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Research article

An A_α-spectral radius for the existence of $\{P_3, P_4, P_5\}$-factors in graphs

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Abstract: Let G be a connected graph of order n with $n \geq 25$. A $\{P_3, P_4, P_5\}$-factor is a spanning subgraph H of G such that every component of H is isomorphic to an element of $\{P_3, P_4, P_5\}$. Nikiforov introduced the A_α-matrix of G as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ [V. Nikiforov, Merging the A- and Q-spectral theories, Appl. Anal. Discrete Math., 11 (2017), 81–107], where $\alpha \in [0, 1]$, $D(G)$ denotes the diagonal matrix of vertex degrees of G and $A(G)$ denotes the adjacency matrix of G. The largest eigenvalue of $A_\alpha(G)$, denoted by $\lambda_\alpha(G)$, is called the A_α-spectral radius of G. In this paper, it is proved that G has a $\{P_3, P_4, P_5\}$-factor unless $G = K_1 \vee (K_{n-2} \cup K_1)$ if $\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$, where α is a real number with $0 \leq \alpha < \frac{2}{3}$.

Keywords: graph; A_α-matrix; A_α-spectral radius; spanning subgraph; $\{P_3, P_4, P_5\}$-factor
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1. Introduction

We deal with finite undirected graphs without loops or multiple edges. Let G denote a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the neighborhood of v and the degree of v in G are denoted by $N_G(v)$ and $d_G(v)$, respectively. Let $i(G)$ denote the number of isolated vertices in G. For a subset $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraphs of G induced by S and $V(G) - S$, respectively. For two vertex disjoint graphs G_1 and G_2, the union of G_1 and G_2 is denoted by $G_1 \cup G_2$. Let tG stand for the disjoint union of t copies of G, where t is a positive integer. The join $G_1 \vee G_2$ is the graph obtained by joining each vertex of G_1 to each vertex of G_2. We denote the path, the cycle, the star and the complete graph of order n by P_n, C_n, $K_{1,n-1}$ and K_n, respectively. Let c be a real number. Recall that $\lfloor c \rfloor$ is the greatest integer with $\lfloor c \rfloor \leq c$.

Let $\mathcal{H}$ denote a set of connected graphs. Then a spanning subgraph H of G is called an $\mathcal{H}$-factor if every component of H is an element of $\mathcal{H}$. If $\mathcal{H} = \{P_3, P_4, P_5\}$, then an $\mathcal{H}$-factor is called a $\{P_3, P_4, P_5\}$-factor. Write $P_{\geq k} = \{P_i | i \geq k\}$. If $\mathcal{H} = P_{\geq k}$, then an $\mathcal{H}$-factor is called a $P_{\geq k}$-factor. If $\mathcal{H} = \{K_2, C_i | i \geq 3\}$, then an $\mathcal{H}$-factor is called a $\{K_2, C_i | i \geq 3\}$-factor. If $\mathcal{H} = \{K_{1,j} | 1 \leq j \leq k\}$, then an $\mathcal{H}$-factor is called a $\{K_{1,j} | 1 \leq j \leq k\}$-factor.

Kano et al. [1] established a connection between the number of isolated vertices and $\{P_3, P_4, P_5\}$-factors in graphs. Akiyama et al. [2] proved that a graph G contains a $P_{\geq 2}$-factor if and only if $i(G-S) \leq 2|S|$ for any subset $S \subseteq V(G)$. Kaneko [3] provided a characterization of a graph having a $P_{\geq 3}$-factor. Liu and Pan [4], Gao et al. [5], and Dai and Hu [6] obtained some sufficient conditions on the existence of $P_{\geq 2}$-factors and $P_{\geq 3}$-factors in graphs. Tutte [7] got a criterion for a graph containing a $\{K_2, C_i | i \geq 3\}$-factor. Klopp and Steffen [8] investigated the properties of $\{K_{1,1}, K_{1,2}, C_i | i \geq 3\}$-factors in graphs. Amahashi and Kano [9] posed a criterion for a graph with a $\{K_{1,j} | 1 \leq j \leq k\}$-factor, where k is an integer with $k \geq 2$. Kano and Saito [10] showed a sufficient condition for a graph to contain a $\{K_{1,j} | k \leq j \leq 2k\}$-factor, where k is an integer with $k \geq 2$. For many other results on spanning subgraphs, we refer the readers to [11–13].

Let $A(G)$ and $D(G)$ denote the adjacency matrix and the degree diagonal matrix of G, respectively. We use $\lambda(G)$ to denote the adjacency spectral radius of G. Let $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of G. The signless Laplacian spectral radius of G is denoted by $q(G)$. For any $\alpha \in [0, 1)$, Nikiforov [14] introduced the A_α-matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

Notice that $A_\alpha(G) = A(G)$ if $\alpha = 0$ and $A_\alpha(G) = \frac{1}{2}Q(G)$ if $\alpha = \frac{1}{2}$. The largest eigenvalue of $A_\alpha(G)$, denoted by $\lambda_\alpha(G)$, is called the A_α-spectral radius of G. Clearly, $\lambda_0(G)$ is the adjacency spectral radius of G and $2\lambda_{\frac{1}{2}}(G)$ is the signless Laplacian spectral radius of G. Thus, $\lambda_\alpha(G)$ generalizes both the adjacency spectral radius and the signless Laplacian spectral radius of G. In recent years, the A_α-matrix of G has attracted a great deal of attention. For details, we refer the readers to [15–17]. We refer the readers to [18, 19] for relevant elementary background on the topic.

Suil [20], Zhao et al. [21], and Zhou et al. [22] provided some spectral conditions for graphs to contain $\{K_2\}$-factors. Li and Miao [23] established a lower bound on the adjacency spectral radius for a connected graph which ensures that this graph has a $P_{\geq 2}$-factor. Zhou [24] showed an adjacency spectral radius and a distance spectral radius condition for a bipartite graph to have a star-factor with given properties, respectively. Miao and Li [25] determined a lower bound on the adjacency spectral radius of a connected graph G to guarantee that G has a $\{K_{1,j} | 1 \leq j \leq k\}$-factor, and presented an upper bound on the distance spectral radius of a connected graph G to ensure that G contains a $\{K_{1,j} | 1 \leq j \leq k\}$-factor. Lv et al. [26] obtained two spectral sufficient conditions on the existence of a $\{K_2, C_{2i+1} : i \geq 1\}$-factor in a graph. Lv et al. [27] established the A_α-spectral radius for graphs to have $\{P_2, C_3, P_5, \mathcal{T}(3)\}$-factors.

