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Research article

Analytical solutions for fractional Navier–Stokes equation using residual power series with ϕ-Caputo generalized fractional derivative

Omar Barkat[1,2,*] and Awatif Muflih Alqahtani[3]

[1] Department of Mathematics, University Center of Barika, Algeria
[2] Laboratory of Science for Mathematics, Computer science and Engineering applications
[3] Department of Mathematics, Shaqra University, Riyadh 11972, Saudi Arabia

* **Correspondence**: Email: omar.barkat@cu-barika.dz.

Abstract: In this study, we aimed to derive analytical solutions for a system of nonlinear time-fractional Navier–Stokes equations in Cartesian coordinates by employing the residual power series method. Moreover, we showed that the ϕ-Caputo fractional derivative describes these equations in time, enabling the Riemann-Liouville, Hadamard, and Katugampola fractional derivatives to be generalized into a unified form. Additionally, we provide results for certain cases that are given in the literature. Therefore, the solutions obtained for the time-fractional Navier–Stokes equations are presented graphically in the Caputo–Hadamard sense.

Keywords: ϕ-Caputo fractional derivative; Caputo–Hadamard fractional derivative; residual power series; fractional Navier–Stokes equation
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Abbreviations:

$\Gamma(.)$: the Gamma function; $\mathcal{J}_a^{\alpha;\phi} f(x,t)$: the left fractional integral; ${}_t^C\mathcal{D}_a^{\alpha;\phi} f(x,t)$: the ϕ-Caputo fractional derivative; ${}_t^C\mathcal{D}_a^{\alpha;ln(t)} f(x,t)$: Caputo Hadamard fractional derivative; $\mathbb{R}$: the set of real numbers, $\Omega \subseteq \mathbb{R}^2$; $\mathbb{N}$: the set of natural numbers; Re: the Reynold's number; u and v: the fluid velocity components along the x and y; P: the fluid pressure function; IC: initial conditions; BC: boundary conditions; Γ_{rigid}: the rigid part of the boundary Γ; α: order derivative

1. Introduction

As known in the literature, an important field of applied mathematics is fractional calculus, which studies differential operators and integrals with non-integer powers. Moreover, due to its wide variety of proven applications, fractional calculus has gained increasing interest from many researchers, such as in physics, electrochemistry, mathematical biology, fluid mechanics, and others. As such, many authors have presented basic works covering various aspects of fractional calculus, such as Herrmann [1], Kiryakova [2], Miller and Ross [3], Kilbas et al. [4], Podlubny [5], Alqahtani et al. [6], Zuo et al. [7,8], and so on. In addition, the approach for solving differential equations of any real order is explained, along with the diverse applications of these methods in multiple domains. However, several analytical and approximate methods have been developed to solve fractional differential equations, like the homotopy analysis method [9–11], modified simple equation method [12], reduced differential transform method [13,14], and many other approaches. Notably, in 2013, the residual power series (RPS) method, invented by the mathematician Omar Abu Arqoub, was used to determine the coefficients of the power series solutions for first and second-order fuzzy differential equations [15]. Power series solutions for linear and nonlinear equations can be constructed with no linearization, perturbation, or discretization, which is intuitive and reliable. The RPS method is being used to address nonlinear ordinary and partial differential equations of different forms and orders. Furthermore, this method offers a simple way to guarantee the convergence of the series solution. Additionally, it can be applied directly to the given problem by selecting the appropriate initial estimate. Also, it has been effectively applied in several studies, such as in [16], where Jaber and Ahmad used it to find the solution of the two-dimensional nonlinear time-fractional Navier–Stokes equation, which is a nonlinear partial differential equation that describes the dynamics of viscous fluids, recording the relationship between external forces applied to the fluid velocity and the fluid pressure [17,18]. Moreover, many researchers have dealt with finding analytical solutions for Navier–Stokes type systems, which have contributed greatly to solving a significant part of real-life problems, particularly, in the context of addressing practical fluid dynamics problems (for instance, see Baranovskii [19,20]).

The time-fractional Navier–Stokes equations have been extensively investigated. Many authors have made great contributions in this regard, including El-Shahed and Salem [21], who extended the classical Navier–Stokes equation by substituting the first-time derivative with a Caputo fractional derivative of an order α, where $0 < \alpha \leq 1$. However, numerical approximations have been suggested for a class of Navier–Stokes equations involving fractional time derivatives by Zhang and Wang [22]. Furthermore, Sawangtong et al. [23] solved the two-dimensional fractional time Navier–Stokes equation using the RPS method, where the fractional derivative used in their research was the Katugambola derivative in the sense of Caputo. In this study, we apply the ϕ-Caputo fractional derivative to solve the time-fractional Navier–Stokes equation using the residual RPS method. This approach is selected because, for specific values of ϕ, the ϕ-Caputo fractional derivative generalizes the Riemann-Liouville, Hadamard, and Katugampola fractional derivatives into a unified form. In other words, we focus on how to derive analytical solutions for a system of nonlinear fractional-time Navier–Stokes equations in Cartesian coordinates using the fractional derivative ϕ as a differential operator. This is crucial for generalizing the fractional derivatives of Riemann-Liouville, Hadamard, and Katugampola into a single formulation. The proposed method offers solutions in the form of rapidly converging series with easy-to-compute components, showing exceptional agreement with exact solutions, as demonstrated by numerical results. Moreover, it decreases the computational effort in comparison to traditional methods, positioning the RPS method as a highly effective and efficient tool for solving both linear and nonlinear fractional partial differential equations. Additionally, we present the results through an example with varying fractional orders to illustrate the outcomes. Furthermore, we provide graphical representations of the solutions to these problems when the Caputo–Hadamard fractional derivative is applied. For further significant works, we direct the reader to references [24,25] on fractional calculus, [26–29] for the fractional Navier–Stokes equation, and [30–32] on Caputo–Hadamard fractional differential equations.

This work is structured as follows. In Section 2, we present specific findings related to fractional calculus, which are used in the research details. In Section 3, we present a new approach to the RPS method. In Section 4, we introduce a generalization of the solutions to the fractional Navier–Stokes equation by applying the ϕ-Caputo fractional derivative. Section 5 is devoted to a study related to the application of the RPS method to the Navier–Stokes equation with initial conditions. The manuscript concludes with final remarks.

2. Preliminaries

In this section, we review key definitions and properties from the theory of fractional calculus, which are utilized throughout this paper.

Definition 2.1 ([6,33]). *Let f be an integrable function defined as $I = [a, b]$, in relation to another function ϕ of an order α, such that $\phi \in C^1(I)$ is a growing function, $\phi'(t) \neq 0$ for each $t \in I$, and $\alpha > 0$. The left fractional integral of f is defined as:*

$$\mathcal{J}_a^{\alpha;\phi}[f(x,t)] = \frac{1}{\Gamma(\alpha)}\int_a^t \phi'(\mu)\big(\phi(t) - \phi(\mu)\big)^{\alpha-1} f(x,\mu)d\mu. \tag{2.1}$$

$$\mathcal{J}_a^{0;\phi}[f(x,t)] = f(x,t). \tag{2.2}$$

Theorem 2.1 ([33]). *Let $\alpha > 0, m \in \mathbb{N}$, I be the interval $-\infty \leq a < \infty$, and f, $\phi \in C^m(I)$ two functions, where ϕ is increasing and $\phi'(t) \neq 0$, for all $t \in I$. The left ϕ-Caputo fractional derivative of f of an order α is given by:*

$${}_t^C\mathcal{D}_a^{\alpha;\phi}[f(x,t)]1 = \mathcal{J}_a^{m-\alpha;\phi}\left(\frac{1}{\phi'(t)}\frac{\partial}{\partial t}\right)^m f(x,t), \tag{2.3}$$

such that $m = [\alpha] + 1$ for $\alpha \notin \mathbb{N}, m = \alpha$ for $\alpha \in \mathbb{N}$.

We utilize the shortened notation to make the notation simpler.

$$f^{[m];\phi}(x,t) = \left(\frac{1}{\phi'(t)}\frac{\partial}{\partial t}\right)^m f(x,t). \tag{2.4}$$

In the event that $\phi(t) = t$, the Caputo fractional derivative is obtained in [34], while the Caputo–Hadamard fractional derivative is obtained if $\phi(t) = ln(t)$ in [31].

