Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 15333–15357.
DOI: 10.3934/math.2025687
Received: 06 November 2024
Revised: 07 May 2025
Accepted: 21 May 2025
Published: 03 July 2025

Research article

On assessing convergence and stability of a novel iterative method for fixed-point problems

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Abstract: Fixed-point theory, a major field of mathematics, analyzes outcomes that remain unchanged under particular operators, featuring multiple applications in mathematics, physics, engineering, computer science, and economics. This study presents the D^* iteration technique, a robust and effective iterative scheme for approximating fixed points in Suzuki generalized nonexpansive mappings. Within the context of uniformly convex Banach spaces, the novel scheme's weak and strong convergence properties were carefully addressed. The efficiency of this approach was demonstrated through detailed theoretical, numerical, and graphical assessments. Additionally, the stability of the iterative process was established. The method is used to generalize and enhance previous findings by approximating solutions for a fractional differential problem.

Keywords: differential equations; weak convergence; strong convergence; iteration process
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

A great number of theoretical developments in fixed-point theory are being employed to address real-world problems. Hazarika et al. [1], for illustration, talked about new developments in functional analysis that improve our comprehension of fixed-point theory and its uses. Younis et al. [2] examined the theoretical underpinnings and practical uses of fixed-point theory, highlighting its applicability in a variety of fields. An essential field of mathematics, fixed-point theory finds use in a wide range of scientific fields and real-world situations. It is crucial for analyzing strategic interactions, creating equilibrium solutions in game theory and economics, and confirming the stability and convergence of iterative algorithms in numerical calculations. Its uses include engineering, where it aids in the analysis of intricate physical systems and the optimization of control devices, and computer science, where it is essential for the creation of algorithms. Furthermore, fixed-point methods are widely used in the study of dynamical systems and differential equations in order to comprehend their long-term behavior and durability. In these fields, the fixed-point method is essential and commonly used to address complex issues and phenomena.

Fixed-point results are often employed to assume the existence of a fixed point. The presence of fixed points is provided by the Banach contraction principle, which Banach invented in the 19th century. This progress has notably accelerated nonlinear analysis research. Since then, several researchers have used other spaces and conditions of contraction to prove the existence of fixed points. But even if their existence is established, finding them usually through iterative procedures remains difficult. There have been several iterative approaches created in past few decades (see [3–8]). These approaches start with an initial point and use iterative algorithms to approximate a fixed point.

Ahmad et al. [9] presented an iterative method based on Green's function, showing how fixed-point techniques can be used practically to solve boundary value issues in symmetric spaces. Shaheen et al. [10] demonstrated the adaptability of fixed-point theory in contemporary computing environments by highlighting current advancements in iterative algorithms, namely in digital metrics. The continuous development and use of fixed-point approaches in nonlinear domains is demonstrated by the convergence results for the split fixed-point problem solution presented by Rashid et al. [11]. Within the fixed-point theory framework, assume $\mathbb{X}$ is a Banach space, and consider a self-map $\mathcal{T}$ defined on a non-empty subset $\mathbb{A}$ defined as $\mathcal{T} : \mathbb{A} \to \mathbb{A}$. Also, define S as an accumulation of the operator $\mathcal{T}$'s fixed points, where $S = \{\rho^* \in \mathbb{A} : \mathcal{T}\rho^* = \rho^*\}$. Furthermore, a contractive operator $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ if $\exists$ $\gamma \in (0, 1)$ also $\forall\ j, k \in \mathbb{A}, \| \mathcal{T} j - \mathcal{T} k \| \leq \gamma \| j - k \|$. If $\gamma = 1$ then $\mathcal{T}$ is demonstrated as nonexpansive mappings and quasi nonexpansive mappings if $\forall\ j \in \mathbb{A}$ and $\rho^* \in$ S , $\| \mathcal{T} j - \rho^* \| \leq \| j - \rho^* \|$. $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ are generalized nonexpansive mappings if $\forall\ j, k \in \mathbb{X}, 1/2 \| j - \mathcal{T} j \| \leq \| j - k \| \Rightarrow \| \mathcal{T} j - \mathcal{T} k \| \leq \| j - k \|$. Let $t \geq 0$ and real sequences $\{\mu_t\}_{t=0}^{\infty}, \{\nu_t\}_{t=0}^{\infty}$, and $\{\omega_t\}_{t=0}^{\infty} \in [0, 1]$. See [4] for further information on the iteration process. Starting from the initial point $j_0 \in \mathbb{A}$, it is given as follows.
In 1890, the founders of [12] represented the Picard iterative process in this way:

$$\left\{\ j_{t+1} = \mathcal{T} j_t. \right. \tag{1.1}$$

The most basic iterative process for predicting fixed points for continuous functions is this one. The founding members of [13] introduced the Mann iterative method in 1953 in the manner described below:

$$\left\{\ j_{t+1} = (1 - \nu_t) j_t + \nu_t \mathcal{T} j_t. \right. \tag{1.2}$$

The study seeks to show that comparable methods can be useful in the theory of divergent iteration processes. The following is how the 1976 Ishikawa iterative process presenter, as cited in [14], expressed it:

$$\begin{cases} k_t = (1 - \mu_t) j_t + \mu_t \mathcal{T} j_t, \\ j_{t+1} = (1 - \nu_t) j_t + \nu_t \mathcal{T} k_t. \end{cases} \tag{1.3}$$

Their study presented the convergence of their iterative procedure toward a fixed point. Particularly, they examined a nonexpansive mapping. The Thakur iterative process was formulated in 1976 by the author of [15] in the following way:

$$\begin{cases} l_t = (1 - \omega_t) j_t + \omega_t \mathcal{T} j_t, \\ k_t = (1 - \mu_t) j_t + \mu_t l_t, \\ j_{t+1} = \mathcal{T} k_t. \end{cases} \tag{1.4}$$

For a specific class of weak-contraction mappings, they analyzed the convergence of their iterative approach and demonstrated a data dependence finding for fixed points of these mappings. In 2000, a Noor iterative approach was provided as follows by the author of [16]:

$$\begin{cases} l_t = (1-\omega_t)j_t + \omega_t \mathcal{T} j_t, \\ k_t = (1-\mu_t)j_t + \mu_t \mathcal{T} l_t, \\ j_{t+1} = (1-\nu_t)j_t + \nu_t \mathcal{T} k_t. \end{cases} \tag{1.5}$$

The authors of this paper offered and explored an innovative class of approximation techniques for common variational constraints. The iterations of Ishikawa and Mann are included in their results as exceptional cases. They also looked at the convergence criteria of various schemes. In 2011, the SP iterative technique was provided as follows by the author of [17]:

$$\begin{cases} l_t = (1-\omega_t)j_t + \omega_t \mathcal{T} j_t, \\ k_t = (1-\mu_t)l_t + \mu_t \mathcal{T} l_t, \\ j_{t+1} = (1-\nu_t)k_t + \nu_t \mathcal{T} k_t. \end{cases} \tag{1.6}$$

For the SP-iteration of continuous functions to converge on any interval, a necessary and sufficient condition is provided. Additionally, they contrast the rates of convergence for SP-iterations with previous research. It is shown that the first SP iteration is equivalent to the other iterations and converges faster. The writers in [18] provided the following introduction to the M iterative process that year:

$$\begin{cases} l_t = (1-\omega_t)j_t + \omega_t \mathcal{T} j_t, \\ k_t = \mathcal{T} l_t, \\ j_{t+1} = \mathcal{T} k_t. \end{cases} \tag{1.7}$$

The following iterative process, generally known as the M^* iterative process, was first introduced by the authors in [19] in 2017.

$$\begin{cases} l_t = (1-\omega_t)j_t + \omega_t \mathcal{T} j_t, \\ k_t = \mathcal{T}((1-\mu_t)j_t + \mu_t \mathcal{T} l_t), \\ j_{t+1} = \mathcal{T} k_t. \end{cases} \tag{1.8}$$

The following is how the authors of [20] designed a new iterative process entitled the K iterative process.

$$\begin{cases} l_t = (1-\omega_t)j_t + \omega_t \mathcal{T} j_t, \\ k_t = \mathcal{T}((1-\mu_t)\mathcal{T} j_t + \mu_t \mathcal{T} l_t), \\ j_{t+1} = \mathcal{T} k_t. \end{cases} \tag{1.9}$$

Through providing concrete examples, they have demonstrated that their iterative process exhibits a superior convergence rate compared to the previous iterative method. They assert that the new iterative process will soon converge, further strengthening their claim. The author of [21] introduced the K^* iterative process in 2018 and highlighted various flaws.

$$\begin{cases} l_t = (1-\omega_t)j_t + \omega_t \mathcal{T} j_t, \\ k_t = \mathcal{T}((1-\mu_t)l_t + \mu_t \mathcal{T} l_t), \\ j_{t+1} = \mathcal{T} k_t. \end{cases} \tag{1.10}$$

They made the convergence analysis more comprehensive. This extension made it possible to comprehend the convergence characteristics of their iterative method in greater detail, which gave important new information about how well it works in a variety of contexts.

Encouraged by these factors, we develop a novel iteration method that, as opposed to the iteration process previously described in previous research, offers an increased convergence rate for contraction mappings.

