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Research article

A general definition of the fractal derivative: Theory and applications

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Abstract: In this paper, we introduce a general definition of the fractal derivative with respect to a function ψ, in the context of the order $0 < \alpha \leq 1$ and the function $\psi(\Theta)$. This novel definition generalizes the classical fractal derivative, which is recovered when $\psi(\Theta) = \Theta$, as described in previous works by Chen et al. [1, 2]. We explored key properties of the ψ-fractal derivative, including the ψ-fractal Laplace transform, which provides a powerful tool for solving complex differential equations in fractal domains. We also derived a generalized ψ-chain rule, extending classical calculus into the fractal domain, and presented fundamental operations related to this unique derivative. We give some applications.

Keywords: fractional derivative; ψ-fractal derivative; ψ-chain rule; ψ-fractal integral; ψ-fractal Laplace transform
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1. Introduction

Fractional calculus (FC) is a powerful mathematical framework that generalizes classical differentiation and integration to arbitrary (non-integer) orders. This extension provides essential tools for modeling phenomena exhibiting long-memory effects and non-local behavior. The applications of FC span diverse scientific domains, including chemistry, physics, electrical engineering, and mechanics, as demonstrated in numerous studies [3, 4]. The literature on FC is extensive, with significant contributions across theoretical and applied perspectives. Foundational works include [5, 6], while applications to real-world problems have been particularly impactful. For instance: Pandey [7] employed the Caputo-Fabrizio fractional derivative to model COVID-19 pandemic dynamics; Evirgen [8] investigated Nipah virus transmission using Caputo fractional derivatives; material science applications include TiO_2 nanopowder synthesis modeling [9]; chemical reaction analysis of alkali-silica systems [10] via Caputo derivatives; Atangana-Baleanu derivatives for COVID-19 modeling [11]; and numerical solutions for matrix fractional differential equations using the FBDF method [12]. Theoretical advances include: [13] Trajectory controllability of higher-order fractional neutral stochastic system with non-instantaneous impulses via state-dependent delay; and controllability and observability analysis of fractal linear systems [14], where one can extend this idea to study the controllability of the Hilfer fractional system. These examples represent only a fraction of the significant contributions in this rapidly evolving field.

The study of fractional differential equations has gained significant attention in recent years due to its widespread applications in various scientific and engineering fields. In particular, researchers have been actively working on parameter estimation, exact solutions, and symmetry analysis of fractional differential equations. In this context, we highlight several important works that have contributed to the advancement of our understanding of fractional differential equations: Smith, J. et al. [15] presented a novel approach to parameter estimation in fractional stochastic SIRD models, incorporating random perturbations. Here, we introduce valuable techniques for modeling and analyzing complex epidemiological systems. Johnson, A. et al. demonstrated the efficacy of the Yang-Laplace transform method in solving the integro-differential equations of fractional order [16]. In [17], Chalishajar et al. studied the Ulam-Hyers stability and controllability for coupled nonlinear fractional stochastic differential systems. In [18], Kasinathan et al. investigated the controllability and observability of stochastic integrodifferential systems. Their method has paved the way for tackling integral equations with fractional derivatives effectively. In 2019, Lee, S. et al. [19] introduced the method for obtaining exact solutions of conformable fractional partial differential equations as described. This approach has been instrumental in finding exact solutions to complex fractional differential equations. Smith, M. et al. [20] conducted a comprehensive study on the complete group classification of time fractional system evolution differential equations with a constant delay. Their work has shed light on the structural properties of time-fractional differential equations with delays. The study of symmetry analysis and conservation laws for boundary value problems associated with time-fractional generalized Burgers' differential equations has been addressed by Adams, R. et al. [21] their findings have provided essential insights into the physical significance and mathematical properties of these equations. In [22], Almeida presented the left Caputo fractional derivative of ϑ with RAF ψ (ψ-CFD) of order α is given by:

$$ {}_a^C D^{\alpha,\psi}\vartheta(\Theta) = \frac{1}{\Gamma(1-\alpha)} \int_a^{\Theta} (\psi(\Theta) - \psi(\tau))^{-\alpha} \vartheta'(\tau) d\tau, \tag{1.1} $$

and the right ψ-CFD of a function ϑ with RAF ψ of order α is given by:

$$ {}^C D_b^{\alpha,\psi}\vartheta(\Theta) = \frac{-1}{\Gamma(1-\alpha)} \int_{\Theta}^{b} (\psi(\tau) - \psi(\Theta))^{-\alpha} \vartheta'(\tau) d\tau. \tag{1.2} $$

Here, it leans toward the Caputo derivative as the order $\psi(\Theta) = \Theta$. [23] Sadek et al. presented the left Caputo–Katugampola derivative of ϑ with RAF ψ (ψ-CKFD) of order α, which is given by:

$$ {}_a^C D^{\alpha,\gamma,\psi}\vartheta(\Theta) = \frac{\gamma^{\alpha}}{\Gamma(1-\alpha)} \int_a^{\Theta} \left(\psi(\Theta)^{\gamma} - \psi(\mu)^{\gamma}\right)^{-\alpha} \vartheta'(\mu) d\mu, \tag{1.3} $$

and the right ψ-CKFD of a function ϑ with RAF ψ of order α is given by:

$$ {}^C D_b^{\alpha,\gamma,\psi}\vartheta(\Theta) = \frac{-\gamma^{\alpha}}{\Gamma(1-\alpha)} \int_{\Theta}^{b} \left(\psi(\mu)^{\gamma} - \psi(\Theta)^{\gamma}\right)^{-\alpha} \vartheta'(\mu) d\mu. \tag{1.4} $$

In [24] Sadek et al. presented the left Θ-fractional derivative of ϑ with RAF ψ (ψ-CKFD) of order α, which is given by:

$$ {}_a^C D^{\alpha,\gamma,\psi}\vartheta(\Theta) = \frac{1}{\Gamma(1-\alpha)} \int_a^{\Theta} \left(\frac{(\psi(\Theta) - \psi(a))^{\gamma} - (\psi(\mu) - \psi(a))^{\gamma}}{\gamma} \right)^{-\alpha} \vartheta'(\mu) d\mu, \tag{1.5} $$

and the right Θ-fractional derivative of a function ϑ with RAF ψ of order α is given by:

$$ {}^C D_b^{\alpha,\gamma,\psi}\vartheta(\Theta) = \frac{-1}{\Gamma(1-\alpha)} \int_{\Theta}^{b} \left(\frac{(\psi(b) - \psi(\Theta))^{\gamma} - (\psi(b) - \psi(\mu))^{\gamma}}{\gamma} \right)^{-\alpha} \vartheta'(\mu) d\mu. \tag{1.6} $$

Here, it leans toward the Caputo-Katugampola fractional derivative as the order $\psi(\Theta) = \Theta$. In [25], Sadek presented the left ψ-conformable fractional derivative of ϑ with RAF ψ of order α, which is given by:

$$ {}_a C^{\alpha,\psi}\vartheta(\Theta) = \lim_{\varepsilon\to 0} \frac{\vartheta(\Theta + \frac{\alpha\varepsilon}{\frac{d}{d\Theta}(\psi(\Theta)-\psi(a))^{\alpha}}) - \vartheta(\Theta)}{\varepsilon}, \tag{1.7} $$

and the right ψ-conformable fractional derivative of a function ϑ with RAF ψ of order α is given by:

$$ C_b^{\alpha,\psi}\vartheta(\Theta) = \lim_{\varepsilon\to 0} \frac{\vartheta(\Theta + \frac{\alpha\varepsilon}{\frac{d}{d\Theta}(\psi(b)-\psi(\Theta))^{\alpha}}) - \vartheta(\Theta)}{\varepsilon}. \tag{1.8} $$

Here, it leans toward the conformable fractional derivative as the order $\psi(\Theta) = \Theta$. Following this trend, some authors presented new types of FD with respect to a function [26, 27]. Within the open literature, fractal derivatives have emerged as a transformative mathematical framework for analyzing complex systems. Originally introduced by Benoit B. Mandelbrot in the late 20th century, these derivatives fundamentally extend classical differentiation by capturing scale-invariant behavior in self-similar structures.

