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Research article

Several constructions of constant dimension code using equal-division method

Yongfeng Niu, Liang Wu, Yizhuo Zhang and Huiling Yu*

The School of Computer Science and Artificial Intelligence and Aliyun School of Big Data, Changzhou University, Changzhou 213159, China

* **Correspondence:** Email: yhl2016@cczu.edu.cn.

Abstract: Constant dimension codes (CDCs) have garnered significant attention in recent years, primarily owing to their crucial applications in random network coding. A central problem in the study of CDCs involves determining the maximum achievable cardinality, denoted as $A_q(n, 2\delta, k)$ for given parameters. In this paper, we propose a new approach to constructing CDCs based on the equal-division method, which we subsequently combine with existing optimal codes from prior literature. The resulting codes yield improved lower bounds for $A_q(n, 2\delta, k)$ compared to previously established results across certain parameters. Furthermore, we extend our approach by incorporating multiple constructions based on distinct ways of equal division. The newly constructed CDCs have larger cardinality under some parameters.

Keywords: equal-division; Ferrers diagram; constant dimension code; rank-metric codes
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1. Introduction

As a special type of subspace code, CDCs have attracted widespread attention because of their utilization in random network coding [1]. In CDCs, one of the main challenges is to pinpoint the maximum probable size of $A_q(n, 2\delta, k)$ with given parameters. Let $\mathbb{F}_q$ be the finite field of order q, and let $\mathbb{F}_q^n$ be the set of all vectors of length n over $\mathbb{F}_q$. For a nonnegative integer $k \leq n$, the set of all k-dimensional subspaces of $\mathbb{F}_q^n$ is called the Grassmannian $\mathcal{G}_q(n, k)$. For any two subspaces $U, V \in \mathcal{G}_q(n, k)$, their subspace distance is defined by

$$d_S(U, V) = \dim U + \dim V - 2\dim(U \cap V) \geq 2\delta. \tag{1.1}$$

A nonempty subset C of $\subseteq \mathcal{G}_q(n, k)$ with d_S is called a CDC. The subspace distance of C is defined as:

$$d_S(C) = \min\{d_S(U, V) \mid U, V \in C, U \neq V\}. \tag{1.2}$$

C is called an $(n, M, 2\delta, k)_q$ code, if $|C| = M$ and $d_S(C) = 2\delta$.

Regarding the construction of CDCs, there are primarily two research directions. One direction employs linear polynomials and Sidon spaces within cyclic subspaces (relevant literature can be found in [2–4]), while the other relies on conventional construction techniques, as discussed below.

Several specialists have made significant contributions to the study of CDCs. There is an extensive literature on CDCs (see [5–7]). Gluesing-Luerssen and Troha in [8] introduced a novel CDC termed the "linkage construction", achieved by linking two smaller CDCs, yielding an improved lower bound. Xu and Chen in [9] proposed the "parallel construction", establishing a new lower bound for $A_q(n, 2\delta, k)$, where $k \geq 2$. Additionally, Chen et al. [10] enhanced the linkage construction, introducing the "parallel linkage construction", resulting in various new lower bounds for $A_q(n, 2\delta, k)$. Li in [11] introduced a notable improvement to linkage construction termed the "multilevel linkage construction", which combined with the multilevel construction, yielded new lower bounds for small-parameter CDCs. On this basis, Lao [12] obtained some relatively good parameters through further optimization of the multilevel linkage construction, thereby enhancing the lower bounds of some CDCs. He [13] and Hong [14] adopted the parallel subcode construction method, achieving excellent results. Niu et al. in [15, 16] proposed a construction method for CDCs by employing the lifted insertion construction, which also improved the lower bounds of some codewords. Liu et al. in [17] and [18] integrated parallel and multilevel constructions to produce numerous CDCs surpassing previously known codes.

Inspired by the results in [17] and [18], we develop a new construction method based on equal-division. When combined with the existing construction in [17], it yields some CDCs with improved parameters. The structure of this paper is delineated as follows: In Section 2, we elaborate on the existing construction methods for CDCs. We also provide related definitions and lemmas, laying a solid foundation for our subsequent construction and computation of CDCs. In Section 3, we present a new construction method and derive theorems from it. Furthermore, we provide some simple examples to substantiate our idea. In Section 4, we expand our idea and make a conjecture. Ultimately, we encapsulate the article's contents through a comprehensive summary.

