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Research article

Hardy-type spaces and Hardy-type inequalities

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Abstract: In the present paper, we defined and then studied Hardy spaces related to spherical mean operators. We proved Hardy-type inequalities, then we showed refined Sobolev-type inequalities between homogeneous Riesz-type spaces, homogeneous Besov-type spaces, and Lorentz-type spaces. Next, we introduced and studied Hausdorff operators on generalized Hardy spaces. Finally, we investigated maximal Bochner-Riesz operators on generalized Hardy spaces.

Keywords: Hardy-type inequalities; Hausdorff operators; maximal Bochner-Riesz operators
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1. Introduction

Let $u(z) = \sum_{n=0}^{\infty} a_n z_n$ be an analytic function in the Hardy space $H^1(D)$ consisting of analytic functions u on the unit disc D such that

$$\|u\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |u(re^{i\theta})| d\theta < \infty. \tag{1.1}$$

Then the coefficients satisfy the well-known Hardy inequality [1, p. 48],

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq C \, \|u\|_{H^1}. \tag{1.2}$$

Hardy's inequality is one of the important inequalities in analysis, and comparable inequalities in the setting of eigenfunction expansions have been studied by many authors. For eigenfunction expansions related to the Laplace-Beltrami operator on compact Riemannian manifolds, a Hardy-type inequality was proved in [2], and different Hardy-type inequalities for the Hermite and Laguerre expansions were studied in [3–9].

For the usual Fourier transform, Hardy's inequality states [10, p. 128],

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|\, |\xi|^{p-2} d\xi \leq C\, \|f\|^p_{H^p(\mathbb{R})}, \tag{1.3}$$

where $H^p(\mathbb{R})$, $0 < p \leq 1$, denotes the real Hardy space consisting of the boundary values of real parts of functions in the Hardy space on the unit disc in the plane. Then the study of holomorphic functions of a single variable on the upper half-plane is the foundation of the theory of classical real Hardy spaces on $\mathbb{R}^N$. We direct the reader to the original publications [11–13] for characterizations of the classical Hardy spaces $H^p(\mathbb{R}^N)$, which have been then extended to spaces of homogeneous type (see, e.g., [14, 15]). Additional details may be found in [10] and its references.

Next we define the one-dimensional Hausdorff operator by

$$\mathcal{H}_\phi u(\xi) = \int_0^\infty \frac{\phi(x)}{x} u(\xi x^{-1}) dx, \tag{1.4}$$

where $\phi \in L^1_{\text{loc}(\mathbb{R})}$. The boundedness of these operators has been studied in [16–18]. The interested reader is directed to [19, 20] for a thorough history and the latest advancements of Hausdorff operators. Several authors have spared their studies for extending the one-dimensional Hausdorff operators to multidimensional spaces. In this regard, some contributions are [21–23].

Since the Fourier transform of an integrable function is not always integrable, integrals that are not completely convergent may be included in the inversion formula for this transform. To overcome this challenge, appropriate summability techniques are presented. Specifically, the following Fourier integral defines the Bochner-Riesz means [24] of order β of functions in $\mathbb{R}^N$,

$$S^\beta_R(u)(x) = \int_{\|\xi\|<R} \left(1 - \frac{\|\xi\|}{R}\right)^\beta \widehat{u}(\xi) e^{i\langle x,\xi\rangle} d\xi, \tag{1.5}$$

where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is the usual norm on $\mathbb{R}^N$. Such operators have been discussed by several authors, see, for instance, [25–27] and the references therein.

Given the notable advancements of harmonic analysis related to spherical mean operators and the fact that the theory of Hardy spaces and Hardy inequalities has not yet been thoroughly studied, our objective in this paper is to advance these topics in the context of the spherical mean and to provide some applications.

There are several significant practical uses for the spherical mean operator, including acoustics and image processing of synthetic aperture radar data [28, 29]. Other applications of spherical means to the theory of partial differential equations were covered in John's seminal book [30].

The behavior of the spherical mean operator and its Fourier transform have been the subject of many authors' recent investigations, including uncertainty principles [31–33], Stockwell theory [34, 35], Weyl-type theorems [36], Gabor theory [37], localization operators [38], Fock spaces [39], wavelet multipliers [40], Littlewood-Paley g-functions [41], wavelet theory [42, 43], and so on.

To be more precise, let us fix some notation: We define the set

$$\Gamma = \mathbb{R}^{N+1} \cup \Big\{(i\rho, \varrho) \ : \ (\rho, \varrho) \in \mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R}^N, \quad |\rho| \leq \|\varrho\|\Big\},$$

and Γ_+ the subset of Γ by

$$\Gamma_+ = \mathbb{R}^{N+1} \cup \Big\{(i\rho, \varrho) \;:\; (\rho, \varrho) \in \mathbb{R}^{N+1}, \quad 0 \le \rho \le \|\varrho\|\Big\}.$$

We denote by $L^p(d\mu)$, $1 \le p \le \infty$, the space of measurable functions on $\Omega = \Big\{(\alpha, \xi) \in \mathbb{R}^{N+1} \;:\; \alpha \ge 0\Big\}$, satisfying,

$$\|u\|_{L^p(d\mu)} = \left(\int_\Omega |u(\alpha, \xi)|^p d\mu(\alpha, \xi)\right)^{1/p} < \infty, \qquad p \ge 1, \tag{1.6}$$

and for $p = \infty$,

$$\|u\|_{L^\infty(d\mu)} = \text{ess} \sup_{(\alpha,\xi)\in\Omega} |u(\alpha, \xi)| < \infty, \tag{1.7}$$

where $d\mu(\alpha, \xi)$ is the weight measure defined on Ω by

$$d\mu(\alpha, \xi) = k_N \alpha^N d\alpha \otimes d\xi, \tag{1.8}$$

with

$$k_N = \frac{1}{2^{(N-1)/2}\Gamma((N+1)/2)(2\pi)^{N/2}}. \tag{1.9}$$

Let $\mathcal{B}_{\Gamma_+}$ be the σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+} = \Big\{\theta^{-1}(B) : \quad B \in \mathcal{B}_{\text{Bor}}(\Omega)\Big\}, \tag{1.10}$$

where θ is defined on the set Γ_+ by

$$\theta(\rho, \varrho) = \left(\sqrt{\rho^2 + \|\varrho\|^2}, \varrho\right). \tag{1.11}$$

The Fourier transform associated to the spherical mean operator is defined on $L^1(d\mu)$ by

$$\mathcal{F}(u)(\rho, \varrho) = \int_\Omega u(\alpha, \xi)\varphi_{\rho,\varrho}(\alpha, \xi)d\mu(\alpha, \xi), \qquad (\rho, \varrho) \in \Gamma, \tag{1.12}$$

where $\varphi_{\rho,\varrho}$ is the spherical mean kernel defined by

$$\varphi_{\rho,\varrho}(\alpha, \xi) = j_{\frac{N-1}{2}}\left(\alpha\sqrt{\rho^2 + \varrho^2}\right)e^{-i\langle\varrho,\xi\rangle}, \qquad (\alpha, \xi) \in \Omega. \tag{1.13}$$

In Section 3, we introduce the Hardy-type spaces $H^p(d\mu)$ in the spherical mean setting and then prove the following results:

Theorem 1. *If $f \in H^p(d\mu)$, $0 < p \le 1$, then f belongs to $L^p(d\mu)$, such that*

$$\|f\|_{L^p(d\mu)} \le C\|f\|_{H^p(d\mu)}. \tag{1.14}$$

Moreover, there exists a positive constant C, such that

$$\forall\, (\rho, \varrho) \in \Gamma_+, \qquad \left|\mathcal{F}(f)(\rho, \varrho)\right| \le C\left\|(\rho, \varrho)\right\|^{(2N+1)(1/p-1)}\left\|f\right\|_{H^p(d\mu)}. \tag{1.15}$$

Consequently, for all $s > 0$,

$$\gamma\left(\Big\{(\rho, \varrho) \in \Gamma_+ : \quad \|(\rho, \varrho)\|^{(2N+1)(1-\frac{2}{p})}\, |\mathcal{F}(f)(\rho, \varrho)| \ge s\Big\}\right) \le C\, s^{-p}\left\|f\right\|^p_{H^p(d\mu)}, \tag{1.16}$$

where $d\gamma$ is the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall\, A \subset \mathcal{B}_{\Gamma_+}, \qquad \gamma(A) = \mu(\theta(A)). \tag{1.17}$$

Section 4 is devoted to prove some Hardy-type inequalities in the spherical mean setting. To do so, we define, for $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the homogeneous spherical mean-Riesz potential spaces $\dot{\mathcal{H}}^s_p(d\mu)$, the generalized homogeneous Besov spaces $\dot{\mathcal{B}}^s_{p,q}(d\mu)$, and the generalized Lorentz spaces $L^{p,q}(d\mu)$. Then we obtain these Hardy-type inequalities:

Theorem 2.

1. *If* $\frac{2N+1}{4} < s < \frac{2N+1}{2}$, *then there is a positive constant C such that, for every* $f \in \dot{\mathcal{H}}^s_2(d\mu)$,

$$\int_\Omega \frac{|f(\alpha,\xi)|^2}{\|(\alpha,\xi)\|^{2s}} d\mu(\alpha,\xi) \le C \left\|f\right\|^2_{\dot{\mathcal{H}}^s_2(d\mu)}. \tag{1.18}$$

2. *If* $0 < s < \frac{2N+1}{q}$, $1 \le q < \infty$, *then*

$$\|f\|_{L^{p,q}(d\mu)} \le C \left\|f\right\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-(2N+1)/q}_{\infty,q}} \left\|f\right\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^s_{q,q}}, \tag{1.19}$$

where $p = \dfrac{(2N+1)q}{2N+1-qs}$.

