(\frac{dH_m^*}{ds^*}\frac{ds^*}{\varphi' ds} + H_{m-1}\frac{k_{m+1}^*}{\varphi^{'}}\right)\frac{\varphi^{'}}{k_{m+2}^*} \\ &= \left(\frac{dH_m^*}{ds^*} + H_{m-1}k_{m+1}^*\right)\frac{1}{k_{m+2}^*} \\ &= H_{m+1}^*. \end{aligned}$$

It is seen that (2.15) also holds for $i = m + 1$. It completes the proof. □

Definition 2.5. *([17]) Let $\alpha(s)$ be a non-degenerate unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{H_0, H_1, H_1, H_2, \ldots, H_{n-2}\}$ denote the Frenet frame and the higher ordered-harmonic curvatures of the curve, respectively. The vector*

$$D = V_1 + H_1 V_3 + \cdots + H_{n-2} V_n$$

is called the generalized Darboux vector of the curve α.

Theorem 2.6. *[17] Let $\alpha(s)$ be a unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$ and $\{H_0, H_1, H_2, \ldots, H_{n-2}\}$ denote the Frenet frame and the higher ordered-harmonic curvatures of the curve, respectively. Then the followings are equivalent:*

i. The curve α is a generalized helix.

ii. D is constant.

iii. $H_{n-2}^{'} + k_{n-3}H_{n-1} = 0.$ (2.16)

Corollary 2.5. *If α is a generalized helix then for all $\beta \in [\alpha]$, the Darboux vectors of α and β are the same.*

Corollary 2.6. *Let α and β be equivalent curves. If α is a generalized helix, then β is also a generalized helix.*

3. Conclusions

In this study, we have extended the equivalence relation for curves in Euclidean space $\mathbb{E}^3$ to $\mathbb{E}^n$ and explored the relationships between the Frenet frame and curvatures of equivalent curves. This structure is closely related to the concept of equivalence groups in the theory of differential equations, where arbitrary functions transform according to rules dictated by underlying (often infinite-dimensional) Lie groups or transformation semigroups. Consequently, the equivalence relation studied here is not only analytically meaningful but also carries a geometric and physical interpretation, reflecting how certain quantities behave under general transformations. Additionally, we have defined the concept of linearly dependent curvatures in $\mathbb{E}^n$. It is well known that curves whose curvatures are linearly dependent are only helices in $\mathbb{E}^3$. However, we have shown that the curvatures of (1,3)-Bertrand curves are linearly dependent. The existence of other curves with linearly dependent curvatures is an open problem in $\mathbb{E}^4$ and higher dimensions. We have expanded this discussion to propose that symbolic computation and numerical methods offer viable pathways to explore such curve families. Future work will implement this pipeline to classify solutions and test conjectured geometric invariants. Moreover, we have established a method for constructing (1,3)-Bertrand curves in $\mathbb{E}^4$. Furthermore, we have analyzed harmonic curvatures and their consistency across equivalent curves, demonstrating that if a curve is helical, all of its equivalent curves retain this helical property. This work contributes to a broader understanding of curve theory, especially in relation to transformations and the equivalence of curves in higher-dimensional Euclidean spaces. Future research could build on these findings to investigate more complex transformations or explore additional applications in geometric modeling and physical systems. The adaptation of the presented theory to non-Euclidean settings, such as curves on Riemannian manifolds or in Minkowski space, is indeed an interesting direction for future research. The general framework may be extendable by carefully replacing Euclidean concepts with their manifold analogues. We believe that a systematic investigation of these generalizations would be a valuable contribution to the field and plan to address these questions in future work.

Author contributions

Ahmet Mollaogullari: Writing-original draft, Writing-review & editing; Mehmet Gumus: Writing-review & editing; Didem Karalarlioglu Camci: Writing-review & editing; Kazim Ilarslan: Methodology, Validation; Cetin Camci: Methodology, Supervision, Conceptualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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