2. Preliminaries

In this section, we will review some basic results. Some parts consist of necessary theoretical background knowledge, and others are key lemmas required for our construction.

2.1. Lifted MRD code

Let $\mathbb{F}_q^{m\times n}$ denote the vector space of all $m \times n$ matrices over the finite field $\mathbb{F}_q$, and consider a matrix $A \in \mathbb{F}_q^{m\times n}$. The rank of A, denoted by $\mathcal{R}(A)$, induces a natural metric on this vector space known as the rank distance. Formally, for any two matrices $A, B \in \mathbb{F}_q^{m\times n}$, the rank distance between them is defined as

$$d_R(A, B) = \mathcal{R}(A - B).$$

An $[m \times n, k, 2\delta]_q$ linear rank-metric code C is a k-dimensional linear subspace of $\mathbb{F}_q^{m\times n}$ equipped with the rank metric, where $|C| = q^k$ and the minimum rank distance is 2δ. When the cardinality of code C attains the upper bound $q^{\max\{m,n\}\times(\min\{m,n\}-2\delta+1)}$, C is said to be a maximum rank-distance (MRD) code [19, 20].

Lemma 2.1. (Lifted MRD code) *Let $n \geq 2k$. The lifted MRD code*

$$C = \{\text{rowspace}(I_k|A) : A \in \mathcal{D}\}$$

is an $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$-CDC, where $\mathcal{D}$ is an MRD $[k \times (n-k), \delta]_q$ code.

2.2. Ferrers diagram rank-metric codes

Let X be a subspace of dimension k of $\mathbb{F}_q^n$, which can be represented by a generator matrix. The basis of X is composed of the k rows of this matrix. Applying the Gaussian elimination algorithm to this generator matrix yields a unique result—a matrix in reduced row echelon form, which we designate as $E(X)$.

Definition 2.1. (Identifying vectors) *[16] For each k-dimensional $\mathbb{F}_q$-subspace X of $\mathbb{F}_q^n$, let $E(X)$ be the reduced row echelon form of X. A binary row vector $v(X)$ of length n and weight k is called the identifying vector of X, where the ones in $v(X)$ are in the positions of columns where $E(X)$ has the leading ones of the rows.*

Remove the zeroes from each row of $E(X)$ to the left of the pivot, and after that remove the columns that contain the pivots. All the remaining entries are shifted to the right. Then we obtain the Ferrers tableaux form of a subspace X, denoted by $\mathcal{F}(X)$. The Ferrers diagram of X, denoted by F_X, is obtained from $\mathcal{F}(X)$ by replacing the entries of $\mathcal{F}(X)$ with dots.

Given an identifying vector, the corresponding subspaces can be constructed easily. All k-dimensional subspaces of $\mathbb{F}_q^n$ with the same identifying vector have the same corresponding Ferrers diagram.

Example 2.1. *Let X be a 3-dimensional subspace of $\mathbb{F}_2^7$ with the following generator matrix in reduced row echelon form:*

$$E(X) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Then the identifying vector of X is $v(X)$ = (1101000). The Ferrers tableaux form $\mathcal{F}(X)$ of X is

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & 0 & 1 & 1 \end{array}.$$

The corresponding Ferrers diagram $\mathcal{F}$ of X is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \end{array}.$$

Definition 2.2. *[21] Let $\mathcal{F}$ be a Ferrers diagram and $C_{\mathcal{F}}$ the corresponding $[\mathcal{F},\rho,\delta]$ Ferrers diagram rank-metric code. Then $\rho \leq \min_{i\in[\delta]} v_i$, where v_i is the number of dots in $\mathcal{F}$ which are not contained in the first i rows and the rightmost $\delta - 1 - i$ columns$(0 \leq i \leq \delta - 1)$, codes that attain this bound are called optimal.*

2.3. Rank-metric code with given ranks and multilevel construction

Definition 2.3. (Rank-metric code with given ranks (GRMC)) *Let $K \subseteq \{0, 1, \ldots, n\}$ and δ be a positive integer. A set $\mathcal{D} \subseteq \mathbb{F}_q^{m\times n}$ is called an $(m \times n, \delta, K)_q$-rank-metric code with given ranks if it satisfies the following two conditions:*

(i) For any $D \in \mathcal{D}$, rank$(D) \in K$;

(ii) For any $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$, the rank distance $d_R(D_1, D_2) = rank(D_1 - D_2) \geq \delta$.