The following is the definition of our new approach:

$$\begin{cases} l_t = \mathcal{T}((1-\omega_t)j_t + \omega_t\mathcal{T}j_t), \\ k_t = \mathcal{T}((1-\mu_t)l_t + \mu_t\mathcal{T}l_t), \\ j_{t+1} = \mathcal{T}((1-\nu_t)k_t + \nu_t\mathcal{T}k_t)). \end{cases} \tag{1.11}$$

This study introduces a refined iterative method for approximating fixed points of contraction operators in Banach spaces, extending prior research. Comparative analysis is performed against the M [18], M^* [19], K [20], and K^* [21] methods. The stability of the approach is verified through a dedicated theorem, reinforcing its dependability. Numerical evaluations indicate that the proposed method outperforms existing schemes, including those of Picard [12], Maan [13], Ishikawa [14], Thakur et al. [15], Noor [16], and SP [17] in terms of convergence speed. The effectiveness of the method is further demonstrated through its application to a nonlinear differential equation.

2. Preliminaries

Presented below are key definitions and lemmas that form the foundation of this study.

Definition 2.1. *[22] The Banach space $\mathbb{X}$ is said to possess uniform convexity if it satisfies the following condition: for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $|\frac{j+k}{2}| \leq \delta$ for each $j,k \in \mathbb{X}$ with $|j| \leq 1$ and $|k| \leq 1$ where $|j-k| > \epsilon$.*

Definition 2.2. *[23] $\mathcal{T}: \mathbb{A} \to \mathbb{A}$ satisfy condition (C) if $\forall\ j,k \in \mathbb{A}$, and the following holds:*
$1/2 \parallel \mathcal{T}j - j \parallel \leq \parallel j-k \parallel \Rightarrow \parallel \mathcal{T}j - \mathcal{T}k \parallel \leq \parallel j-k \parallel$.

Definition 2.3. *[24] A Banach space $\mathbb{X}$ fulfills the Opial property [25] if $\forall\{j_t\} \in \mathbb{X}$, it is weakly convergent to $j \in \mathbb{X}$, and $lim_{t\to\infty} \sup \|j_t - j\| < lim_{t\to\infty} \sup \|j_t - k\|$, $\forall\ k \in \mathbb{X}$ with $k \neq j$.*

Lemma 2.4. *[13] Let us say there is an Opial property satisfying on a $\mathcal{T}$ Suzuki generalized nonexpansive mapping. If $\{j_t\}$ is weakly convergent to ρ^* and $lim_{t\to\infty} \parallel \mathcal{T}j_t - j_t \parallel = 0 \Rightarrow \mathcal{T}(\rho^*) = \rho^*$, then as a result, the demiclosedness property at zero holds for the operator $I - \mathcal{T}$.*

Lemma 2.5. *[13] (Theorem 5). Suppose $\mathbb{A}$ is a weakly compact convex subset of $\mathbb{X}$, and let $\mathcal{T}$ be the same as aforementioned. Then $\rho^* \in S$ for $\mathcal{T}$.*

Definition 2.6. *[26] Assume the sequences $\{j_t\}_{t=0}^{\infty}$ and $\{o_t\}_{t=0}^{\infty}$ approximated by the iterative process converge to $\rho^* \in S$, $\|j_t - \rho^*\| \leq c_t$ and $\|o_t - \rho^*\| \leq d_t$, $\forall\ t \geq 0$. If $\{c_t\}_{t=0}^{\infty}$ and $\{d_t\}_{t=0}^{\infty}$ rapidly converge to c and d, respectively, then*

$$lim_{t\to\infty}\frac{\|c_t - c\|}{\|d_t - d\|} = 0.$$

Thus, $\{j_t\}_{t=0}^{\infty}$ has a better convergence rate rather than $\{o_t\}_{t=0}^{\infty}$ to ρ^*.

Definition 2.7. *[27] Let $\{j_t\}_{t=0}^{\infty} \in \mathbb{A}$. Then an iterative process $j_{t+1} = f(\mathcal{T}, j_t)$ approaching ρ^* is $\mathcal{T}$-stable, if for $\epsilon_t = \|l_t + 1 - f(\mathcal{T} : j_t)\|, t \in N \Rightarrow lim_{t\to\infty}\epsilon = 0$ iff $lim_{t\to\infty} j_t = \rho^*$.*

Lemma 2.8. *[28] If $\{j_t\}_{t=0}^{\infty}$ and $\{l_t\}_{t=0}^{\infty}$ satisfy the relation $j_{t+1} \leq (1 - l_t)j_t + l_t$, where $l_t \in (0, 1)\ \forall\ t \in N$, $\Sigma_{t=0}^{\infty} l_t = \infty$ and $\frac{j_t}{l_t} \to 0$ as $n \to \infty$. Then $lim_{t\to\infty} j_t = 0$.*

Lemma 2.9. *[29] Given the context, let us suppose that $\mathbb{X}$ meets the criterion of being a uniformly convex Banach space, and let $0 < p \leq u_n \leq q < 1$ represent a real sequence for all $t \geq 1$. Let $\{j_t\}$ and $\{k_t\}$ be sequences of $\mathbb{X}$ such that $lim_{t\to\infty} \sup \|j_t\| \leq j, lim_{t\to\infty} \sup \|k_t\| \leq j$, and $lim_{t\to\infty} \sup \|u_t j_t + (1 - u_t k_t)\| = j$ are true for some $j \geq 0 \Rightarrow lim_{t\to\infty}\|j_t - k_t\| = 0$.*

Lemma 2.10. *[23] Let $\mathbb{A} \neq \phi \subset \mathbb{X}$ and $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ be as defined above. Then, $\forall\ j, k \in \mathbb{X}$,*

$$\| \mathcal{T} j - \mathcal{T} k \| \quad \leq 3 \| \mathcal{T} j - j \| + \| j - k \| .$$

Let $\mathbb{A} \neq \phi \subset \mathbb{X}$, and $\{j_t\} \in \mathbb{X}$ is bounded. For $j \in \mathbb{X}$,

$$r(j, \{j_t\}) = limsup_{t\to\infty} \| j_t - j \| .$$

The asymptotic radius of the sequence $\{j(t)\}$ about $\mathbb{A}$ is the subsequent calculation.

$$r(\mathbb{A}, \{j_t\}) = inf\{r(j, \{j_t\}) : j \in \mathbb{A}\}$$

This illustrates the set that, with regard to $\mathbb{X}$, denotes the asymptotic center of $\{j_t\}$:

$$A(\mathbb{A}, \{j_t\}) = \{j \in \mathbb{A} : r(j, \{j_t\}) = r(\mathbb{A}, \{j_t\}).$$

3. Convergence and stability results

This section demonstrates the unique convergence of the suggested iterative method, followed by a proof of its stability. It focuses on both weak and strong convergence results for fixed points of Suzuki generalized nonexpansive mappings in uniformly convex Banach spaces. First of all, we begin with the following proof, which is directly related to this fundamental result.

Theorem 3.1. *Let $\mathcal{T} : \mathbb{A} \to \mathbb{A}$, $\{j_t\}_{t=0}^{\infty}$ be an iterative process defined by D^* as in equation 1.11 (using real sequences $\{\mu_t\}_{t=0}^{\infty}$ and $\{\omega_t\}_{t=0}^{\infty}$ in $[0\ 1]$ satisfying $\Sigma_{t=0}^{\infty}\omega_t = \infty$ or $\Sigma_{t=0}^{\infty}\mu_t = \infty$. Then $\{j_t\}_{t=0}^{\infty}$ is strongly convergent to a distinct point $\rho^* \in \mathcal{T}$.*

Proof. Within the Banach space, $\mathcal{T}$ is characterized as a contraction mapping, also having $\rho^* \in \mathbb{A}$. From the D^* iterative process,

$$\begin{aligned} \|l_t - \rho^*\| &= \|\mathcal{T}((1 - \omega_t)j_t + \omega_t\mathcal{T} j_t) - \mathcal{T}\rho^*\| \\ &\leq \gamma\|(1 - \omega_t)j_t + \omega_t\mathcal{T} j_t - \rho^*\| \\ &\leq \gamma\|(1 - \omega_t)(j_t - \rho^*) + \omega_t(\mathcal{T} j_t - \rho^*)\| \\ &\leq \gamma(1 - \omega_t)\|(j_t - \rho^*)\| + \omega_t\|(\mathcal{T} j_t - \rho^*)\| \end{aligned}$$

$$
\begin{aligned}
&\leq \gamma\Big\{(1-\omega_t)\|(j_t-\rho^*)\| + \gamma\omega_t\|(j_t-\rho^*)\|\Big\} \\
&\leq \gamma\Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\|. \\
\|k_t-\rho^*\| &= \|\mathcal{T}((1-\mu_t)l_t+\mu_t\mathcal{T}l_t)-\mathcal{T}\rho^*\| \\
&\leq \gamma\|(1-\mu_t)l_t+\mu_t\mathcal{T}l_t-\rho^*\| \\
&\leq \gamma\|(1-\mu_t)(l_t-\rho^*)+\mu_t(\mathcal{T}l_t-\rho^*)\| \\
&\leq \gamma(1-\mu_t)\|(l_t-\rho^*)\| + \mu_t\|(\mathcal{T}l_t-\rho^*)\| \\
&\leq \gamma\Big\{(1-\mu_t)\|(l_t-\rho^*)\| + \gamma\mu_t\|(l_t-\rho^*)\|\Big\} \leq \gamma\|l_t-\rho^*\| \\
&\leq \gamma^2\Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\|. \\
\|j_{t+1}-\rho^*\| &= \|\mathcal{T}((1-\nu_t)k_t+\nu_t\mathcal{T}k_t)-\mathcal{T}\rho^*\| \\
&\leq \gamma\|(1-\nu_t)k_t+\nu_t\mathcal{T}k_t-\rho^*\| \\
&\leq \gamma\|(1-\nu_t)(k_t-\rho^*)+\nu_t(\mathcal{T}k_t-\rho^*)\| \\
&\leq \gamma(1-\nu_t)\|(k_t-\rho^*)\| + \nu_t\|(\mathcal{T}k_t-\rho^*)\| \\
&\leq \gamma\Big\{(1-\nu_t)\|(k_t-\rho^*)\| + \gamma\nu_t\|(k_t-\rho^*)\|\Big\} \leq \gamma\|k_t-\rho^*\| \\
&\leq \gamma^3\Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\|.
\end{aligned}
$$