3. *If* $s \in \left(0, \frac{2N+1}{q}\right)$, $q \in [1, \infty)$, *then there exists a positive constant C such that, for every* $f \in \dot{\mathcal{B}}^s_{q,q}(d\mu)$,

$$\left(\int_\Omega \frac{|f(\alpha,\xi)|^q}{\|(\alpha,\xi)\|^{sq}}\, d\mu(\alpha,\xi)\right)^{\frac{1}{q}} \le C \left\|f\right\|^{\theta}_{\dot{\mathcal{B}}^s_{q,q}} \left\|f\right\|^{1-\theta}_{\dot{\mathcal{B}}^{s-\frac{2N+1}{q}}_{\infty,q}}, \tag{1.20}$$

where $\theta = 1 - \frac{qs}{2N+1}$.

4. *For all* $f \in H^p(d\mu)$, $p \in (0, 1]$, *we have*

$$\int_{\Gamma_+} \frac{|\mathcal{F}(f)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) \le C \left\|f\right\|^p_{H^p(d\mu)}, \tag{1.21}$$

provided that

$$(2N+1)(2-p) \le s < 2N+1 + p(D(N,p)+1), \qquad D(N,p) = \left\lfloor \frac{(2N+1)}{p} - 1 \right\rfloor. \tag{1.22}$$

In Section 5, we introduce the multivariate generalized Hausdorff operator $\mathcal{H}_N = \mathcal{H}_N(\Phi; \mathcal{M})$,

$$\mathcal{H}_N f(\alpha,\xi) := \int_\Omega \Phi(s,t) f\left(\mathcal{M}(s,t)(\alpha,\xi)\right) d\mu(s,t), \tag{1.23}$$

where Φ is a Borel measurable function and $\mathcal{M} = \mathrm{diag}(a_1, \ldots, a_{N+1})$ is a nonsingular diagonal matrix. Let

$$\eta\left(\mathcal{M}(s,t)\right) = |a_1(s,t)|^N \prod_{j=2}^{N+1} |a_j(s,t)|. \tag{1.24}$$

We show that the adjoint of $\mathcal{H}_N = \mathcal{H}_N(\Phi; \mathcal{M})$ is given by

$$\mathcal{H}_N^* := \mathcal{H}_N\left(\psi(s,t); \mathcal{M}^{-1}\right) \tag{1.25}$$

where $\psi(s,t) = \Phi(s,t)\eta\left(\mathcal{M}^{-1}(s,t)\right)\left|\det(\mathcal{M}^{-1}(s,t)\right|$. Moreover we introduce another version of generalized Hardy spaces via the generalized transforms as

$$H^1_{\text{Riesz}} := \left\{f \in L^1(d\mu) \ : \ \|\mathbf{R}_j f\|_{L^1(d\mu)} < \infty, \qquad j = 1, \dots, N+1\right\}, \tag{1.26}$$

where $\mathbf{R}_j$ are generalized Riesz transforms associated with the spherical mean operator defined by

$$\mathcal{F}(\mathbf{R}_j f)(\rho, \varrho) = -i\frac{(\rho,\varrho)_j}{\|(\rho,\varrho)\|}\mathcal{F}(f)(\rho,\varrho), \quad j = 1, \dots, N+1. \tag{1.27}$$

We provide this space with the norm

$$\|f\|_{H^1_{\text{Riesz}}} := \|f\|_{L^1(d\mu)} + \sum_{j=1}^{N+1} \|\mathbf{R}_j f\|_{L^1(d\mu)}. \tag{1.28}$$

Then we study the boundedness of $\mathcal{H}_N$ on the spaces $L^p(d\mu)$, $1 \le p \le \infty$, and on H^1_{Riesz}. More precisely we prove the following results:

Theorem 3. *Let $f \in L^p(d\mu)$. If for some $1 \le p \le \infty$,*

$$\int_\Omega |\Phi(s,t)|\left(\eta\left(\mathcal{M}^{-1}(s,t)\right)\left|\det\left((\mathcal{M}^{-1}(s,t)\right)\right|\right)^{1/p} d\mu(s,t) := K(p,N,\mathcal{H}_N) < \infty, \tag{1.29}$$

then

$$\|\mathcal{H}_N f\|_{L^p(d\mu)} \le K(p,N,\mathcal{H}_N)\|f\|_{L^p(d\mu)}. \tag{1.30}$$

Moreover, if the condition (1.29) *is satisfied for some $1 \le p \le \infty$, then the operator $\mathcal{H}_N^*$ is bounded on the conjugate space $L^{p'}(d\mu)$, such that*

$$\|\mathcal{H}_N^* f\|_{L^{p'}(d\mu)} \le K(p,N,\mathcal{H}_N)\|f\|_{L^{p'}(d\mu)}. \tag{1.31}$$

Furthermore:

1. *If* (1.29) *is valid for $p = 1$, then*

$$\mathcal{F}\left(\mathcal{H}_N f\right) = \mathcal{H}_N^*\left(\mathcal{F}(f)\right) \quad \text{and} \quad \mathcal{H}_N(\Phi)(\mathbf{R}_j f) = \mathbf{R}_j(\mathcal{H}_N(\widetilde{\Phi})f), \tag{1.32}$$

   where $\widetilde{\Phi}(s,t) = \operatorname{sgn}(a(s,t))\,\Phi(s,t)$, for $(s,t) \in \Omega$.

2. *If* (1.29) *is valid for $p = \infty$, then*

$$\mathcal{F}\left(\mathcal{H}_N^* f\right) = \mathcal{H}_N\left(\mathcal{F}(f)\right) \quad \text{and} \quad \mathcal{H}_N^*(\mathbf{R}_j f) = \mathbf{R}_j(\mathcal{H}(\widetilde{\Phi})_N^* f). \tag{1.33}$$

3. *The Hausdorff operator $\mathcal{H}_N$ is bounded on the Hardy-type space H^1_{Riesz} provided that*

$$\|\Phi\|_{L^2_{N,\mathcal{M}^{-1}}} := \int_\Omega \|\Phi(\alpha,\xi)\|\,|a(\alpha,\xi)|^{-2N-1} d\mu(\alpha,\xi) < \infty, \tag{1.34}$$

   and there exists a positive constant C, such that

$$\|\mathcal{H}_N f\|_{H^1_{\text{Riesz}}} \le C\,\|\Phi\|_{L^2_{N,\mathcal{M}^{-1}}}\,\|f\|_{H^1_{\text{Riesz}}}\,. \tag{1.35}$$

In Section 6, we introduce the Bochner-Riesz mean operator $\sigma^z_{N,t}$, $t > 0$, $z > \frac{N}{2}$, on $\mathcal{S}_*(\mathbb{R}^{N+1})$ by

$$\sigma^z_{N,t}(u)(\alpha,\xi) := \int_{\|(\rho,\varrho)\|<t} \left(1 - \frac{\|(\rho,\varrho)\|^2}{t^2}\right)^z \left(1 - \frac{\|\varrho\|^2}{t^2}\right)^z \overline{\varphi_{\rho,\varrho}(\alpha,\xi)}\mathcal{F}(u)(\rho,\varrho)d\gamma(\rho,\varrho), \tag{1.36}$$

where $\mathcal{S}_*(\mathbb{R}^{N+1})$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{N+1}$, even with respect to the first variable. Then the maximal operators ς^z_N, $z > \frac{N}{2}$, associated with the Bochner-Riesz mean operators are defined by

$$\varsigma^z_N(u) := \sup_{t>0} \left|\sigma^z_{N,t}(u)\right|. \tag{1.37}$$

The aim of this section is to study the boundedness of $\sigma^z_{N,t}$ and ς^z_N on the spaces $L^p(d\mu)$ and $H^p(d\mu)$. Precisely we prove:

Theorem 4.

1. *The operator $\sigma^z_{N,t}$ is bounded from $L^p(d\mu)$, $1 \le p \le \infty$, onto itself.*
2. *The operator $\sigma^z_{N,t}$ is extended to a bounded operator from $H^p(d\mu)$, $0 < p \le 1$, onto $\mathcal{S}'_*(\mathbb{R}^{N+1})$.*
3. *For $\frac{2N+1}{N+2z+\frac{3}{2}} < p \le 1$, the maximal Bochner-Riesz operator ς^z_N is bounded from $H^p(d\mu)$ onto $L^p(d\mu)$, provided that $\frac{N}{2} < z < \frac{N}{2} + \frac{1}{4}$.*

2. Preliminaries

In this section, we will recall all the properties of the spherical mean operator necessary for this article. For further information, we refer the reader to [36, 44, 45].

Definition 1. *The spherical mean operator [36] is defined on $C_*(\mathbb{R}^{N+1})$ by*

$$\forall(\alpha,\xi) \in \Omega, \qquad \mathcal{R}(u)(\alpha,\xi) = \int_{\mathbb{S}^N} u(\alpha\rho, \xi + \alpha\varrho)d\sigma(\rho,\varrho), \tag{2.1}$$

where $d\sigma$ is the normalized surface measure on the unit sphere

$$\mathbb{S}^N = \left\{(\rho,\varrho) \in \mathbb{R}^{N+1} = \mathbb{R}\times\mathbb{R}^N \ : \ \rho^2 + \|\varrho\|^2 = 1\right\}. \tag{2.2}$$

Here $C_*(\mathbb{R}^{N+1})$ is the space of continuous functions on $\mathbb{R}^{N+1}$, even with respect to the first variable. Then the spherical mean kernel $\varphi_{\rho,\varrho}$, $(\rho,\varrho) \in \mathbb{C}^{N+1}$, is the function given by

$$\varphi_{\rho,\varrho}(\alpha,\xi) = \mathcal{R}\left(\cos(\rho\ \cdot)e^{-i\langle\varrho,\cdot\rangle}\right)(\alpha,\xi) = j_{\frac{N-1}{2}}\left(\alpha\sqrt{\rho^2+\varrho^2}\right)e^{-i\langle\varrho,\xi\rangle}, \qquad (\alpha,\xi) \in \Omega, \tag{2.3}$$

where $j_{\frac{N-1}{2}}$ is the spherical Bessel function given by,

$$j_{(N-1)/2}(\lambda) = \Gamma((N+1)/2) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma((2j+1+N)/2)}(\lambda/2)^{2j}. \tag{2.4}$$

Proposition 1.