If $|\mathcal{D}| = M$, we denote it as an $(m \times n, M, \delta, K)_q$-GRMC for simplicity.

The following lemmas are crucial to present the multilevel construction. We use $\mathcal{P}_q(n)$ to denote the set of all subspaces of $\mathbb{F}_q^n$.

Lemma 2.2. *[21] For $X, Y \in \mathcal{P}_q(n)$, then*

$$d_S(X, Y) \geq d_H(v(X), v(Y)),$$

where d_H is the Hamming metric.

Lemma 2.3. *[21] For $X, Y \in \mathcal{P}_q(n)$. If $v(X) = v(Y)$, then*

$$d_S(X, Y) = 2d_R(C_X, C_Y),$$

where C_X and C_Y denote the submatrices of X and Y without the columns of their pivots, respectively.

The following is an introduction to multilevel construction: Let $\mathcal{A}$ be a binary code of length n, constant weight k, and minimum Hamming distance 2δ. For each codeword $v \in \mathcal{A}$, let $EF(v)$ be its echelon Ferrers form with corresponding Ferrers diagram $\mathcal{F}_v$. If for every $v \in \mathcal{A}$ there exists an $[\mathcal{F}_v, k_v, \delta]_q$ code $\mathcal{D}_v$, then the row spaces of the matrices in $\bigcup_{v\in\mathcal{A}} \mathcal{D}_v$ form an $(n, 2\delta, k)_q$-CDC, as established by Lemmas 2.2 and 2.3.

Lemma 2.4. *[17] Let $\mathcal{D}_1$ be a $(k \times (n - k), M_1, \delta, [0, k - \delta])_q$-GRMC, set $C_1 = \{rowspace(B \mid I_k) : B \in \mathcal{D}_1\}$. If $k \geq 2\delta$, then C_1 is an $(n, M_1, 2\delta, k)_q$-CDC, where*

$$M_1 = \sum_{i=\delta}^{k-\delta} A_i(\delta) + 1,$$

and $A_i(\delta)$ denotes the number of codewords with rank i in an $[m \times n, \delta]_q$-MRD code.

Lemma 2.5. *[18] Let $n \geq 2k + \delta$ and $k \geq 2\delta$. Let q be any prime power and*

$$M_2 = q^{(n-k)(k-\delta+1)} \frac{1 - q^{-\lfloor \frac{k}{\delta} \rfloor \delta^2}}{1 - q^{-\delta^2}} + q^{(n-k-\delta)(k-\delta+1)}.$$

Then there exists an $(n, M_2, 2\delta, k)_q$*-CDC constructed via the multilevel construction satisfying that for any of its identifying vectors* $\boldsymbol{v}_2 = (\underbrace{\boldsymbol{v}_2^{(1)}}_{n-k} \mid \underbrace{\boldsymbol{v}_2^{(2)}}_{k})$*, it holds that the weight of* $\boldsymbol{v}_2^{(1)}$ *is greater than or equal to k, and this CDC contains a lifted MRD code* $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$*-CDC as a subset.*

Lemma 2.6. *[17] Let n, k, and δ be positive integers satisfying* $n = 2k + \delta$ *and* $k \geq 2\delta$*. Let* C_1 *and* C_2 *be the codes defined in Lemmas 2.4 and 2.5, respectively. Then the code* $C_1 \cup C_2$ *forms an* $(n, M_{12}, 2\delta, k)_q$*-CDC, its cardinality is*

$$M_{12} = q^{(n-k)(k-\delta+1)} \frac{1 - q^{-\lfloor \frac{k}{\delta} \rfloor \delta^2}}{1 - q^{-\delta^2}} + q^{(n-k-\delta)(k-\delta+1)} + \sum_{i=\delta}^{k-\delta} A_i(\delta) + 1.$$

3. Main results

In this section, we will show our constructions. Inspired by the constructions above, the core of our methodology is equal division. Lemma 3.1 presents the foundational case of 2 equal divisions with subsequent extension. Lemma 3.3 develops the 3 equal division technique. A subsequent remark generalizes this to 5 equal divisions and 7 equal divisions. Next, we provide a detailed exposition of the proposed methodology.