The development of fractal derivatives has opened new avenues in multiple disciplines:

- Foundational applications in fractal geometry and multi-fractal analysis.
- Novel approaches to fractal time series characterization.
- Advanced modeling of scale-dependent phenomena across diverse domains.

We advance the field through three key contributions:

(1) Extension of the theoretical foundations of fractal calculus.
(2) Development of novel applications across scientific domains.
(3) Creation of educational frameworks for wider dissemination.

Fractal derivatives provide a powerful lens to reveal hidden structural patterns in complex systems. Our research harnesses this potential to address contemporary challenges, offering new perspectives on phenomena ranging from natural structures to socio-economic dynamics. By bridging theoretical innovation with practical applications, we aim to expand the boundaries of this transformative mathematical paradigm.

The concept of the fractal derivative was first introduced by Wen Chen [1, 2]. Since its inception, fractal calculus has found significant applications in modeling real-world phenomena [28]. The literature contains numerous important contributions to this field, including: Wang [29] applied fractal derivatives to model tsunami propagation across irregular boundaries; He [30] developed fundamental principles for temperature variation in fractal media, subsequently deriving laws for fluid mechanics and heat conduction in fractal spaces; Additional significant works include [31, 32]. The fractal derivative of a function $\vartheta(\Theta)$ of order $0 < \alpha \leq 1$, as defined in [1, 2], is given by:

$$\Delta^{\alpha}(\vartheta)(\Theta) = \lim_{\xi \to \Theta} \frac{\vartheta(\xi) - \vartheta(\Theta)}{\xi^{\alpha} - \Theta^{\alpha}}, \tag{1.9}$$

if ϑ is differentiable, then

$$\Delta^{\alpha}(\vartheta)(\Theta) = \frac{\vartheta'(\Theta)}{\Theta^{\alpha-1}\alpha}.$$

Motivated by the above-mentioned background and the generalization of the Caputo fractional derivative in Eqs (1.1) and (1.2), we introduce a new generalized fractal derivative. This generalized fractal derivative depends on both parameter α and function ψ, providing greater flexibility in applications.

The structure of this paper is as follows: In Section 2, we delve into the concept of fractal derivatives concerning one functional relative to another function, accompanied by its integral. Within this section, we explore key properties such as the ψ-chain rule. In, Section 3 , we delve into the realm of ψ-fractal integrals, while in Section 4, we tackle the acquisition of the Laplace transform and delve into the interplay between ψ-fractal derivatives and integrals. To bring our discussion to a close, we reserve Section 5 for the exploration of practical applications.

2. The general fractal derivative

In this section, the general fractal derivative with respect to another function is defined. To demonstrate the validity of the given novel definition, some examples and theorems are provided.

Definition 2.1. *Let $\psi \in C^1(a,b)$ such that $\psi'(\Theta) > 0$ and $a \geq 0$. Then the left ψ-fractal derivative of order $0 < \alpha \leq 1$ of the $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ with respect to ψ is*

$$ {}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta) = \lim_{\xi \to \Theta} \frac{\vartheta(\xi) - \vartheta(\Theta)}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}}, \tag{2.1} $$

the right ψ-fractal derivative $\Delta_b^{\alpha,\psi}$ of $\vartheta : (-\infty, b] \longrightarrow \mathbb{R}$ with respect to ψ of order α is:

$$ \Delta_b^{\alpha,\psi}(\vartheta)(\Theta) = -\lim_{\xi \to \Theta} \frac{\vartheta(\xi) - \vartheta(\Theta)}{(\psi(b) - \psi(\xi))^{\alpha} - (\psi(b) - \psi(\Theta))^{\alpha}}. \tag{2.2} $$

Remark 2.1.

- *If ${}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)$ exists on (a,b), then*

$$ {}^{a}\Delta^{\alpha,\psi}(\vartheta)(a) = \lim_{\Theta \to a^{+}} {}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta). $$

- *If $\Delta_b^{\alpha,\psi}(\vartheta)(\Theta)$ exists on (a,b), then*

$$ \Delta_b^{\alpha,\psi}(\vartheta)(b) = \lim_{\Theta \to b^{-}} \Delta_b^{\alpha,\psi}(\vartheta)(\Theta). $$

- *If ${}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)$ and $\Delta_b^{\alpha,\psi}(\vartheta)(\Theta)$ and both exist for every $\Theta \in [a,b]$, then ϑ is α-ψ-differentiable.*
- *If $a = 0$ and $\psi(\Theta) = \Theta$, then we recover the definition in Eq (1.9).*

It should be noted that the expression given in Eq (2.1) corresponds to the fractal derivative defined in [1, 2]. Furthermore, if parameter α is set to 1, Eq (2.1) simply transforms into the standard first derivative definition.

Lemma 2.1. *Given that λ is a constant, $0 < \alpha \leq 1$, p is a positive, and $\psi'(\Theta) > 0$, with the left ψ-fractal derivative of some functions with respect ψ, we have*

(1) ${}^{a}\Delta^{\alpha,\psi}(\lambda) = 0$.
(2) ${}^{a}\Delta^{\alpha,\psi}(\alpha(\psi(\Theta) - \psi(a))^{p}) = p(\psi(\Theta) - \psi(a))^{p-\alpha}$.
(3) ${}^{a}\Delta^{\alpha,\psi}(\psi(\Theta) - \psi(a))^{\alpha}) = 1$.
(4) ${}^{a}\Delta^{\alpha,\psi}(e^{\lambda\psi(\Theta)-\psi(a))^{\alpha}}) = \lambda e^{\lambda\psi(\Theta)-\psi(a))^{\alpha}}$.