Theorem 2.2 ([31]). *Let $\Re(\alpha) \geq 0, m = [\Re(\alpha)] + 1$. If $f \in C_\delta^m([a, b])$, with $(0 < a < b < \infty)$, and*

$$AC_\delta^m([a,b]) = \left\{g: [a,b] \to \mathbb{C}: \delta^{m-1}g(x) \in AC[a,b], \delta = x\frac{d}{dx}\right\}, \tag{2.5}$$

then ${}_t^C\mathcal{D}_a^{\alpha;ln(t)} f(t)$ exist everywhere on $[a, b]$:

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;ln(t)} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \left(ln \frac{t}{\xi}\right)^{m-\alpha-1} \left(t\frac{d}{dt}\right)^{m} f(\xi) \frac{d\xi}{\xi}. \quad (2.6) $$

Lemma 2.1 ([31]). *Let $f \in AC^{m}([a,b])$, and $\alpha \in \mathbb{C}$, then*

$$ \mathcal{I}_{a}^{\alpha;ln(t)} {}_{t}^{C}\mathcal{D}_{a}^{\alpha;ln(t)} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{\delta^{k} f(a)}{k!} \left(ln \frac{t}{a}\right)^{k}. \quad (2.7) $$

Theorem 2.3 ([31]). *Let $f \in C^{m}([a,b])$, and $m-1 < \alpha \leq m, m \in \mathbb{N}, \eta > 0$, then*

1) $f(t) = \left(\phi(t) - \phi(a)\right)^{\eta-1}$, *then* $\mathcal{I}_{a}^{\alpha;\phi}[f(t)] = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\eta)} \left(\phi(t) - \phi(a)\right)^{\alpha+\eta-1}$, (2.8)

2) $\mathcal{I}_{a}^{\alpha;\phi} {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi}[f(x,t)] = f(x,t) - \sum_{k=0}^{n-1} \frac{f^{[k],\phi}(x,a)}{k!} \left(\phi(t) - \phi(a)\right)^{k}$. (2.9)

3. Residual power series (RPS) method

In this section, we discuss the RPS method through a new approach.

Definition 3.1. *A power series representation of the form*

$$ \sum_{m=0}^{\infty} H_{m} \left(\phi(t) - \phi(a)\right)^{m\alpha} = C_{0} + C_{1}\left(\phi(t) - \phi(a)\right)^{\alpha} + C_{2}\left(\phi(t) - \phi(a)\right)^{2\alpha} + \cdots, \quad (3.1) $$

is called a fractional power series around a, *such that t is a variable,* $H'_{m}s$ *are constants called the coefficients of the series, where* $0 \leq n-1 < \alpha \leq n, n \in N$, *and* $\phi(t) \geq \phi(a)$.

Theorem 3.1. *Suppose that f has a fractional power series (FPS) representation at* $\phi(a)$ *of the form*

$$ f(t) = \sum_{m=0}^{\infty} H_{m} \left(\phi(t) - \phi(a)\right)^{m\alpha}, $$

$$ where\ 0 \leq n-1 < \alpha \leq n, \phi(a) \leq \phi(t) \leq \phi(a) + \mathcal{R}. \quad (3.2) $$

If $f(t)$, ${}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} f(t) \in C[\phi(a), \phi(a) + \mathcal{R}]$ *where* ${}_{t}^{C}\mathcal{D}_{a}^{m\alpha;\phi} \in C[\phi(a), \phi(a) + \mathcal{R}]$ *for* $m = 1,2,3, \ldots$, *then the coefficients* H_{m} *in the equation will take the form*

$$ H_{m} = \frac{{}_{t}^{C}\mathcal{D}_{a}^{m\alpha;\phi} f(a)}{\Gamma(m\alpha + 1)}, $$

$$ where\ {}_{t}^{C}\mathcal{D}_{a}^{m\alpha;\phi} = {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} . {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} \ldots {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} (m - times), \quad (3.3) $$

with $\mathcal{R}$ *is the radius of convergence.*

Theorem 3.2. *A power series of the form* $\sum_{m=0}^{\infty} f_{m}(t) \left(\phi(t) - \phi(a)\right)^{m\alpha}$ *is called a multiple FPS about* $\phi(t) = \phi(a)$ *of the form*

$$ u(x,t) = \sum_{m=0}^{\infty} f_{m}(x) \left(\phi(t) - \phi(a)\right)^{m\alpha}, x \in I, \phi(a) \leq \phi(t) \leq \phi(a) + \mathcal{R}. \quad (3.4) $$

If ${}_{t}^{C}\mathcal{D}_{a}^{m\alpha;\phi} u(x,t), m = 0,1,2,3, \ldots$ *are continuous on* $I \times (\phi(a), \phi(a) + \mathcal{R})$, *then*

$$f_m(x) = \frac{{}_t^C\mathcal{D}_a^{m\alpha;\phi} u(x,t)}{\Gamma(m\alpha+1)}, \tag{3.5}$$

where ${}_t^C\mathcal{D}_a^{m\alpha;\phi} = \frac{\partial^{m\alpha}}{\partial t} = \frac{\partial^\alpha}{\partial t} . \frac{\partial^\alpha}{\partial t} \dots \frac{\partial^\alpha}{\partial t}(m - times)$, *and* $\mathcal{R} = min_{C\in I}\, \mathcal{R}_H$ *with* $\mathcal{R}_H$ *is the radius of convergence of the FPS* $\sum_{m=0}^{\infty} f_m(H) \left(\phi(t) - \phi(a)\right)^{m\alpha}$.

It is evident from the last theorem that $am + 1$ *dimensional function can be obtained in the same way as the following corollary* $\phi(t) = \phi(a)$.

Corollary 3.1. *Suppose that* $u(x,y,t)$ *has a multiple FPS representation at* $\phi(t) = \phi(a)$ *of the form:*

$$u(x,y,t) = \sum_{m=0}^{\infty} f_m(x,y) \left(\phi(t) - \phi(a)\right)^{m\alpha},$$

$$(x,y) \in I_1 \times I_2, \phi(a) \le \phi(t) \le \phi(a) + \mathcal{R}. \tag{3.6}$$

If ${}_t^C\mathcal{D}_a^{m\alpha;\phi} u(x,y,t), m = 0,1,2,3,\dots$ *are continuous on* $I_1 \times I_2 \times (\phi(a), \phi(a) + \mathcal{R})$, *then*

$$f_m(x,y) = \frac{{}_t^C\mathcal{D}_a^{m\alpha;\phi} u(x,y,t)}{\Gamma(m\alpha+1)}, \tag{3.7}$$

where ${}_t^C\mathcal{D}_a^{m\alpha;\phi} = \frac{\partial^{m\alpha}}{\partial t} = \frac{\partial^\alpha}{\partial t} . \frac{\partial^\alpha}{\partial t} \dots \frac{\partial^\alpha}{\partial t}(m - times)$, *and* $\mathcal{R} = min_{C\in I_1\times I_2} \mathcal{R}_{H,K}$ *in which* $\mathcal{R}_{H,K}$ *is the radius of convergence of the FPS* $\sum_{m=0}^{\infty} f_m(H,K) \left(\phi(t) - \phi(a)\right)^{m\alpha}$.

4. Generalization of the solutions of the fractional Navier–Stokes equation using the ϕ-Caputo fractional derivative

In this section, we apply the RPS method for solving the Navier–Stokes equation for the nonlinear two-dimensional time fractional ϕ-Caputo fractional derivative in the following form:

For every $(x,y,t) \in (0,a] \times (0,b] \times (0,T]$, for any positive constants $a, b, and\ T$,

$${}_t^C\mathcal{D}_a^{\alpha;\phi} u(x,y,t) + Re(u(x,y,t).\nabla)u(x,y,t) = \nabla^2 u(x,y,t) - \nabla P, \tag{4.1a}$$

$$\nabla . u(x,y,t) = 0,\ \Omega \times (0,T], \tag{4.1b}$$

$$u(x,0,t) = u(y,t), on\ \Gamma_{rigid} \times (0,T],\ \text{is the boundary conditions,} \tag{4.2}$$

$$u(x,y,0) = f_0(x,y),\ \text{in}\ \Omega, \text{is the initial conditions,}\ \Omega \subseteq \mathbb{R}^2, \tag{4.3}$$

where ${}_t^C\mathcal{D}_a^{\alpha;\phi}$ signifies the ϕ-Caputo fractional derivative of the fractional order α with $0 < \alpha \le 1, \Omega \subseteq \mathbb{R}^2, on\ \Gamma_{rigid}$ is the rigid part of the boundary Γ, u and v are the fluid velocity components along the x and y axis (ms^{-1}), t is the time (s), $P = P(x,y,t)$ is the fluid pressure function (Pa), and Re is Reynolds's number.