Motivated by [1, 20] directly, we investigate the existence of $\{P_3, P_4, P_5\}$-factors in connected graphs, and establish a relationship between the A_α-spectral radius and $\{P_3, P_4, P_5\}$-factors in connected graphs. Our main result is shown as follows.

Theorem 1.1. Let α be a real number with $0 \leq \alpha < \frac{2}{3}$, and let G be a connected graph of order n with $n \geq 25$. If G satisfies

$$\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)),$$

then G has a $\{P_3, P_4, P_5\}$-factor unless $G = K_1 \vee (K_{n-2} \cup K_1)$.

In fact, a $\{P_3, P_4, P_5\}$-factor is also a $P_{\geq 3}$-factor. Then the following corollary holds.

Corollary 1.2. Let α be a real number with $0 \leq \alpha < \frac{2}{3}$, and let G be a connected graph of order n with $n \geq 25$. If G satisfies

$$\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)),$$

then G has a $P_{\geq 3}$-factor unless $G = K_1 \vee (K_{n-2} \cup K_1)$.

2. Some preliminaries

In 2010, Kano et al. [1] provided a sufficient condition for the existence of $\{P_3, P_4, P_5\}$-factors in graphs.

Lemma 2.1. [1] If a graph G satisfies

$$i(G - S) \leq \frac{2}{3}|S|$$

for any subset $S \subset V(G)$, then G contains a $\{P_3, P_4, P_5\}$-factor.

Lemma 2.2. [14] For any $\alpha \in [0, 1)$ and a complete graph K_n, we conclude

$$\lambda_\alpha(K_n) = n - 1.$$

Lemma 2.3. [14] If G is a connected graph, and H is a proper subgraph of G, then we have

$$\lambda_\alpha(G) > \lambda_\alpha(H),$$

where $\alpha \in [0, 1)$.

Let M be a real symmetric matrix of order n whose columns and rows are indexed by $V = \{1, 2, \ldots, n\}$, where $V = V_1 \cup V_2 \cup \cdots \cup V_t$, $|V_i| = n_i$ and $n = \sum_{i=1}^{t} n_i$. Assume that M is a matrix with the partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_t$, that is,

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in V_i and columns in V_j. The average row sum of M_{ij} is denoted by m_{ij}. Then the matrix $M_\pi = (m_{ij})$ is called the quotient matrix of M. In particular, if the row sum of each block M_{ij} is a constant, then the partition is called equitable.

Lemma 2.4. [28] Let M be a real matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is a nonnegative matrix, then the largest eigenvalue of M is equal to the largest eigenvalue of M_π.

Lemma 2.5. [29] Let M be a Hermitian matrix of order s, and let N be a principal submatrix of M of order t. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$ are the eigenvalues of M and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t$ are the eigenvalues of N, then $\lambda_i \geq \mu_i \geq \lambda_{s-t+i}$ for $1 \leq i \leq t$.

3. The proof of Theorem 1.1

Proof of Theorem 1.1. Suppose, to the contrary, that G contains no $\{P_3, P_4, P_5\}$-factor. Then it follows from Lemma 2.1 that $i(G-S) > \frac{2}{3}|S|$ for some nonempty subset $S \subset V(G)$. Let $|S| = s$ and $i(G-S) = i$. According to the integrity of $i(G-S)$, we get $i \geq \lfloor \frac{2}{3}s \rfloor + 1$. From the above discussion, we easily see that G is a spanning subgraph of $G_1 = K_s \vee (K_{n-\lfloor \frac{5}{3}s \rfloor -1} \cup (\lfloor \frac{2}{3}s \rfloor + 1)K_1)$. Combining this with Lemma 2.3, we deduce

$$\lambda_\alpha(G) \leq \lambda_\alpha(G_1), \tag{3.1}$$

with equality holding if and only if $G = G_1$. The following proof will be divided into three cases by the value of n.

Case 1. $n \geq \lfloor \frac{5}{3}s \rfloor + 3$.

Recall that $G_1 = K_s \vee (K_{n-\lfloor \frac{5}{3}s \rfloor -1} \cup (\lfloor \frac{2}{3}s \rfloor + 1)K_1)$. The quotient matrix of $A_\alpha(G_1)$ by the partition $V(G_1) = V(K_s) \cup V(K_{n-\lfloor \frac{5}{3}s \rfloor -1}) \cup V((\lfloor \frac{2}{3}s \rfloor + 1)K_1)$ can be written as

$$B_1 = \begin{pmatrix} \alpha n - \alpha s + s - 1 & (1-\alpha)(n - \lfloor \frac{5}{3}s \rfloor - 1) & (1-\alpha)(\lfloor \frac{2}{3}s \rfloor + 1) \\ (1-\alpha)s & n + \alpha s - \lfloor \frac{5}{3}s \rfloor - 2 & 0 \\ (1-\alpha)s & 0 & \alpha s \end{pmatrix}.$$

By a direct calculation, the characteristic polynomial of B_1 is

$$\begin{aligned}\varphi_{B_1}(x) =& x^3 - \Big(\alpha n + n + \alpha s - \Big\lfloor \frac{2}{3}s \Big\rfloor - 3\Big)x^2 \\ &+ \Big(\alpha n^2 + \alpha^2 sn + \alpha sn - \alpha n\Big\lfloor \frac{2}{3}s \Big\rfloor - 2\alpha n - n - s\Big\lfloor \frac{2}{3}s \Big\rfloor - 2\alpha s - s + \Big\lfloor \frac{2}{3}s \Big\rfloor + 2\Big)x \\ &- \alpha^2 sn^2 + 2\alpha^2 sn\Big\lfloor \frac{2}{3}s \Big\rfloor - 2\alpha sn\Big\lfloor \frac{2}{3}s \Big\rfloor + 3\alpha^2 sn - \alpha sn + sn\Big\lfloor \frac{2}{3}s \Big\rfloor + sn \\ &- 2\alpha^2 s^2\Big\lfloor \frac{2}{3}s \Big\rfloor - \alpha^2 s\Big\lfloor \frac{2}{3}s \Big\rfloor^2 + 3\alpha s^2\Big\lfloor \frac{2}{3}s \Big\rfloor + 2\alpha s\Big\lfloor \frac{2}{3}s \Big\rfloor^2 - s^2\Big\lfloor \frac{2}{3}s \Big\rfloor - s\Big\lfloor \frac{2}{3}s \Big\rfloor^2 - 2\alpha^2 s^2 \\ &- 3\alpha^2 s\Big\lfloor \frac{2}{3}s \Big\rfloor + 3\alpha s^2 + 5\alpha s\Big\lfloor \frac{2}{3}s \Big\rfloor - s^2 - 3s\Big\lfloor \frac{2}{3}s \Big\rfloor - 2\alpha^2 s + 2\alpha s - 2s. \end{aligned} \tag{3.2}$$