1. *For every $(\alpha,\xi) \in \mathbb{R}^{N+1}$, $z = (\rho,\varrho) \in \mathbb{C}^{N+1}$, and $\mu \in \mathbb{N}^{N+1}$, we have*

$$\left|D^\mu_z \varphi_{\rho,\varrho}(\alpha,\xi)\right| \le \|(\alpha,\xi)\|^{|\mu|} \exp\left(2\|(\alpha,\xi)\|\,\|\mathrm{Im} z\|\right). \tag{2.5}$$

2. *For all* $(\rho,\varrho)\in\Gamma$,

$$\sup_{(\alpha,\xi)\in\mathbb{R}^{N+1}}\left|\varphi_{\rho,\varrho}(\alpha,\xi)\right| = 1, \tag{2.6}$$

where

$$D_z^{\mu} = \frac{\partial^{|\mu|}}{\partial z_1^{\mu_1}\cdots\partial z_{N+1}^{\mu_{N+1}}} \quad \text{and} \quad |\mu| = \mu_1+\cdots+\mu_{N+1}. \tag{2.7}$$

In the following, we denote by:

1. $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+} = \left\{\theta^{-1}(B): \quad B\in\mathcal{B}_{\mathrm{Bor}}(\Omega)\right\}, \tag{2.8}$$

where θ is defined on the set Γ_+ by

$$\theta(\rho,\varrho) = \left(\sqrt{\rho^2+\|\varrho\|^2},\varrho\right). \tag{2.9}$$

2. $d\gamma$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall\, A\subset\mathcal{B}_{\Gamma_+},\quad \gamma(A) = \mu(\theta(A)). \tag{2.10}$$

3. $L^p(d\gamma)$, $1\le p\le\infty$, the space of measurable functions defined on Γ_+, such that

$$\|\tilde{u}\|_{L^p(d\gamma)} = \left(\int_{\Gamma_+}|\tilde{u}(\rho,\varrho)|^p d\gamma(\rho,\varrho)\right)^{1/p} < \infty, \quad p\in[1,\infty), \tag{2.11}$$

$$\|\tilde{u}\|_{L^\infty(d\gamma)} = \operatorname*{ess\,sup}_{(\rho,\varrho)\in\Gamma_+}|\tilde{u}(\rho,\varrho)| < \infty, \quad p=\infty. \tag{2.12}$$

Definition 2. *The Fourier transform associated with* $\mathcal{R}$ *is defined on* $L^1(d\mu)$ *by*

$$\mathcal{F}(u)(\rho,\varrho) = \int_\Omega u(\alpha,\xi)\varphi_{\rho,\varrho}(\alpha,\xi)d\mu(\alpha,\xi), \qquad (\rho,\varrho)\in\Gamma. \tag{2.13}$$

In what follows, we recall some of its properties [45].

Properties 1.

1. *For all* $u\in L^1(d\mu)$*, we have*

$$\forall\,(\rho,\varrho)\in\Gamma, \qquad \mathcal{F}(u)(\rho,\varrho) = \widetilde{\mathcal{F}}(u)\circ\theta(\rho,\varrho), \tag{2.14}$$

where

$$\forall\,(\rho,\varrho)\in\Omega, \qquad \widetilde{\mathcal{F}}(u)(\rho,\varrho) = \int_\Omega u(\alpha,\xi)j_{\frac{N-1}{2}}(\alpha\rho)e^{-i\langle\varrho,\xi\rangle}d\mu(\alpha,\xi). \tag{2.15}$$

2. *For all* $u\in L^1(d\mu)$*,*

$$\|\mathcal{F}(u)\|_{L^\infty(d\gamma)} \le \|u\|_{L^1(d\mu)}. \tag{2.16}$$

3. *Inversion Formula: For* $u\in L^1(d\mu)$ *such that* $\mathcal{F}(u)\in L^1(d\gamma)$*,*

$$u(\alpha,\xi) = \int_{\Gamma_+}\mathcal{F}(u)(\rho,\varrho)\overline{\varphi_{\rho,\varrho}(\alpha,\xi)}d\gamma(\rho,\varrho), \qquad \text{a.e. } (\alpha,\xi)\in\Omega. \tag{2.17}$$

4. *Plancherel-type formula: For all u in* $\mathcal{S}_*(\mathbb{R}^{N+1})$,

$$\int_\Gamma |\mathcal{F}(u)(\rho,\varrho)|^2 d\gamma(\rho,\varrho) = \int_\Omega |u(\alpha,\xi)|^2 d\mu(\alpha,\xi). \tag{2.18}$$

5. *The Fourier transform* $\mathcal{F}$ *can be extended to an isometric isomorphism from* $L^2(d\mu)$ *onto* $L^2(d\gamma)$.

Definition 3. *For* $(\alpha,\xi)\in\Omega$, *we define the generalized translation operator* $\tau_{(\alpha,\xi)}$ *by*

$$\tau_{(\alpha,\xi)}(u)(s,y) = \frac{\Gamma(\frac{N+1}{2})}{\pi^{1/2}\Gamma(N/2)} \int_0^\pi u\left(\sqrt{\alpha^2+s^2+2s\alpha\cos\theta}, y+\xi\right)(\sin\theta)^{N-1}d\theta, \tag{2.19}$$

and then, we define the generalized convolution product by

$$\forall\,(\alpha,\xi)\in\Omega,\quad u*v(\alpha,\xi) = \int_\Omega u(s,y)\tau_{(\alpha,\xi)}v(s,-y)d\mu(s,y),\qquad u,\, v\in D_*(\mathbb{R}^{N+1}). \tag{2.20}$$

Here $D_*(\mathbb{R}^{N+1})$ is the space of C^∞-functions on $\mathbb{R}^{N+1}$ which are of compact support and even with respect to the first variable.

Proposition 2. *The translation operator is bounded from* $L^p(d\mu)$ *into itself, such that*

$$\left\|\tau_{(\alpha,\xi)}u\right\|_{L^p(d\mu)} \leqslant \|u\|_{L^p(d\mu)},\qquad p\in[1,\infty]. \tag{2.21}$$

The convolution is associative and commutative, and fulfills the following results [43].

Properties 2.

1. *For* $u\in L^1(d\mu)$ *and* $v\in L^1(d\mu)\cup L^2(d\mu)$,

$$\mathcal{F}(u*v) = \mathcal{F}(u)\mathcal{F}(v). \tag{2.22}$$

2. *Let* $p,q,r\in[1,\infty]$ *such that* $\frac{1}{p}+\frac{1}{q}-\frac{1}{r}=1$. *If* $u\in L^p(d\mu)$ *and* $v\in L^q(d\mu)$, *then* $u*v\in L^r(d\mu)$, *such that*

$$\|u*v\|_{L^r(d\mu)} \leq \|u\|_{L^p(d\mu)}\|v\|_{L^q(d\mu)}\,. \tag{2.23}$$

Definition 4.

1. *The tempered distribution associated with* $u\in L^p(d\mu)$ *is defined by*

$$\forall\,\varphi\in\mathcal{S}_*(\mathbb{R}^{N+1}),\qquad \langle\mathcal{T}_u,\varphi\rangle = \int_\Omega u(\alpha,\xi)\varphi(\alpha,\xi)d\mu(\alpha,\xi) = \langle u,\varphi\rangle_{L^2(d\mu)}. \tag{2.24}$$

2. *The spherical mean transform* $\mathcal{F}(D)$ *of a distribution* $D\in\mathcal{S}'_*(\mathbb{R}^{N+1})$ *is defined by*

$$\forall\,\varphi\in\mathcal{S}_*(\mathbb{R}^{N+1}),\qquad \langle\mathcal{F}(D),\varphi\rangle = \langle D,\mathcal{F}(\varphi)\rangle. \tag{2.25}$$

3. *The generalized convolution product of* $D\in\mathcal{S}'_*(\mathbb{R}^{N+1})$ *and* $\varphi\in\mathcal{S}_*(\mathbb{R}^{N+1})$ *is defined by*

$$D*\varphi(\alpha,\xi) = \left\langle D_{(s,y)},\tau_{(s,-y)}\varphi(\alpha,\xi)\right\rangle. \tag{2.26}$$

Then we have the following properties.

Properties 3.

1. *For $u \in L^p(d\mu)$, we have for every $\varphi \in \mathcal{S}_*(\mathbb{R}^{N+1})$,*

$$\langle \mathcal{F}(u), \varphi \rangle = \langle \mathcal{F}(\mathcal{T}_u), \varphi \rangle = \langle \mathcal{T}_u, \mathcal{F}^{-1}(\varphi) \rangle = \langle u, \mathcal{F}^{-1}(\varphi) \rangle_{L^2(d\mu)}. \tag{2.27}$$

2. *The transformation $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}'_*(\mathbb{R}^{N+1})$ onto $\mathcal{S}'_*(\mathbb{R}^{N+1})$.*
3. *For $p \in [1, \infty]$, if $u \in L^p(d\mu)$ and $\varphi \in \mathcal{S}_*(\mathbb{R}^{N+1})$, then $\mathcal{T}_u * \varphi$ is a distribution belonging to $L^p(d\mu)$, and is given by the function $u * \varphi$.*
4. *For every $\vartheta \in \mathcal{S}_*(\mathbb{R}^{N+1})$,*

$$\langle \mathcal{T}_u * \varphi, \vartheta \rangle = \langle \check{u}, \varphi * \check{\vartheta} \rangle, \tag{2.28}$$

and

$$\mathcal{F}(\mathcal{T}_u * \varphi) = \mathcal{F}(\mathcal{T}_u)\mathcal{F}(\varphi), \tag{2.29}$$

where $\check{\vartheta}(\alpha, \xi) = \vartheta(\alpha, -\xi)$.