Equation (4.4), is expressed as follows in Cartesian coordinates based on x, y, and z:

$${}_t^C\mathcal{D}_a^{\alpha;\phi} u(x,y,z,t) + Reu\frac{\partial u}{\partial x} + Rev\frac{\partial u}{\partial y} + Rew\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{4.4a}$$

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} v(x,y,z,t) + Reu\frac{\partial v}{\partial x} + Rev\frac{\partial v}{\partial y} + Rew\frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{4.4b} $$

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} w(x,y,z,t) + Reu\frac{\partial w}{\partial x} + Rev\frac{\partial w}{\partial y} + Rew\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{4.4c} $$

From Eq (4.4), $2D$ Navier–Stokes equations of fractional order, may be written as:

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} u(x,y,t) + Reu\frac{\partial u}{\partial x} + Rev\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.5a} $$

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} v(x,y,t) + Reu\frac{\partial v}{\partial x} + Rev\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{4.5b} $$

$$ \frac{\partial u}{\partial x}(x,y,t) + \frac{\partial v}{\partial y}(x,y,t) = 0, \tag{4.6} $$

$$ u(x,y,t) = ub, (BC);\ u(x,y,0) = f_i(x,y), (IC). \tag{4.7} $$

According to the RPS approach, the solution for system (4.5) is a *FPS* about the beginning point $\phi(t) = 0$.

$$ u(x,y,t) = \sum_{m=0}^{\infty} f_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{4.8a} $$

$$ v(x,y,t) = \sum_{m=0}^{\infty} g_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{4.8b} $$

$$ P(x,y,t) = \sum_{m=0}^{\infty} h_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{4.8c} $$

The starting criteria (4.7) are satisfied by *u* and *v,* and they can be rewritten as:

$$ u(x,y,0) = f(x,y), \tag{4.9a} $$

$$ v(x,y,0) = g(x,y). \tag{4.9b} $$

As a result, we may get the first estimate of u and v as:

$$ u_0(x,y,0) = f_0(x,y) = f(x,y), \tag{4.10a} $$

$$ v_0(x,y,0) = f_0(x,y) = g(x,y). \tag{4.10b} $$

Therefore, Eqs (4.8a) and (4.8b) could be rewritten as:

$$ u(x,y,t) = f(x,y) + \sum_{m=1}^{\infty} f_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{4.11a} $$

$$ v(x,y,t) = g(x,y) + \sum_{m=1}^{\infty} g_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{4.11b} $$

Since there are no initial conditions for P, we move the index *m* from 0 to 1 in order to become

$$ P(x,y,t) = \sum_{m=1}^{\infty} h_{m-1}(x,y)\frac{(\phi(t)-\phi(a))^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)}. \tag{4.11c} $$

We use u_i, v_i and P_i to represent the $i-$th truncated series of u_i, v_i and P_i, respectively, in the next step:

$$ u_i(x,y,t) = f(x,y) + \sum_{m=1}^{i} f_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{4.12a} $$

$$v_i(x,y,t) = g(x,y) + \sum_{m=1}^{i} g_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{4.12b}$$

$$P_i(x,y,t) = \sum_{m=1}^{i} h_{m-1}(x,y)\frac{(\phi(t)-\phi(a))^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)}, \tag{4.12c}$$

for $i = 1,2,3,\ldots$.

For Eq (4.5), we define the residual functions Res_u and Res_v as follows:

$$Res_u = {}_t^C\mathcal{D}_a^{\alpha;\phi} u(x,y,t) + Reu\frac{\partial u}{\partial x} + Rev\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.13a}$$

$$Res_v = {}_t^C\mathcal{D}_a^{\alpha;\phi} v(x,y,t) + Reu\frac{\partial v}{\partial x} + Rev\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{4.13b}$$

The *i*-th truncated residual functions are thus

$$Res_{u_i} = {}_t^C\mathcal{D}_a^{\alpha;\phi} u_i(x,y,t) + Reu_i\frac{\partial u_i}{\partial x} + Rev_i\frac{\partial u_i}{\partial y} + \frac{\partial P_i}{\partial x} - \left(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}\right), \tag{4.14a}$$

$$Res_{v_i} = {}_t^C\mathcal{D}_a^{\alpha;\phi} v_i(x,y,t) + Reu_i\frac{\partial v_i}{\partial x} + Rev_i\frac{\partial v_i}{\partial y} + \frac{\partial P_i}{\partial y} - \left(\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial y^2}\right), \tag{4.14b}$$

Based on [35–37], $\lim_{i\to\infty} Res_i = Res(x,y,t)$, and $Res(x,y,t) = 0$ for each $\phi(t) \in [\phi(a), \phi(a) + R]$ and $x, y \in \mathbb{R}$, with R is a non -negative real number representing the radius of convergence. Hence, ${}_t^C\mathcal{D}_a^{\alpha;\phi} Res(x,y,t) = 0$. Given that a constant function's fractional derivative in the Caputo sense is zero, the fractional derivative ${}_t^C\mathcal{D}_a^{r\alpha;\phi}$ of $Res(x,y,t)$ and $Res_i(x,y,t)$ are correspond to $\phi(t) = \phi(a)$ for each $i = 0,1,2,\ldots$.

If we set $\phi(a) = 0$, and $r = i - 1$, we get

$${}_t^C\mathcal{D}_a^{(i-1)\alpha;\phi} Res_{u_i}(x,y,0) = 0, \tag{4.15a}$$

$${}_t^C\mathcal{D}_a^{(i-1)\alpha;\phi} Res_{v_i}(x,y,0) = 0. \tag{4.15b}$$

Now, we use the RPS technique to obtain the form of the coefficients $f_m(x,y)$, $g_m(x,y)$, or $h_{m-1}(x,y)$, where $m = 1,2,3,\ldots i$ in Eq (4.5).

First, we enter the *i*-th shortened *u*, *v*, and *P* series into Eq (4.14). Second, we determine the formula for the fractional derivative of ${}_t^C\mathcal{D}_a^{(i-1)\alpha;\phi}$ for both $Res_{u_i}(x,y,t)$ and $Res_{v_i}(x,y,t)$,where $i = 1,2,3,\ldots$. Last, we solve the algebraic system (4.14) that was acquired.

5. Application of the RPS method on the Navier–Stokes equation with initial conditions

In this section, we apply the RPS method shown above to a classical test problem from [38,39] and turn it into a fractional one by substituting a fractional derivative of order $0 < \alpha \leq 1$for the first time derivative. Afterward, we discuss the graphics and numerical results. System (4.5) in which $0 \leq x, y \leq \pi$ is the subject of our problem, and the initial conditions are

$$u(x,y,0) = -cos(x)\,sin(y), \tag{5.1a}$$

$$v(x,y,0) = sin(x)\,co\,s(y). \tag{5.1b}$$

The boundary conditions are

$$u(x,0,t)=0, \tag{5.2a}$$

$$v(0,y,t)=0. \tag{5.2b}$$

The following approach will be used by the RPS method:
Assume that the following is how the problem is resolved.

$$u(x,y,t)=\sum_{m=0}^{\infty} f_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.3a}$$

$$v(x,y,t)=\sum_{m=0}^{\infty} g_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.3b}$$

$$P(x,y,t)=\sum_{m=0}^{\infty} h_{m-1}(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.3c}$$

with initial conditions (5.1), we can get the initial guess for $m=0$. Since the pressure has no beginning condition, we obtain

$$u(x,y,t)=-cos(x)\,sin(y)\;+\sum_{m=1}^{\infty} f_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.4a}$$

$$v(x,y,t)=sin(x)\,co\,s(y)+\sum_{m=1}^{\infty} g_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.4b}$$

$$P(x,y,t)=\sum_{m=1}^{\infty} h_{m-1}(x,y)\frac{(\phi(t)-\phi(a))^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)}. \tag{5.4c}$$

The abbreviated series of the suggested solutions will now be built as follows:

$$u_i(x,y,t)=-cos(x)\,sin\,(y)+\sum_{m=1}^{i} f_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.5a}$$

$$v_i(x,y,t)=sin(x)\,cos(y)+\sum_{m=1}^{i} g_m(x,y)\frac{(\phi(t)-\phi(a))^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{5.5b}$$

$$P_i(x,y,t)=\sum_{m=1}^{i} h_{m-1}(x,y)\frac{(\phi(t)-\phi(a))^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)}, \tag{5.5c}$$

The residual functions are going to be defined by

$$Res_u={}_{t}^{C}\mathcal{D}_a^{\alpha;\phi}u(x,y,t)+Reu\frac{\partial u}{\partial x}+Rev\frac{\partial u}{\partial y}+\frac{\partial P}{\partial x}-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{5.6a}$$

$$Res_v={}_{t}^{C}\mathcal{D}_a^{\alpha;\phi}v(x,y,t)+Reu\frac{\partial v}{\partial x}+Rev\frac{\partial v}{\partial y}+\frac{\partial P}{\partial y}-\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{5.6b}$$