Proof. We provide only result 2, and the rest can be proven similar.

$$ \begin{aligned} {}^{a}\Delta^{\alpha,\psi}(\alpha(\psi(\Theta) - \psi(a))^{p}) &= \lim_{\xi \to \Theta} \frac{\alpha(\psi(\xi) - \psi(a))^{p} - \alpha(\psi(\Theta) - \psi(a))^{p}}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}} \\ &= \alpha \lim_{\xi \to \Theta} \frac{\frac{(\psi(\xi)-\psi(a))^{p}-(\psi(\Theta)-\psi(a))^{p}}{\xi-\Theta}}{\frac{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}{\xi-\Theta}} \\ &= \alpha \frac{p(\psi(\Theta) - \psi(a))^{p-1}}{\alpha(\psi(\Theta) - \psi(a))^{\alpha-1}} \\ &= p(\psi(\Theta) - \psi(a))^{p-\alpha}. \end{aligned} $$

□

Lemma 2.2. *Let* $0 < \alpha \leq 1$.

(a) *If the function* $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ *is differentiable at* Θ*, then*

$$ {}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta) = \frac{\vartheta'(\Theta)}{((\psi(\Theta) - \psi(a))^{\alpha})'}. \tag{2.3} $$

(b) *If the function* $\vartheta : (-\infty, b] \longrightarrow \mathbb{R}$ *is differentiable at* Θ*, then*

$$ \Delta_{b}^{\alpha,\psi}(\vartheta)(\Theta) = -\frac{\vartheta'(\Theta)}{((\psi(b) - \psi(\Theta))^{\alpha})'}. \tag{2.4} $$

Proof. **(a)**

$$ \begin{aligned} {}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta) &= \lim_{\xi\to\Theta} \frac{\vartheta(\xi) - \vartheta(\Theta)}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}} \\ &= \lim_{\xi\to\Theta} \frac{\vartheta(\xi) - \vartheta(\Theta)}{\xi - \Theta} \lim_{\xi\to\Theta} \frac{\xi - \Theta}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}} \\ &= \lim_{\xi\to\Theta} \frac{\vartheta(\xi) - \vartheta(\Theta)}{\xi - \Theta} \frac{1}{\lim_{\xi\to\Theta} \frac{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}}{\xi - \Theta}} \\ &= \frac{\vartheta'(\Theta)}{\alpha\psi'(\Theta)(\psi(\Theta) - \psi(a))^{\alpha-1}}. \end{aligned} $$

Eq (2.4) can be proved in a similar way. □

2.1. Useful results

Theorem 2.1. *Let* $0 < \alpha \leq 1$.

(1) If a function $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ *is left fractal* α*-*ψ*-differentiable at* Θ_0*, then* ϑ *is continuous at* Θ_0.
(2) If a function $\vartheta : (-\infty, b] \longrightarrow \mathbb{R}$ *is right fractal* α*-*ψ*-differentiable at* Θ_0*, then* ϑ *is continuous at* Θ_0.

Proof. Since

$$ \vartheta(\xi) - \vartheta(\Theta_0) = \frac{\vartheta(\xi) - \vartheta(\Theta_0)}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}}((\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}). $$

Then,

$$ \lim_{\xi\to\Theta_0} [\vartheta(\xi) - \vartheta(\Theta_0)] = \lim_{\xi\to\Theta_0} \frac{\vartheta(\xi) - \vartheta(\Theta_0)}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}} \lim_{\xi\to\Theta_0} ((\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}), $$

so

$$ \lim_{\xi\to\Theta_0} [\vartheta(\xi) - \vartheta(\Theta_0)] =^{a} \Delta^{\alpha,\psi}(\vartheta)(\Theta_0) \times 0 = 0, $$

which implies that $\lim_{\xi\to\Theta_0} \vartheta(\xi) = \vartheta(\Theta_0)$, hence, ϑ is continuous at Θ_0. □

Theorem 2.2. *Let* $\lambda, \beta \in \mathbb{R}$ *and* $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ *and* $w : [a, +\infty) \longrightarrow \mathbb{R}$ *are left fractal* α*-*ψ*-differentiable. Then*

(1) ${}^{a}\Delta^{\alpha,\psi}(\lambda\vartheta+\beta w)=\lambda^{a}\Delta^{\alpha,\psi}(\vartheta)+\beta^{a}\Delta^{\alpha,\psi}(w)$.
(2) ${}^{a}\Delta^{\alpha,\psi}(\vartheta w)(\Theta)={}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)w(\Theta)+\vartheta(\Theta)^{a}\Delta^{\alpha,\psi}(w)(\Theta)$.
(3) ${}^{a}\Delta^{\alpha,\psi}\left(\frac{\vartheta}{w}\right)(\Theta)=\frac{{}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)w(\Theta)-{}^{a}\Delta^{\alpha,\psi}(w)(\Theta)\vartheta(\Theta)}{w(\Theta)^{2}}$.

Proof. Parts (1) using the Definition 2.1. For (2): Now, for fixed Θ,

$$\begin{aligned}
{}^{a}\Delta^{\alpha,\psi}(\vartheta w)(\Theta) &= \lim_{\xi\to\Theta}\frac{\vartheta(\xi)w(\xi)-\vartheta(\Theta)w(\Theta)}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}\\
&= \lim_{\xi\to\Theta}\frac{\vartheta(\xi)w(\xi)-\vartheta(\Theta)w(\xi)+\vartheta(\Theta)w(\xi)-\vartheta(\Theta)w(\Theta)}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}\\
&= \lim_{\xi\to\Theta}\frac{\vartheta(\xi)-\vartheta(\Theta)}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}w(\xi)\\
&\quad+\lim_{\xi\to\Theta}\vartheta(\Theta)\frac{w(\xi)-w(\Theta)}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}\\
&= {}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)w(\xi)+{}^{a}\Delta^{\alpha,\psi}(w)(\Theta)\vartheta(\Theta).
\end{aligned}$$

For (4): Now, for fixed Θ,

$$\begin{aligned}
{}^{a}\Delta^{\alpha,\psi}\left(\frac{\vartheta}{w}\right)(\Theta) &= \lim_{\xi\to\Theta}\frac{\frac{\vartheta(\xi)}{w(\xi)}-\frac{\vartheta(\Theta)}{w(\Theta)}}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}\\
&= \lim_{\xi\to\Theta}\frac{\frac{\vartheta(\xi)w(\Theta)-w(\xi)\vartheta(\Theta)}{w(\xi)w(\Theta)}}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}\\
&= \lim_{\xi\to\Theta}\frac{\vartheta(\xi)w(\Theta)-\vartheta(\Theta)w(\Theta)+\vartheta(\Theta)w(\Theta)-w(\xi)\vartheta(\Theta)}{w(\xi)w(\Theta)((\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha})}\\
&= \lim_{\xi\to\Theta}\frac{\vartheta(\xi)-\vartheta(\Theta)}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}}\frac{w(\Theta)}{w(\xi)w(\Theta)}\\
&\quad-\lim_{\xi\to\Theta}\frac{\vartheta(\Theta)}{w(\xi)w(\Theta)}\frac{w(\xi)-w(\Theta)}{(\psi(\xi)-\psi(a))^{\alpha}-(\psi(\Theta)-\psi(a))^{\alpha}},
\end{aligned}$$

since ϑ and w are left fractal α-ψ-differentiable so

$$\begin{aligned}
{}^{a}\Delta^{\alpha,\psi}(\frac{\vartheta}{w})(\Theta) &= {}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)\frac{w(\Theta)}{w(\Theta)w(\Theta)}-\frac{\vartheta(\Theta)}{w(\Theta)w(\Theta)}{}^{a}\Delta^{\alpha,\psi}(w)(\Theta)\\
&= \frac{{}^{a}\Delta^{\alpha,\psi}(\vartheta)(\Theta)w(\Theta)-{}^{a}\Delta^{\alpha,\psi}(w)(\Theta)\vartheta(\Theta)}{w(\Theta)^{2}}.
\end{aligned}$$