Notice that the partition $V(G_1) = V(K_s) \cup V(K_{n-\lfloor \frac{5}{3}s \rfloor -1}) \cup V((\lfloor \frac{2}{3}s \rfloor + 1)K_1)$ is equitable. By virtue of Lemma 2.4, $\lambda_\alpha(G_1)$ is the largest root of $\varphi_{B_1}(x) = 0$, that is, $\varphi_{B_1}(\lambda_\alpha(G_1)) = 0$. Let $\theta_1 = \lambda_\alpha(G_1) \geq \theta_2 \geq \theta_3$ be the three roots of $\varphi_{B_1}(x) = 0$ and $Q = \mathrm{diag}(s, n - \lfloor \frac{5}{3}s \rfloor - 1, \lfloor \frac{2}{3}s \rfloor + 1)$. We easily see that

$$Q^{\frac{1}{2}} B_1 Q^{-\frac{1}{2}} = \begin{pmatrix} \alpha n - \alpha s + s - 1 & (1-\alpha)s^{\frac{1}{2}}(n - \lfloor \frac{5}{3}s \rfloor - 1)^{\frac{1}{2}} & (1-\alpha)s^{\frac{1}{2}}(\lfloor \frac{2}{3}s \rfloor + 1)^{\frac{1}{2}} \\ (1-\alpha)s^{\frac{1}{2}}(n - \lfloor \frac{5}{3}s \rfloor - 1)^{\frac{1}{2}} & n + \alpha s - \lfloor \frac{5}{3}s \rfloor - 2 & 0 \\ (1-\alpha)s^{\frac{1}{2}}(\lfloor \frac{2}{3}s \rfloor + 1)^{\frac{1}{2}} & 0 & \alpha s \end{pmatrix}$$

is symmetric, and

$$\begin{pmatrix} n + \alpha s - \lfloor \frac{5}{3}s \rfloor - 2 & 0 \\ 0 & \alpha s \end{pmatrix}$$

is a submatrix of $Q^{\frac{1}{2}}B_1Q^{-\frac{1}{2}}$. Since $Q^{\frac{1}{2}}B_1Q^{-\frac{1}{2}}$ and B_1 have the same eigenvalues, Lemma 2.5 (the Cauchy interlacing theorem) leads to

$$\theta_2 \leq n + \alpha s - \left\lfloor \frac{5}{3}s \right\rfloor - 2 < \begin{cases} n-4, & \text{if } s \equiv 0 \ (\mathrm{mod}\ 3); \\ n-2, & \text{if } s \equiv 1 \ (\mathrm{mod}\ 3); \\ n-3, & \text{if } s \equiv 2 \ (\mathrm{mod}\ 3). \end{cases} \tag{3.3}$$

For the graph $G_* = K_1 \vee (K_{n-2} \cup K_1)$, its adjacency matrix $A(G_*)$ admits the quotient matrix B_* which is derived by replacing s with 1 in B_1, and B_* admits the characteristic polynomial $\varphi_{B_*}(x)$ which is derived by replacing s with 1 in $\varphi_{B_1}(x)$. Hence, we have

$$\begin{aligned}\varphi_{B_*}(x) =& x^3 - (\alpha n + n + \alpha - 3)x^2 + (\alpha n^2 + \alpha^2 n - \alpha n - n - 2\alpha + 1)x \\ &- \alpha^2 n^2 + 3\alpha^2 n - \alpha n + n - 4\alpha^2 + 5\alpha - 3.\end{aligned}$$

In view of Lemma 2.4, $\lambda_\alpha(G_*)$ is the largest root of $\varphi_{B_*}(x) = 0$, that is, $\varphi_{B_*}(\lambda_\alpha(G_*)) = 0$. If $s = 1$, then $G_1 = G_*$, and so $\lambda_\alpha(G_1) = \lambda_\alpha(G_*) = \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$. Combining this with (3.1), we deduce $\lambda_\alpha(G) \leq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$, where the equality holds if and only if $G = K_1 \vee (K_{n-2} \cup K_1)$. This contradicts the assumption that G is not the indicated graph. Next, we consider $s \geq 2$.

Since K_{n-1} is a proper subgraph of $G_* = K_1 \vee (K_{n-2} \cup K_1)$, it follows from (3.3) and Lemmas 2.2 and 2.3 that

$$\lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)) > \lambda_\alpha(K_{n-1}) = n - 2 > \theta_2. \tag{3.4}$$

We shall consider three subcases by the value of s.

Subcase 1.1. $s \equiv 0 \ (\mathrm{mod}\ 3)$.

In this subcase, $s \geq 3$, $\lfloor \frac{2}{3}s \rfloor = \frac{2}{3}s$ and $n \geq \lfloor \frac{5}{3}s \rfloor + 3 = \frac{5}{3}s + 3$. According to (3.2), we admit

$$\begin{aligned}\varphi_{B_1}(x) =& x^3 - \Big(\alpha n + n + \alpha s - \frac{2}{3}s - 3\Big)x^2 \\ &+ \Big(\alpha n^2 + \alpha^2 sn + \frac{1}{3}\alpha sn - 2\alpha n - n - \frac{2}{3}s^2 - 2\alpha s - \frac{1}{3}s + 2\Big)x \\ &- \alpha^2 sn^2 + \frac{4}{3}\alpha^2 s^2 n - \frac{4}{3}\alpha s^2 n + 3\alpha^2 sn - \alpha sn + \frac{2}{3}s^2 n + sn - \frac{16}{9}\alpha^2 s^3 \\ &+ \frac{26}{9}\alpha s^3 - \frac{10}{9}s^3 - 4\alpha^2 s^2 + \frac{19}{3}\alpha s^2 - 3s^2 - 2\alpha^2 s + 2\alpha s - 2s.\end{aligned}$$

Let $G_2 = K_3 \vee (K_{n-6} \cup 3K_1)$. Then its adjacency matrix $A(G_2)$ has the quotient matrix B_2 which is derived by replacing s with 3 in B_1, and B_2 has the characteristic polynomial $\varphi_{B_2}(x)$ which is obtained by replacing s with 3 in $\varphi_{B_1}(x)$. Therefore, we obtain

$$\begin{aligned}\varphi_{B_2}(x) =& x^3 - (\alpha n + n + 3\alpha - 5)x^2 + (\alpha n^2 + 3\alpha^2 n - \alpha n - n - 6\alpha - 5)x \\ &- 3\alpha^2 n^2 + 21\alpha^2 n - 15\alpha n + 9n - 90\alpha^2 + 141\alpha - 63.\end{aligned}$$

Using Lemma 2.4, $\lambda_\alpha(G_2)$ is the largest root of $\varphi_{B_2}(x) = 0$, that is, $\varphi_{B_2}(\lambda_\alpha(G_2)) = 0$. We are to verify $\lambda_\alpha(G_1) \leq \lambda_\alpha(G_2)$.