The i-th truncated residual functions are thus

$$Res_{u_i}={}_{t}^{C}\mathcal{D}_a^{\alpha;\phi}u_i(x,y,t)+Reu_i\frac{\partial u_i}{\partial x}+Rev_i\frac{\partial u_i}{\partial y}+\frac{\partial P_i}{\partial x}-\left(\frac{\partial^2 u_i}{\partial x^2}+\frac{\partial^2 u_i}{\partial y^2}\right), \tag{5.7a}$$

$$Res_{v_i}={}_{t}^{C}\mathcal{D}_a^{\alpha;\phi}v_i(x,y,t)+Reu_i\frac{\partial v_i}{\partial x}+Rev_i\frac{\partial v_i}{\partial y}+\frac{\partial P_i}{\partial y}-\left(\frac{\partial^2 v_i}{\partial x^2}+\frac{\partial^2 v_i}{\partial y^2}\right), \tag{5.7b}$$

Substituting Eq (5.5) in Eq (5.7) gives

$$Res_{u_i} = {}_{t}^{C}\mathcal{D}_a^{\alpha;\phi}\left(\sum_{m=1}^{i} f_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$-Re\cos(x)\sin(y)\sum_{m=1}^{i}\frac{\partial f_m}{\partial x}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$+Re\sin(x)\sin(y)\sum_{m=1}^{i} f_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$+Re\left(\sum_{m=1}^{i} f_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)\left(\sum_{m=1}^{i}\frac{\partial f_m}{\partial x}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$+Re\sin(x)\cos(y)\sum_{m=1}^{i}\frac{\partial f_m}{\partial y}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$-Re\cos(x)\cos(y)\sum_{m=1}^{i} g_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$+Re\left(\sum_{m=1}^{i} g_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)\left(\sum_{m=1}^{i}\frac{\partial f_m}{\partial y}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$+\sum_{m=1}^{i}\frac{\partial h_{m-1}}{\partial x}\times\frac{\left(\phi(t)-\phi(a)\right)^{(m-1)\alpha}}{\Gamma\left((m-1)\alpha+1\right)}$$

$$-2\cos(x)\sin(y)-Re\sin(x)\cos(x)$$

$$-\left(\sum_{m=1}^{i}\frac{\partial^2 f_m}{\partial x^2}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}+\sum_{m=1}^{i}\frac{\partial^2 f_m}{\partial y^2}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right), \tag{5.8a}$$

$$Res_{v_i} = {}_{t}^{C}\mathcal{D}_a^{\alpha;\phi}\left(\sum_{m=1}^{i} g_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$-Re\cos(x)\sin(y)\sum_{m=1}^{i}\frac{\partial g_m}{\partial x}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$+Re\cos(x)\cos(y)\sum_{m=1}^{i} f_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$+Re\left(\sum_{m=1}^{i} f_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)\left(\sum_{m=1}^{i}\frac{\partial g_m}{\partial x}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$+Re\sin(x)\cos(y)\sum_{m=1}^{i}\frac{\partial g_m}{\partial y}\times\frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$-Re\sin(x)\sin(y)\sum_{m=1}^{i} g_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$+Re\left(\sum_{m=1}^{i} g_m \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)\left(\sum_{m=1}^{i} \frac{\partial g_m}{\partial y} \times \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$+\sum_{m=1}^{i} \frac{\partial h_{m-1}}{\partial y} \times \frac{\left(\phi(t)-\phi(a)\right)^{(m-1)\alpha}}{\Gamma\left((m-1)\alpha+1\right)}$$

$$+2\sin(x)\cos(y) - Re\sin(y)\cos(y)$$

$$-\left(\sum_{m=1}^{i} \frac{\partial^2 g_m}{\partial x^2} \times \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)} + \sum_{m=1}^{i} \frac{\partial^2 g_m}{\partial y^2} \times \frac{\left(\phi(t)-\phi(a)\right)^{m\alpha}}{\Gamma(m\alpha+1)}\right). \quad (5.8b)$$

For *i*=1. The truncated series (5.5) after setting i=1 is

$$u_1(x,y,t) = -\cos(x)\sin(y) + f_1(x,y)\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}, \quad (5.9a)$$

$$v_1(x,y,t) = \sin(x)\cos(y) + g_1(x,y)\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}, \quad (5.9b)$$

$$P_1(x,y,t) = h_0(x,y). \quad (5.9c)$$

The first residual functions must then be found by substituting Eq (5.9) in Eq (5.8) as follows:

$$Res_{u_1} = f_1 - Re\cos(x)\sin(y)\frac{\partial f_1}{\partial x}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}$$

$$+Re\sin(x)\sin(y) f_1 \frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}$$

$$+Re f_1 \frac{\partial f_1}{\partial x}\frac{\left(\phi(t)-\phi(a)\right)^{2\alpha}}{\Gamma^2(\alpha+1)} + Re\sin(x)\cos(y)\frac{\partial f_1}{\partial y}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}$$

$$-Re\cos(x)\cos(y) g_1 \frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)} + Re g_1 \frac{\partial f_1}{\partial y}\frac{\left(\phi(t)-\phi(a)\right)^{2\alpha}}{\Gamma^2(\alpha+1)} + \frac{\partial h_0}{\partial x}$$

$$-2\cos(x)\sin(y) - Re\sin(x)\cos(x)$$

$$-\left(\frac{\partial^2 f_1}{\partial x^2}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\partial^2 f_1}{\partial y^2}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}\right). \quad (5.10a)$$

$$Res_{v_1} = g_1 - Re\cos(x)\sin(y)\frac{\partial g_1}{\partial x}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ Re\cos(x)\cos(y) f_1 \frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}$$

$$+Re f_1 \frac{\partial g_1}{\partial x}\frac{\left(\phi(t)-\phi(a)\right)^{2\alpha}}{\Gamma^2(\alpha+1)} + Re\sin(x)\cos(y)\frac{\partial g_1}{\partial y}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)}$$

$$-Re\sin(x)\sin(y) g_1 \frac{\left(\phi(t)-\phi(a)\right)^{\alpha}}{\Gamma(\alpha+1)} + Re g_1 \frac{\partial g_1}{\partial y}\frac{\left(\phi(t)-\phi(a)\right)^{2\alpha}}{\Gamma^2(\alpha+1)}$$

$$+\frac{\partial h_0}{\partial y}+2\,sin(x)\,cos(y)-Resin(y)\,cos(y)$$

$$-\left(\frac{\partial^2 g_1}{\partial x^2}\frac{(\phi(t)-\phi(a))^\alpha}{\Gamma(\alpha+1)}+\frac{\partial^2 g_1}{\partial y^2}\frac{(\phi(t)-\phi(a))^\alpha}{\Gamma(\alpha+1)}\right). \tag{5.10b}$$

The truncated residual functions at $\phi(t)=0$ are then computed to obtain:

$$Res_{u_1}=f_1-2cos(x)\,si\,n(y)-Resin(x)\,cos(x)+\frac{\partial h_0}{\partial x}, \tag{5.11a}$$

$$Res_{v_1}=g_1+2sin(x)\,cos(y)-Resin(y)\,cos(y)+\frac{\partial h_0}{\partial y}, \tag{5.11b}$$

and by Eq (4.14), we know that

$$Res_{u_1}(x,y,0)=0,\ Res_{v_1}(x,y,0)=0. \tag{5.12}$$

When we solve these equations for f_1 and g_1, we obtain:

$$f_1=2cos(x)\,si\,n(y)+Resin(x)\,cos(x)-\frac{\partial h_0}{\partial x}, \tag{5.13a}$$

$$g_1=-2sin(x)\,cos(y)+Resin(y)\,cos(y)-\frac{\partial h_0}{\partial y}. \tag{5.13b}$$