Lemma 3.1. *Let n, k, and δ be non-negative integers satisfying* $n \geq 2k + \delta$ *and* $k \geq 2\delta$*. There exists an* $(n, M_3^2, 2\delta, k)_q$*-CDC* C_3^2*.*

Proof. We construct the set of identifying vectors of length n as follows:

$$\mathcal{A}_2 = \{(\underbrace{1 \cdots 1}_{k-2\delta} \mid u \mid u \mid \underbrace{0 \cdots 0}_{n-k-2\delta}) : u \in \mathcal{B}_2\},$$

where $\mathcal{B}_2 = \{(\mathcal{B}_{2,1}, \mathcal{B}_{2,2}, \cdots, \mathcal{B}_{2,s_2})\}$ and

$$\mathcal{B}_{2,1} = (\underbrace{0 \cdots 0}_{\frac{\delta}{2}} \underbrace{1 \cdots 1}_{\frac{\delta}{2}} \underbrace{0 \cdots 0}_{\frac{\delta}{2}} \underbrace{1 \cdots 1}_{\frac{\delta}{2}}\} ,$$

$$\mathcal{B}_{2,2} = (\overbrace{\underbrace{0 \cdots 0}_{\frac{\delta}{2^2}} \underbrace{1 \cdots 1}_{\frac{\delta}{2^2}}, \cdots, \underbrace{0 \cdots 0}_{\frac{\delta}{2^2}} \underbrace{1 \cdots 1}_{\frac{\delta}{2^2}}}^{2\delta}) ,$$

$$, \cdots ,$$

$$\mathcal{B}_{2,s_2} = (\overbrace{\underbrace{0 \cdots 0}_{\frac{\delta}{2^{s_2}}} \underbrace{1 \cdots 1}_{\frac{\delta}{2^{s_2}}}, \cdots, \underbrace{0 \cdots 0}_{\frac{\delta}{2^{s_2}}} \underbrace{1 \cdots 1}_{\frac{\delta}{2^{s_2}}}}^{2\delta}).$$

In $\mathcal{B}_2$, we divide the 2δ in first u into 2-equal parts, 2^2-equal parts, ... , 2^{s_2}-equal parts ($2^i(1 \leq i \leq s_2)$ is a factor of δ). Accordingly, there are 2, 2^2, ... , 2^{s_2} groups, such that for any two of these vectors, the Hamming distance of u is exactly δ. Similarly, this applies to the second u as well. By doing so, we ensure that the Hamming distance between any two elements in set $\mathcal{A}_2$ is at least 2δ.

□

Applying Lemmas 2.6 and 3.1, we can arrive at the following theorem.

Theorem 3.1. *Let $C_1 \cup C_2$ and C_3^2 be the codes defined in Lemmas 2.6 and 3.1, respectively. Then the code $C = C_1 \cup C_2 \cup C_3^2$ forms an $(n, M, 2\delta, k)_q$-CDC.*

Proof. We need to prove that the minimum subspace distance is 2δ for any two subspaces $\mathcal{U}, \mathcal{V} \in C$, $U = \text{rowspace}(\mathcal{U})$ and $V = \text{rowspace}(\mathcal{V})$.

(1) When $\mathcal{U} \in C_1, \mathcal{V} \in C_2$. From Lemma 2.6, we can readily obtain $d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta$.

(2) When $\mathcal{U} \in C_1, \mathcal{V} \in C_3^2$. The construction of C_3^2 involves dividing a specific identifying vector from C_2 into equal parts. Therefore, we can similarly prove this by splitting it into two segments. Firstly, let us examine the first $n-k-\delta$ terms. By equally dividing $(\underbrace{0\cdots0}_{\delta}\underbrace{1\cdots1}_{\delta})$ in C_3^2 into 2^{s_2} parts, we can determine that the Hamming distance between C_1 and C_3^2 is $k-\delta$ (where $k-\delta \geq \delta$). Next, we repeat this procedure for the subsequent $k+\delta$ terms, obtaining another Hamming distance of k. This allows us to derive $d_S(\mathcal{U}, \mathcal{V}) \geq (2k-\delta) \geq 2\delta$ by Lemmas 2.2 and 2.3.