□

Theorem 2.3. *Let $\lambda,\beta\in\mathbb{R}$ and $\vartheta:(-\infty,b)\longrightarrow\mathbb{R}$ and $w:(-\infty,b)\longrightarrow\mathbb{R}$ are right fractal α-differentiable. Then*

(1) $\Delta_{b}^{\alpha,\psi}(\lambda\vartheta+\beta w)=\lambda\Delta_{b}^{\alpha,\psi}(\vartheta)+\beta\Delta_{b}^{\alpha,\psi}(w)$.
(2) $\Delta_{b}^{\alpha,\psi}(\vartheta w)(\Theta)=\Delta_{b}^{\alpha,\psi}(\vartheta)(\Theta)w(\Theta)+\vartheta(\Theta)\Delta_{b}^{\alpha,\psi}(w)(\Theta)$.
(3) $\Delta_{b}^{\alpha,\psi}\left(\frac{\vartheta}{w}\right)(\Theta)=\frac{\Delta_{b}^{\alpha,\psi}(\vartheta)(\Theta)w(\Theta)-\Delta_{b}^{\alpha,\psi}(w)(\Theta)\vartheta(\Theta)}{w(\Theta)^{2}}$.

Proof. The proof can be proved in a similar method of the Theorem 2.2. □

Theorem 2.4. *Let $\vartheta : J \subset \mathbb{R} \longrightarrow \mathbb{R}$ differentiable and $w : [a, +\infty) \longrightarrow J$ left fractal α-ψ-differentiable. Then the function $\vartheta o w : [a, +\infty) \longrightarrow \mathbb{R}$ is left fractal α-differentiable and we have (ψ-Chain rule):*

$$^{a}\Delta^{\alpha,\psi}(\vartheta o w)(\Theta) = \vartheta'(w(\Theta))^{a}\Delta^{\alpha,\psi}(w)(\Theta). \tag{2.5}$$

Proof. For Eq (2.5), we have,

$$\begin{aligned} ^{a}\Delta^{\alpha,\psi}(\vartheta o w)(\Theta) &= \lim_{\xi\to\Theta} \frac{\vartheta(w(\xi)) - \vartheta(w(\Theta))}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}} \\ &= \lim_{\xi\to\Theta} \frac{\vartheta(w(\xi)) - \vartheta(w(\Theta))}{w(\xi) - w(\Theta)} \frac{w(\xi) - w(\Theta)}{(\psi(\xi) - \psi(a))^{\alpha} - (\psi(\Theta) - \psi(a))^{\alpha}} \\ &= \vartheta'(w(\Theta))^{a}\Delta^{\alpha,\psi}(w)(\Theta). \end{aligned}$$

□

Theorem 2.5. *Let $\vartheta : J \subset \mathbb{R} \longrightarrow \mathbb{R}$ differentiable and $w : [a, +\infty) \longrightarrow J$ right fractal α-ψ-differentiable. Then the function $\vartheta o w : [a, +\infty) \longrightarrow \mathbb{R}$ is right fractal α-ψ-differentiable and we have:*

$$\Delta_{b}^{\alpha,\psi}(\vartheta o w)(\Theta) = \vartheta'(w(\Theta))\Delta_{b}^{\alpha,\psi}(w)(\Theta). \tag{2.6}$$

Proof. The proof can be proved in a similar way of Theorem 2.4. □

3. The general fractal integral

In this section, we present the general fractal integral concerning another function, as well as two important theorems.

Definition 3.1. *Assume $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ is a continuous function. Then, for order $0 < \alpha \leq 1$, the left ψ-fractal integral ${}_{a}I_{\Theta}^{\alpha,\psi}$ of ϑ is:*

$${}_{a}I_{t}^{\alpha,\psi}(\vartheta)(\Theta) = \alpha \int_{a}^{\Theta} \vartheta(s)\psi'(s)(\psi(s) - \psi(a))^{\alpha-1} ds.$$

Similarly, the right ψ-fractal integral ${}^{b}I_{\Theta}^{\alpha,\psi}$ of $\vartheta : (-\infty, b] \longrightarrow \mathbb{R}$ is:

$${}^{b}I_{\Theta}^{\alpha,\psi}(\vartheta)(\Theta) = -\alpha \int_{\Theta}^{b} \vartheta(s)\psi'(s)(\psi(b) - \psi(s))^{\alpha-1} ds.$$

Theorem 3.1. *If $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ be continuous function. Then,*

$$^{a}\Delta^{\alpha,\psi}{}_{a}I_{\Theta}^{\alpha,\psi}(\vartheta)(\Theta) = \vartheta(\Theta). \tag{3.1}$$

Similarly, for continuous function $\vartheta : (-\infty, b] \longrightarrow \mathbb{R}$, we have

$$\Delta_{b}^{\alpha,\psi\, b}I_{\Theta}^{\alpha,\psi}(\vartheta)(\Theta) = \vartheta(\Theta). \tag{3.2}$$

Proof. For Eq (3.1), let $\Theta \in [a, +\infty)$. ${}_aI_\Theta^{\alpha,\psi}(\vartheta)(\Theta)$ is clearly differentiable. Hence,

$$\begin{aligned} {}^a\Delta^{\alpha,\psi}\left({}_aI_\Theta^{\alpha,\psi}(\vartheta)\right)(\Theta) &= \frac{1}{\alpha\psi'(\Theta)(\psi(\Theta)-\psi(a))^{\alpha-1}} \frac{d}{d\Theta} {}_aI_\Theta^{\alpha,\psi}(\vartheta)(\Theta) \\ &= \frac{1}{\alpha\psi'(\Theta)(\psi(\Theta)-\psi(a))^{\alpha-1}} \frac{d}{d\Theta} \int_a^\Theta \alpha\psi'(s)(\psi(s)-\psi(a))^{\alpha-1}\vartheta(s)ds \\ &= \frac{1}{\alpha\psi'(\Theta)(\psi(\Theta)-\psi(a))^{\alpha-1}} \alpha\psi'(\Theta)(\psi(\Theta)-\psi(a))^{\alpha-1}\vartheta(\Theta) \\ &= \vartheta(\Theta). \end{aligned}$$

□

Theorem 3.2. *Let* $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ *be differentiable and* $0 < \alpha \leq 1$*. Then,*

$$ {}_aI_\Theta^{\alpha,\psi}({}^a\Delta^{\alpha,\psi}(\vartheta)(\Theta)) = \vartheta(\Theta) - \vartheta(a), \tag{3.3}$$

and for $\vartheta : (-\infty, b] \longrightarrow \mathbb{R}$*, we have*

$$ {}^bI_\Theta^{\alpha,\psi}(\Delta_b^{\alpha,\psi}(\vartheta)(\Theta)) = \vartheta(b) - \vartheta(\Theta). \tag{3.4}$$