Now, to determine h_0, we apply the following boundary conditions:

$$u_1(x,0,t)=0, v_1(0,y,t)=0. \tag{5.14}$$

Using the boundary conditions (5.14), we obtain the following by substituting Eq (5.13) in Eq (5.9):

$$\frac{\partial h_0}{\partial x}=Resin(x)\,cos(x), \tag{5.15a}$$

$$\frac{\partial h_0}{\partial y}=Resin(y)\,cos(y). \tag{5.15b}$$

By integrating Eq (4.15a) with respect to *x*, we obtain:

$$h_0(x,y)=-\frac{Re}{4}cos(2x)+c(y). \tag{5.16}$$

Then, the function

$$c(y)=-\frac{Re}{4}cos(2y). \tag{5.17}$$

After entering this equation into Eq (5.16), we obtain:

$$h_0(x,y)=-\frac{Re}{4}(cos(2x)+cos(2y)). \tag{5.18}$$

Consequently, the final forms of the functions *f* and *g* are as follows:

$$f_1=2cos(x)\,si\,n(y), \tag{5.19a}$$

$$g_1=-2sin(x)\,cos(y). \tag{5.19b}$$

Finally, the following is the first estimated RPS solution:

$$u_1(x,y,t)=-cos(x)\sin(y)+2cos(x)\,si\,n(y)\frac{(\phi(t)-\phi(a))^\alpha}{\Gamma(\alpha+1)}, \tag{5.20a}$$

$$v_1(x,y,t) = sin(x)\,cos(y) - 2sin(x)\,cos(y)\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}, \quad (5.20b)$$

$$P_1(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y)). \quad (5.20c)$$

For *i*=2. The truncated series (5.5), after setting *i*=2, is

$$u_2(x,y,t) = -cos(x)\,si\,n(y) + f_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)} + f_2\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (5.21a)$$

$$v_2(x,y,t) = sin(x)\,cos(y) + g_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}) + g_2\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (5.21b)$$

$$P_2(x,y,t) = h_0 + h_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}. \quad (5.21c)$$

Therefore, the following residual truncated functions can be obtained by substituting these equations into Eq **(**5.8) as:

$$Res_{u_2} = f_1 + f_2\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)} - Recos(x)\,si\,n(y)\frac{\partial f_1}{\partial x}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$-Recos(x)si\,n(y)\frac{\partial f_2}{\partial x}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} + Resin(x)\,sin(y)\,f_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$+Ref_1\frac{\partial f_1}{\partial x}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma^2(\alpha+1)} + Ref_1\frac{\partial f_2}{\partial x}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}$$

$$+Resin(x)\,sin(y)\,f_2\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} + Ref_2\frac{\partial f_1}{\partial x}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}$$

$$+Ref_2\frac{\partial f_2}{\partial x}\frac{(\phi(t)-\phi(a))^{4\alpha}}{\Gamma^2(\alpha+1)} + Re\,sin(x)\,cos(y)\frac{\partial f_1}{\partial y}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$+Re\,sin(x)\,cos(y)\frac{\partial f_2}{\partial y}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} - Recos(x)\,cos(y)\,g_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$+Reg_1\frac{\partial f_1}{\partial y}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma^2(\alpha+1)} + Reg_1\frac{\partial f_2}{\partial y}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}$$

$$-Recos(x)\,cos(y)\,g_2\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} + Reg_2\frac{\partial f_1}{\partial y}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}$$

$$+Reg_2\frac{\partial f_2}{\partial y}\frac{(\phi(t)-\phi(a))^{4\alpha}}{\Gamma^2(2\alpha+1)} - Re\,sin(x)\,cos(x)$$

$$-2\,cos(x)\,sin(y) + \frac{\partial h_0}{\partial x} + \frac{\partial h_1}{\partial x}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$-\frac{\partial^2 f_1}{\partial x^2}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)} - \frac{\partial^2 f_2}{\partial x^2}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$-\frac{\partial^2 f_1}{\partial y^2}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}-\frac{\partial^2 f_2}{\partial y^2}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{5.22a}$$

$$Res_{v_2}=g_1+g_2\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}-Recos(x)\,sin(y)\frac{\partial g_1}{\partial x}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$-Recos(x)sin(y)\frac{\partial g_2}{\partial x}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+Recos(x)\,cos(y)\,f_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}+Ref_1\frac{\partial g_1}{\partial x}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma^2(\alpha+1)}$$

$$+Ref_1\frac{\partial g_2}{\partial x}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}+Recos(x)\,cos(y)\,f_2\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+Ref_2\frac{\partial g_1}{\partial x}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}+Ref_2\frac{\partial g_2}{\partial x}\frac{(\phi(t)-\phi(a))^{4\alpha}}{\Gamma^2(2\alpha+1)}$$

$$+Re\,sin(x)\,cos(y)\frac{\partial g_1}{\partial y}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}+Re\,sin(x)\,cos(y)\frac{\partial g_2}{\partial y}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$-Re\,sin(x)\,sin(y)g_1\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}+Reg_1\frac{\partial g_1}{\partial y}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma^2(\alpha+1)}$$

$$+Reg_1\frac{\partial g_2}{\partial y}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}-Re\,sin(x)\,sin(y)\;g_2\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+Reg_2\frac{\partial g_1}{\partial y}\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}+Reg_2\frac{\partial g_2}{\partial y}\frac{(\phi(t)-\phi(a))^{4\alpha}}{\Gamma^2(2\alpha+1)}$$

$$-Re\,sin(y)\,cos(y)+2\,sin(x)\,cos(y)+\frac{\partial h_0}{\partial y}+\frac{\partial h_1}{\partial y}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$-\frac{\partial^2 g_1}{\partial x^2}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}-\frac{\partial^2 g_2}{\partial x^2}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\partial^2 g_1}{\partial y^2}\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$-\frac{\partial^2 g_2}{\partial y^2}\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{5.22b}$$

Next, applying operator ${}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi}$ into Eq (5.22), and then substituting $\phi(t)=0$, we get

$${}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi}Res_{u_2}(x,y,0)=f_2-Recos(x)\,sin(y)\frac{\partial f_1}{\partial x}+Re\,sin(x)\,sin(y)\,f_1$$

$$+Re\,sin(x)\,cos(y)\frac{\partial f_1}{\partial y}-Re\,cos(x)\,cos(y)g_1-\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial h_1}{\partial x}, \tag{5.23a}$$

$${}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi}Res_{v_2}(x,y,0)=g_2-Recos(x)\,sin(y)\frac{\partial g_1}{\partial x}+Re\,cos(x)\,cos(y)\,f_1$$

$$+Re\,sin(x)\,cos(y)\frac{\partial g_1}{\partial y}-Re\,sin(x)\,sin(y)g_1-\frac{\partial^2 g_1}{\partial x^2}-\frac{\partial^2 g_1}{\partial y^2}+\frac{\partial h_1}{\partial y}. \tag{5.23b}$$

Based on Eq (4.15), then

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} Res_{u_2}(x,y,0)=0, \tag{5.24a}$$

$$ {}_{t}^{C}\mathcal{D}_{a}^{\alpha;\phi} Res_{v_2}(x,y,0)=0. \tag{5.24b}$$

This fact enables us to obtain the initial formulas of f_2 and g_2 by inserting f_1 and g_1 and their partial derivatives in Eq (5.23) as:

$$f_2=-4Resin(x)\cos(x)-4\cos(x)\sin(y)-\frac{\partial h_1}{\partial x}, \tag{5.25a}$$

$$g_2=-4Resin(y)\cos(y)+4\sin(x)\cos(y)-\frac{\partial h_1}{\partial y}. \tag{5.25b}$$

To determine h_1, we need to apply the following boundary conditions:

$$u_2(x,0,t)=0,\ v_2(0,y,t)=0. \tag{5.26}$$

By replacing Eq (5.25) with Eq (5.22), and then applying the boundary conditions to the resulting equations, we obtain:

$$\frac{\partial h_1}{\partial x}=-4Resin(x)\cos(x), \tag{5.27a}$$

$$\frac{\partial h_1}{\partial y}=-4Resin(y)\cos(y). \tag{5.27b}$$

The integration of Eq (5.27) with regard to x provides

$$h_1(x,y)=Re\cos(2x)+c(y). \tag{5.28}$$

Then, the function

$$c(y)=Re\cos(2y). \tag{5.29}$$

After entering this equation into Eq (5.28), we obtain:

$$h_1(x,y)=Re(\cos(2x)+\cos(2y)). \tag{5.30}$$

Consequently, the final forms of the functions f_2 and g_2 are as follows:

$$f_2=-4\cos(x)\sin(y), \tag{5.31a}$$

$$g_2=4\sin(x)\cos(y). \tag{5.31b}$$

Finally, the following is the first estimated RPS solution:

$$u_2(x,y,t)=-\cos(x)\sin(y)+2\cos(x)\sin(y)\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$-4\cos(x)\sin(y)\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{5.32a}$$

$$v_2(x,y,t)=\sin(x)\cos(y)-2\sin(x)\cos(y)\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}$$

$$+4\sin(x)\cos(y)\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{5.32b}$$

$$P_2(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y)) + Re(cos(2x) + cos(2y))\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)}. \quad (5.32c)$$