By repetition,

$$
\begin{aligned}
\|j_t-\rho^*\| &\leq \gamma^3\Big\{1-\omega_{t-1}(1-\gamma)\Big\}\|(j_{t-1}-\rho^*)\| \\
\|j_{t-1}-\rho^*\| &\leq \gamma^3\Big\{1-\omega_{t-2}(1-\gamma)\Big\}\|(j_{t-2}-\rho^*)\| \\
\|j_{t-2}-\rho^*\| &\leq \gamma^3\Big\{1-\omega_{t-3}(1-\gamma)\Big\}\|(j_{t-3}-\rho^*)\| \\
&\vdots \\
\|j_1-\rho^*\| &\leq \gamma^3\Big\{1-\omega_0(1-\gamma)\Big\}\|(j_0-\rho^*)\|.
\end{aligned}
$$

Thus,

$$
\|j_{t+1}-\rho^*\| \leq \gamma^{3(t+1)}\|(j_0-\rho^*)\| \prod_{i=0}^{t}\Big\{1-\omega_i(1-\gamma)\Big\}.
$$

Now, $\gamma < 1$ so $(1-\gamma) > 0$ and $\omega_t \leq 1$ $\forall$ t $\in$ N. Thus, $1-\omega_t(1-\gamma) < 1$ $\forall$ $n \in N$. Also, in general $1 - r \leq e^{-r}$, for all r $\in$ [0,1]. So, $\|j_{t+1}-\rho^*\| \leq \gamma^{3(t+1)}\|(j_0-\rho^*)\|e^{-(1-\gamma)\sum_{i=0}^{n}\omega_i}$. Now, $lim_{t\to\infty}\|j_t-\rho^*\| = 0$. □

Remark 3.2. *By changing the condition in the aforementioned theorem to* $\Sigma_{t=0}^{\infty}\omega_t=\infty$ *from* $\Sigma_{t=0}^{\infty}\mu_t=\infty$*, consider the inequality* $\|l_t-\rho^*\| \leq \gamma\|j_t-\rho^*\|$*. From this it follows that* $\|k_t-\rho^*\| \leq \gamma^2\{1-\mu_t(1-k)\}\|(j_t-\rho^*)\|$*. Thus,*

$$
\|j_{t+1}-\rho^*\| \leq \gamma^{3(t+1)}\|(j_0-\rho^*)\| \prod_{i=0}^{t}\Big\{1-\mu_i(1-\gamma)\Big\}.
$$

An iteration process needs to be stable to get precise and consistent results. Before proceeding further, stability is analyzed as a crucial aspect in the following theorem.

Theorem 3.3. *Let* $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ *,* $\rho^* \in S$*,* $j_0 = o_0$*, and* $\{j_t\}_{t=0}^{\infty}$ *be as defined in equation 1.11, and also* $\{\mu_t\}_{t=0}^{\infty}$*,*$\{\omega_t\}_{t=0}^{\infty}$*, and* $\{\nu_t\}_{t=0}^{\infty}$ $\in$ *[0 1] satisfy all axioms of Theorem 3.1. Then the* D^* *iterative process is* $\mathcal{T}$*-stable.*

Proof. Suppose we have a randomly selected sequence $j_t{}^{\infty}t = 0 \subset \mathbb{X}$ from set $\mathbb{A}$. Additionally, examine the sequence obtained from the D^* iterative process, denoted as $j_{n+1} = f(\mathcal{T}; j_t)$. It converges to $\rho \in S$ (as stated in Theorem 3.1). Let $\epsilon_t = |j_{t+1} - f(\mathcal{T}; j_t)|$. The aim is to demonstrate that when $\lim_{t\to\infty} \epsilon_t = 0$, it implies $\lim_{t\to\infty} j_t = \rho$. Conversely, if $\lim_{t\to\infty} j_t = \rho$, then $\lim_{t\to\infty} \epsilon_t = 0$. Let $lim_{t\to\infty}\epsilon_t = 0$.
Then,

$$\|j_{t+1} - \rho^*\| \leq \|j_{t+1} - f(\mathcal{T}, j_t)\| + \|f(\mathcal{T}, j_t - \rho^*\| = \epsilon_t + \|j_(t+1) - \rho^*\|.$$

From the aforementioned result, it follows that $\epsilon_t \leq +k^3\{1 - \omega_t(1-\gamma)\}\|(j_t - \rho^*)\|$. Since $0 < \gamma < 1$ and $0 \leq \omega_t \leq 1$, $\forall$ t $t \in N$ $lim_{t\to\infty}\epsilon_t = 0$, and using Lemma 2.5, we conclude that $lim_{t\to\infty}\|j_t - \rho^*\| = 0$. Hence $lim_{t\to\infty} j_n = p$. Conversly, let $lim_{t\to\infty} j_t = \rho^*$.
This leads to the conclusion that

$$\begin{aligned}
\epsilon_n &= \|j_(t+1) - f(\mathcal{T}, j_t)\| \\
&\leq \|j_{t+1} - \rho^*\| + \|f(\mathcal{T}, j_t - \rho^*\| \\
&\leq \|j_{t+1} - \rho^*\| + \gamma^3\{1 - \omega_t(1-\gamma)\}\|(j_t - \rho^*)\|.
\end{aligned}$$

Therefore, $lim_{t\to\infty}\epsilon_t = 0$, and the D^* iterative process demonstrates $\mathcal{T}$-stability.

□

The next part focuses on strong and weak convergence results for fixed points of generalized non-expansive mappings in uniformly convex Banach spaces. To achieve this, the following lemma is first established as a crucial step before further discussions.

Lemma 3.4. *Consider $\mathcal{T} : \mathbb{A} \to \mathbb{A}$, $\rho^* \in S$, and $\{j_t\}_{t=0}^{\infty}$ be as defined in Theorem 3.1 then for any $\rho^* \in S$, $\exists$ $lim_{t\to\infty}\|j_t - \rho^*\|$.*

Proof. As $\mathcal{T}$ being hold condition (C). Also, Assume that $\rho^* \in$ S and t $\in \mathbb{A}$ So
$1/2\|\rho^* - \mathcal{T}t\| = 0 \leq \|\rho^* - t\| \Rightarrow \|\mathcal{T}\rho^* - \mathcal{T}t\| \leq \|\rho^* - t\|$.
So, by using the definition,

$$\begin{aligned}
\|l_t - \rho^*\| &= \|(1-\omega_t)j_t + \omega_t\mathcal{T}j_t - \rho^*\| \\
&\leq (1-\omega_t)\|j_t - \rho^*\| + \omega_t)\|\mathcal{T}j_t - \rho^*\| \\
&\leq (1-\omega_t)\|j_t - \rho^*\| + \omega_t)\|j_t - \rho^*\| \\
&= \|j_t - \rho^*\|. \\
\|k_t - \rho^*\| &= \|\mathcal{T}((1-\mu_t)l_t + \mu_t\mathcal{T}l_t) - \rho^*\| \\
&\leq \|(1-\mu_t)\mathcal{T}l_t + \mu_t\mathcal{T}l_t - \rho^*\| \\
&\leq (1-\mu_t)\|\mathcal{T}l_t - \rho^*\| + \mu_t\|\mathcal{T}l_t - \rho^*\| \\
&\leq (1-\mu_t\|l_t - \rho^*\| + \mu_t\|l_t - \rho^*\| \\
&\leq (1-\mu_t)\|l_t - \rho^*\| + \mu_t\|l_t - \rho^*\| \\
&\leq \|l_t - \rho^*\| \leq \|j_t - \rho^*\|.
\end{aligned}$$

Then,

$$\begin{aligned}
||j_{t+1} - \rho^*|| &= ||\mathcal{T}((1-\nu_t)k_t + \nu_t \mathcal{T} k_t) - \mathcal{T}\rho^*|| \\
&\leq ||(1-\nu_t)k_t + \nu_t \mathcal{T} k_t - \rho^*|| \\
&\leq ||(1-\nu_t)(k_t - \rho^*) + \nu_t(\mathcal{T} k_t - \rho^*)|| \\
&\leq (1-\nu_t)||(k_t - \rho^*)|| + \nu_t||(\mathcal{T} k_t - \rho^*)|| \\
&\leq (1-\nu_t)||(k_t - \rho^*)|| + \nu_t||(k_t - \rho^*)|| \\
&\leq ||k_t - \rho^*|| \\
&\leq ||j_t - \rho^*||.
\end{aligned}$$