(3) When $\mathcal{U} \in C_2, \mathcal{V} \in C_3^2$. It is easy to prove that $d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta$ by Lemmas 2.2 and 2.3.

This completes the proof. □

Next, we give the cardinality of Theorem 3.1 through a corollary.

Corollary 3.1. *Let $n = 2k + \delta$ with integers $k \geq 2\delta > 0$. Then the code C attains the maximum possible cardinality, i.e.,*

$$A_q(n, 2\delta, k) \geq M_1 + M_2 + \sum_{l=1}^{s_2} q^{k\times(n-k)-2\times\delta^2(2+2^{-l})}.$$

Proof. The cardinality of $C_1 \cup C_2$ is in Lemma 2.6, here we determine the cardinality of C_3^2. Since our construction is based on the identifying vectors by equal division, we can determine our cardinality as follow: First, we give the form of the codeword,

$$\begin{pmatrix} * & \mathcal{F} \\ \boldsymbol{O} & * \end{pmatrix},$$

where $\mathcal{F}$ is $k \times (n-k)$ echelon Ferrers diagram, $\boldsymbol{O}$ is $2\delta \times 2\delta$ echelon diagram, and $\mathcal{F}$ contains many small full Ferrers diagrams, so we secondly determine the total number of full Ferrers diagrams inside is $(k-2\delta)\times(n-k)+(n-k-2\delta)\times 2\delta+\frac{\delta^2}{2^{2\times s_2}}\times(2^{(s_2+1)}-1)$; next, the number of zeros in O is $\frac{\delta^2}{2^{2\times s_2}}\times(2^{(s_2+1)}+1)$. Therefore, according to Definition 2.2, there exists an optimal code $[\mathcal{H}, \delta]_q$ ($\mathcal{H}$ is a Ferrers diagrams with $k \times (n-k)$) whose dimension is the same as the number of points in $\mathcal{H}$ that do not contain the first $\delta - 1$ rows. Here, we provide specific calculations: $(k-2\delta)\times(n-k)+(n-k-2\delta)\times 2\delta + \frac{\delta^2}{2^{2\times s_2}} \times (2^{(s_2+1)}-1) - \frac{\delta^2}{2^{2\times s_2}} \times (2^{(s_2+1)}+1) = k\times(n-k) - 2\times\delta^2(2+2^{-s_2})$. □

Regarding the construction of Lemma 3.1, we examined the case where zeros precede ones in the equal division. We observe that interchanging the positions of zeros and ones yields another codeword. Below, we provide a detailed discussion of this case.

Lemma 3.2. *Let n, k, and δ be non-negative integers satisfying $n \geq 2k + \delta$ and $k \geq 2\delta$. There exists an $(n, M_3^{2'}, 2\delta, k)_q$-CDC $C_3^{2'}$, where*

$$|M_3^{2'}| = \sum_{l=1}^{s_2} q^{(k-\delta+1)\times(n-2k+\frac{\delta}{2^l})+\delta\times(k-\frac{3\delta}{2}-\frac{\delta}{2^{l+1}}+1)}.$$

Proof. We construct the set of identifying vectors of length n as follows:

$$\mathcal{A}_2' = \{(\underbrace{1\cdots1}_{k-2\delta} \mid \overline{u} \mid \overline{u} \mid \underbrace{0\cdots0}_{n-k-2\delta}) : u \in \mathcal{B}_2\},$$

where $\mathcal{B}_2$ is as shown in Lemma 3.1 above, $\overline{u}$ denotes the binary inversion of u, obtained by interchanging all zero and one positions. Since the proof technique and the cardinality determination follow similar approaches to those in Lemma 3.1 and Corollary 3.1, we omit the detailed exposition here.