Let Δ_N be the generalized Laplace operator given by

$$\Delta_N := -\left(\frac{\partial^2}{\partial \alpha^2} + \frac{N}{\alpha}\frac{\partial}{\partial \alpha} + \sum_{j=1}^{N} \frac{\partial^2}{\partial \xi_j^2} \right). \tag{2.30}$$

Then for each $u \in \mathcal{S}'_*(\mathbb{R}^{N+1})$, we define the distribution $\Delta_N u$ by

$$\langle \Delta_N u, \psi \rangle = \langle u, \Delta_N \psi \rangle. \tag{2.31}$$

It satisfies the following property:

$$\mathcal{F}(\Delta_N u) = -\left(\rho^2 + 2\|\varrho\|^2\right)\mathcal{F}(u). \tag{2.32}$$

In the following we denote $\mathcal{T}_u$ given by (2.24) by u for simplicity, and throughout this paper, C will be a positive constant, which can change from line to line.

3. Hardy-type spaces

We start this section with the following definition.

Definition 5. *Let $0 < p \leq 1$. A Lebesgue measurable function a on Ω is a (p, N)-atom if it satisfies:*

1. *There exists $t > 0$ for which*

$$\operatorname{supp}(a) \subset B_{N+1}(0, t) := \{x \in \Omega \ : \ \|(\alpha, \xi)\| < t\}. \tag{3.1}$$

2. *We have*

$$\|a\|_{L^\infty(d\mu)} \leq t^{-\frac{(2N+1)}{p}}. \tag{3.2}$$

3. *For* $|\mu| \leq D(N,p)$,

$$\int_\Omega (s,\xi)^\mu a(s,\xi)d\mu(s,\xi) < \infty, \tag{3.3}$$

where $D(N,p) = \left\lfloor \frac{(2N+1)}{p} - 1 \right\rfloor$ *is the largest integer less than or equal to* $\frac{(2N+1)}{p} - 1$.

The generalized Hardy space $H^p(d\mu)$, $0 < p \leq 1$, is the subspace of distributions that can be expressed as

$$f = \sum_{i=0}^{\infty} t_i a_i, \tag{3.4}$$

where for any $i \in \mathbb{N}$, a_i is a (p,N)-atom, the sequence $\{t_i\}_{i\in\mathbb{N}}$ belongs to $\mathbb{C}$ such that $\sum_{i=0}^{\infty} |t_i|^p < \infty$, and the series in (3.4) is convergent in $\mathcal{S}'_*(\mathbb{R}^{N+1})$.

Let $\|\cdot\|_{H^p(d\mu)}$ be the norm on $H^p(d\mu)$ given by:

$$\|f\|_{H^p(d\mu)} = \inf \left(\sum_{i=0}^{\infty} |t_i|^p \right)^{\frac{1}{p}}, \tag{3.5}$$

where (3.5) is calculated over all complex numbers $\{t_i\}$ satisfying (3.4).

For the space $H^p(d\mu)$, we prove the following embedding result.

Proposition 3. *If* $f \in H^p(d\mu)$, $0 < p \leq 1$, *then* f *belongs to* $L^p(d\mu)$, *such that*

$$\|f\|_{L^p(d\mu)} \leq C(N)\|f\|_{H^p(d\mu)}, \tag{3.6}$$

where $C(N) = \Big(\mu(B_{N+1}(0,1))\Big)^{\frac{1}{p}}$.

Proof. Let a be a (p,N)-atom, $0 < p \leq 1$. From Definition 5, there exists $t > 0$ such that

$$a(\alpha,\xi) = 0, \quad \text{if } \|(\alpha,\xi)\| > t, \qquad \|a\|_{L^\infty(d\mu)} \leq t^{-\frac{(2N+1)}{p}}. \tag{3.7}$$

Then

$$\|a\|_{L^p(d\mu)} = \left(\int_\Omega |a(\alpha,\xi)|^p d\mu(\alpha,\xi) \right)^{\frac{1}{p}} \leq \|a\|_{L^\infty(d\mu)} \Big(\mu(B_{N+1}(0,t))\Big)^{\frac{1}{p}} \leq \Big(\mu(B_{N+1}(0,1))\Big)^{\frac{1}{p}}. \tag{3.8}$$

If $f \in H^p(d\mu)$, then $f = \sum_{j=0}^{\infty} t_j a_j$. Thus $f \in L^p(d\mu)$ such that

$$\|f\|_{L^p(d\mu)} \leq C(N) \sum_{j=0}^{\infty} |t_j|. \tag{3.9}$$

Since $\sum_{j=0}^{\infty} |t_j| \leq \left(\sum_{i=0}^{\infty} |t_i|^p \right)^{\frac{1}{p}}$, then the result follows. □

Theorem 5. *If* $f \in H^p(d\mu)$, $0 < p \leq 1$, *then there exists a constant* $C > 0$, *such that*

$$\forall\, (\rho,\varrho) \in \Gamma_+, \quad |\mathcal{F}(f)(\rho,\varrho)| \leq C\|(\rho,\varrho)\|^{(2N+1)(\frac{1}{p}-1)}\|f\|_{H^p(d\mu)}. \tag{3.10}$$

Proof. We have $f = \sum_{j=0}^{\infty} t_j a_j$. Then $\mathcal{F}(f) = \sum_{j=0}^{\infty} t_j \mathcal{F}(a_j)$. Hence

$$|\mathcal{F}(f)(\rho,\varrho)| \le \sum_{j=0}^{\infty} |t_j||\mathcal{F}(a_j)(\rho,\varrho)| \le \sup_{j\in\mathbb{N}} |\mathcal{F}(a_j)(\rho,\varrho)| \sum_{j=0}^{\infty} |t_j|. \tag{3.11}$$

Moreover, as

$$\sum_{j=0}^{\infty} |t_j| \le \left(\sum_{j=0}^{\infty} |t_j|^p \right)^{\frac{1}{p}}, \tag{3.12}$$

we derive that

$$|\mathcal{F}(f)(\rho,\varrho)| \le \sup_{j\in\mathbb{N}} |\mathcal{F}(a_j)(\rho,\varrho)| \left(\sum_{j\in\mathbb{N}} |t_j|^p \right)^{\frac{1}{p}}. \tag{3.13}$$

For $j \in \mathbb{N}$, we have

$$\begin{aligned} |\mathcal{F}(a_j)(\rho,\varrho)| &= \left| \int_{B_{N+1}(0,t)} a_j(s,y)\varphi_{\rho,\varrho}(s,y)d\mu(s,y) \right| \\ &= \left| \int_{B_{N+1}(0,t)} \Big(\varphi_{\rho,\varrho}(s,y) - \mathcal{K}_{D(N,p)}(s,y;\rho,\varrho)\Big) a_j(s,y)d\mu(s,y) \right| \\ &\le \int_{B_{N+1}(0,t)} \Big|\varphi_{\rho,\varrho}(s,y) - \mathcal{K}_{D(N,p)}(s,y;\rho,\varrho)]\Big| |a_j(s,y)|d\mu(s,y), \end{aligned}$$

where

$$\mathcal{K}_{D(N,p)}(s,y;\rho,\varrho) := \sum_{j=0}^{D(N,p)} \frac{i^j}{2j!} \mathcal{R}\Big(\langle (s,y),(\alpha,\xi)\rangle^j + \langle (-s,y),(\alpha,\xi)\rangle^j \Big).$$

Therefore

$$\Big|\varphi_{\rho,\varrho}(s,y) - \mathcal{K}_{D(N,p)}(s,y;\rho,\varrho)\Big| \le C\|(\rho,\varrho)\|^{D(N,p)+1}\|(s,y)\|^{D(N,p)+1}.$$

Thus

$$\begin{aligned} |\mathcal{F}(a_j)(\rho,\varrho)| &\le C\|(\rho,\varrho)\|^{D(N,p)+1} \int_{B_{N+1}(0,t)} \|(s,y)\|^{D(N,p)+1}|a_j(s,y)|d\mu(s,y) \\ &\le C\|(\rho,\varrho)\|^{D(N,p)+1}\|a_j\|_{L^\infty(d\mu)} \int_{B_{N+1}(0,t)} \|(s,y)\|^{D(N,p)+1}d\mu(s,y) \\ &\le C\|(\rho,\varrho)\|^{D(N,p)+1} t^{2N+1+D(N,p)}\|a_j\|_{L^\infty(d\mu)}. \end{aligned}$$

As

$$\|a_j\|_{L^\infty(d\mu)} \le t^{-\frac{(2N+1)}{p}},$$

we obtain

$$|\mathcal{F}(a_j)(\rho,\varrho)| \le C\|(\rho,\varrho)\|^{D(N,p)+1} t^{(2N+1)(1-\frac{1}{p})+D(N,p)+1}, \quad (\rho,\varrho) \in \Gamma_+. \tag{3.14}$$

On the other hand

$$|\mathcal{F}(a_j)(\rho,\varrho)| \le \int_{B_{N+1}(0,t)} |a_j(s,y)|d\mu(s,y)$$

$$\leq C(N)\|a_j\|_{L^\infty(d\mu)}t^{2N+1}$$
$$\leq Ct^{(2N+1)(1-\frac{1}{p})}. \tag{3.15}$$

We claim that

$$|\mathcal{F}(a_j)(\rho,\varrho)| \leq C\|(\rho,\varrho)\|^{-(2N+1)(1-\frac{1}{p})}, \quad \forall\,(\rho,\varrho) \in \Gamma_+. \tag{3.16}$$

Indeed, if $\|(\rho,\varrho)\| \leq t^{-1}$ and as $(2N+1)(1-\frac{1}{p}) + D(N,p) + 1 > 0$, we obtain

$$\|(\rho,\varrho)\|^{D(N,p)+1}t^{(2N+1)(1-\frac{1}{p})+D(N,p)+1} \leq C\|(\rho,\varrho)\|^{-(2N+1)(1-\frac{1}{p})}.$$

Thus (3.14) implies (3.16).