As a result, the norm $|j_t - \rho|$ is bounded and exhibits non-increasing behavior with respect to all $\rho \in S$. □

Theorem 3.5. *Let $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ be as defined above, also fulfilling the axiom of condition (C). For randomly selected $j_0 \in \mathbb{A}$, j_t is generated by the suggested iterative process $\forall\ t\ n \geq 1$ where μ_t and ω_t are the same as in the above theorems. Then $S \neq \emptyset$ iff j_t is bounded and $lim_{t\to\infty}||\mathcal{T} j_t - j_t|| = 0$.*

Proof. Consider $\rho^* \in S$ by assuming $S \neq \phi$. Thus, by the aforementioned Lemma 3.4, $lim_{n\to\infty}||j_t - \rho^*||$ has a definied value and j_t is bounded.
Let $\quad lim_{n\to\infty}||j_t - \rho^*|| = j.$
Based on Lemma 3.4, it follows that

$$lim_{t\to\infty} \sup ||l_t - \rho^*|| \leq lim_{t\to\infty} \sup ||j_t - \rho^*|| = j.$$

$$lim_{t\to\infty} \sup ||\mathcal{T} j_t - \rho^*|| \leq lim_{t\to\infty} \sup ||j_t - \rho^*|| = j.$$

Also,

$$\begin{aligned}
||j_{t+1} - \rho^*|| &= ||\mathcal{T} k_t - \rho^*|| \\
||j_{t+1} - \rho^*|| &= ||\mathcal{T}((1-\nu_t)k_t + \nu_t \mathcal{T} k_t) - \mathcal{T}\rho^*|| \\
&\leq ||(1-\nu_t)k_t + \nu_t \mathcal{T} k_t - \rho^*|| \\
&\leq ||(1-\nu_t)(k_t - \rho^*) + \nu_t(\mathcal{T} k_t - \rho^*)|| \\
&\leq (1-\nu_t)||(k_t - \rho^*)|| + \nu_t||(\mathcal{T} k_t - \rho^*)|| \\
&\leq (1-\nu_t)||(k_t - \rho^*)|| + \nu_t||(k_t j_t - \rho^*)|| \\
&\leq ||k_t - \rho^*||.
\end{aligned}$$

Therefore,

$$r \leq lim_{t\to\in\mathcal{T}ty} \in \mathcal{T}||l_t - \rho^*||.$$

Thus,

$$r = lim_{t\to\in\mathcal{T}ty}||l_t - \rho^*||$$

$$= \quad lim_{t\to\in\mathcal{T}ty}\|(1-\omega_t)j_t + \omega_t\mathcal{T}j_t - \rho^*\|$$
$$= \quad lim_{t\to\in\mathcal{T}ty}\|(\omega_t)(\mathcal{T}j_t - \rho^*) + (1-\omega_t)(j_t - \rho^*)\|.$$

Thus,

$$lim_{t\to\in\mathcal{T}ty}\|\mathcal{T}j_t - j_t\| = 0.$$

Conversely, we assume j_t is bounded and $lim_{t\to\infty}\|\mathcal{T}j_t - j_t\| = 0$.
Let $\rho^* \in A(M, j_t)$.
Now, by Lemma 2.5,

$$\begin{aligned} j(\mathcal{T}\rho^*, \{j_t\}) = \limsup_{t\to\infty} \|j_t - \mathcal{T}\rho^*\| & \\ \leq & \limsup_{t\to\infty}(3\|\mathcal{T}j_t - j_t\| + \|j_t - \rho^*\|) \\ \leq & \limsup_{t\to\infty} \|j_t - \rho^*\| \\ = r & \quad (\rho^*, \{j_t\}). \end{aligned}$$

Hence, $\mathcal{T}\rho^* \in A(C, \{j_t\})$. As X is uniformly convex, $A(C, \{j_t\})$ is a singleton, Thus, $\mathcal{T}\rho^* = \rho^*$. Consequentially, $S \neq \emptyset$.

□

Theorem 3.6. *Consider the mapping $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ defined in Theorem 3.5, with μ_t and ω_t as specified. Assume that S is not empty and that $\mathbb{X}$ satisfies the Opial property. Also, $j_0 \in \mathbb{A}$, and j_t is as defined in equation 1.11. Then j_t is weakly convergent to $\rho^* \in S$.*

Proof. $\mathcal{T}$ must have a point in S. Also, j_t satisfies Theorem 3.5. As $\mathbb{X}$ is reflexive being uniformly covex, under Eberlin's theorem, by applying the appropriate arguments, it can be concluded that a subsequence j_{n_m} of j_t weakly converges to $\rho^1 \in \mathbb{X}$. Moreover, due to the closed and uniformly convex nature of $\mathbb{A}$, Mazur's theorem guarantees that ρ_1 lies within $\mathbb{A}$. Furthermore, Lemma 2.5 implies that $\rho_1^* \in S$. In the subsequent discussion, the focus will be on j_t being weakly convergent to ρ_1^*. If this statement is false, then $\exists$ a subsequence $\{j_{n_m}\}$ for j_t such that j_{n_m} is weakly convergent to $\rho_2^* \in \mathbb{A}$ and $\rho_2^* \neq \rho_1^*$. By Lemma 3.5, $\rho_2^* \in S$, since $\lim_{t\to\infty} \|j_t - \rho^*\|$ exists $\forall\, \rho^* \in S$. By the previous theorem and Opial's property, it can be concluded that

$$\begin{aligned} \lim_{t\to\infty} \|j_t - \rho_1^*\| &= \lim_{j\to\infty} \|r_{n_j} - \rho_1^*\| \\ &< \lim_{j\to\infty} \|r_{n_j} - \rho_2^*\| \\ &= \lim_{t\to\infty} \|j_t - \rho_2^*\| \\ &= \lim_{k\to\infty} \|r_{n_k} - \rho_2^*\| \\ &< \lim_{k\to\infty} \|r_{n_k} - \rho_1^*\| \\ &= \lim_{t\to\infty} \|j_t - \rho_1^*\|. \end{aligned}$$

This conclusion contradicts the premises established in this work.
So, $\rho_1^* = \rho_2^* \Rightarrow j_t$ is weakly convergent to $\rho^* \in S$. □

Theorem 3.7. *Consider the mapping $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ defined in Theorem 3.5, with μ_t and ω_t as specified. Assume that S is not empty and that $\mathbb{X}$ satisfies the Opial property. Also, $j_0 \in \mathbb{A}$, and j_t is as defined in equation 1.11. Then j_t is strongly convergent to $\rho^* \in S$.*

Proof. Since $\mathbb{A}$ is compact therefore there exist a subsequence j_{n_m} of j_t which is strongly convergent to $\rho^* \in S$. Lemma 2.5 implies that $\|j_{k_m} - \mathcal{T}\rho^*\| \leq 3\|\mathcal{T} j_{k_m} - j_{k_m}\| + \|j_{k_m} - \rho^*\|, \forall k \geq 1$. Letting $k \to \infty$ yields $\rho^* \in S$.
Based on Lemma 3.4, it can be concluded that as t tends to infinity, the limit of $|j_t - \rho|$ exists for all $\rho \in S$. Consequently, j_t strongly converges to ρ^*. □

4. Analytical and numerical comparison

This section provides both analytical and numerical comparisons between the proposed iteration process and the existing methods 1.1−1.10. It is shown that the iteration sequence obtained from the D^* iteration process trends towards the $\mathcal{T}$ fixed point as it converges, following a pattern similar to that of [8]. Instead of applying the condition $\sum_{t=0}^{\infty} \omega_t\mu_t = \infty$, the condition $\sum_{t=0}^{\infty} \omega_t = \infty$ is considered. The analytical comparison is presented first, supported by a theorem proving that the proposed method, as defined in equation 1.11, outperforms the existing methods 1.7−1.10. Below, a rigorous proof establishes the theoretical superiority of the proposed iteration process over these methods, ensuring a stronger convergence result.