Proof. For Eq (3.3), we have

$$\begin{aligned} {}_aI_\Theta^{\alpha,\psi}({}^a\Delta^{\alpha,\psi}(\vartheta)(\Theta)) &= \int_a^\Theta \alpha\psi'(s)(\psi(s)-\psi(a))^{\alpha-1}({}^a\Theta^{\alpha,\psi}(\vartheta))(s)ds. \\ &= \int_a^\Theta \alpha\psi'(s)(\psi(s)-\psi(a))^{\alpha-1}\frac{1}{\alpha\psi'(s)(\psi(s)-\psi(a))^{\alpha-1}}\vartheta'(s)ds \\ &= \int_a^\Theta \vartheta'(s)ds \\ &= \vartheta(\Theta) - \vartheta(a). \end{aligned}$$

For Eq (3.4), can be proved in a similar method. □

4. The ψ-fractal Laplace transform

Definition 4.1. *Let* $0 < \alpha \leq 1$ *and* $\vartheta, \psi : [a, +\infty) \longrightarrow \mathbb{R}$ *be real valued functions such that* ψ *is continuous and* $\psi'(\Theta) > 0$ *on* $[a, +\infty)$*. Thus,*

(a) *The left* ψ*-fractal Laplace transform of* ϑ *is:*

$$ {}_aL_\alpha^\psi\{\vartheta(\Theta)\}(s) = \int_a^{+\infty} e^{-s(\psi(\Theta)-\psi(a))^\alpha} \frac{\alpha\vartheta(\Theta)\psi'(\Theta)d\Theta}{(\psi(\Theta)-\psi(a))^{1-\alpha}}.$$

(b) *The right* ψ*-fractal Laplace transform of* ϑ *is:*

$$ {}^bL_\alpha^\psi\{\vartheta(\Theta)\}(s) = \int_{-\infty}^b e^{s(\psi(b)-\psi(\Theta))^\alpha} \frac{\alpha\vartheta(\Theta)\psi'(\Theta)d\Theta}{(\psi(b)-\psi(\Theta))^{1-\alpha}}.$$

Definition 4.2. *Let $0 < \alpha \leq 1$, $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$. ϑ is said to be of ψ-fractal-exponential order if there exist non-negative constants c, M, T such that $|\vartheta(\Theta)| \leq Me^{c(\psi(\Theta)-\psi(a))^{\alpha}}$ for $\Theta \geq T$.*

Now, we present the conditions for the existence of the ψ-fractal Laplace transform.

Theorem 4.1. *If $\vartheta : [a, +\infty) \longrightarrow \mathbb{R}$ is a piecewise continuous function and is of ψ-fractal-exponential order, then its fractal Laplace transform exists for $s > c$.*

Proof. The proof is similar to the proof of Theorem 3.4 in [33]. □

We pose $X_\alpha(s) :=_a L_\alpha^\psi\{\vartheta(\Theta)\}(s)$ and $X_\alpha^b(s) :=^b L_\alpha^\psi\{\vartheta(\Theta)\}(s)$.

Theorem 4.2. *Let $0 < \alpha \leq 1$ and the function ϑ be differentiable. Then,*

$$ {}_aL_\alpha^\psi\left\{{}_a\Delta^{\alpha,\psi}(\vartheta)(\Theta)\right\}(s) = sX_\alpha(s) - \vartheta(a), \tag{4.1}$$

and

$$ {}^bL_\alpha^\psi\left\{\Delta_b^{\alpha,\psi}(\vartheta)(\Theta)\right\}(s) = sX_\alpha^b(s) + \vartheta(b). \tag{4.2}$$

Proof. For Eq (4.1), we have

$$\begin{aligned}
{}_aL_\alpha^\psi\left\{a_\Delta^{\alpha,\psi}(\vartheta)(\Theta)\right\}(s) &= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha}\Delta^{\alpha,\psi}(\vartheta)(\Theta)\frac{\alpha\vartheta(\Theta)\psi'(\Theta)d\Theta}{(\psi(\Theta)-\psi(a))^{1-\alpha}} \\
&= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha}\frac{(\psi(\Theta)-\psi(a))^{1-\alpha}}{\alpha\psi'(\Theta)}\vartheta'(\Theta)\frac{\alpha\vartheta(\Theta)\psi'(\Theta)d\Theta}{(\psi(\Theta)-\psi(a))^{1-\alpha}} \\
&= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha}\vartheta'(\Theta)d\Theta \\
&= \left[e^{-s(\psi(\Theta)-\psi(a))^\alpha}\vartheta(\Theta)\right]_a^\infty + s\int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha}\vartheta(\Theta)d\mu_\alpha^\psi(\Theta) \\
&= -\vartheta(a) + sX_\alpha(s).
\end{aligned}$$

□

Theorem 4.3. *Let $0 < \alpha \leq 1$. Then, the left ψ-fractal Laplace transform with RAF ψ of order α of some functions:*

(1) ${}_aL_\alpha^\psi\{1\}(s) = \frac{1}{s}$, $s > 0$.
(2) ${}_aL_\alpha^\psi\left\{e^{(\psi(\Theta)-\psi(a))^\alpha}\right\}(s) = \frac{1}{s-1}$, $s > 1$.
(3) ${}_aL_\alpha^\psi\left\{e^{\lambda(\psi(\Theta)-\psi(a))^\alpha}\right\}(s) = \frac{1}{s-\lambda}$, $s > \lambda$.

Proof. (1) Let $s > 0$, we have

$$\begin{aligned}
{}_aL_\alpha^\psi\{1\}(s) &= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha}\frac{\alpha\psi'(\Theta)d\Theta}{(\psi(\Theta)-\psi(a))^{1-\alpha}} \\
&= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha}((\psi(\Theta)-\psi(a))^\alpha)'d\Theta \\
&= \left[-\frac{1}{s}e^{-s(\psi(\Theta)-\psi(a))^\alpha}\right]_a^\infty \\
&= \frac{1}{s}.
\end{aligned}$$

(2) Let $s > 1$, so we have

$$
\begin{aligned}
{}_aL_\alpha^\psi\left\{e^{(\psi(\Theta)-\psi(a))^\alpha}\right\}(s) &= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha} e^{(\psi(\Theta)-\psi(a))^\alpha} \frac{\alpha\psi'(\Theta)d\Theta}{(\psi(\Theta)-\psi(a))^{1-\alpha}} \\
&= \int_a^\infty e^{(1-s)\psi(\Theta)^\alpha}((\psi(\Theta)-\psi(a))^\alpha)'d\Theta \\
&= \left[\frac{1}{1-s}e^{(1-s)(\psi(\Theta)-\psi(a))^\alpha}\right]_a^\infty \\
&= \frac{1}{s-1}.
\end{aligned}
$$