Since K_{n-3} is a proper subgraph of $G_2 = K_3 \vee (K_{n-6} \cup 3K_1)$, it follows from (3.3) and Lemmas 2.2 and 2.3 that

$$\lambda_\alpha(K_3 \vee (K_{n-6} \cup 3K_1)) > \lambda_\alpha(K_{n-3}) = n - 4 > \theta_2. \tag{3.5}$$

Write $\beta = \lambda_\alpha(K_3 \vee (K_{n-6} \cup 3K_1))$. Notice that $\varphi_{B_2}(\beta) = 0$. By a direct computation, we get

$$\varphi_{B_1}(\beta) = \varphi_{B_1}(\beta) - \varphi_{B_2}(\beta) = \frac{1}{9}(s-3)f_1(\beta), \tag{3.6}$$

where $f_1(\beta) = (6-9\alpha)\beta^2 + (9\alpha^2 n + 3\alpha n - 6s - 18\alpha - 21)\beta - 9\alpha^2 n^2 + 3\alpha^2 n(4s+21) - 3\alpha n(4s+15) + 3n(2s+9) - 2\alpha^2(8s^2+42s+135) + \alpha(26s^2+135s+423) - 10s^2 - 57s - 189$. Notice that

$$-\frac{9\alpha^2 n + 3\alpha n - 6s - 18\alpha - 21}{2(6-9\alpha)} < n - 4 < \beta \tag{3.7}$$

by (3.5), $s \geq 6$, and $n \geq \frac{5}{3}s + 3$. Since the symmetry axis of $f_1(\beta)$ is $\beta = -\frac{9\alpha^2 n+3\alpha n-6s-18\alpha-21}{2(6-9\alpha)}$, it follows from (3.7) that

$$\begin{aligned} f_1(\beta) >& f_1(n-4) \\ =&(6-6\alpha)n^2 + (12\alpha^2 s - 12\alpha s + 27\alpha^2 - 3\alpha - 42)n \\ &- 2\alpha^2(8s^2+42s+135) + \alpha(26s^2+135s+351) - 10s^2 - 33s - 9. \end{aligned} \tag{3.8}$$

Let $f_2(n) = (6-6\alpha)n^2 + (12\alpha^2 s - 12\alpha s + 27\alpha^2 - 3\alpha - 42)n - 2\alpha^2(8s^2+42s+135) + \alpha(26s^2+135s+351) - 10s^2 - 33s - 9$. Note that

$$-\frac{12\alpha^2 s - 12\alpha s + 27\alpha^2 - 3\alpha - 42}{2(6-6\alpha)} < \frac{5}{3}s + 3 \leq n$$

by $s \geq 15$ and $0 \leq \alpha < \frac{2}{3}$. Thus, we deduce

$$\begin{aligned} f_2(n) \geq& f_2\Big(\frac{5}{3}s+3\Big) \\ =&\frac{1}{3}((12s^2-9s-567)\alpha^2 + (-32s^2+102s+864)\alpha + 20s^2 - 129s - 243) \\ >&\frac{1}{3}\Big(\frac{4}{9}(12s^2-9s-567) + \frac{2}{3}(-32s^2+102s+864) + 20s^2 - 129s - 243\Big) \\ =&\frac{1}{9}(12s^2 - 195s + 243) \\ >&0, \end{aligned} \tag{3.9}$$

where the last two inequalities hold from $\frac{32s^2-102s-864}{2(12s^2-9s-567)} > \frac{2}{3} > \alpha \geq 0$ and $s \geq 15$, respectively.

If $s \in \{6, 9, 12\}$, then

$$-\frac{12\alpha^2 s - 12\alpha s + 27\alpha^2 - 3\alpha - 42}{2(6-6\alpha)} = \begin{cases} \frac{14+25\alpha-33\alpha^2}{4-4\alpha}, & \text{if } s = 6, \\ \frac{14+37\alpha-45\alpha^2}{4-4\alpha}, & \text{if } s = 9, \\ \frac{14+49\alpha-57\alpha^2}{4-4\alpha}, & \text{if } s = 12, \end{cases}$$

$$<25 \leq n$$

by $0 \leq \alpha < \frac{2}{3}$. Thus, we obtain

$$\begin{aligned} f_2(n) \geq & f_2(25) \\ =&(-16s^2 + 216s + 405)\alpha^2 + (26s^2 - 165s - 3474)\alpha - 10s^2 - 33s + 2691 \\ =&\begin{cases} 1125\alpha^2 - 3528\alpha + 2133, & \text{if } s = 6, \\ 1053\alpha^2 - 2853\alpha + 1584, & \text{if } s = 9, \\ 693\alpha^2 - 1710\alpha + 855, & \text{if } s = 12, \end{cases} \\ >&0 \end{aligned} \tag{3.10}$$

by $0 \leq \alpha < \frac{2}{3}$.

From (3.9) and (3.10), we infer $f_2(n) > 0$ for $s \geq 6$ and $s \equiv 0 \pmod 3$. Combining this with (3.6) and (3.8), we conclude

$$\varphi_{B_1}(\beta) = \frac{1}{9}(s-3)f_1(\beta) \geq \frac{1}{9}(s-3)f_1(n-4) = \frac{1}{9}(s-3)f_2(n) \geq 0 \tag{3.11}$$

for $s \geq 3$ and $s \equiv 0 \pmod 3$. Recall that $\lambda_\alpha(G_1)$ is the largest root of $\varphi_{B_1}(x) = 0$. As $\theta_2 < n - 4 < \lambda_\alpha(K_3 \vee (K_{n-6} \cup 3K_1)) = \beta$ (see (3.5)), we deduce

$$\lambda_\alpha(G_1) \leq \beta = \lambda_\alpha(K_3 \vee (K_{n-6} \cup 3K_1)) = \lambda_\alpha(G_2) \tag{3.12}$$

by (3.11).