Theorem 4.1. *Let $\mathcal{T} : \mathbb{A} \to \mathbb{A}$ be as defined above, with $\rho^* \in S$. The iteration processes given in 1.7, 1.8, 1.9, and 1.10 demonstrate a comparatively slower rate of convergence than the iteration process formulated in 1.11. The scheme defined in 1.11 attains a more rapid approach to the unique fixed point ρ^* of the contraction mapping S, establishing its superiority in terms of convergence efficiency, on the condition that the sequences $\{\mu_t\}_{t=0}^{\infty}$, $\{\omega_t\}_{t=0}^{\infty}$, and $\{\nu_t\}_{t=0}^{\infty} \in$ [0 1] hold for all axioms mentioned in 3.1.*

Proof. Using the iterative scheme in 1.11 along with the proof method from Theorem 3.1, the following expression is derived:

$$\|j_{t+1} - \rho^*\| \leq \gamma^{3(t+1)}\|(j_0 - \rho^*)\| \prod_{i=0}^{t} \Big\{1 - \mu_i(1-\gamma)\Big\}.$$

Furthermore, $\mu \leq \mu_t \ \forall \ t \in N$:

$$\|j_{t+1} - \rho^*\| \leq \gamma^{3(t+1)}\|(j_0 - \rho^*)\|\Big\{1 - \mu_i(1-\gamma)\Big\}^{t+1}. \tag{4.1}$$

Applying the iterative procedure in 1.7, the following expression is derived:

$$\begin{aligned}
\|l_t - \rho^*\| &= |((1-\omega_t)j_t + \omega_t\mathcal{T} j_t) - \mathcal{T}\rho^*\| \\
&\leq \|(1-\omega_t)j_t + \omega_t\mathcal{T} j_t - \rho^*\| \\
&\leq \|(1-\omega_t)(j_t - \rho^*) + \omega_t(\mathcal{T} j_t - \rho^*)\| \\
&\leq (1-\omega_t)\|(j_t - \rho^*)\| + \omega_t\|(\mathcal{T} j_t - \rho^*)\| \\
&\leq \Big\{(1-\omega_t)\|(j_t - \rho^*)\| + \gamma\omega_t\|(j_t - \rho^*)\|\Big\} \\
&\leq \gamma\Big\{1 - \omega_t(1-\gamma)\Big\}\|(j_t - \rho^*)\|.
\end{aligned}$$

Now,

$$\|k_t-\rho^*\| \le \|\mathcal{T}l_t-\rho^*\| \le \gamma\|l_t-\rho^*\| \le \gamma\Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\|.$$

Therefore,

$$\|j_{t+1}-\rho^*\| \le \|\mathcal{T}k_t-\rho^*\| \le \gamma\|k_t-\rho^*\| \le \gamma^2\Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\|.$$

After some repetition,

$$\begin{aligned}
\|j_t-\rho^*\| &\le \gamma^2\Big\{1-\omega_{t-1}(1-\gamma)\Big\}\|(j_{t-1}-\rho^*)\| \\
\|j_{t-1}-\rho^*\| &\le \gamma^2\Big\{1-\omega_{t-2}(1-\gamma)\Big\}\|(j_{t-2}-\rho^*)\| \\
\|j_{t-2}-\rho^*\| &\le \gamma^2\Big\{1-\omega_{t-3}(1-\gamma)\Big\}\|(j_{t-3}-\rho^*)\| \\
&\vdots \\
\|j_1-\rho^*\| &\le \gamma^2\Big\{1-\omega_0(1-\gamma)\Big\}\|(u_0-\rho^*)\|.
\end{aligned}$$

Thus,

$$\|j_{t+1}-\rho^*\| \le \gamma^{2(t+1)}\|(j_0-\rho^*)\| \prod_{i=0}^{t}\Big\{1-\omega_i(1-\gamma)\Big\}.$$

Now, since $\omega \le \omega_t\ \forall t, lt \in N$, it follows that

$$\|j_{t+1}-\rho^*\| \le \gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\omega\mu_i(1-\gamma)\Big\}^{t+1}. \tag{4.2}$$

Take $c_t = \gamma^{3(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{t+1}$ by 4.1 and $d_t = \gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{t+1}$ by 4.1. It follows that $lim_{t\to\infty}\frac{c_t}{d_t} = 0$, in view of the fact that $\frac{c_t}{d_t} = \frac{\|j_{t+1}-\rho^*\| \le \gamma^{3(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{n+1}}{\gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\omega\mu_i(1-\gamma)\Big\}^{t+1}}$ with $\gamma\frac{1-\mu_i(1-\gamma)}{1-\omega\mu_i(1-\gamma)} < 1$.

In conclusion, Definition 2.6 indicates that the iteration procedure 1.11 converges more efficiently than the iteration 1.7.

Applying the iterative procedure defined in 1.8, the following computation is made:

$$\begin{aligned}
\|l_t-\rho^*\| &= \|((1-\omega_t)j_t+\omega_t\mathcal{T}j_t)-\mathcal{T}\rho^*\| \\
&\le \|(1-\omega_t)j_t+\omega_t\mathcal{T}j_t-\rho^*\| \\
&\le \|(1-\omega_t)(j_t-\rho^*)+\omega_t(\mathcal{T}j_t-\rho^*)\| \\
&\le (1-\omega_t)\|(j_t-\rho^*)\|+\omega_t\|(\mathcal{T}j_t-\rho^*)\| \\
&\le (1-\omega_t)\|(j_t-\rho^*)\|+\gamma\omega_t\|(j_t-\rho^*)\| \\
&\le \Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\|.
\end{aligned}$$

$$\|k_t-\rho^*\| = \|\mathcal{T}((1-\mu_t)j_t+\mu_t\mathcal{T}l_t)-\mathcal{T}\rho^*\|$$

$$
\begin{aligned}
&\leq \gamma\|(1-\mu_t)j_t + \mu_t\mathcal{T}l_t - \rho^*\| \\
&\leq \gamma\|(1-\mu_t)(j_t-\rho^*) + \mu_t(\mathcal{T}l_t - \rho^*)\| \\
&\leq \gamma(1-\mu_t)\|(j_t-\rho^*)\| + \mu_t\|(\mathcal{T}l_t - \rho^*)\| \\
&\leq \gamma(1-\mu_t)\|(j_t-\rho^*)\| + \gamma\mu_t\|(l_t - \rho^*)\| \\
&\leq \gamma(1-\mu_t)\|(j_t-\rho^*)\| + \gamma\mu_t\Big\{1-\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\| \\
&\leq \gamma\Big\{(1-\mu_t) + \gamma\mu_t - \gamma\mu_t\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\| \\
&\leq \gamma\Big\{(1-(1-\gamma)\mu_t - \gamma\mu_t\omega_t(1-\gamma)\Big\}\|(j_t-\rho^*)\| \\
&\leq \gamma\Big\{(1-\mu_t(1-\gamma)(1-\gamma\omega_t)\Big\}\|(j_t-\rho^*)\|.
\end{aligned}
$$

Thus,

$$\|j_{t+1}-\rho^*\| \leq \|\mathcal{T}k_t - \rho^*\| \leq \gamma^2\Big\{(1-\mu_t(1-\gamma)(1-\gamma\omega_t)\Big\}\|(j_t-\rho^*)\|.$$

By repetition,

$$
\begin{aligned}
\|j_t-\rho^*\| &\leq \gamma^2\Big\{1-\mu_{t-1}(1-\gamma)(1-\gamma\omega_{t-1})\Big\}\|(j_{t-1}-\rho^*)\| \\
\|j_{t-1}-\rho^*\| &\leq \gamma^2\Big\{1-\mu_{t-2}(1-\gamma)(1-\gamma\omega_{t-2})\Big\}\|(j_{t-2}-\rho^*)\| \\
\|j_{t-2}-\rho^*\| &\leq \gamma^2\Big\{1-\mu_{t-3}(1-\gamma)(1-\gamma\omega_{t-3})\Big\}\|(j_{t-3}-\rho^*)\| \\
&\vdots \\
\|u_1-\rho^*\| &\leq \gamma^2\Big\{1-\mu_0(1-\gamma)(1-\gamma\omega_0)\Big\}\|(j_0-\rho^*)\|.
\end{aligned}
$$

Thus,

$$\|j_{t+1}-\rho^*\| \leq \gamma^{2(t+1)}\|(j_0-\rho^*)\| \prod_{i=0}^{t} \Big\{(1-\mu_i(1-\gamma)(1-\gamma\omega_i)\Big\}.$$

Now, since $\mu \leq \mu_t\omega \leq \omega_t \ \forall\ t \in N$:

$$\|j_{t+1}-\rho^*\| \leq \gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)(1-\gamma\omega_i)\Big\}^{t+1}. \tag{4.3}$$

Considering $d_t = \gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)(1-\gamma\omega_i)\Big\}^{t+1}$, it follows that $lim_{t\to\infty}\frac{c_t}{d_t} = 0$, in view of the fact that $\frac{c_t}{d_t} = \frac{\|j_{t+1}-\rho^*\|\leq\gamma^{3(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{n+1}}{\gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)(1-\gamma\omega_i)\Big\}^{t+1}}$ with $\gamma\frac{1-\mu_i(1-\gamma)}{1-\mu_i(1-\gamma)(1-\gamma\omega_i)} < 1$.

In conclusion, Definition 2.6 indicates that the iteration procedure 1.11 converges more efficiently than the iteration 1.8.