Our time-fractal problem can be solved by repeating the same process for $i = 3, 4, 5, \ldots$ as:

$$u(x,y,t) = -cos(x)\, sin(y)\left[1 - 2\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)} + 4\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} - 8\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(2\alpha+1)} + 16\frac{(\phi(t)-\phi(a))^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right]$$

$$= -cos(x)\, sin(y)\left[\sum_{i=0}^{\infty}\frac{\left(-2(\phi(t)-\phi(a))^{\alpha}\right)^{i}}{\Gamma(i\alpha+1)}\right] \quad (5.33a)$$

$$v(x,y,t) = sin(x)\, cos(y)\left[1 - 2\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)} + 4\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} - 8\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} + 16\frac{(\phi(t)-\phi(a))^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right]$$

$$= sin(x)\, cos(y)\left[\sum_{i=0}^{\infty}\frac{\left(-2(\phi(t)-\phi(a))^{\alpha}\right)^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.33b)$$

$$P(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y))\left[1 - 4\frac{(\phi(t)-\phi(a))^{\alpha}}{\Gamma(\alpha+1)} + 16\frac{(\phi(t)-\phi(a))^{2\alpha}}{\Gamma(2\alpha+1)} - 64\frac{(\phi(t)-\phi(a))^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$

$$= -\frac{Re}{4}(cos(2x) + cos(2y))\left[\sum_{i=0}^{\infty}\frac{\left(-4(\phi(t)-\phi(a))^{\alpha}\right)^{i}}{\Gamma(i\alpha+1)}\right]. \quad (5.33c)$$

This solution leads us to conclude that Eq (5.33) contains two important special cases.

Initially, assuming $\phi(t) = t$ (this is the fractional integral of Riemann–Liouville) and $a = 0$. The exact solution in this case is as follows:

$$u(x,y,t) = -cos(x)\, sin(y)\left[\sum_{i=0}^{\infty}\frac{(-2t^{\alpha})^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.34a)$$

$$v(x,y,t) = sin(x)\, cos(y)\left[\sum_{i=0}^{\infty}\frac{(-2t^{\alpha})^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.34b)$$

$$P(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y))\left[\sum_{i=0}^{\infty}\frac{(-4t^{\alpha})^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.34c)$$

which is in full agreement with the results acquired by Sawangtong et al. [23] using the generalized Shehu residual power series. This solution leads us to the conclusion that Eq (5.34) has two significant special instances. Initially, assuming that $\phi(t) = \frac{t^{\rho}}{\rho}$ (Katugampola fractional derivative in the sense of Caputo) and $a = 0$, the exact solution in this case is as follows:

$$u(x,y,t) = -cos(x)\, sin(y)\left[\sum_{i=0}^{\infty}\frac{\left(-2\frac{t^{\rho\alpha}}{\rho\alpha}\right)^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.35a)$$

$$v(x,y,t) = sin(x)\, cos(y)\left[\sum_{i=0}^{\infty}\frac{\left(-2\frac{t^{\rho\alpha}}{\rho\alpha}\right)^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.35b)$$

$$P(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y))\left[\sum_{i=0}^{\infty}\frac{\left(-4\frac{t^{\rho\alpha}}{\rho\alpha}\right)^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.35c)$$

and are alike in agreement with the solutions found by Sawangtong et al. [23] using the generalized Shehu residual power series. On the other hand, if $\phi(t) = ln(t)$ and $a = 1$ (we have the Hadamard fractional integral), Eq (5.34) yields a solution that becomes:

$$u(x,y,t) = -cos(x)\,si\,n(y)\left[\sum_{i=0}^{\infty}\frac{(-2(ln\,t)^{\alpha})^{i}}{\Gamma(i\alpha+1)}\right]. \quad (5.36a)$$

The Mittag–Leffler function enables us to express the above answer in closed form as:

Based on the Hadamard derivative in the sense of Caputo (4.5), along with IC (5.1) and BC (5.2), the analytical solutions u and v of the two-dimensional time fractional Navier–Stokes equation, as well as the fluid pressure, are thus provided by:

$$v(x,y,t) = sin(x)\,co\,s(y)\left[\sum_{i=0}^{\infty}\frac{(-2(ln\,t)^{\alpha})^{i}}{\Gamma(i\alpha+1)}\right], \quad (5.36b)$$

$$P(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y))\left[\sum_{i=0}^{\infty}\frac{(-4(ln\,t)^{\alpha})^{i}}{\Gamma(i\alpha+1)}\right]. \quad (5.36c)$$

$$u(x,y,t) = -cos(x)\,si\,n(y)E_{\alpha}(-2(ln\,t)^{\alpha}), \quad (5.37a)$$

$$v(x,y,t) = sin(x)\,co\,s(y)\,E_{\alpha}(-2(ln\,t)^{\alpha}), \quad (5.37b)$$

$$P(x,y,t) = -\frac{Re}{4}(cos(2x) + cos(2y))E_{\alpha}(-4(ln\,t)^{\alpha}). \quad (5.37c)$$

6. Results and discussion

In the following, we present the graphical fadings of the nonlinear two-dimensional time fractional equation solution Navier–Stokes Eq (4.5), with IC (5.1) and BC (5.2), under the Caputo-Hadamard memory. We demonstrate the plots of the three solutions u, v and P for the solutions at $\alpha = 0.95, 0.75, 0.55, 0.35$, where $x, y \in [0, \pi], t \in [1,2]$ and the Reynold's number $Re = 40$ are shown in Figures 1–4. In Figure 1, it can be seen from those figures that $u(x, y, t)$ and $P(x, y, t)$ are generally growing as t grows, even though the increase decreases as α increases. The second component of velocity, $v(x, y, t)$, where $x = \frac{\pi}{4}$, has numerical results that indicate that it decreases as t grows, but it becomes less decreasing, as α increases. In Figure 2, it is observed that $v(x, y, t)$ and $P(x, y, t)$ grow generally with x up to $x = \frac{\pi}{2}$ and gradually decrease starting from $x = \frac{\pi}{2}$. For the second component of the velocity, $u(x, y, t)$, with $x = \frac{\pi}{4}$, numerical results indicate that it decreases with the growth of y up to $y = \frac{\pi}{2}$, and then grows generally with increasing y. The numerical results for the first component of velocity $u(x, y, t)$ of the fractional Navier–Stokes equation obtained by the RPS method are shown in Figure 3, for a range of values of t, x, and with $y = \frac{\pi}{4}$. It is evident from those figures that $u(x, y, t)$ generally increases as t and x increase, but decreases as α increases. Finally, for the pressure in Figure 4, where $y = \frac{\pi}{4}$, it is shown that $P(x, y, t)$ increases around $x = \frac{\pi}{4}$, and then decreases when $x\ and\ t$ increase, where this increase decreases with increasing α.

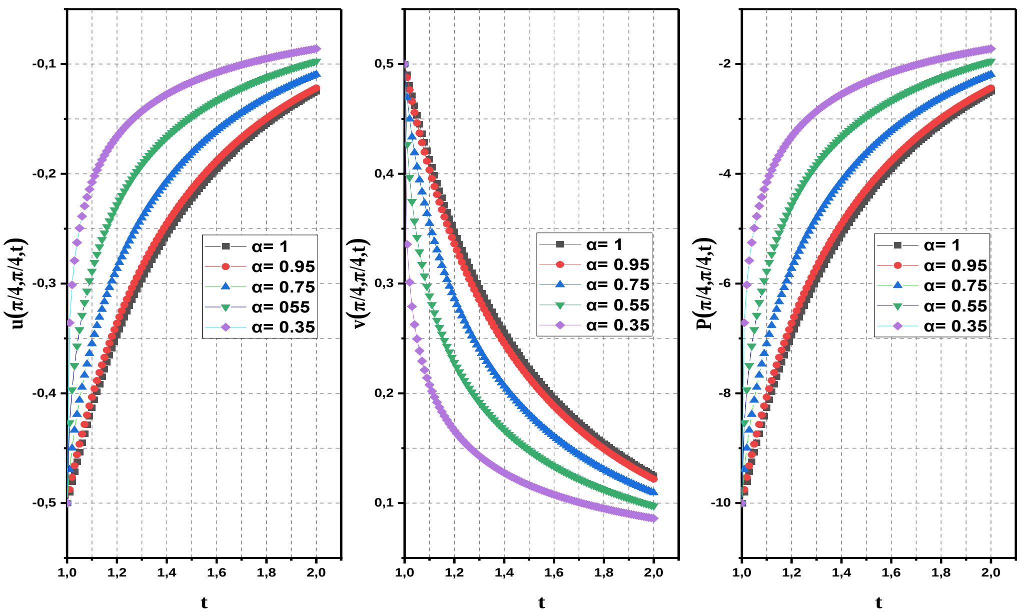

Figure 1. The graphs of Eq (5.36) of different values of parameters $t \in [1,2]$, α $with$ $a = 1, \alpha = 1, 0.95, 0.75, 0.55, 0.35$.