In what follows, we are to show $\lambda_\alpha(G_2) < n - 2$. By a direct calculation, we get

$$\begin{aligned} \varphi_{B_2}(n-2) =&(n-2)^3 - (\alpha n + n + 3\alpha - 5)(n-2)^2 \\ &+ (\alpha n^2 + 3\alpha^2 n - \alpha n - n - 6\alpha - 5)(n-2) \\ &- 3\alpha^2 n^2 + 21\alpha^2 n - 15\alpha n + 9n - 90\alpha^2 + 141\alpha - 63 \\ =&(2-2\alpha)n^2 + (15\alpha^2 - 11\alpha - 6)n - 90\alpha^2 + 141\alpha - 41 \\ \geq&(2-2\alpha)(25)^2 + 25(15\alpha^2 - 11\alpha - 6) - 90\alpha^2 + 141\alpha - 41 \\ =&285\alpha^2 - 1384\alpha + 1059 \\ >&0, \end{aligned}$$

where the last two inequalities hold from $-\frac{15\alpha^2-11\alpha-6}{2(2-2\alpha)} < 25 \leq n$ and $\frac{1384}{2\times 285} > \frac{2}{3} > \alpha \geq 0$, respectively. Hence, we infer

$$\lambda_\alpha(G_2) < n - 2. \tag{3.13}$$

According to (3.1), (3.4), (3.12), and (3.13), we have

$$\lambda_\alpha(G) \leq \lambda_\alpha(G_1) \leq \lambda_\alpha(G_2) < n - 2 < \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)),$$

which contradicts $\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$.

Subcase 1.2. $s \equiv 1 \pmod 3$.

In this subcase, $s \geq 4$, $\lfloor \frac{2}{3}s \rfloor = \frac{2s-2}{3}$ and $n \geq \lfloor \frac{5}{3}s \rfloor + 3 = \frac{5s+7}{3}$. In view of (3.2), we obtain

$$\begin{aligned}\varphi_{B_1}(x) =& x^3 - \Big(\alpha n + n + \alpha s - \frac{2}{3}s - \frac{7}{3}\Big)x^2 \\ &+ \Big(\alpha n^2 + \alpha^2 sn + \frac{1}{3}\alpha sn - \frac{4}{3}\alpha n - n - \frac{2}{3}s^2 - 2\alpha s + \frac{1}{3}s + \frac{4}{3}\Big)x \\ &- \alpha^2 sn^2 + \frac{4}{3}\alpha^2 s^2 n - \frac{4}{3}\alpha s^2 n + \frac{5}{3}\alpha^2 sn + \frac{1}{3}\alpha sn + \frac{2}{3}s^2 n + \frac{1}{3}sn - \frac{16}{9}\alpha^2 s^3 \\ &+ \frac{26}{9}\alpha s^3 - \frac{10}{9}s^3 - \frac{16}{9}\alpha^2 s^2 + \frac{23}{9}\alpha s^2 - \frac{13}{9}s^2 - \frac{4}{9}\alpha^2 s - \frac{4}{9}\alpha s - \frac{4}{9}s.\end{aligned}$$

Write $\gamma = \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$. Notice that $\varphi_{B_*}(\gamma) = 0$. A simple calculation yields that

$$\varphi_{B_1}(\gamma) = \varphi_{B_1}(\gamma) - \varphi_{B_*}(\gamma) = \frac{1}{9}(s-1)g_1(\gamma), \tag{3.14}$$

where $g_1(\gamma) = (6-9\alpha)\gamma^2 + (9\alpha^2 n + 3\alpha n - 6s - 18\alpha - 3)\gamma - 9\alpha^2 n^2 + 3\alpha^2 n(4s+9) - 3\alpha n(4s+3) + 3n(2s+3) - 4\alpha^2(4s^2+8s+9) + \alpha(26s^2+49s+45) - 10s^2 - 23s - 27$. Note that

$$-\frac{9\alpha^2 n + 3\alpha n - 6s - 18\alpha - 3}{2(6-9\alpha)} < n-2 < \gamma$$

by (3.4), $s \geq 4$ and $n \geq \frac{5s+7}{3}$. Thus, we have

$$\begin{aligned}g_1(\gamma) >& g_1(n-2) \\ =& (6-6\alpha)n^2 + (12\alpha^2 s - 12\alpha s + 9\alpha^2 + 3\alpha - 18)n \\ &- 4\alpha^2(4s^2+8s+9) + \alpha(26s^2+49s+45) - 10s^2 - 11s + 3.\end{aligned} \tag{3.15}$$

Let $g_2(n) = (6-6\alpha)n^2 + (12\alpha^2 s - 12\alpha s + 9\alpha^2 + 3\alpha - 18)n - 4\alpha^2(4s^2+8s+9) + \alpha(26s^2+49s+45) - 10s^2 - 11s + 3$. It follows from $s \geq 4$ and $0 \leq \alpha < \frac{2}{3}$ that

$$-\frac{12\alpha^2 s - 12\alpha s + 9\alpha^2 + 3\alpha - 18}{2(6-6\alpha)} < \frac{5s+7}{3} \leq n,$$

and so

$$\begin{aligned}g_2(n) \geq& g_2\Big(\frac{5s+7}{3}\Big) \\ =& \frac{1}{3}((12s^2+33s-45)\alpha^2 - (32s^2+62s-58)\alpha + 20s^2 + 17s - 19) \\ >& \frac{1}{3}\Big(\frac{4}{9}(12s^2+33s-45) - \frac{2}{3}(32s^2+62s-58) + 20s^2 + 17s - 19\Big) \\ =& \frac{1}{9}(12s^2 - 29s - 1) \\ >& 0,\end{aligned} \tag{3.16}$$

where the last two inequalities hold from $\frac{32s^2+62s-58}{2(12s^2+33s-45)} > \frac{2}{3} > \alpha \geq 0$ and $s \geq 4$, respectively.

By virtue of (3.14)–(3.16), we obtain

$$\varphi_{B_1}(\gamma) = \frac{1}{9}(s-1)g_1(\gamma) > \frac{1}{9}(s-1)g_1(n-2) = \frac{1}{9}(s-1)g_2(n) > 0 \tag{3.17}$$

for $s \geq 4$ and $s \equiv 1 \pmod 3$. Notice that $\lambda_\alpha(G_1)$ is the largest root of $\varphi_{B_1}(x) = 0$. As $\theta_2 < n-2 < \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)) = \gamma$ (see (3.4)), we conclude

$$\lambda_\alpha(G_1) < \gamma = \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$$

by (3.17). Combining this with (3.1), we have

$$\lambda_\alpha(G) \leq \lambda_\alpha(G_1) < \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)),$$

which is a contradiction to $\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$.

Subcase 1.3. $s \equiv 2 \pmod 3$.