Applying the iterative procedure defined in 1.9, the following computation is made:

$$\|j_{t+1}-\rho^*\| \leq \gamma^{2(t+1)}\|(j_0-\rho^*)\| \prod_{i=0}^{t} \Big\{1-\mu_i(1-\gamma)\Big\}.$$

Since $\mu \leq \mu_t \forall\ t \in N$, it follows that

$$\|j_{t+1} - \rho^*\| \leq \gamma^{2(t+1)} \|(j_0 - \rho^*)\| \Big\{1 - \mu_i(1-\gamma)\Big\}^{t+1}.$$

Considering $d_t = \gamma^{2(t+1)}\|(j_0 - \rho^*)\|\Big\{1 - \mu_i(1-\gamma)\Big\}^{t+1}$, it follows that $lim_{t\to\infty}\frac{c_t}{d_t} = 0$, In view of the fact that $\frac{c_t}{d_t} = \frac{\|j_{t+1}-\rho^*\| \leq \gamma^{3(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{n+1}}{\gamma^{2(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{t+1}}$ with $\gamma\frac{1-\mu_i(1-\gamma)}{1-\mu_i(1-\gamma)} < 1$.
In conclusion, Definition 2.6 indicates that the iteration procedure 1.11 converges more efficiently than the iteration 1.9.
Applying the iterative procedure defined in 1.10, the following computation is made:

$$\|j_{t+1} - \rho^*\| \leq \gamma^{3(t+1)} \|(j_0 - \rho^*)\| \prod_{i=0}^{t} \Big\{1 - \omega\mu_i(1-k)\Big\}.$$

Since, $\omega \leq \omega_t\ \forall\ t \in N$, it follows that

$$\|j_{t+1} - \rho^*\| \leq \gamma^{3(t+1)} \|(j_0 - \rho^*)\| \Big\{1 - \omega\mu_i(1-\gamma)\Big\}^{t+1}.$$

Considering $d_t = \gamma^{3(t+1)}\|(j_0 - \rho^*)\|\Big\{1 - \omega\mu_i(1-\gamma)\Big\}^{t+1}$, it follows that $lim_{t\to\infty}\frac{c_t}{d_t} = 0$, in view of the fact that $\frac{c_t}{d_t} = \frac{\|j_{t+1}-\rho^*\| \leq \gamma^{3(t+1)}\|(j_0-\rho^*)\|\Big\{1-\mu_i(1-\gamma)\Big\}^{n+1}}{\gamma^{3(t+1)}\|(j_0-\rho^*)\|\Big\{1-\omega\mu_i(1-\gamma)\Big\}^{t+1}}$ with $\frac{1-\mu_i(1-\gamma)}{1-\omega\mu_i(1-\gamma)} < 1$.
In conclusion, Definition 2.6 indicates that the iteration procedure 1.11 converges more efficiently than the iteration 1.10. □

Numerical evidence is provided to demonstrate that the iteration process defined in 1.11 converges faster than the previously introduced methods defined in 1.1−1.6. A comparative analysis is conducted using a numerical example to evaluate the effectiveness of the proposed iteration process against these existing methods. The results are presented through tables and graphical representations, clearly showcasing the improved convergence rate and efficiency of the new iteration process.

Example 4.2. *Let $\mathcal{T} : R \to R$ be a self mapping defined by $\mathcal{T}(j) = \sqrt{4x+5}$. So, $\mathcal{T}$ is a contraction mapping. Let $\mu_t = \frac{n}{n+1}$, $\nu_t = \frac{n+1}{2n+1}$, and $\omega_t = \frac{2n+1}{3n+1}$. Table 1 displays the iterative values for $j_0 = 7.1$. Figure 1 graphically depicts the convergence of the suggested iterative algorithm. The efficiency of D^* as in equation 1.11 is clear.*

From Table 1, it is evident that the proposed method converges to the fixed point faster than the existing methods, demonstrating its superior efficiency.
To further illustrate the convergence behavior, Figure 1 and Figure 2 provide a visual comparison of the error bounds for the D^* method and the other iterative methods. The graph clearly shows that the D^* method converges to the fixed point more rapidly, with the error decreasing exponentially compared to the other methods.

Table 1. The Maan, Ishikawa, Noor, and D^* irritative methods are compared numerically.

I.N	Maan	Ishikawa	Noor	Picard	Thakur	SP	D^*
1	5.779273	5.779273	5.779273	5.779273	5.779273	5.302555	5.019008
2	5.461461	5.401009	5.383490	5.302555	5.234694	5.069169	5.000281
3	5.293363	5.217212	5.197551	5.119592	5.066449	5.015141	5.000004
4	5.192013	5.120142	5.103682	5.047610	5.018159	5.003172	5.000000
5	5.127686	5.067205	5.054966	5.019008	5.004846	5.000643	5.000000
6	5.085757	5.037856	5.029323	5.007597	5.001272	5.000127	5.000000
7	5.057988	5.021425	5.015710	5.003038	5.000330	5.000025	5.000000
8	5.039403	5.012167	5.008443	5.001215	5.000085	5.000005	5.000000
9	5.026874	5.006927	5.004548	5.000486	5.000022	5.000001	5.000000
10	5.018381	5.003952	5.002454	5.000194	5.000005	5.000000	5.000000
11	5.012601	5.002258	5.001326	5.000078	5.000001	5.000000	5.000000
12	5.008655	5.001292	5.000718	5.000031	5.000000	5.000000	5.000000
13	5.005954	5.000740	5.000389	5.000012	5.000000	5.000000	5.000000
14	5.004102	5.000424	5.000211	5.000005	5.000000	5.000000	5.000000
15	5.002828	5.000243	5.000114	5.000002	5.000000	5.000000	5.000000
16	5.001953	5.000140	5.000062	5.000001	5.000000	5.000000	5.000000
17	5.001349	5.000080	5.000034	5.000000	5.000000	5.000000	5.000000
18	5.000933	5.000046	5.000018	5.000000	5.000000	5.000000	5.000000
19	5.000645	5.000027	5.000010	5.000000	5.000000	5.000000	5.000000
20	5.000447	5.000015	5.000005	5.000000	5.000000	5.000000	5.000000

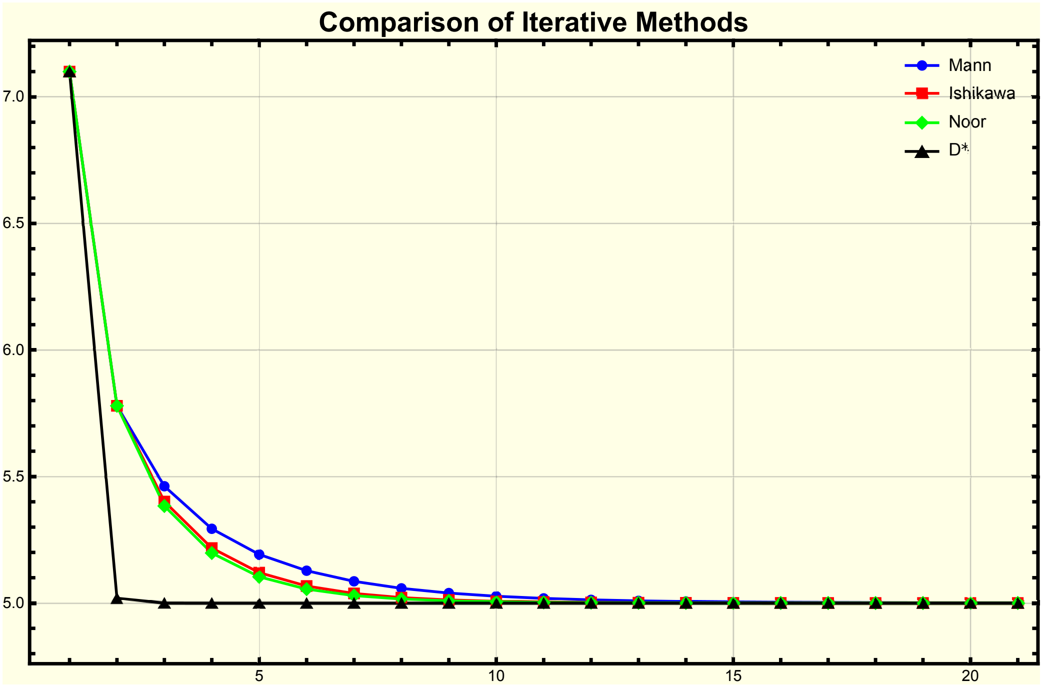

Figure 1. The Maan, Ishikawa, Noor, and D^* Iterative methods are compared graphically.

The graphical representations in Figures 1 and 2 highlight the superior convergence characteristics of the proposed D^* iteration method compared to existing iterative approaches. The results demonstrate that the D^* method exhibits a significantly faster convergence rate, minimizing the error more efficiently than traditional schemes such as Maan, Ishikawa, Noor, Picard, SP, and Thakur iterations.

The theoretical findings align with the numerical results, affirming that the D^* iteration process not only converges faster but also ensures a more stable approximation to the exact solution. This establishes the practical relevance of the proposed approach in solving complex mathematical models.

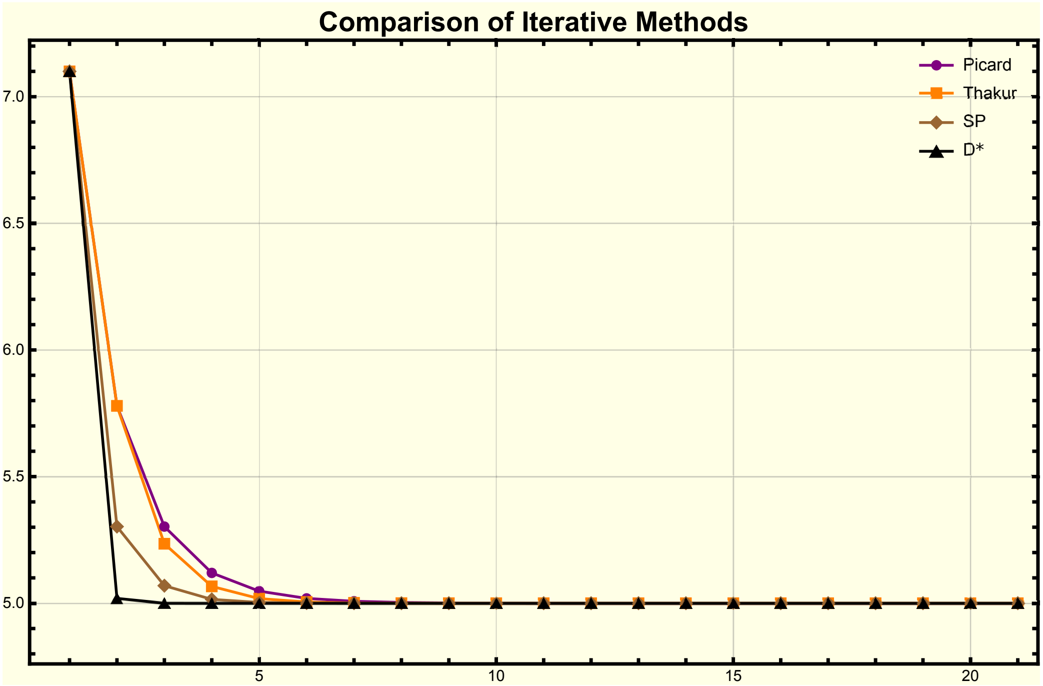

Figure 2. The Picard, Thakur, SP, and D^* Iterative methods are compared graphically.