□

Theorem 3.2. *Let $C_1 \cup C_2$, C_3^2 and $C_3^{2'}$ be the codes defined in Lemmas 2.6, 3.1, and 3.2, respectively. Then the code $C = C_1 \cup C_2 \cup C_3^2 \cup C_3^{2'}$ forms an $(n, M, 2\delta, k)_q$-CDC, where*

$$|M| = M_1 + M_2 + \sum_{l=1}^{s_2} q^{k(n-k)-2\delta^2(2+2^{-l})} + \sum_{l=1}^{s_2'} q^{(k-\delta+1)(n-2k+\frac{\delta}{2^l})+\delta\times(k-\frac{3\delta}{2}-\frac{\delta}{2^{l+1}}+1)}.$$

Proof. From the proof of Theorem 3.1 and the construction of Lemma 3.2, it follows directly that $C_3^{2'}$ can be merged with both C_1 and C_2, while ensuring that the distance between any two of them remains greater than or equal to 2δ. Next, we demonstrate that the distance between $C_3^{2'}$ and C_3^2 is also greater than or equal to 2δ. For any two subspaces $\mathcal{U}, \mathcal{V} \in C$, $U = \text{rowspace}(\mathcal{U})$ and $V = \text{rowspace}(\mathcal{V})$. To visually demonstrate our proof, the partial identifying vectors of C_3^2 and $C_3^{2'}$ are presented as follows:

$$v(u) = (\overbrace{\underbrace{0\cdots0}_{\frac{\delta}{2^{s_2}}}\underbrace{1\cdots1}_{\frac{\delta}{2^{s_2}}}, \cdots, \underbrace{0\cdots0}_{\frac{\delta}{2^{s_2}}}\underbrace{1\cdots1}_{\frac{\delta}{2^{s_2}}}}^{2\delta}),$$

$$v(\overline{u}) = (\overbrace{\underbrace{1\cdots1}_{\frac{\delta}{2^{s_2}}}\underbrace{0\cdots0}_{\frac{\delta}{2^{s_2}}}, \cdots, \underbrace{1\cdots1}_{\frac{\delta}{2^{s_2}}}\underbrace{0\cdots0}_{\frac{\delta}{2^{s_2}}}}^{2\delta}).$$

From the identifying vectors above, since the zeros and ones in the two vectors are misaligned, each pair of u's has a Hamming distance of δ. As we have two such u distances, it is easy to prove that $d_S(\mathcal{U}, \mathcal{V}) \geq d_H(\mathcal{U}, \mathcal{V}) \geq 2\delta$ by Lemmas 2.2 and 2.3.

□

Following the 2-equal division analysis, we consider its natural generalization to 3-equal division, with the following lemma:

Lemma 3.3. *Let n, k, and δ be non-negative integers satisfying $n \geq 2k + \delta$ and $k \geq 2\delta$. Then there exists an $(n, M_3^3, 2\delta, k)_q$-CDC, denoted by C_3^3.*

Proof. We construct the set of identifying vectors of length n as follows:

$$\mathcal{A}_3 = \{(\underbrace{1\cdots1}_{k-2\delta} \mid u \mid u \mid \underbrace{0\cdots0}_{n-k-2\delta}) : u \in \mathcal{B}_3\},$$

where $\mathcal{B}_3 = \{(\mathcal{B}_{3,1}, \mathcal{B}_{3,2}, \cdots, \mathcal{B}_{3,s_3})\}$ and

$$\mathcal{B}_{3,s_3} = (\overbrace{\underbrace{0\cdots0}_{\frac{\delta}{3^{s_3}}}\underbrace{1\cdots1}_{\frac{\delta}{3^{s_3}}},\cdots,\underbrace{0\cdots0}_{\frac{\delta}{3^{s_3}}}\underbrace{1\cdots1}_{\frac{\delta}{3^{s_3}}}}^{2\delta}).$$

From two u above, we respectively derive δ, which yields $d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta$. □

Similar to Theorem 3.1, the 3 equal divisions construction may replace the 2 equal divisions in combination with $C_1 \cup C_2$ above; next, we give its relevant theorem.

Theorem 3.3. *Let $C_1 \cup C_2$ and C_3^3 be the codes defined in Lemmas 2.6 and 3.3, respectively. Then the code $C = C_1 \cup C_2 \cup C_3^3$ forms an $(n, M, 2\delta, k)_q$-CDC, where*

$$A_q(n, 2\delta, k) \geq M_1 + M_2 + \sum_{l=1}^{s} q^{k\times(n-k)-2\times\delta^2(2+3^{-l})}.$$

The proof is similar to that of 2 equal division; we omit the detailed exposition here. Next, we give some examples.