In this subcase, $s \geq 2$, $\lfloor \frac{2}{3}s \rfloor = \frac{2s-1}{3}$ and $n \geq \lfloor \frac{5}{3}s \rfloor + 3 = \frac{5s+8}{3}$. Using (3.2), we possess

$$\begin{aligned}\varphi_{B_1}(x) =& x^3 - \Big(\alpha n + n + \alpha s - \frac{2}{3}s - \frac{8}{3}\Big)x^2 \\ &+ \Big(\alpha n^2 + \alpha^2 sn + \frac{1}{3}\alpha sn - \frac{5}{3}\alpha n - n - \frac{2}{3}s^2 - 2\alpha s + \frac{5}{3}\Big)x \\ &- \alpha^2 sn^2 + \frac{4}{3}\alpha^2 s^2 n - \frac{4}{3}\alpha s^2 n + \frac{7}{3}\alpha^2 sn - \frac{1}{3}\alpha sn + \frac{2}{3}s^2 n + \frac{2}{3}sn - \frac{16}{9}\alpha^2 s^3 \\ &+ \frac{26}{9}\alpha s^3 - \frac{10}{9}s^3 - \frac{26}{9}\alpha^2 s^2 + \frac{40}{9}\alpha s^2 - \frac{20}{9}s^2 - \frac{10}{9}\alpha^2 s + \frac{5}{9}\alpha s - \frac{10}{9}s.\end{aligned}$$

Let $G_3 = K_2 \vee (K_{n-4} \cup 2K_1)$. Then its adjacency matrix $A(G_3)$ has the quotient matrix B_3 which is obtained by replacing s with 2 in B_1, and B_3 admits the characteristic polynomial $\varphi_{B_3}(x)$ which is derived by replacing s with 2 in $\varphi_{B_1}(x)$. Hence, we get

$$\begin{aligned}\varphi_{B_3}(x) =& x^3 - (\alpha n + n + 2\alpha - 4)x^2 + (\alpha n^2 + 2\alpha^2 n - \alpha n - n - 4\alpha - 1)x \\ &- 2\alpha^2 n^2 + 10\alpha^2 n - 6\alpha n + 4n - 28\alpha^2 + 42\alpha - 20.\end{aligned}$$

In terms of Lemma 2.4, $\lambda_\alpha(G_3)$ is the largest root of $\varphi_{B_3}(x) = 0$, that is, $\varphi_{B_3}(\lambda_\alpha(G_3)) = 0$. We are to verify $\lambda_\alpha(G_1) \leq \lambda_\alpha(G_3)$.

Note that K_{n-2} is a proper subgraph of $G_3 = K_2 \vee (K_{n-4} \cup 2K_1)$. By means of (3.3) and Lemmas 2.2 and 2.3, we conclude

$$\lambda_\alpha(K_2 \vee (K_{n-4} \cup 2K_1)) > \lambda_\alpha(K_{n-2}) = n-3 > \theta_2. \tag{3.18}$$

Write $\eta = \lambda_\alpha(K_2 \vee (K_{n-4} \cup 2K_1))$. Note that $\varphi_{B_3}(\eta) = 0$. By a direct computation, we possess

$$\varphi_{B_1}(\eta) = \varphi_{B_1}(\eta) - \varphi_{B_3}(\eta) = \frac{1}{9}(s-2)h_1(\eta), \tag{3.19}$$

where $h_1(\eta) = (6-9\alpha)\eta^2 + (9\alpha^2 n + 3\alpha n - 6s - 18\alpha - 12)\eta - 9\alpha^2 n^2 + 3\alpha^2 n(4s+15) - 3\alpha n(4s+9) + 6n(s+3) - 2\alpha^2(8s^2 + 29s + 63) + \alpha(26s^2 + 92s + 189) - 10s^2 - 40s - 90$. According to (3.18), $s \geq 5$, and $n \geq \frac{5s+8}{3}$, we deduce

$$-\frac{9\alpha^2 n + 3\alpha n - 6s - 18\alpha - 12}{2(6-9\alpha)} < n-3 < \eta,$$

and so

$$\begin{aligned}h_1(\eta) >&h_1(n-3)\\ =&(6-6\alpha)n^2+(12\alpha^2 s-12\alpha s+18\alpha^2-30)n\\ &-2\alpha^2(8s^2+29s+63)+\alpha(26s^2+92s+162)-10s^2-22s. \end{aligned} \tag{3.20}$$

Let $h_2(n) = (6-6\alpha)n^2+(12\alpha^2 s-12\alpha s+18\alpha^2-30)n-2\alpha^2(8s^2+29s+63)+\alpha(26s^2+92s+162)-10s^2-22s$. It follows from $s \geq 11$ and $0 \leq \alpha < \frac{2}{3}$ that

$$-\frac{12\alpha^2 s-12\alpha s+18\alpha^2-30}{2(6-6\alpha)} < \frac{5s+8}{3} \leq n,$$

and so

$$\begin{aligned}h_2(n) \geq& h_2\Big(\frac{5s+8}{3}\Big)\\ =&\frac{1}{3}((12s^2+12s-234)\alpha^2+(-32s^2+20s+358)\alpha+20s^2-56s-112)\\ >&\frac{1}{3}\Big(\frac{4}{9}(12s^2+12s-234)+\frac{2}{3}(-32s^2+20s+358)+20s^2-56s-112\Big)\\ =&\frac{1}{9}(12s^2-112s+68)\\ >&0, \end{aligned} \tag{3.21}$$

where the last two inequalities hold from $\frac{32s^2-20s-358}{2(12s^2+12s-234)} > \frac{2}{3} > \alpha \geq 0$ and $s \geq 11$, respectively.

If $s \in \{5, 8\}$, then

$$\begin{aligned}-\frac{12\alpha^2 s-12\alpha s+18\alpha^2-30}{2(6-6\alpha)} =& \begin{cases} \frac{5+10\alpha-13\alpha^2}{2-2\alpha}, & \text{if } s=5,\\ \frac{5+16\alpha-19\alpha^2}{2-2\alpha}, & \text{if } s=8,\end{cases}\\ <&25 \leq n\end{aligned}$$

by $0 \leq \alpha < \frac{2}{3}$. Hence, we infer

$$\begin{aligned}h_2(n) \geq& h_2(25)\\ =&(-16s^2+242s+324)\alpha^2+(26s^2-208s-3588)\alpha-10s^2-22s+3000\\ =&\begin{cases} 1134\alpha^2-3978\alpha+2640, & \text{if } s=5,\\ 1236\alpha^2-3588\alpha+2184, & \text{if } s=8,\end{cases}\\ >&0\end{aligned} \tag{3.22}$$

by $0 \leq \alpha < \frac{2}{3}$.