Moreover, if $\|(\rho,\varrho)\| \geq t^{-1}$, then since $(2N+1)(1-\frac{1}{p}) < 0$, we have

$$t^{(2N+1)(1-\frac{1}{p})} \leq C\|(\rho,\varrho)\|^{-(2N+1)(1-\frac{1}{p})}.$$

Hence (3.15) implies (3.16). Now by (3.13), we have

$$|\mathcal{F}(f)(\rho,\varrho)| \leq C\|(\rho,\varrho)\|^{-(2N+1)(1-\frac{1}{p})}\left(\sum_{j=0}^{\infty}|t_j|^p\right)^{\frac{1}{p}},$$

as desired. □

Corollary 1. *If $f \in H^p(d\mu)$, $0 < p \leq 1$, then there is a constant $C > 0$ such that*

$$\gamma\left(\left\{(\rho,\varrho) \in \Gamma_+ : \quad \|(\rho,\varrho)\|^{(2N+1)(1-\frac{2}{p})}|\mathcal{F}(f)(\rho,\varrho)| \geq s\right\}\right) \leq C\,s^{-p}\,\|f\|^p_{H^p(d\mu)}, \quad s > 0. \tag{3.17}$$

Proof. According to Theorem 5,

$$\begin{aligned}\gamma\left(\left\{(\rho,\varrho) \in \Gamma_+ : \quad \|(\rho,\varrho)\|^{(2N+1)(1-\frac{2}{p})}|\mathcal{F}(f)(\rho,\varrho)| \geq s\right\}\right) &\leq \mu\left(B\left(0,\left(Cs^{-1}\|f\|_{H^p(d\mu)}\right)^{\frac{p}{2N+1}}\right)\right)\\ &\leq C\,s^{-p}\,\|f\|^p_{H^p(d\mu)},\end{aligned}$$

as desired. □

4. Hardy-type inequalities

We denote by C the ring of center 0 of small radius $\frac{1}{2}$ and great radius 2. There exist two radial functions ϕ and ψ belonging to $D_*(\mathbb{R}^{N+1})$, with values in $[0,1]$ and satisfying:

1. $\operatorname{supp}\phi \subset C$, $\operatorname{supp}\psi \subset B_{N+1}(0,1)$.
2. $\forall(\rho,\varrho) \in \mathbb{R}^{N+1}$, we have $\psi(\rho,\varrho) + \sum_{j\geq 0}\phi(2^{-j}(\rho,\varrho)) = 1$.
3. $\forall(\rho,\varrho) \in C$, we have $\sum_{j\in\mathbb{Z}}\phi(2^{-j}(\rho,\varrho)) = 1$.
4. If $|n-m| \geq 2$, then $\operatorname{supp}\phi(2^{-n}\,\cdot) \cap \operatorname{supp}\phi(2^{-m}\,\cdot) = \emptyset$.

5. $\forall j \geq 1$, $\operatorname{supp}\psi \cap \operatorname{supp}\phi(2^{-j}\,\cdot) = \emptyset$.

Let

$$\Delta_j u = \mathcal{F}^{-1}\left(\phi\left(\frac{(\rho,\varrho)}{2^j}\right)\mathcal{F}(u)\right), \qquad S_j u = \sum_{n\leq j-1}\Delta_n u, \quad \forall\, j\in\mathbb{Z}, \tag{4.1}$$

and let $\mathcal{S}'_{h,*}(\mathbb{R}^{N+1})$ be the subspace of tempered distribution satisfying:

$$\lim_{j\to-\infty} S_j u = 0 \quad \text{in} \quad \mathcal{S}'_*(\mathbb{R}^{N+1}). \tag{4.2}$$

In the remainder of this paper, we establish the convention that the notation $\left(\sum_q a_q^r\right)^{\frac{1}{r}}$ represents $\sup_q a_q$, when $r=\infty$ for all positive sequences $\{a_q\}_{q\in\mathbb{Z}}$.

For $p,q\in[1,\infty]$, we define the generalized homogeneous Besov space $\dot{\mathcal{B}}^s_{p,q}(d\mu)$, $s\in\mathbb{R}$, as the space of distribution in $\mathcal{S}'_{h,*}(\mathbb{R}^{N+1})$ such that

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}} := \left(\sum_{j\in\mathbb{Z}}\left(2^{sj}\|\Delta_j f\|_{L^p(d\mu)}\right)^q\right)^{\frac{1}{q}} < \infty. \tag{4.3}$$

This definition is independent of the pair (ϕ,ψ).

We define the operator $\Re^s_N$, $s\in\mathbb{R}$, from $\mathcal{S}'_{h,*}(\mathbb{R}^{N+1})$ onto itself by:

$$\Re^s_N(f) = \mathcal{F}^{-1}(\|\cdot\|^s\mathcal{F}(f)). \tag{4.4}$$

These will be called spherical mean-Riesz potentials.

For $p\in[1,\infty]$ and $s\in\mathbb{R}$, the homogeneous spherical mean-Riesz potential space $\dot{\mathcal{H}}^s_p(d\mu)$ is given by $\Re^{-s}_N(L^p(d\mu))$, such that

$$\|f\|_{\dot{\mathcal{H}}^s_p(d\mu)} = \|\mathcal{R}^s_N(f)\|_{L^p(d\mu)}. \tag{4.5}$$

Theorem 6. *Let $\frac{2N+1}{4} < s < \frac{2N+1}{2}$. Then there is a constant $C>0$ such that, for every $f\in\dot{\mathcal{H}}^s_2(d\mu)$,*

$$\int_\Omega \frac{|f(\alpha,\xi)|^2}{\|(\alpha,\xi)\|^{2s}}d\mu(\alpha,\xi) \leq C\|f\|^2_{\dot{\mathcal{H}}^s_2(d\mu)}. \tag{4.6}$$

For the proof of this theorem, we need the following lemma.

Lemma 1. *Let s be in $(0, N+\frac{1}{2})$. Then the function $(\alpha,\xi)\mapsto\|(\alpha,\xi)\|^{-2s}$ is in $\dot{\mathcal{B}}^{2N+1-2s}_{1,\infty}$.*

Proof. We write

$$\|(\alpha,\xi)\|^{-2s} = u_1 + u_2,$$

where

$$u_1 = \theta\|(\alpha,\xi)\|^{-2s} \quad \text{and} \quad u_2 = (1-\theta)\|(\alpha,\xi)\|^{-2s}. \tag{4.7}$$

Here $\theta\in D_*(\mathbb{R}^{N+1})$ is identically equal to 1 near the unit ball $B_{N+1}(0,1)$. Using the homogeneity of the function $\|(\alpha,\xi)\|^{-2s}$, we derive that

$$\left\|\Delta_n\|(\alpha,\xi)\|^{-2s}\right\|_{L^1(d\mu)} \leq 2^{n(2s-2N-1)}\|\Delta_0 u_1\|_{L^1(d\mu)} + 2^{n(2s-2N-1)}\|\Delta_0 u_2\|_{L^1(d\mu)}. \tag{4.8}$$

It is clear that $u_1 \in L^1(d\mu)$. Therefore $\Delta_0 u_1 \in L^1(d\mu)$. On the other hand by (2.32),

$$\|\Delta_0 u_2\|_{L^1(d\mu)} \leq C_j \left\|(-\triangle_N)^j u_2\right\|_{L^1(d\mu)} < \infty, \tag{4.9}$$

provided that $j > N + \frac{1}{2} - s$. Thus we obtain

$$\left\|\|(\alpha,\xi)\|^{-2s}\right\|_{\mathcal{B}^{2N+1-2s}_{1,\infty}} := \left\| 2^{n(2N+1-2s)} \left\|\Delta_n \|(\alpha,\xi)\|^{-2s}\right\|_{L^1(d\mu)}\right\|_{\ell^\infty(\mathbb{Z})} < \infty, \tag{4.10}$$

a desired. □

Proof of Theorem 6. Let us define

$$I_s(f) := \int_\Omega \frac{|f(\alpha,\xi)|^2}{\|(\alpha,\xi)\|^{2s}} d\mu(\alpha,\xi) = \left\langle \|\cdot\|^{-2s}, f^2 \right\rangle. \tag{4.11}$$

Since $f^2 \in \mathcal{S}'_{h,*}(\mathbb{R}^{N+1})$, then

$$\begin{aligned} I_s(f) &= \sum_{|n-m|\leq 2} \left\langle \Delta_n(\|\cdot\|^{-2s}), \Delta_m(f^2)\right\rangle \\ &\leq C \sum_{|n-m|\leq 2} \left\langle 2^{n\left(\frac{2N+1}{2}-2s\right)}\Delta_n(\|\cdot\|^{-2s}), 2^{-m\left(\frac{2N+1}{2}-2s\right)}\Delta_m(f^2)\right\rangle. \end{aligned}$$