(3) Let $s > \lambda$, so we have

$$
\begin{aligned}
{}_aL_\alpha^\psi\left\{e^{\lambda(\psi(\Theta)-\psi(a))^\alpha}\right\}(s) &= \int_a^\infty e^{-s(\psi(\Theta)-\psi(a))^\alpha} e^{\lambda(\psi(\Theta)-\psi(a))^\alpha} \frac{\alpha\psi'(\Theta)d\Theta}{(\psi(\Theta)-\psi(a))^{1-\alpha}} \\
&= \int_a^\infty e^{(\lambda-s)(\psi(\Theta)-\psi(a))^\alpha}((\psi(\Theta)-\psi(a))^\alpha)'d\Theta \\
&= \left[\frac{1}{\lambda-s}e^{(1-s)(\psi(\Theta)-\psi(a))^\alpha}\right]_a^\infty \\
&= \frac{1}{s-\lambda}.
\end{aligned}
$$

□

Proposition 4.1. *Let the functions ϑ and y are left and right transformable, then*

(1) ${}_aL_\alpha^\psi\{\vartheta + y\} ={}_a L_\alpha^\psi\{\vartheta\} +_a L_\alpha^\psi\{y\}$,
(2) ${}^bL_\alpha^\psi\{\vartheta + y\} ={}^b L_\alpha^\psi\{\vartheta\} +^b L_\alpha^\psi\{y\}$,
(3) ${}_aL_\alpha^\psi\{\lambda\vartheta\} = \lambda {}_aL_\alpha^\psi\{\vartheta\}$, $\lambda \in \mathbb{R}$,
(4) ${}^bL_\alpha^\psi\{\lambda\vartheta\} = \lambda {}^bL_\alpha^\psi\{\vartheta\}$, $\lambda \in \mathbb{R}$.

Proof. By the Definition 4.1, we say that ${}_aL_\alpha^\psi$ and ${}^bL_\alpha^\psi$ are linear operators. □

5. The applications

In this section, we present some applications, including the solution of linear ψ-fractal differential systems, which is important in control theory ([14] and nonlinear systems in TiO2 nanopowder synthesis via the sol-gel method [9]. We use the software MATLAB applied to obtain the required results (see Figures 1–11).

Example 5.1. Let the ψ-fractal initial value problem:

$$
\begin{cases} {}^0\Delta^{\alpha,\psi}(\vartheta)(\Theta) = K\vartheta(\Theta), & \Theta > a, \\ \vartheta(a) = \vartheta_a, \end{cases} \tag{5.1}
$$

so the exact solution is $\vartheta(\Theta) = e^{K(\psi(\Theta)-\psi(a))^\alpha}\vartheta_a$. Using the ψ-fractal Laplace Transform to both sides of Eq (5.1), so

$$
{}_aL_\alpha^\psi\{{}^a\Delta^{\alpha,\psi}(\vartheta)(\Theta)\}(s) ={}_a L_\alpha^\psi\{K\vartheta(\Theta)\}(s),
$$

from Theorem 4.2 and Proposition 4.1, we have

$$s_a L_\alpha^\psi\{\vartheta(\Theta)\}(s) - \vartheta_a = K_a L_\alpha^\psi\{\vartheta(\Theta)\}(s).$$

Simplifying this, we get

$$_a L_\alpha^\psi\{\vartheta(\Theta)\}(s) = \frac{1}{s-K}\vartheta_a, \tag{5.2}$$

so from 4.3 in Theorem 4.3, we get

$$_a L_\alpha^\psi\{\vartheta(\Theta)\}(s) =_a L_\alpha^\psi\left\{e^{K(\psi(\Theta)-\psi(a))^\alpha}\right\}(s)\vartheta_a. \tag{5.3}$$

Then, we have

$$\vartheta(\Theta) = e^{K(\psi(\Theta)-\psi(a))^\alpha}\vartheta_a. \tag{5.4}$$

Figures 1–8 compare the behavior of the solution $\vartheta(\Theta)$ of Eq (5.1) for different values of fractional order α and different choices of $\psi(\Theta)$. Key observations across all figures:

- **Effect of fractional order α:**
  - As α decreases from 1 (classical derivative) to smaller values (e.g., 0.2,...,0.7, 0.8, 0.9), the solution exhibits slower growth or decay. This reflects the memory-dependent nature of fractional/fractal derivatives.
  - For $\alpha = 1$, the solution reduces to the classical exponential function $e^{K(\psi(\Theta)-\psi(a))}$.
- **Effect of $\psi(\Theta)$:**
  - The choice of $\psi(\Theta)$ determines how the solution scales with Θ. Different $\psi(\Theta)$ functions lead to distinct curvature and growth/decay rates.

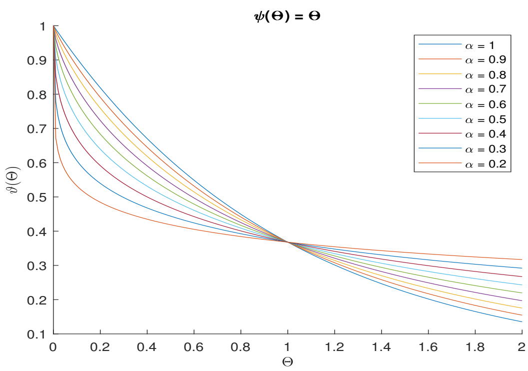

Figure 1. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$ and $\psi(\Theta) = \Theta$.

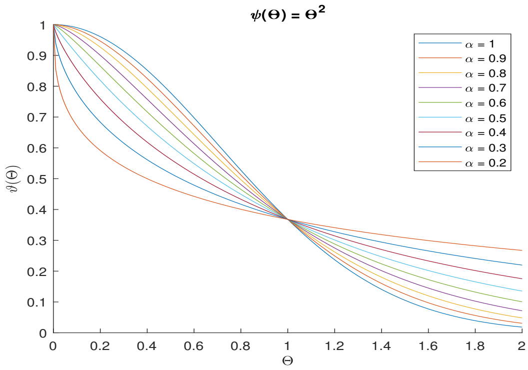

Figure 2. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$ and $\psi(\Theta) = \Theta^2$.

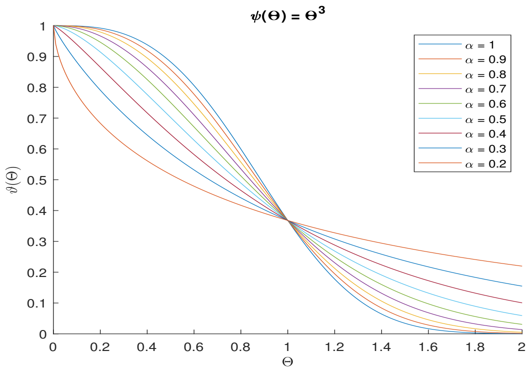

Figure 3. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$ and $\psi(\Theta) = \Theta^3$.

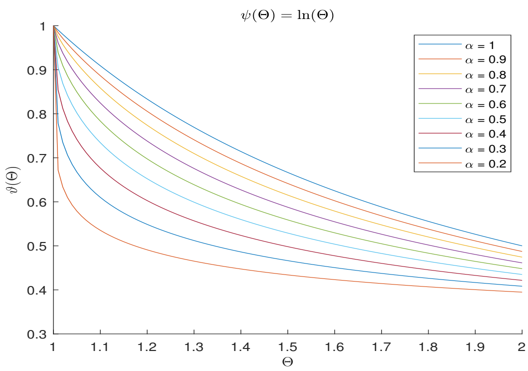

Figure 4. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$ and $\psi(\Theta) = \ln(\Theta)$.