According to (3.21) and (3.22), we conclude $h_2(n) > 0$ for $s \geq 5$ and $s \equiv 2 \pmod 3$. Together with (3.19) and (3.20), we get

$$\varphi_{B_1}(\eta) = \frac{1}{9}(s-2)h_1(\eta) \geq \frac{1}{9}(s-2)h_1(n-3) = \frac{1}{9}(s-2)h_2(n) \geq 0 \tag{3.23}$$

for $s \geq 2$ and $s \equiv 2 \pmod 3$. Recall that $\lambda_\alpha(G_1)$ is the largest root of $\varphi_{B_1}(x) = 0$. As $\theta_2 < n-3 < \lambda_\alpha(K_2 \vee (K_{n-4} \cup 2K_1)) = \eta$ (see (3.18)), we obtain

$$\lambda_\alpha(G_1) \leq \eta = \lambda_\alpha(K_2 \vee (K_{n-4} \cup 2K_1)) = \lambda_\alpha(G_3) \tag{3.24}$$

by (3.23).

Next, we prove $\lambda_\alpha(G_3) < n-2$. A direct computation yields that

$$\begin{aligned}
\varphi_{B_3}(n-2) =&(n-2)^3 - (\alpha n + n + 2\alpha - 4)(n-2)^2 \\
&+ (\alpha n^2 + 2\alpha^2 n - \alpha n - n - 4\alpha - 1)(n-2) \\
&- 2\alpha^2 n^2 + 10\alpha^2 n - 6\alpha n + 4n - 28\alpha^2 + 42\alpha - 20 \\
=&(1-\alpha)n^2 + (6\alpha^2 - 4\alpha - 3)n - 28\alpha^2 + 42\alpha - 10 \\
\geq&(1-\alpha)(25)^2 + 25(6\alpha^2 - 4\alpha - 3) - 28\alpha^2 + 42\alpha - 10 \\
=&122\alpha^2 - 683\alpha + 540 \\
>&0,
\end{aligned}$$

where the last two inequalities hold from $-\frac{6\alpha^2-4\alpha-3}{2(1-\alpha)} < 25 \leq n$ and $\frac{683}{2\times 122} > \frac{2}{3} > \alpha \geq 0$, respectively. Consequently, we deduce

$$\lambda_\alpha(G_3) < n-2. \tag{3.25}$$

It follows from (3.1), (3.4), (3.24), and (3.25) that

$$\lambda_\alpha(G) \leq \lambda_\alpha(G_1) \leq \lambda_\alpha(G_3) < n-2 < \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1)),$$

which contradicts $\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$.

Case 2. $n = \lfloor \frac{5}{3}s \rfloor + 2$.

In this case, $G_1 = K_s \vee (\lfloor \frac{2}{3}s \rfloor + 2)K_1$. The quotient matrix of $A(G_1)$ with respect to the partition $V(G_1) = V(K_s) \cup V((\lfloor \frac{2}{3}s \rfloor + 2)K_1)$ equals

$$B_4 = \begin{pmatrix} \alpha n - \alpha s + s - 1 & (1-\alpha)(\lfloor \frac{2}{3}s \rfloor + 2) \\ (1-\alpha)s & \alpha s \end{pmatrix},$$

for which we calculate the characteristic polynomial

$$\varphi_{B_4}(x) = x^2 - (\alpha n + s - 1)x + \alpha^2 sn - \alpha^2 s^2 + \alpha s^2 - (1-\alpha)^2 s\left\lfloor \frac{2}{3}s \right\rfloor - 2\alpha^2 s + 3\alpha s - 2s.$$

Since the partition $V(G_1) = V(K_s) \cup V((\lfloor \frac{2}{3}s \rfloor + 2)K_1)$ is equitable, it follows from Lemma 2.4 that $\lambda_\alpha(G_1)$ is the largest root of $\varphi_{B_4}(x) = 0$. Hence, we conclude

$$\lambda_\alpha(G_1) = M, \tag{3.26}$$

where $M = \frac{\alpha n + s - 1 + \sqrt{(\alpha n + s - 1)^2 - 4(\alpha^2 sn - \alpha^2 s^2 + \alpha s^2 - (1-\alpha)^2 s\lfloor \frac{2}{3}s \rfloor - 2\alpha^2 s + 3\alpha s - 2s)}}{2}$. We are to prove $\lambda_\alpha(G_1) < n-2$. According to $n = \lfloor \frac{5}{3}s \rfloor + 2$, we have

$$(2(n-2) - \alpha n - s + 1)^2 - (\alpha n + s - 1)^2 + 4(\alpha^2 sn - \alpha^2 s^2 + \alpha s^2 - (1-\alpha)^2 s\left\lfloor \frac{2}{3}s \right\rfloor - 2\alpha^2 s + 3\alpha s - 2s)$$

$$=(4-4\alpha)n^2+(4\alpha^2s-4s+8\alpha-12)n-4\alpha^2s^2+4\alpha s^2-4(1-\alpha)^2s\left\lfloor\frac{2}{3}s\right\rfloor-8\alpha^2s+12\alpha s+8$$

$$=\begin{cases}\frac{4}{9}((4-4\alpha)s^2-(3\alpha+3)s), & \text{if } s\equiv 0 \pmod 3;\\ \frac{4}{9}((4-4\alpha)s^2+(5\alpha-11)s+8\alpha-2), & \text{if } s\equiv 1 \pmod 3;\\ \frac{4}{9}((4-4\alpha)s^2+(\alpha-7)s+5\alpha-2), & \text{if } s\equiv 2 \pmod 3.\end{cases} \tag{3.27}$$

Subcase 2.1. $s \equiv 0 \pmod 3$.

Obviously, $n = \frac{5}{3}s + 2 \geq 25$. Then $s \geq 15$. Let $\psi_1(s) = (4-4\alpha)s^2 - (3\alpha+3)s$. Note that $\frac{3\alpha+3}{2(4-4\alpha)} < 15 \leq s$. Hence, we deduce

$$\psi_1(s) \geq \psi_1(15) = 9(95 - 105\alpha) > 0. \tag{3.28}$$

Subcase 2.2. $s \equiv 1 \pmod 3$.

It is obvious that $n = \frac{5s+4}{3} \geq 25$. Then $s \geq 16$. Let $\psi_2(s) = (4-4\alpha)s^2 + (5\alpha-11)s + 8\alpha - 2$. Since $0 \leq \alpha < \frac{2}{3}$ and $-\frac{5\alpha-11}{2(4-4\alpha)} < 16 \leq s$, we obtain

$$\psi_2(s) \geq \psi_2(16) = 18(47 - 52\alpha) > 0. \tag{3.29}$$

Subcase 2.3. $s \equiv 2 \pmod 3$.