Lemma 1 claims that $\|\cdot\|^{-2s}$ belongs to $\dot{\mathcal{B}}_{2,\infty}^{\frac{2N+1}{2}-2s}$. Moreover, it is easy to see that

$$\|f^2\|_{\dot{\mathcal{B}}_{2,1}^{2s-\frac{2N+1}{2}}} \leq C\|f\|^2_{\dot{\mathcal{H}}^s_2(d\mu)}. \tag{4.12}$$

Thus $I_s(f) \leq C\|f\|^2_{\dot{\mathcal{H}}^s_2(d\mu)}$. □

Let u be a function defined on Ω. Then its distribution is given by

$$d_{u,N}(t) := \mu\left(\{|u| \geq t\}\right), \tag{4.13}$$

and its rearrangement is given by

$$u^*_N(s) := \inf\{t \;:\; d_{u,N}(t) \leq s\}. \tag{4.14}$$

For $q \in [1,\infty]$ and $p \in [1,\infty)$, let

$$\|f\|_{L^{p,q}(d\mu)} = \begin{cases} \left(\int_0^\infty \left(s^{\frac{1}{p}} f^*_N(s)\right)^q \dfrac{ds}{s}\right)^{\frac{1}{q}}, & \text{if} \quad q < \infty, \\ \sup_{s>0} s^{\frac{1}{p}} f^*_N(s), & \text{if} \quad q = \infty. \end{cases} \tag{4.15}$$

The set of all measurable functions f such that $\|f\|_{L^{p,q}(d\mu)} < \infty$ is known as the generalized Lorentz space $L^{p,q}(d\mu)$. Notice that $L^{p,p}(d\mu) = L^p(d\mu)$. Moreover, by real interpolation methods, any space $L^{p,q}(d\mu)$ can be obtained from $L^p(d\mu)$. Particularly, for $p \in (1,\infty)$, we have $L^{p,q}(d\mu) = \left[L^1(d\mu), L^\infty(d\mu)\right]_{\kappa,q}$, such that $p^{-1} = 1 - \kappa$.

Theorem 7. *Let s be a real number such that $0 < s < \frac{2N+1}{q}$, where $1 \leq q < \infty$. Then*

$$\|f\|_{L^{p,q}(d\mu)} \leq C\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-(2N+1)/q}_{\infty,q}} \|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^{s}_{q,q}}, \tag{4.16}$$

where $p = \dfrac{(2N+1)q}{2N+1-qs}$.

Proof. If $f \in \mathcal{S}_*(\mathbb{R}^{N+1})$, then

$$\|f\|^q_{L^{p,q}(d\mu)} = p\int_0^\infty t^q\Big(d_{f,N}(t)\Big)^{\frac{q}{p}}\frac{dt}{t}. \tag{4.17}$$

For $A > 0$, we put $f = f_{1,A} + f_{2,A}$ with

$$f_{1,A} = A^{2N+1}\psi(A\,\cdot) * f, \qquad f_{2,A} = A^{2N+1}\phi(A\,\cdot) * f.$$

We proceed as [47] to prove that

$$\int_0^\infty A^{sq-2N-2}\|f_{1,A}\|^q_{L^\infty(d\mu)}dA \leq C\|f\|^q_{\dot{\mathcal{B}}^{s-\frac{2N+1}{q}}_{\infty,q}}, \tag{4.18}$$

and

$$\int_0^\infty A^{sq-1}\|f_{2,A}\|^q_{L^\infty(d\mu)}dA \leq C\|f\|^q_{\dot{\mathcal{B}}^{s}_{q,q}}. \tag{4.19}$$

For all $t > 0$, we have

$$\{|f| \geq t\} \subset \{|f_{1,A}| \geq t/2\} \cup \{|f_{2,A}| \geq t/2\}. \tag{4.20}$$

We have that $t = t(A)$ such that $\|f_{1,A}\|_{L^\infty(d\mu)} = \dfrac{t}{4}$. Then $d_{f,N}(t) \leq d_{f_{2,A_t},N}(t/2)$.

From the Bienaymene-Tchebytchev relation, one has

$$d_{f_{2,A_t},N}(t/2) \leq 2^q t^{-q}\|f_{2,A_t}\|^q_{L^q(d\mu)}. \tag{4.21}$$

Moreover

$$\begin{aligned}\|f\|^q_{L^{p,q}(d\mu)} &= p\int_0^\infty t^q\Big(d_{f,N}(t)\Big)^{\frac{q}{p}}\frac{dt}{t}\\ &\leq p\int_0^\infty t(A)^{q-1}t'(A)\Big(d_{f_{2,A_t},N}(t/2)\Big)^{\frac{q}{p}}dA.\end{aligned}$$

We deduce that

$$\begin{aligned}\|f\|^q_{L^{p,q}(d\mu)} &\leq C\Bigg(\int_0^\infty A^{(2N+1)q}\|\psi(A\,\cdot) * f\|^q_{L^\infty(d\mu)}\Big(d_{f_{2,A_t},N}(\varrho/2)\Big)^{\frac{q}{p}}\frac{dA}{A}\\ &+ \int_0^\infty A^{(2N+1)(q-1)}\|\psi(A\,\cdot) * f\|^{q-1}_{L^\infty(d\mu)}\|\Theta(A\,\cdot) * f\|_{L^\infty(d\mu)}\Big(d_{f_{2,A_t},N}(\varrho/2)\Big)^{\frac{q}{p}}dA\Bigg)\\ &= I_1 + I_2,\end{aligned}$$

where

$$\Theta(A(\alpha,\xi)) = \langle\nabla_{N+1}\psi(A(\alpha,\xi)), (\alpha,\xi)\rangle.$$

Applying Hölder's inequality, we obtain

$$\begin{aligned} I_1 &\leq C\left(\int_0^\infty A^{qs-(2N+1)}\|f_{1,A}\|^q_{L^\infty(d\mu)}\frac{dA}{A}\right)^{1-\frac{q}{p}}\left(\int_0^\infty A^{qs}\|f_{2,A}\|^q_{L^q(d\mu)}\frac{dA}{A}\right)^{\frac{q}{p}} \\ &\leq C\left(\|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^s_{q,q}}\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-\frac{2N+1}{q}}_{\infty,q}}\right)^q. \end{aligned}$$

As for I_1, we have

$$\begin{aligned} I_2 &\leq C\left(\int_0^\infty A^{(2N+1)(q-1)}\|\psi(A\cdot)*f\|^{q-1}_{L^\infty(d\mu)}\|\Theta(A\cdot)*f\|_{L^\infty(d\mu)}\frac{dA}{A}\right)^{\frac{q}{p}} \\ &\times \left(\int_0^\infty A^{qs}\|f_{2,A}\|^q_{L^q(d\mu)}\frac{dA}{A}\right)^{\frac{q}{p}}. \end{aligned}$$

By simple calculations, we have

$$I_2 \leq C\left(\|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^s_{q,q}}\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-\frac{2N+1}{q}}_{\infty,q}}\right)^q.$$

Combining our estimates for I_1 and I_2 we obtain (4.16). □

Corollary 2. *Let $s \in (0, \frac{2N+1}{q})$ and let $q \in [1,\infty)$. Then there exists a constant $C > 0$ such that, for every $f \in \dot{\mathcal{B}}^s_{q,q}(d\mu)$, the following inequality holds:*

$$\left(\int_\Omega \frac{|f(\alpha,\xi)|^q}{\|(\alpha,\xi)\|^{sq}}\, d\mu(\alpha,\xi)\right)^{\frac{1}{q}} \leq C\|f\|^\theta_{\dot{\mathcal{B}}^s_{q,q}}\|f\|^{1-\theta}_{\dot{\mathcal{B}}^{s-\frac{2N+1}{q}}_{\infty,q}}, \tag{4.22}$$

where $\theta = 1 - \frac{qs}{2N+1}$.

Lemma 2. *Let $q_1, q_2, p_1, p_2 \in [1,\infty]$. If $f \in L^{p_1,q_1}(d\mu)$ and $g \in L^{p_2,q_2}(d\mu)$, then*

$$\|fg\|_{L^{p,q}(d\mu)} \leq C\|f\|_{L^{p_1,q_1}(d\mu)}\|g\|_{L^{p_2,q_2}(d\mu)}, \tag{4.23}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Proof. The proof of this lemma is similar to that of the Euclidean setting [44] and the Dunkl setting [46]. □

Proof of Corollary 2. Let $1 < p < \infty$ and $s \in \left(0, \frac{2N+1}{q}\right)$ with $\frac{1}{p} = \frac{1}{q} - \frac{s}{2N+1}$. We take $g(\alpha,\xi) = \frac{1}{\|(\alpha,\xi)\|^s}$ and apply (4.23) in the following form:

$$\|fg\|_{L^{q,q}(d\mu)} \leq C\|f\|_{L^{p,q}(d\mu)}\|g\|_{L^{k,\infty}(d\mu)}$$

where $k = \frac{2N+1}{s}$ and $p = \frac{q(2N+1)}{2N+1-qs}$. As $g \in L^{k,\infty}(d\mu)$, we have

$$\left(\int_\Omega \frac{|f(\alpha,\xi)|^q}{\|(\alpha,\xi)\|^{sq}} d\mu(\alpha,\xi)\right)^{\frac{1}{q}} \leq C\|f\|_{L^{p,q}(d\mu)}.$$