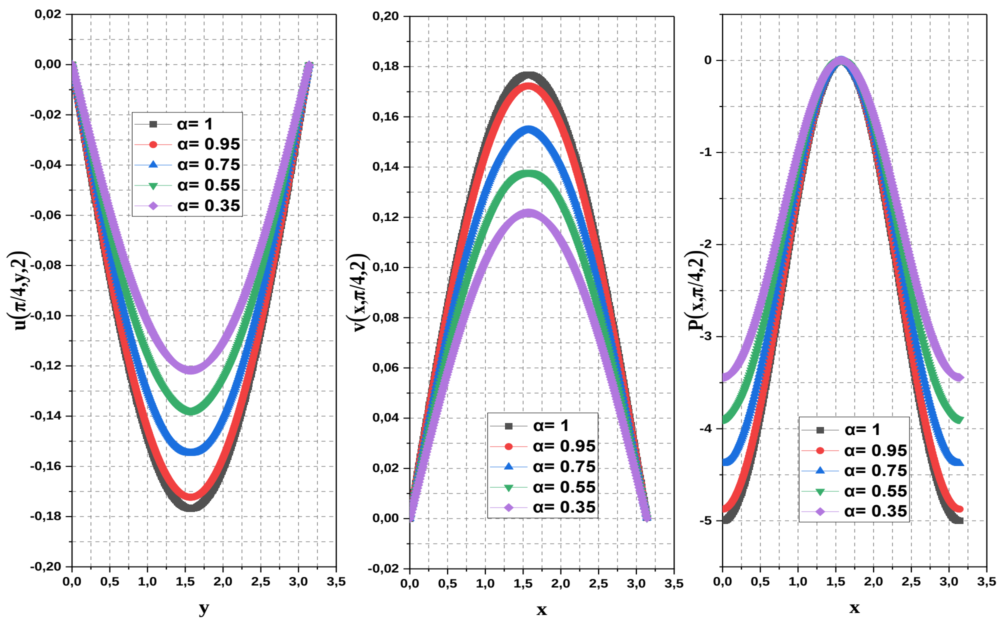

Figure 2. The graphs of Eq (5.36) for different values of parameters $x, y \in [0, \pi], \alpha$ $with$ $a = 1, \alpha = 1, 0.95, 0.75, 0.55, 0.35$.

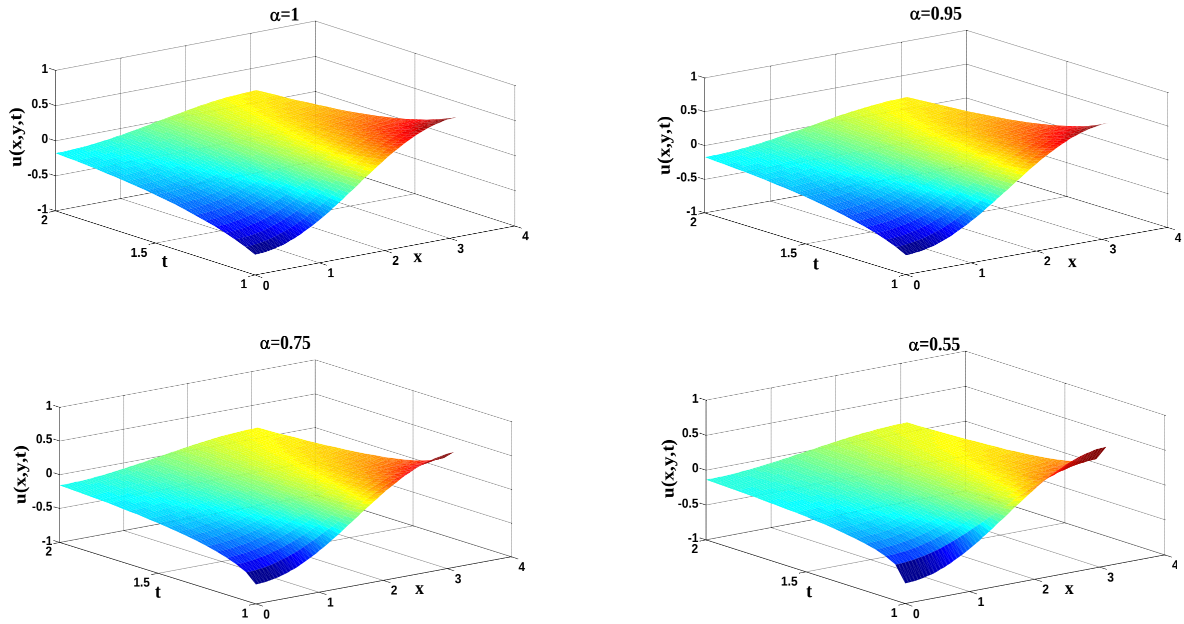

Figure 3. The surface shows the behavior of solution $u(x, y, t)$ of the application using Eq (5.36) with respect to t and x with $\alpha = 1, \alpha = 0.95, \alpha = 0.75, \alpha = 0.55,\ y = \frac{\pi}{4}, Re = 40$.

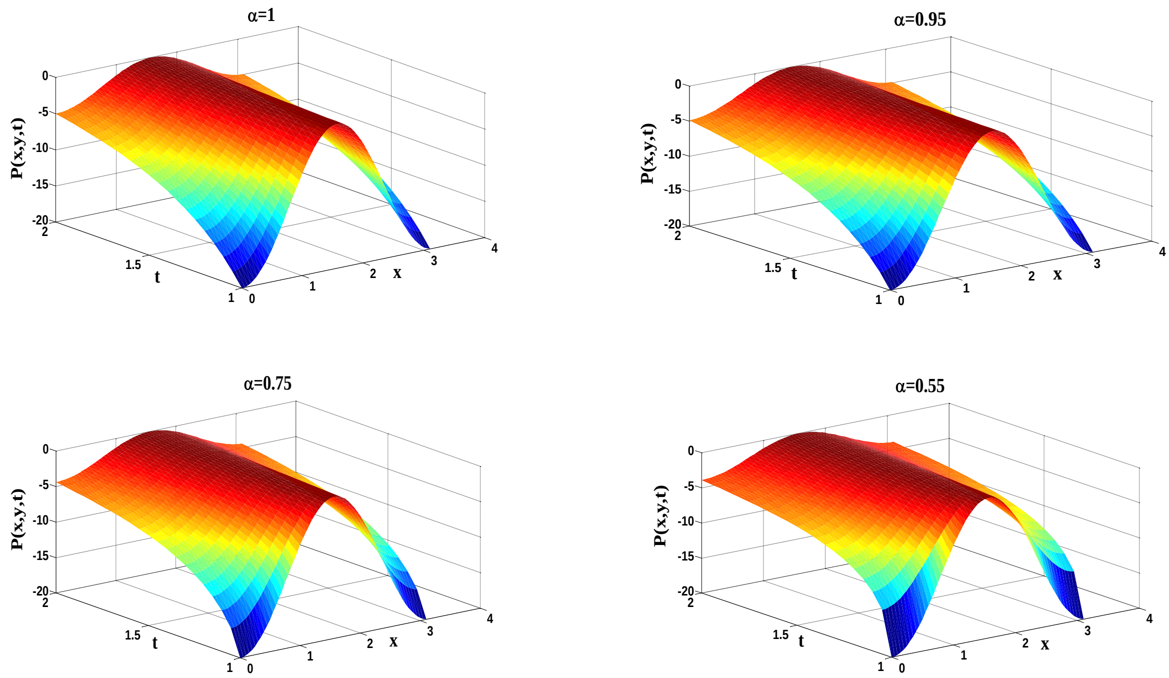

Figure 4. The surface shows the behavior of solution $P(x, y, t)$ of the application using Eq (5.36) with respect to t and x with $\alpha = 1, \alpha = 0.95, \alpha = 0.75, \alpha = 0.55,\ y = \frac{\pi}{4}, Re = 40$.