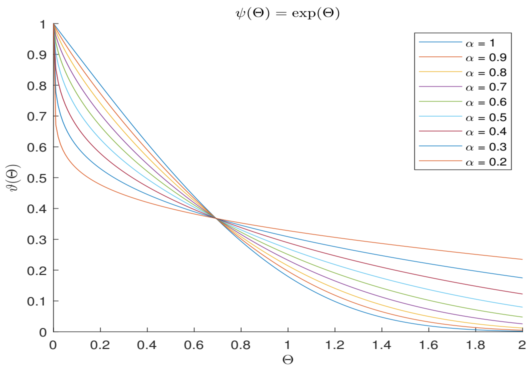

Figure 5. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$, and $\psi(\Theta) = e^{\Theta}$.

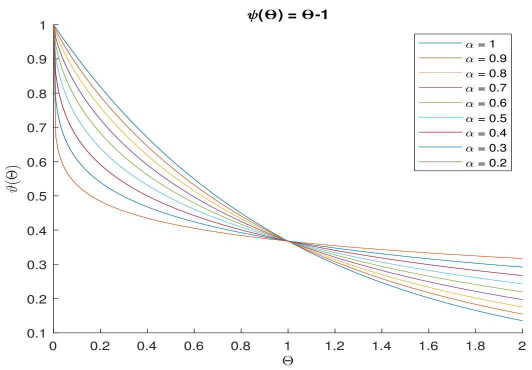

Figure 6. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$, and $\psi(\Theta) = \Theta - 1$.

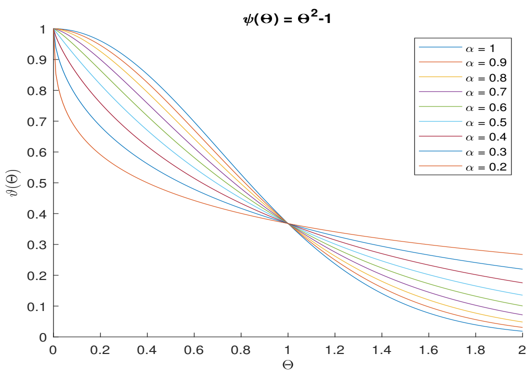

Figure 7. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$, and $\psi(\Theta) = \Theta^2 - 1$.

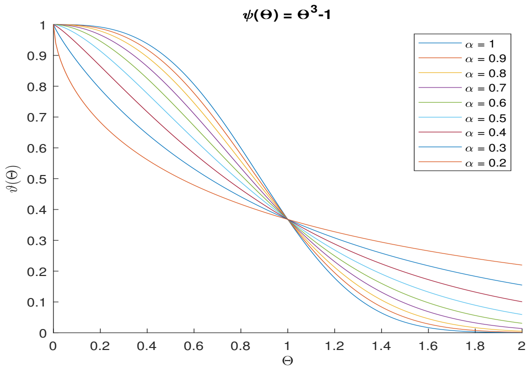

Figure 8. The solution of Eq (5.1) for different values of α with $K = -1$, $\vartheta_a = 1$, and $\psi(\Theta) = \Theta^3 - 1$.

Example 5.2. This example of linear ψ-fractal differential system has great importance in the field of control theory (see [14]). Let the linear ψ-fractal differential system

$$\begin{cases} {}^{0}\Delta^{\alpha,\psi}(\vartheta)(\Theta) = M\vartheta(\Theta) + y(\Theta), \quad \Theta \geq 0, \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{5.5}$$

where ϑ, y are vector functions and M is an $m \times m$ matrix. Using Lemma 2.2, the function ϑ is defined by

$$\vartheta(\Theta) = e^{M(\psi(\Theta)-\psi(0))^{\alpha}}\vartheta_0 + \int_0^{\Theta} e^{M((\psi(\Theta)-\psi(0))^{\alpha}-(\psi(s)-\psi(0))^{\alpha})}y(s)\, d\mu_{\alpha}^{\psi}\psi(s),$$

where

$$d\mu_{\alpha}^{\psi}\psi(s) = \alpha\psi'(s)(\psi(s) - \psi(a))^{\alpha-1}\, ds.$$

Example 5.3. In this example, we consider the nonlinear system in TiO_2 nanopowder synthesis by sol-gel method in [9], it describes the formation of TiO_2 over time:

$$\begin{cases} D^{\alpha}(\vartheta_1)(\Theta) = -k_1\vartheta_1(\Theta)\vartheta_2(\Theta) - k_2\vartheta_1(\Theta)\vartheta_3(\Theta), \\ D^{\alpha}(\vartheta_2)(\Theta) = -k_1\vartheta_1(\Theta)\vartheta_2(\Theta) + k_3\vartheta_3(\Theta)^2, \\ D^{\alpha}(\vartheta_3)(\Theta) = k_1\vartheta_1(\Theta)\vartheta_2(\Theta) - k_2\vartheta_1(\Theta)\vartheta_3(\Theta) - k_3\vartheta_3(\Theta)^2, \\ D^{\alpha}(\vartheta_4)(\Theta) = k_1\vartheta_1(\Theta)\vartheta_2(\Theta) + k_2\vartheta_1(\Theta)\vartheta_3(\Theta), \\ D^{\alpha}(\vartheta_5)(\Theta) = k_2\vartheta_1(\Theta)\vartheta_3(\Theta) + k_3\vartheta_3(\Theta)^2, \end{cases} \tag{5.6}$$

with initial conditions:

$$\vartheta_1(1) = 12.35, \vartheta_2(1) = 24.7, \vartheta_3(1) = 0, \vartheta_4(1) = 0, \vartheta_5(1) = 0,$$

where D^α is the Caputo fractional derivative. Replacing this D^α with this $\Delta^{\alpha,\psi}$, we have the following ψ-fractal dynamical system:

$$\begin{cases} \Delta^{\alpha,\psi}(\vartheta_1)(\Theta) = -k_1\vartheta_1(\Theta)\vartheta_2(\Theta) - k_2\vartheta_1(\Theta)\vartheta_3(\Theta), \\ \Delta^{\alpha,\psi}(\vartheta_2)(\Theta) = -k_1\vartheta_1(\Theta)\vartheta_2(\Theta) + k_3\vartheta_3(\Theta)^2, \\ \Delta^{\alpha,\psi}(\vartheta_3)(\Theta) = k_1\vartheta_1(\Theta)\vartheta_2(\Theta) - k_2\vartheta_1(\Theta)\vartheta_3(\Theta) - k_3\vartheta_3(\Theta)^2, \\ \Delta^{\alpha,\psi}(\vartheta_4)(\Theta) = k_1\vartheta_1(\Theta)\vartheta_2(\Theta) + k_2\vartheta_1(\Theta)\vartheta_3(\Theta), \\ \Delta^{\alpha,\psi}(\vartheta_5)(\Theta) = k_2\vartheta_1(\Theta)\vartheta_3(\Theta) + k_3\vartheta_3(\Theta)^2. \end{cases} \tag{5.7}$$

From Lemma 2.2, we get de ODE and solve it using ode23s in MATLAB. In the following figures, we draw the solution of this model to see how it is affected by ψ and α. Figures 9–11 compare the behavior of the solution $\vartheta(\Theta)$ of Eq (5.7) and its corresponding system when replacing $\Delta^{\alpha,\psi}$ for different values of α (fractional order) and different choices of $\psi(\Theta)$.