Combining this with (4.16), we obtain (4.22). □

Theorem 8. *For all $f \in H^p(d\mu)$, $p \in (0,1]$,*

$$\int_{\Gamma_+} \frac{|\mathcal{F}(f)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) \leq C\|f\|^p_{H^p(d\mu)}, \tag{4.24}$$

provided that

$$(2N+1)(2-p) \leq s < 2N+1+p(D(N,p)+1). \tag{4.25}$$

Proof. We have $f = \sum_{j=0}^{\infty} t_j a_j$. Then $\mathcal{F}(f) = \sum_{j=0}^{\infty} t_j \mathcal{F}(a_j)$. Therefore

$$\int_{\Gamma_+} \frac{|\mathcal{F}(f)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) \leq \sum_{j=0}^{\infty} |t_j|^p \int_{\Gamma_+} \frac{|\mathcal{F}(a_j)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho). \tag{4.26}$$

Thus it is enough to estimate

$$\int_{\Gamma_+} \frac{|\mathcal{F}(a_j)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho).$$

If $R > 0$, then

$$\begin{aligned} \int_{\Gamma_+} \frac{|\mathcal{F}(a_j)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) &= \int_{B(0,R)} \frac{|\mathcal{F}(a_j)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) + \int_{\Gamma_+ \setminus B(0,R)} \frac{|\mathcal{F}(a_j)(\rho,\varrho)|^p}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) \\ &= I_1 + I_2. \end{aligned}$$

From (3.14) and (4.25), we have

$$\begin{aligned} I_1 &\leq Ct^{p(2N+2+D(N,p))-(2N+1)))} \int_{B(0,R)} \frac{\|(\rho,\varrho)\|^{p(D(N,p)+1)}}{\|(\rho,\varrho)\|^s} d\gamma(\rho,\varrho) \\ &\leq Ct^{p(2N+2+D(N,p))-(2N+1)))} R^{(2N+1+p(D(N,p)+1)-s}. \end{aligned}$$

Now by (2.18) and (4.25), we have

$$\begin{aligned} I_2 &\leq \left(\int_{\Gamma_+} |\mathcal{F}(a_j)(\rho,\varrho)|^2 d\gamma(\rho,\varrho) \right)^{\frac{p}{2}} \left(\int_{\Gamma_+ \setminus B(0,R)} \|(\rho,\varrho)\|^{\frac{2s}{p-2}} d\gamma(\rho,\varrho) \right)^{\frac{2-p}{2}} \\ &\leq C\|a_j\|^p_{L^2(d\mu)} R^{\frac{(2N+1)(2-p)}{2}-s}. \end{aligned}$$

Moreover, we have

$$\|a_j\|^p_{L^2(d\mu)} \leq Ct^{-\frac{(2N+1)(2-p)}{2}}.$$

Thus

$$I_2 \leq Ct^{-\frac{(2N+1)(2-p)}{2}} R^{\frac{(2N+1)(2-p)}{2}-s}. \tag{4.27}$$

We have two cases:

In the case when $s = (2N+1)(2-p)$, we choose $R = \frac{1}{t}$, and then we get $I_1 \leq C$ and $I_2 \leq C$.

In the case when $(2N+1)(2-p) < s < 2N+1+p(D(N,p)+1)$, we will discuss the two cases $0 < t < 1$ and $t \geq 1$.

For $0 < t < 1$, we derive from the condition

$$(2N + 1)(2 - p) < s < 2N + 1 + p(D(N, p) + 1)$$

that there exists $R > 0$ satisfying

$$t^{p(2N+2+D(N,p))-(2N+1))}R^{(2N+1+p(D(N,p)+1)-s} \leq C, \quad \text{and} \quad t^{-\frac{(2N+1)(2-p)}{2}}R^{\frac{(2N+1)(2-p)}{2}-s} \leq C.$$

Thus we derive $I_1 \leq C$ and $I_2 \leq C$.

If $t \geq 1$, then we choose $R = t^{\frac{2N+1-p(D(N,p)+2+2N)}{2N+1+p(D(N,p)+1)-s}}$. It follows that $I_1 \leq C$ and $I_2 \leq C$, which completes the proof. □

5. Hausdorff operator

We define the multivariate generalized Hausdorff operator $\mathcal{H}_N = \mathcal{H}_N(\Phi, \mathcal{M})$ as

$$f : \Omega \to \mathcal{H}_N f(\alpha, \xi) := \int_\Omega \Phi(s, t) f(\mathcal{M}(s, t)(\alpha, \xi)) d\mu(s, t), \tag{5.1}$$

where Φ is a Borel measurable function, and $\mathcal{M}$ is a nonsingular diagonal matrix such that $\mathcal{M}(\Omega) \subset \Omega$.

In this section $\mathcal{M} = \text{diag}(a_1, \ldots, a_{N+1})$ will be a nonsingular diagonal matrix. Let

$$\eta(\mathcal{M}(s, t)) = |a_1(s, t)|^N \prod_{j=2}^{N+1} |a_j(s, t)|. \tag{5.2}$$

In this section, we will define the adjoint of $\mathcal{H}_N$, and study the boundedness of the multivariate generalized Hausdorff operator on $L^p(d\mu)$, and then in generalized Hardy spaces.

Theorem 9. *Let $f \in L^p(d\mu)$. If for some $1 \leq p \leq \infty$,*

$$\int_\Omega |\Phi(s, t)| \left(\eta(\mathcal{M}^{-1}(s, t)) \left|\det((\mathcal{M}^{-1}(s, t))\right|\right)^{1/p} d\mu(s, t) := K(p, N, \mathcal{H}_N) < \infty, \tag{5.3}$$

then

$$\|\mathcal{H}_N f\|_{L^p(d\mu)} \leq K(p, N, \mathcal{H}_N) \|f\|_{L^p(d\mu)}. \tag{5.4}$$

Notice that $K(p, N, \mathcal{H}_N)$ does not depend on f, but depends only on p, N, $\mathcal{H}_N$, Φ and $\mathcal{M}$.

Proof. Let $p \in (1, \infty)$. Then by Minkowski's inequality, we have

$$\begin{aligned} \|\mathcal{H}_N f\|_{L^p(d\mu)} &= \left(\int_\Omega \left| \int_\Omega \Phi(s, t) f(\mathcal{M}(s, t)(\alpha, \xi)) d\mu(s, t) \right|^p d\mu(\alpha, \xi)\right)^{1/p} \\ &\leq \int_\Omega \left(\int_\Omega |\Phi(s, t) f(\mathcal{M}(s, t)(\alpha, \xi))|^p d\mu(\alpha, \xi)\right)^{1/p} d\mu(s, t) \\ &= \int_\Omega |\Phi(s, t)| \left(\int_\Omega |f(k, y)|^p \frac{|\det\left(\mathcal{M}^{-1}(s, t)\right)|}{|a_1(s, t)|^N \prod_{j=2}^{N+1} |a_j(s, t)|} d\mu(k, y)\right)^{1/p} d\mu(s, t) \end{aligned}$$

$$
\begin{aligned}
&= \|f\|_{L^p(d\mu)} \int_\Omega |\Phi(s,t)| \left(\eta(\mathcal{M}^{-1}(s,t)) \left|\det(\mathcal{M}^{-1}(s,t))\right|\right)^{1/p} d\mu(s,t) \\
&:= K(p,N,\mathcal{H}_N)\|f\|_{L^p(d\mu)},
\end{aligned}
$$

which proves (5.4).

The case $p = \infty$ is obvious, and if $p = 1$, then we proceed as above, without the need of Minkowski's inequality. □

Example 1. *By choosing $\Phi(s_1, s_2) = \chi_{[0,1]^2}(s_1, s_2)$ and $\mathcal{M}(s_1, s_2) = \operatorname{diag}(s_1, s_2)$ we obtain the following bivariate Cesàro-type operator:*

$$
\begin{aligned}
C_2 f(\xi_1, \xi_2) &= \int_0^1 \int_0^1 f(s_1\xi_1, s_2\xi_2) d\mu(s_1, s_2) \\
&= \frac{1}{\xi_1^2 \xi_2} \int_0^{\xi_1} \int_0^{\xi_2} f(t_1, t_2) d\mu(t_1, t_2), \qquad \xi_1\xi_2 \neq 0. \qquad (5.5)
\end{aligned}
$$

Theorem 10. *We assume that the condition* (5.3) *is satisfied for some $1 \le p \le \infty$. Let $\psi(s,t) = \Phi(s,t)\eta\left(\mathcal{M}^{-1}(s,t)\right)\left|\det(\mathcal{M}^{-1}(s,t)\right|$. Then the operator $\mathcal{H}_N^* := \mathcal{H}_N\left(\psi(s,t); \mathcal{M}^{-1}\right)$ is bounded on the conjugate space $L^{p'}(d\mu)$. Moreover, we have*

$$
\|\mathcal{H}_N^* f\|_{L^{p'}(d\mu)} \le K(p,N,\mathcal{H}_N)\|f\|_{L^{p'}(d\mu)}. \qquad (5.6)
$$

Before proving this theorem, we need the following lemmas.