Example 3.1. *Let $n = 15$, $\delta = 3$, $k = 6$. According to Theorem 3.3, we can derive: $A_q(n, 2\delta, k) \geq q^{36} \times \frac{1-q^{-18}}{1-q^{-9}} + q^{24} + (q^9 - 1)\begin{bmatrix}6\\3\end{bmatrix}_q + 1 + q^{12}$. In this context, we can set $q = 3$. By doing so, we obtain: $A_3(15, 6, 6) \geq 150102543991378192$, which exceeds the known bound 150102543990846750 in [17].*

Example 3.2. *Let $n = 17$, $\delta = 3$, $k = 7$. According to Theorem 3.3, we can derive: $A_q(n, 2\delta, k) \geq q^{50} \times \frac{1-q^{-18}}{1-q^{-9}} + q^{35} + (q^{10} - 1)\begin{bmatrix}7\\3\end{bmatrix}_q + 1 + q^{28}$. In this context, we can set $q = 3$. By doing so, we obtain: $A_3(17, 6, 7) \geq 717934513968483599669410$, which exceeds the known bound 717934513945606807214448 in [17].*

Moreover, for $q \in \{4, 5, 7, 8, 9\}$, the proposed bounds improve upon all previously known results. These improvements are explicitly demonstrated in Table 1.

Table 1. Constant dimension codes from Theorem 3.3 and [17] lower bounds for $A_q(n, 2\delta, k)$.

$A_q(n, 2\delta, k)$	Theorem 3.3	[17]
$A_3(15, 6, 6)$	150102543991378192	150102543990846750
$A_4(15, 6, 6)$	4722384795619125200909	4722384778841908199452
$A_5(15, 6, 6)$	1455192298270279374501 8646	1455192273855709068298 8320
$A_7(15, 6, 6)$	26517310501764710246156 34220748	26517309117635990108176 16918746
$A_8(15, 6, 6)$	32451924327576809345232 073287057	32451855608100075344505 3205203320
$A_9(15, 6, 6)$	2252840242746404289780 89100290010744	2252839960316867802912 9780303636252
$A_3(17, 6, 7)$	717934513968483599669 410	717934513945606807214 448
$A_4(17, 6, 7)$	1267655509196312537842 640333617	1267655437138714999914 722735680
$A_5(17, 6, 7)$	8882158900424487874238 54508575432501	8881788744768854329316 6694301204500
$A_7(17, 6, 7)$	1798925878903094903167 05509743702491795 4049	1798465087215432610771 2869516684569121 86048
$A_8(17, 6, 7)$	1427267046152938338384 08160550032211179782 3617	1427247703339824504317 28630278602373997963 7248
$A_9(17, 6, 7)$	5153790001507475655223 26862058741306517240 108301	5153775220622933028375 56974561316380272313 825860

Remark 3.1. *Similarly, we can extend the above construction method to p equal divisions (where p is a prime number). Analogous to Lemmas 3.1 and 3.3, we can obtain the codes C_3^5 and C_3^7 constructed from 5 equal divisions and 7 equal divisions. The relevant construction forms are shown below:*

$$\mathcal{B}_{5,1} = (\underbrace{1\cdots1}_{\frac{\delta}{5}}\underbrace{0\cdots0}_{\frac{\delta}{5}},\cdots,\underbrace{1\cdots1}_{\frac{\delta}{5}}\underbrace{0\cdots0}_{\frac{\delta}{5}}),$$

$$\mathcal{B}_{7,1} = (\underbrace{1\cdots\cdots1}_{\frac{3\delta}{7}}\underbrace{0\cdots\cdots0}_{\frac{4\delta}{7}}\underbrace{1\cdots\cdots1}_{\frac{4\delta}{7}}\underbrace{0\cdots\cdots0}_{\frac{3\delta}{7}}).$$

For the 5 equal divisions, we obtain by interchanging the positions of zeros and ones, such that ones precede zeros. For the 7 equal divisions, we establish a special construction form to facilitate subsequent constructions.

4. *m* equal-division construction

4.1. Specific construction

Building upon the previous constructions, we further investigate whether multiple equal divisions schemes can be effectively combined. Specifically, we integrate C_3^p corresponding to the above cases of p equal divisions with $C_1 \cup C_2$ in Lemma 2.6.