Clearly, $n = \frac{5s+5}{3} \geq 25$. Then $s \geq 14$. Let $\psi_3(s) = (4-4\alpha)s^2 + (\alpha-7)s + 5\alpha - 2$. Since $0 \leq \alpha < \frac{2}{3}$ and $-\frac{\alpha-7}{2(4-4\alpha)} < 14 \leq s$, we get

$$\psi_3(s) \geq \psi_3(14) = 9(76 - 85\alpha) > 0. \tag{3.30}$$

According to (3.26)–(3.30), we conclude $\lambda_\alpha(G_1) < n-2$. Combining this with (3.1) and (3.4), we have $\lambda_\alpha(G) \leq \lambda_\alpha(G_1) < n-2 < \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$, which contradicts $\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$.

Case 3. $n = \lfloor\frac{5}{3}s\rfloor + 1$.

In this case, $G_1 = K_s \vee (\lfloor\frac{2}{3}s\rfloor + 1)K_1$. Consider the partition $V(G_1) = V(K_s) \cup V((\lfloor\frac{2}{3}s\rfloor + 1)K_1)$. The corresponding quotient matrix of $A(G_1)$ equals

$$B_5 = \begin{pmatrix} \alpha n - \alpha s + s - 1 & (1-\alpha)(\lfloor\frac{2}{3}s\rfloor + 1) \\ (1-\alpha)s & \alpha s \end{pmatrix}.$$

Then, the characteristic polynomial of B_5 is

$$\varphi_{B_5}(x) = x^2 - (\alpha n + s - 1)x + \alpha^2 sn - \alpha^2 s^2 + \alpha s^2 - (1-\alpha)^2 s\left\lfloor\frac{2}{3}s\right\rfloor - \alpha^2 s + \alpha s - s.$$

Since the partition $V(G_1) = V(K_s) \cup V((\lfloor\frac{2}{3}s\rfloor+1)K_1)$ is equitable, $\lambda_\alpha(G_1)$ is the largest root of $\varphi_{B_5}(x) = 0$ by Lemma 2.4. Thus, we obtain

$$\lambda_\alpha(G_1) = N, \tag{3.31}$$

where $N = \frac{\alpha n+s-1+\sqrt{(\alpha n+s-1)^2-4(\alpha^2 sn-\alpha^2 s^2+\alpha s^2-(1-\alpha)^2 s\lfloor\frac{2}{3}s\rfloor-\alpha^2 s+\alpha s-s)}}{2}$. We are to show $\lambda_\alpha(G_1) < n-2$. In terms of $n = \lfloor\frac{5}{3}s\rfloor + 1$, we get

$$(2(n-2) - \alpha n - s + 1)^2 - (\alpha n + s - 1)^2 + 4(\alpha^2 sn - \alpha^2 s^2 + \alpha s^2 - (1-\alpha)^2 s\left\lfloor\frac{2}{3}s\right\rfloor - \alpha^2 s + \alpha s - s)$$

$$
\begin{aligned}
&=(4-4\alpha)n^2+(4\alpha^2 s-4s+8\alpha-12)n-4\alpha^2 s^2+4\alpha s^2-4(1-\alpha)^2 s\left\lfloor\frac{2}{3}s\right\rfloor\\
&\quad-4\alpha^2 s+4\alpha s+4s+8\\
&=\begin{cases}\frac{4}{9}((4-4\alpha)s^2+(9\alpha-15)s+9\alpha), & \text{if } s\equiv 0 \pmod 3;\\ \frac{4}{9}((4-4\alpha)s^2+(17\alpha-23)s+5\alpha+10), & \text{if } s\equiv 1 \pmod 3;\\ \frac{4}{9}((4-4\alpha)s^2+(13\alpha-19)s+8\alpha+4), & \text{if } s\equiv 2 \pmod 3.\end{cases}
\end{aligned}
\tag{3.32}
$$

Subcase 3.1. $s \equiv 0 \pmod 3$.

We easily see $n = \frac{5}{3}s + 1 \geq 25$, and so $s \geq 15$. Write $\Phi_1(s) = (4-4\alpha)s^2 + (9\alpha-15)s + 9\alpha$. Since $0 \leq \alpha < \frac{2}{3}$ and $-\frac{9\alpha-15}{2(4-4\alpha)} < 15 \leq s$, we possess

$$\Phi_1(s) \geq \Phi_1(15) = 9(75-84\alpha) > 0. \tag{3.33}$$

Subcase 3.2. $s \equiv 1 \pmod 3$.

Obviously, $n = \frac{5s+1}{3} \geq 25$, and so $s \geq 16$. Let $\Phi_2(s) = (4-4\alpha)s^2 + (17\alpha-23)s + 5\alpha + 10$. Since $0 \leq \alpha < \frac{2}{3}$ and $-\frac{17\alpha-23}{2(4-4\alpha)} < 16 \leq s$, we infer

$$\Phi_2(s) \geq \Phi_2(16) = 9(74-83\alpha) > 0. \tag{3.34}$$

Subcase 3.3. $s \equiv 2 \pmod 3$.

Clearly, $n = \frac{5s+2}{3} \geq 25$, and so $s \geq 17$. Let $\Phi_3(s) = (4-4\alpha)s^2+(13\alpha-19)s+8\alpha+4$. Since $0 \leq \alpha < \frac{2}{3}$ and $-\frac{13\alpha-19}{2(4-4\alpha)} < 17 \leq s$, we deduce

$$\Phi_3(s) \geq \Phi_3(17) = 9(93-103\alpha) > 0. \tag{3.35}$$

It follows from (3.31)–(3.35) that $\lambda_\alpha(G_1) < n-2$. Together with (3.1) and (3.4), we conclude $\lambda_\alpha(G) \leq \lambda_\alpha(G_1) < n-2 < \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$, which contradicts $\lambda_\alpha(G) \geq \lambda_\alpha(K_1 \vee (K_{n-2} \cup K_1))$. This completes the proof of Theorem 1.1. □

4. Conclusions

In this paper, we establish a relationship between the A_α-spectral radius and $\{P_3, P_4, P_5\}$-factors in connected graphs and provide a tight A_α-spectral radius condition for the existence of $\{P_3, P_4, P_5\}$-factors in connected graphs. Inspired by the work above, it is natural to ask whether there are other types of factors that can be considered by using the A_α-spectral radius. On the other hand, there are very few results on $\{P_3, P_4, P_5\}$-factors of graphs. Hence, it is natural to establish some new sufficient conditions to ensure that a graph contains a $\{P_3, P_4, P_5\}$-factor.

Author contributions

Yuli Zhang: Writing-original draft preparation, review and editing; Sizhong Zhou: Writing-original draft preparation, review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest to this work.

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