7. Conclusions

In this paper, we present an analytical solution to the proposed problem of the time-fractional Navier–Stokes equation using the RPS method within the fractional derivative (ϕ-Caputo). These equations are described in time, enabling the generalization of the Riemann–Liouville, Hadamard, and Katugampola fractional derivatives into a unified form. In addition, we demonstrate the results through an example with different fractional orders of α, illustrating the outcomes. Moreover, we provide graphical representations of the solutions to these problems when the Caputo–Hadamard fractional derivative is utilized, noting that Matlab is used to generate these graphs. Moreover, this method reduces the amount of computational work compared to traditional methods. We hope that this work will serve as a step in extending the applications of the RPS method to solve fractional problems with boundary conditions at infinity, an area in which we expect this method to be very applicable. Moreover, the values of α of the Hadamard derivative affect the velocity magnitude of the fluid flow, as shown in the graphical results section.

Author contributions

O.B. and A.M.A: Methodology, Investigation, Formal analysis, Data curation, conceptualization, Writing-review and editing; O.B: writing-original draft, Visualization; A.M.A: Supervision, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. R. Herrmann, *Fractional calculus: an introduction for physicists*, Singapore: World Scientific Publishing Company, 2011. https://doi.org/10.1142/8072
2. V. S. Kiryakova, *Generalized fractional calculus and applications*, Pitman research notes in mathematics series, Longman Scientific & Technical, Harlow, 1994.
3. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Willey & Sons, 1993.
4. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier, 2006.
5. I. Podlubny, *Fractional differential equations calculus*, an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Vol. 198, New, York: Academic, Press, 1999.
6. A. M. Alqahtani, H. Mihoubi, Y. Arioua, B. Bouderah, Analytical solutions of time-fractional Navier–Stokes equations employing homotopy Perturbation–Laplace transform method, *Fractal Fract.*, **9** (2024), 23. https://doi.org/10.3390/fractalfract9010023

7. J. Zuo, C. Liu, C. Vetro, Normalized solutions to the fractional Schrödinger equation with potential, *Mediterr. J. Math.*, **20** (2023), 216. https://doi.org/10.1007/s00009-023-02422-1
8. J. Zuo, T. An, G. Ye, Z. Qiao, Nonhomogeneous fractional p-Kirchhoff problems involving a critical nonlinearity, *Electron. J. Qual. Theory Differ. Eq.*, **41** (2019), 1–15. https://doi.org/10.14232/ejqtde.2019.1.41
9. T. Öziş, D. Ağırseven, He's homotopy perturbation method for solving heat-like and wave-like equations with variable coefficients, *Phys. Lett. A*, **372** (2008), 5944–5950. https://doi.org/10.1016/j.physleta.2008.07.060
10. R. M. Jena, S. Chakraverty, Solving time-fractional Navier–Stokes equations using homotopy perturbation Elzaki transform, *SN. Appl. Sci.*, **1** (2019), 16. https://doi.org/10.1007/s42452-018-0016-9
11. S. J. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, 1 Ed., New York: Chapman and Hall/CRC Press, 2003. https://doi.org/10.1201/9780203491164
12. G. Bakicierler, S. Alfaqeih, E. Mısırlı, Application of the modified simple equation method for solving two nonlinear time-fractional long water wave equations, *Rev. Mex. Fís.*, **67** (2021), 1–7. https://doi.org/10.31349/revmexfis.67.060701
13. R. Abazari, B. Soltanalizadeh, Reduced differential transform method and its application on Kawahara equations, *Thai J. Math.*, **11** (2000), 199–216.
14. Y. Keskin, G. Oturanc, The Reduced differential transform method: a new approach to fractional partial differential equations, *Nonlinear Sci. Lett. A*, **1** (2010), 207–217.
15. O. A. Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability, *J. Adv. Res. Appl. Math.*, **5** (2013), 31–52.
16. K. K. Jaber, R. S. Ahmad, Analytical solution of the time fractional Navier–Stokes equation, *Ain Shams Eng. J.*, **9** (2018), 1917–1927. https://doi.org/10.1016/j.asej.2016.08.021
17. A. Arafa, G. Elmahdy, Application of residual power series method to fractional coupled physical equations arising in fluids flow, *Int. J. Differ. Equ.*, **10** (2018), 7692849. https://doi.org/10.1155/2018/7692849
18. T. Ogawa, S. V. Rajopadhye, M. E. Schooner, Energy decay for a weak solution of the Navier–Stokes equation with slowly varying external forces, *Energy J. Funct. Anal.*, **144** (1997), 325–358. https://doi.org/10.1006/jfan.1996.3011
19. E. S. Baranovskii, Exact solutions for non-isothermal flows of second grade fluid between parallel plates, *Nanomaterials*, **13** (2023), 1409. https://doi.org/10.3390/nano13081409
20. E. S. Baranovskii, Analytical unsteady poiseuille flow of a second grade fluid with slip boundary conditions, *Polymers*, **16** (2024), 179. https://doi.org/10.3390/polym16020179
21. M. El-Shahed, A. Salem, On the generalized Navier–Stokes equations, *Appl. Math. Comput.*, **156** **(**2004), 287–293. https://doi.org/10.1016/j.amc.2003.07.022
22. J. Zhang, J. Wang, Numerical analysis for Navier–Stokes equations with time fractional derivatives, *Appl. Math. Comput.*, **336** (2018), 481–489. https://doi.org/10.1016/j.amc.2018.04.036
23. W. Sawangtong, P. Dunnimit, B. Wiwatanapataphe, P. Sawangtong, An analytical solution to the time fractional Navier–Stokes equation based on the Katugampola derivative in Caputo sense by the generalized Shehu residual power series approach, *Partial Differ. Equ. Appl. Math.*, **11** (2024), 100890. https://doi.org/10.1016/j.padiff.2024.100890
24. T. M. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica, *Fractional calculus with applications in mechanics*, vibrations and diffusion processes, John Wiley & Sons, Inc., 2014. https://doi.org/10.1002/9781118577530

25. C. G. Eliana, E. C. De Oliveira, Fractional versions of the fundamental theorem of calculus, *Appl. Math.*, **4** (2013), 23–33. https://doi.org/10.4236/am.2013.47A006
26. S. R. Bistafa, On the development of the Navier-Stokes equation by Navier, *Rev. Bras. Ensino Fís.*, **40** (2017), e2603. https://doi.org/10.1590/1806-9126-rbef-2017-0239
27. V. B. L. Chaurasia, D. Kumar, Solution of the time-fractional Navier–Stokes equation, *Gen. Math. Notes*, **4** (2011), 49–59.
28. C. Dou, Z. Zhao, Analytical solution to 1D compressible Navier-Stokes equations, *J. Func. Spaces*, **2021** (2021), 6. https://doi.org/10.1155/2021/6339203
29. H. Mihoubi, A. M. Alqahtani, Y. Arioua, B. Bouderah, T. Tayebi, Homotopy perturbation ρ-Laplace transform approach for numerical simulation of fractional Navier–Stokes equations, *Contemp. Math.*, **6** (2025), 1–29. https://doi.org/10.37256/cm.6320255978
30. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **142** (2012), 8. https://doi.org/10.1186/1687-1847-2012-142
31. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2014** (2014), 10. https://doi.org/10.1186/1687-1847-2014-10
32. A. M. Alqahtani, H. Mihoubi, Analytical solutions for fractional Black-Scholes European option pricing equation by using Homotopy perturbation method with Caputo fractional derivative, *Math. Model. Eng. Prob.*, **12** (2025), 1562–1570. https://doi.org/10.18280/mmep.120510
33. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
34. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach Science Publishers, 1993.
35. A. M. Alqahtani, A. Shukla, Computational analysis of multi-layered Navier–Stokes system by Atangana–Baleanu derivative, *Appl. Math. Sci. Eng.*, **32** (2024), 2290723. https://doi.org/10.1080/27690911.2023.2290723
36. R. Company, L. Jódar, J. R. Pintos, A numerical method for European Option Pricing with transaction costs nonlinear equation, *Math. Comput. Model.*, **50** (2009), 910–920. https://doi.org/10.1016/j.mcm.2009.05.019
37. R. K. Gazizov, N. H. Ibragimov, Lie symmetry analysis of differential equations in finance, *Nonlinear Dyn.*, **17**(1998), 387–407. https://doi.org/10.1023/A:1008304132308
38. P. Amster, C. G. Averbuj, M. C. Mariani, Stationary solutions for two nonlinear Black–Scholes type equations, *Appl. Numer. Math.*, **47** (2003), 275–280. https://doi.org/10.1016/S0168-9274(03)00070-9
39. R. M. Jena, S. Chakraverty, Solving time-fractional Navier–Stokes equations using homotopy perturbation Elzaki transform, *SN. Appl. Sci.*, **1** (2019), 16. https://doi.org/10.1007/s42452-018-0016-9