Theorem 4.1. *Let $n = 2k + \delta$ with $k \geq 2\delta$. Consider the code $C_{\mathcal{M}} = C_1 \cup C_2 \cup C_3^2 \cup C_3^3 \cup C_3^5 \cup C_3^7$, where $C_1 \cup C_2$ is from Lemma 2.6, while C_3^p corresponds to the above cases of p equal divisions with $p \in \{2, 3, 5, 7\}$ and $s_2 = s_3 = s_5 = s_7 = 1$. Then $C_{\mathcal{M}}$ forms an $(n, M, 2\delta, k)_q$-CDC, where*

$$|M| = M_1 + M_2 + \sum_{p \in \{2,3,5,7\}} M_3^p,$$

where M_3^p denotes the cardinality corresponding to the equal-divisions above.

Proof. The minimum distances among C_1, C_2, and each equal-division construction have been established previously. Here we focus on proving the minimum distances between different equal divisions constructions themselves.

When $\mathcal{U} \in C_3^2, \mathcal{V} \in C_3^3$. Based on the above construction, we can extract $\frac{\delta}{3}$ and $\frac{\delta}{6}$ respectively from the two sets of δ elements within the first k elements in identifying vectors, which means the Hamming distance of the first k elements is δ. Similarly, we can prove that the Hamming distance of the remaining $n - k$ elements is also δ. According to Lemmas 2.2 and 2.3, we can conclude that $d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta$.

Following the same method, we can also demonstrate that the Hamming distance between C_3^2 and C_3^5, as well as between C_3^3 and C_3^5, is also 2δ. Since 7 equal divisions is a special form that we set up, we will prove it separately from the other equal divisions:

We first determine the distance between C_3^2 and C_3^7; in the construction we can take out the distance of $\frac{13\delta}{7}$ from the first k elements; the subsequent $n - k$ items are the same. So the total distance is greater than 2δ. Similarly, we can calculate that the distance between C_3^3 and C_3^7 is $\frac{16\delta}{7}$, and the distance between C_3^5 and C_3^7 is $\frac{16\delta}{7}$. In summary, the distances between the various equal divisions discussed above are all greater than or equal to 2δ; therefore, they can be combined to form a CDC. Proof complete. □

4.2. General construction

Following the same methodology, we can make a conjecture: according to the fundamental theorem of arithmetic [22], we can get $\delta = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$($p_h$ is a prime number, $1 \leq h \leq k$). By rationally arranging the positions of zeros and ones, so that for any i, j, ($i \in \{p_1, p_2, \cdots, p_k\}$, $1 \leq j \leq s_k$) the elements in C_3^i are pairwise disjoint, and their Hamming distance $d_H(\boldsymbol{v}_x, \boldsymbol{v}_y) \geq 2\delta$ ($\boldsymbol{v}_x$ and $\boldsymbol{v}_y$ are any two arbitrary identifying vectors from the codes mentioned above). Let the CDCs from different equal divisions be denoted as $\bigcup_{i \in \{p_1, p_2, \cdots, p_k\}} C_3^i$. Regarding the above idea, we can summarize it into the following conjecture:

Conjecture 4.1. *Let n, k, and δ be non-negative integers satisfying $n = 2k + \delta$ and $k \geq 2\delta$. Then the code $C_1 \cup C_2 \bigcup_{i \in \{p_1, p_2, \cdots, p_k\}} C_3^i$ is an $(n, M, 2\delta, k)_q$-CDC, where the maximal possible cardinality is given by:*

$$|M| = M_1 + M_2 \quad + \sum_{i\in\{p_1,p_2,\cdots,p_k\}} \sum_{j=1}^{s_k} M_3^{ij}.$$

5. Conclusions

In this paper, we present a new construction of CDCs based on the equal-division method. This method can improve the lower bound of CDCs on some parameters. However, it is still an open and difficult problem whether different equal-division methods will lead to good or bad results. We find that through strategic distribution of zeros and ones in the identifying vectors across different equal-division constructions, these equal-division constructions admit effective combinations that enhance code cardinality. In this article, we have focused on several fundamental prime cases but conjecture that additional primes may satisfy the construction requirements. These conjectures are derived from our analytical results.

Author contributions

Yongfeng Niu: Conceptualization, supervision, writing review and editing, funding acquisition; Liang Wu: Conceptualization, methodology, writing-original draft; Yizhuo Zhang: Supervision, writing-review; Huiling Yu: Supervision, writing-review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (Al) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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