The following Mathematica code implements the D^* iteration method for Example 4.2 and compares it with other standard iterative methods in Table 1, Figures 1 and 2:

```
(* Define the contraction mapping *)
T[j_] := Sqrt[4*j + 5];

(* Fixed point of the mapping *)
fixedPoint = 5;

(* Parameters for the iterative methods *)
mu[t_] := t/(t + 1);
nu[t_] := (t + 1)/(2*t + 1);
omega[t_] := (2*t + 1)/(3*t + 1);

(* Initial value *)
j0 = 7.1;
```

```
(* Define the D^{*} iteration method *)
DStarIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0},
  Do[l = T[(1 - omega[t])*j + omega[t]*T[j]];
   k = T[(1 - mu[t])*l + mu[t]*T[l]];
   j = T[(1 - nu[t])*k + nu[t]*T[k]];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
  Prepend[results, j0]]

(* Define other iteration methods *)
MannIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0},
  Do[j = (1 - nu[t])*j + nu[t]*T[j];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
  Prepend[results, j0]]

IshikawaIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0, k},
  Do[k = (1 - mu[t])*j + mu[t]*T[j];
   j = (1 - nu[t])*j + nu[t]*T[k];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
  Prepend[results, j0]]

NoorIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0, k, l},
  Do[l = (1 - omega[t])*j + omega[t]*T[j];
   k = (1 - mu[t])*j + mu[t]*T[l];
   j = (1 - nu[t])*j + nu[t]*T[k];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
  Prepend[results, j0]]

PicardIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0}, Do[j = T[j];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
  Prepend[results, j0]]

SPIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0, k, l},
  Do[l = (1 - omega[t])*j + omega[t]*T[j];
   k = (1 - mu[t])*l + mu[t]*T[l];
   j = (1 - nu[t])*k + nu[t]*T[k];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
```

```
  Prepend[results, j0]]

ThakurIteration[j0_, maxIter_] :=
 Module[{results = Table[0, {maxIter}], j = j0, k, l},
  Do[l = (1 - omega[t])*j + omega[t]*T[j];
   k = (1 - mu[t])*j + mu[t]*l;
   j = T[k];
   results[[t + 1]] = j;, {t, 0, maxIter - 1}];
  Prepend[results, j0]]

(* Generate results for 20 iterations *)
maxIterations = 20;
mannResults = MannIteration[j0, maxIterations];
ishikawaResults = IshikawaIteration[j0, maxIterations];
noorResults = NoorIteration[j0, maxIterations];
thakurResults = ThakurIteration[j0, maxIterations];
picardResults = PicardIteration[j0, maxIterations];
spResults = SPIteration[j0, maxIterations];
dstarResults = DStarIteration[j0, maxIterations];

(* Create Table 1 *)
table1 = TableForm[
  Table[{t, mannResults[[t + 1]], ishikawaResults[[t + 1]],
    noorResults[[t + 1]], picardResults[[t + 1]],
    thakurResults[[t + 1]], spResults[[t + 1]],
    dstarResults[[t + 1]]}, {t, 1, maxIterations}],
  TableHeadings -> {Range[maxIterations], {"I.N", "Mann", "Ishikawa",
     "Noor", "Picard", "Thakur", "SP", "D^{*}"}},
  TableAlignments -> Right]

(* Create Figure 1 - Comparison of Mann, Ishikawa, Noor, and D^{*} *)
figure1 =
 ListLinePlot[{mannResults, ishikawaResults, noorResults,
   dstarResults}, PlotRange -> All,
  PlotLabel -> Style["Comparison of Iterative Methods", 16, Bold],
  AxesLabel -> {Style["Iteration", 14], Style["Value", 14]},
  GridLines -> Automatic, PlotMarkers -> Automatic,
  Background -> Lighter[Yellow, 0.9],
  PlotStyle -> {Blue, Red, Green, Black},
  PlotLegends -> Placed[{"Mann", "Ishikawa", "Noor", "D^{*}"}, {Right, Top}],
  ImageSize -> 600, Frame -> True,
  FrameStyle -> Directive[Black, Thick]]
```

```
(* Create Figure 2 - Comparison of Thakur, SP, Picard, and D^{*} *)
figure2 =
 ListLinePlot[{picardResults, thakurResults, spResults, dstarResults},
   PlotRange -> All,
  PlotLabel -> Style["Comparison of Iterative Methods", 16, Bold],
  AxesLabel -> {Style["Iteration", 14], Style["Value", 14]},
  GridLines -> Automatic, PlotMarkers -> Automatic,
  Background -> Lighter[Yellow, 0.9],
  PlotStyle -> {Purple, Orange, Brown, Black},
  PlotLegends -> Placed[{"Picard", "Thakur", "SP", "D^{*}"}, {Right, Top}],
  ImageSize -> 600, Frame -> True,
  FrameStyle -> Directive[Black, Thick]]
```

5. Solving fractional differential equations using iterative algorithms

Fractional differential equations extend the framework of conventional differential equations to include fractional−order derivatives, providing a substantial structure aimed at simulating processes. This broadening makes fractional differential equations particularly relevant in numerous disciplines, like mechanics, computational biology, money management, and engineering. Differing from conventional derivatives, fractional derivatives reflect the comprehensive background regarding a procedure, allowing for more precise portrayals associated with complicated patterns. Fractional differential equations are utilized in numerous domains, including biological systems, fluid dynamics, image processing, and finance. This research focused on abstract considerations, including their presence, singularity, and robustness. About results for effective solving, see [30–32]. This study applies the iterative procedure stated as in 1.5 to demonstrate the presence of a unique solution to the ensuing Caputo-type nonlinear fractional differential equations with boundary conditions:

$$\begin{cases} {}^cD^\alpha v(t) + g(t, v(t)) = 0, & 1 < \alpha < 2, \\ v(0) = v(1) = 0, & 0 \le t \le 1, \end{cases} \tag{5.1}$$

where $(g : [0,1] \times \mathbb{R} \to \mathbb{R})$ is a continuous function, while ${}^cD^\alpha$ r signifies the fractional derivative of order α in the sense of Caputo.
The analysis takes place in the Banach space $\mathbb{B} = \mathbb{Y}[0,1]$, the space of continuous functions on $[0,1]$ under the supremum norm. Within this functional space, the following integral equation is examined:

$$v(t) = \int_0^1 h(s,t) g(t, v(t))\, ds, \tag{5.2}$$

where $h : [0,1] \times [0,1] \to \mathbb{R}$ is Green's function given by

$$h(s,t) = \begin{cases} \frac{s(1-t)^{\alpha-1} - (s-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \\ \frac{s(1-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1. \end{cases}$$

The following result ensures that the iterative method in 1.5 has a solution to the given problem.

Theorem 5.1. *Let $\mathbb{B} = \mathbb{Y}[0,1]$ be a Banach space, and consider the mapping $\mathfrak{S} : \mathbb{B} \to \mathbb{B}$ defined by*

$$\mathfrak{S}v(t) = \int_0^1 h(s,t)g(t,v(t))\,ds,$$

for all $v(t) \in \mathbb{B}$, and $t \in [0,1]$. Assume that the continuous function $g : [0,1] \times [0,1] \to \mathbb{R}$ satisfies

$$|g(t,v_1) - g(t,v_2)| \leq \beta|v_1 - v_2|,$$

$\forall\ v_1, v_2 \in \mathbb{B}$, $t \in [0,1]$, where $0 < \beta < 1$. Then the Caputo-type nonlinear fractional differential equation 5.1 has a unique solution.