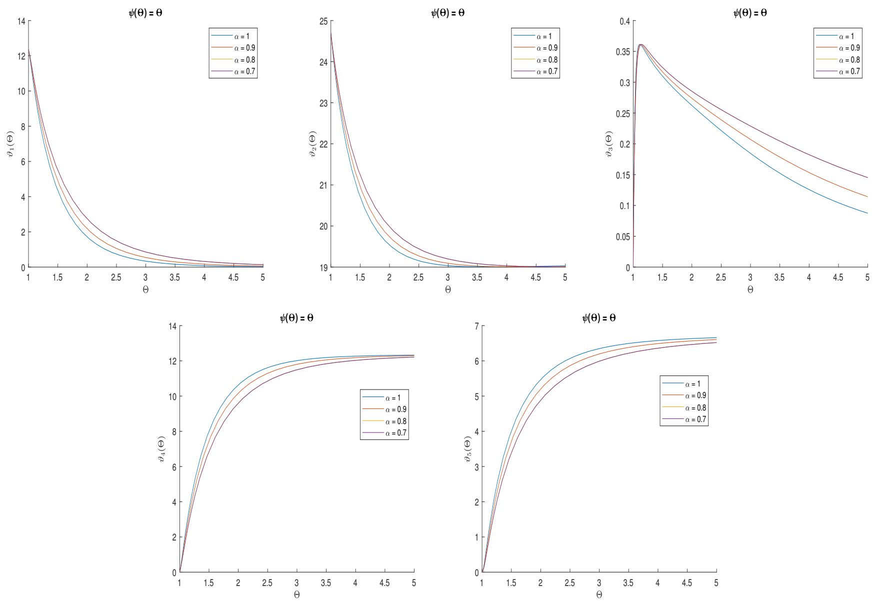

Figure 9. Comparisons of numerical solutions of nonlinear dynamical system with $\psi(\Theta) = \Theta$.

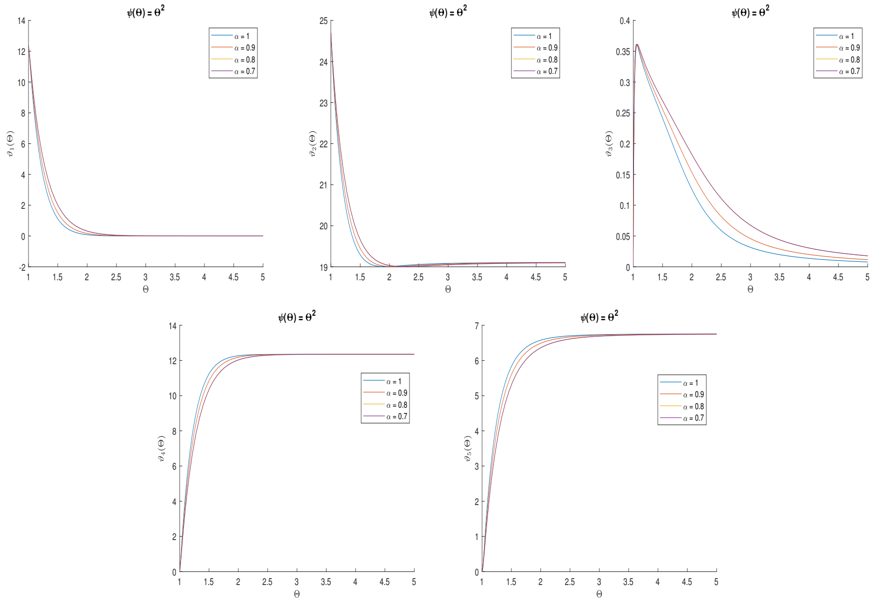

Figure 10. Comparisons of numerical solutions of nonlinear dynamical system with $\psi(\Theta) = \Theta^2$.

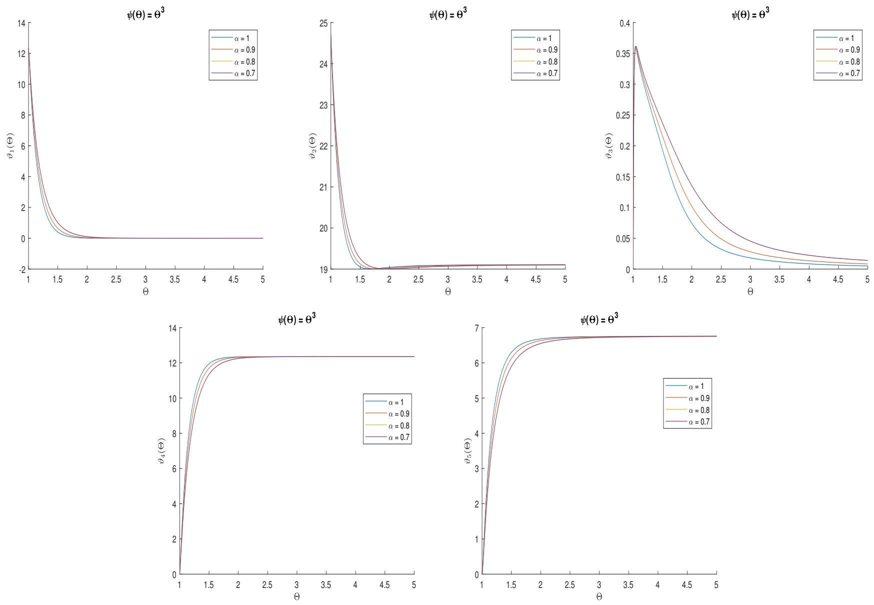

Figure 11. Comparisons of numerical solutions of a nonlinear dynamical system with $\psi(\Theta) = \Theta^3$.

6. Conclusions

We introduced a new concept in this study: the fractal derivative of a function with respect to another function, along with its integral and associated theorems. Our discoveries have yielded important insights and potential applications, particularly in the field of non-integer order differential equations. Our findings open new avenues for future research. We intend to investigate practical applications of this new definition, with an emphasis on addressing complex real-world problems and furthering our understanding of fractal calculus. Moreover, the continuation of this research will combine fractional and ψ-fractal operators, leading to ψ-fractal fractional operators, which generalize existing fractal-fractional operators. In future work, we will focus on studying the optimal control of fractional neutral stochastic integrodifferential systems with infinite delay, as discussed in [34], using this new definition.

Author contributions

Lakhlifa Sadek: Conceptualization, formal analysis, software, writing – original draft, writing – review & editing; Ahmad Sami Bataineh: Formal analysis, writing – review & editing; El Mostafa Sadek: Formal analysis, visualization; Ishak Hashim: Validation, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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