Proof. A function $v \in B$ satisfies the Caputo-type nonlinear fractional differential equation 5.1 if and only if it is a solution of the integral equation 5.2. To demonstrate this, let $v_1, v_2 \in B$ for $t \in [0,1]$. It follows that

$$|\mathfrak{S}v_1(t) - \mathfrak{S}v_2(t)| = \left|\int_0^1 h(s,t)g(t,v_1(t))\,ds - \int_0^1 h(s,t)g(t,v_2(t))\,ds\right|.$$

This can be simplified to

$$|\mathfrak{S}v_1(t) - \mathfrak{S}v_2(t)| \leq \int_0^1 h(s,t)|g(t,v_1(t)) - g(t,v_2(t))|\,ds.$$

Applying the Lipschitz condition of g, it follows that

$$|\mathfrak{S}v_1(t) - \mathfrak{S}v_2(t)| \leq \int_0^1 h(s,t)\beta|v_1(t) - v_2(t)|\,ds.$$

Utilizing the supremum norm in $\mathbb{B} = \mathbb{Y}[0,1]$, this leads to

$$|\mathfrak{S}v_1(t) - \mathfrak{S}v_2(t)| \leq \beta\|v_1 - v_2\|.$$

In the Banach space $\mathbb{Y}[0,1]$, where $\mathbb{Y}[0,1]$ is a convex closed subset of itself, $\mathfrak{S}$ becomes a contractive mapping for $\beta \in (0,1)$. Therefore, by Theorem 3.1, the iteration sequence $\{j_p\}$ defined in 1.5 converges to a unique fixed point of $\mathfrak{S}$, implying that $\{j_p\}$ converges to the unique solution of the problem in 5.1. This proves the existence and uniqueness of the solution to the nonlinear fractional differential equation 5.1 of the Caputo type. □

The following example illustrates the validity of the preceding result.

Example 5.2. *Consider the fractional differential equation:*

$$\begin{cases} {}^cD^\gamma j(y) + y^3 = 0, \quad 0 \leq y \leq 1, \quad \gamma = 1.8, \\ j(0) = j(1) = 0. \end{cases} \tag{5.3}$$

The exact solution of this problem is given by:

$$j(y) = \frac{1}{\Gamma(\gamma)}\int_0^y \left[y(1-z)^{\gamma-1} - (y-z)^{\gamma-1}\right] z^3\,dz + \frac{y}{\Gamma(\gamma)}\int_y^1 (1-z)^{\gamma-1} z^3\,dz. \tag{5.4}$$

Define the operator $\mathfrak{S} : \mathbb{A}[0,1] \to \mathbb{A}[0,1]$ *as:*

$$\mathfrak{S} : j(y) = \frac{1}{\Gamma(\gamma)} \int_0^y \left[y(1-z)^{\gamma-1} - (y-z)^{\gamma-1} \right] z^3 \, dz + \frac{y}{\Gamma(\gamma)} \int_y^1 (1-z)^{\gamma-1} z^3 \, dz. \tag{5.5}$$

Alternatively, in terms of Green's function $g(y,z)$*:*

$$S\,u(y) = \int_0^1 g(y,z) z^3 \, dz. \tag{5.6}$$

For the fractional differential equation 5.3, Green's function $g(y,z)$ *is derived from the boundary value problem's structure. Given the Caputo fractional derivative* ${}^cD^\gamma$ *and boundary conditions* $j(0) = j(1) = 0$*, the Green's function satisfies:*

$$g(y,z) = \begin{cases} \frac{y(1-z)^{\gamma-1} - (y-z)^{\gamma-1}}{\Gamma(\gamma)}, & 0 \le z \le y \le 1, \\ \frac{y(1-z)^{\gamma-1}}{\Gamma(\gamma)}, & 0 \le y \le z \le 1. \end{cases}$$

This choice ensures that the integral representation:

$$j(y) = \int_0^1 g(y,z) z^3 \, dz$$

solves differential equation 5.3 by construction, as $g(y,z)$ *encodes the non-local effects of the fractional derivative and homogeneous boundary conditions. Using the initial guess* $j_0(y) = y^2(1-y)$*, it satisfies* $j_0(0) = j_0(1) = 0$*, matching the boundary conditions. Its quadratic form provides a smooth, computationally tractable starting point for iteration. For problems of the type in equation 5.3, the form* $y^2(1-y)$ *reflects physical systems where the solution vanishes at the boundaries and peaks in the interior. The control sequences are:*

$$\alpha_t = 0.9, \quad \beta_t = 0.8, \quad \gamma_t = 0.6, \quad n = 0, 1, 2, \ldots, \tag{5.7}$$

and the iterative sequence $\{j_t\}$ *generated via the contraction mapping converges to the exact solution of the problem.*

The results in Table 2 demonstrate the remarkable efficiency of the D^* iteration method in approximating the solution of the fractional differential equation 5.3. Even with a small number of iterations (e.g., $t = 3$), the approximate solutions $j_3(y)$ align perfectly with the exact solution $j(y)$, achieving machine-level precision by $t = 50$. This rapid convergence underscores the superiority of the D^* method over classical approaches, as it attains high accuracy with minimal computational steps. Such efficiency is particularly valuable for solving complex fractional models in applied sciences, where both speed and precision are critical.

t	$j(t)$	j_3	j_7	j_{19}	j_{50}
0.0	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.1	0.007006455648	0.007006455648	0.007006455648	0.007006455648	0.007006455648
0.2	0.013984193194	0.013984193194	0.013984193194	0.013984193194	0.013984193194
0.3	0.020806053535	0.020806053535	0.020806053535	0.020806053535	0.020806053535
0.4	0.027168368481	0.027168368481	0.027168368481	0.027168368481	0.027168368481
0.5	0.032522337660	0.032522337660	0.032522337660	0.032522337660	0.032522337660
0.6	0.036010176112	0.036010176112	0.036010176112	0.036010176112	0.036010176112
0.7	0.036404499420	0.036404499420	0.036404499420	0.036404499420	0.036404499420
0.8	0.032050143668	0.032050143668	0.032050143668	0.032050143668	0.032050143668
0.9	0.020807924580	0.020807924580	0.020807924580	0.020807924580	0.020807924580
1.0	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000

Table 2. Comparison between exact and approximate solutions.

The following Mathematica code implements the D^* iteration method for Example 5.2 and compares it with the exact solution in Table 2:

```
(* Define the parameters and functions *)
(* Gamma parameter *)
gamma = 1.8;

(* Green's function *)
g[y_, z_] :=
  If[z <= y, (y*(1 - z)^(gamma - 1) - (y - z)^(gamma - 1))/
    Gamma[gamma], (y*(1 - z)^(gamma - 1))/Gamma[gamma]];

(* Exact solution *)
uExact[y_] := (1/
     Gamma[gamma])*(Integrate[(y*(1 - z)^(gamma - 1) - (y -
            z)^(gamma - 1))*z^3, {z, 0, y}] +
     Integrate[y*(1 - z)^(gamma - 1)*z^3, {z, y, 1}]);

(* Operator T *)
T[u_, y_] := Integrate[g[y, z]*z^3, {z, 0, 1}];

(* Initial guess *)
j0[y_] := y^2*(1 - y);

(* Control sequences *)
?[n_] := 0.9;  (* Similar to alpha *)
[n_] := 0.8;   (* Similar to beta *)
?[n_] := 0.6;   (* Similar to gammaSeq *)
```

```
(* New three-step iterative method *)
(* Base case *)
j[0, y_] := j0[y];

(* Iterative step *)
j[t_, y_] :=
  j[t, y] =
   Module[{lt, kt},
    (* First step *)
    lt = T[(1 - ?[t])*j[t - 1, #] + ?[t]*T[j[t - 1, #]] &, y];
    (* Second step *)
    kt = T[(1 - [t])*lt + [t]*T[lt] &, y];
    (* Third step *)
    T[(1 - ?[t])*kt + ?[t]*T[kt] &, y]];

(* Generate the table for comparison *)
yValues = Range[0, 1, 0.1];
table = TableForm[
   Table[{y, uExact[y], j[3, y], j[7, y], j[19, y], j[50, y]}, {y,
     yValues}],
   TableHeadings -> {None, {"y", "u(y)", "j3", "j7", "j19", "j50"}}];

(* Display the table *)
table
```

6. Conclusions

Fixed-point theory is a fundamental area of mathematics with extensive applications across disciplines such as physics, engineering, economics, and computer science. It provides a powerful framework for analyzing the existence and properties of solutions to various mathematical equations and complex problems. Among these, fractional 1-Fredholm integro-differential equations hold particular significance. These equations incorporate fractional derivatives and integrals alongside a Fredholm integral operator, making them highly relevant in mathematical modeling, particularly in scenarios involving memory effects and long-range interactions. In this study, the D^* iteration method is introduced as an effective tool for approximating fixed points within the domain of Suzuki generalized nonexpansive mappings and nearly contractive mappings. A thorough investigation is conducted on the convergence properties of the D^* iteration scheme when applied to the fixed points of Suzuki generalized nonexpansive mappings in uniformly convex Banach spaces. To demonstrate the efficacy of the proposed iterative method, comprehensive analytical, numerical, and graphical evaluations are provided. Additionally, the stability aspects of this novel iteration process are explored in depth. As a practical application of the key theoretical insights, this approach is employed to approximate solutions for a fractional differential equation, further emphasizing its utility. The findings presented in this work refine and extend several existing results, contributing greater depth and precision to the field. Future research may focus on developing a more generalized iterative approach that surpasses the proposed method in terms of efficiency and applicability in fixed−point approximation.

Conflict of interest

The authors declare no conflict of interest in this paper.

Use of Generative-AI tools declaration

During the preparation of this work, the authors did not used any tool or service, and they take full responsibility for the content of the publication.

Authors contributions

Danish Ali conceived the main idea, formulated the problem, developed the methodology, performed the analysis, conducted the numerical experiments, and wrote the manuscript. Aftab Hussain supervised the research, provided guidance throughout the study, and critically reviewed the manuscript. Amer Hassan Albargi contributed through valuable discussions and supported the final revision of the manuscript. All authors read and approved the final version.

Acknowledgements

The authors give thanks to their universities.

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