Lemma 3. *We assume that the condition* (5.3) *is satisfied for some $1 \le p \le \infty$. The operator $\mathcal{H}_N^*$ is the adjoint of $\mathcal{H}_N$, that is, for any $f \in L^p(d\mu)$ and $g \in L^{p'}(d\mu)$,*

$$
\int_\Omega g(\alpha,\xi)\mathcal{H}_N f(\alpha,\xi) d\mu(\alpha,\xi) = \int_\Omega f(\alpha,\xi)\mathcal{H}_N^* g(\alpha,\xi) d\mu(\alpha,\xi). \qquad (5.7)
$$

Proof. From Theorem 9, both integrals in (5.7) exist. Thus, we involve Fubini's Theorem twice as follows:

$$
\begin{aligned}
&\int_\Omega g(\alpha,\xi)\mathcal{H}_N f(\alpha,\xi) d\mu(\alpha,\xi) \\
&= \int_\Omega \int_\Omega \Phi(s,t) f\left(\mathcal{M}(s,t)(\alpha,\xi)\right) d\mu(s,t) g(\alpha,\xi) d\mu(\alpha,\xi) \\
&= \int_\Omega \Phi(s,t) \left(\int_\Omega f(\mathcal{M}(s,t)(\alpha,\xi)) g(\alpha,\xi) d\mu(s,t) \right) d\mu(\alpha,\xi) \\
&= \int_\Omega \Phi(s,t) \left(\int_\Omega f(y,k) g(\mathcal{M}^{-1}(s,t)(y,k)) \frac{|\det \mathcal{M}^{-1}(s,t)|}{|a_1(s,t)|^N \prod_{j=2}^{N+1} |a_j(s,t)|} d\mu(y,k) \right) d\mu(s,t) \\
&= \int_\Omega f(y,k) \left(\int_\Omega \Phi(s,t)\eta\left(\left(\mathcal{M}^{-1}(s,t)\right)\right) |\det(\mathcal{M}^{-1}(s,t))| g(\mathcal{M}^{-1}(s,t)(y,k)) d\mu(s,t) \right) d\mu(y,k) \\
&= \int_\Omega f(y,k)\mathcal{H}_N^* g(y,k) d\mu(y,k),
\end{aligned}
$$

which shows (5.7). □

Lemma 4. *We assume that, for some* $1 \leq p \leq \infty$*, the condition* (5.3) *is satisfied. Then*

$$K(p', N, \mathcal{H}_N^*) \quad = \quad K(p, N, \mathcal{H}_N). \tag{5.8}$$

Proof. Let $1 < p \leq \infty$. By Theorem 9, the operator $\mathcal{H}_N$ is bounded on $L^p(d\mu)$.
For $\psi(s,t) = \Phi(s,t)\eta(\mathcal{M}^{-1}(s,t))\Big|\det(\mathcal{M}^{-1}(s,t)\Big|$, we have

$$\begin{aligned} K(p', N, \mathcal{H}_N^*) &= \int_\Omega |\Phi(s,t)| \eta(\mathcal{M}^{-1}(s,t)) \left|\det(\mathcal{M}^{-1}(s,t))\right| (\eta(\mathcal{M}(s,t)) \left|\det(\mathcal{M}(s,t))\right|)^{1/p'} d\mu(s,t) \\ &= \int_\Omega |\Phi(s,t)| \left(\eta(\mathcal{M}^{-1}(s,t)) \left|\det(\mathcal{M}^{-1}(s,t))\right|\right)^{1/p} d\mu(s,t) \\ &:= K(p, N, \mathcal{H}_N). \end{aligned}$$

This implies the result in the case when $1 < p \leq \infty$. The case when $p = 1$ is trivial. □

Proof of Theorem 10. Involving (5.7), Theorem 9, and (5.8), we derive the result. □

Corollary 3. *We assume that, for some* $1 \leq p \leq \infty$*, the condition* (5.3) *is satisfied. Then*

$$\|\mathcal{H}_N^*\|_p = \|\mathcal{H}_N\|_p. \tag{5.9}$$

Example 2. *The adjoint of the generalized Cesàro operator* C_2*, defined in* (5.5)*, is given by*

$$C_2^* g(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} \frac{1}{x_1^2 x_2} g(x_1, x_2) d\mu(x_1, x_2), \qquad t_1 t_2 \neq 0. \tag{5.10}$$

In the remainder of this section, we will assume that the matrix satisfies: $\mathcal{M} = a(s,t) I_{N+1}$, with $a(s,t) \neq 0$ for almost everywhere $(s,t) \in \Omega$.

Definition 6. *The generalized Riesz transforms associated with the spherical mean operator are defined on* $L^2(d\mu)$ *by*

$$\mathcal{F}(\mathbf{R}_j f)(\rho, \varrho) = -i \frac{(\rho, \varrho)_j}{\|(\rho, \varrho)\|} \mathcal{F}(f)(\rho, \varrho), \quad j = 1, \ldots, N+1. \tag{5.11}$$

We extend the definition of these generalized Riesz transforms on $L^1(d\mu)$ and we introduce another version of the generalized Hardy spaces via the generalized transforms as

$$H^1_{\text{Riesz}} := \Big\{ f \in L^1(d\mu) \; : \; \|\mathbf{R}_j f\|_{L^1(d\mu)} < \infty, \quad j = 1, \ldots, N+1 \Big\}.$$

We provide this space by the following norm:

$$\|f\|_{H^1_{\text{Riesz}}} := \|f\|_{L^1(d\mu)} + \sum_{j=1}^{N+1} \|\mathbf{R}_j f\|_{L^1(d\mu)}. \tag{5.12}$$

Theorem 11.

1. *If* (5.3) *is valid for* $p = 1$*, then*

$$\mathcal{F}(\mathcal{H}_N f)(\rho,\varrho) = \mathcal{H}_N^*(\mathcal{F}(f))(\rho,\varrho). \tag{5.13}$$

2. *If* (5.3) *is valid for* $p = \infty$*, then*

$$\mathcal{F}(\mathcal{H}_N^* f)(\rho,\varrho) = \mathcal{H}_N(\mathcal{F}(f))(\rho,\varrho). \tag{5.14}$$

Proof. Involving (2.13) and (5.3), we obtain

$$\begin{aligned}
\mathcal{F}(\mathcal{H}_N f)(\rho,\varrho) &= \int_\Omega \mathcal{H}_N f(\alpha,\xi)\varphi_{\rho,\varrho}(\alpha,\xi)d\mu(\alpha,\xi)\\
&= \int_\Omega\left(\int_\Omega \Phi(s,t)f(\mathcal{M}(s,t)(\alpha,\xi))d\mu(s,t)\right)\varphi_{\rho,\varrho}(\alpha,\xi)d\mu(\alpha,\xi)\\
&= \int_\Omega \Phi(s,t)\int_\Omega f(\mathcal{M}(s,t)(\alpha,\xi))\varphi_{\rho,\varrho}(\alpha,\xi)d\mu(\alpha,\xi)d\mu(s,t)\\
&= \int_\Omega \Phi(s,t)\left(\int_\Omega f(y,k)\Big(|a(s,t)|^{2N+1}\Big)^{-1}\varphi_{\rho,\varrho}\left(\frac{y,k}{a(s,t)}\right)d\mu(y,k)\right)d\mu(s,t)\\
&= \int_\Omega \Phi(s,t)\,(|a(s,t)|)^{-2N-1}\left(\int_\Omega f(y,k)\varphi_{\frac{(\rho,\varrho)}{a(s,t)}}(y,k)d\mu(y,k)\right)d\mu(s,t)\\
&= \int_\Omega \Phi(s,t)\eta\left(\mathcal{M}^{-1}(s,t)\right)\left|\det((\mathcal{M}^{-1}(s,t))\right|\mathcal{F}(f)\left(\mathcal{M}^{-1}(s,t)(\rho,\varrho)\right)d\mu(s,t)\\
&= \mathcal{H}_N^*(\mathcal{F}(f))(\rho,\varrho).
\end{aligned}$$

As for (5.13), the proof of (5.14) is analogous. □

Corollary 4.

1. *If* (5.3) *is valid for* $p = 1$*, then*

$$\mathcal{H}_N(\Phi)(\mathbf{R}_j f) = \mathbf{R}_j(\mathcal{H}_N(\widetilde{\Phi})f), \tag{5.15}$$

where $\widetilde{\Phi}(s,t) = \operatorname{sgn}(a(s,t))\,\Phi(s,t)$*, for all* $(s,t)\in\Omega$.

2. *If* (5.3) *is valid for* $p = \infty$*, then*

$$\mathcal{H}_N^*(\mathbf{R}_j f) = \mathbf{R}_j(\mathcal{H}(\widetilde{\Phi})_N^* f). \tag{5.16}$$

Theorem 12. *The Hausdorff operator* $\mathcal{H}_N$ *is bounded on the Hardy space* H^1_{Riesz} *provided that*

$$\|\Phi\|_{L^2_{N,\mathcal{M}^{-1}}} := \int_\Omega \|\Phi(\alpha,\xi)\|\,|a(\alpha,\xi)|^{-2N-1}d\mu(\alpha,\xi) < \infty, \tag{5.17}$$

and there exists a positive constant C *such that*

$$\|\mathcal{H}_N f\|_{H^1_{\mathrm{Riesz}}} \le C\|\Phi\|_{L^2_{N,\mathcal{M}^{-1}}}\|f\|_{H^1_{\mathrm{Riesz}}}. \tag{5.18}$$

Proof. First, we note that $|\widetilde{\Phi}(s,t)| = |\Phi(s,t)|$. Involving (5.15) and (5.4), we have

$$\begin{aligned} \|\mathcal{H}_N f\|_{H^1_{\mathrm{Riesz}}} &= \|\mathcal{H}_N f\|_{L^1(d\mu)} + \sum_{j=1}^{N+1} \left\|\mathbf{R}_j \mathcal{H}_N(\Phi) f\right\|_{L^1(d\mu)} \\ &= \|\mathcal{H}_N f\|_{L^1(d\mu)} + \sum_{j=1}^{N+1} \int_\Omega \left|\mathbf{R}_j \mathcal{H}_N(\Phi) f(\alpha,\xi)\right| d\mu(\alpha,\xi) \\ &\leq \|\mathcal{H}_N f\|_{L^1(d\mu)} + \sum_{j=1}^{N+1} \int_\Omega |\widetilde{\Phi}(s,t)| \left|\int_\Omega \mathbf{R}_j f(\mathcal{M}(s,t)(\alpha,\xi)) d\mu(\alpha,\xi)\right| d\mu(s,t) \\ &\leq \|\mathcal{H}_N f\|_{L^1(d\mu)} + \int_\Omega |\Phi(s,t)| \sum_{j=1}^{N+1} \left\|\mathbf{R}_j f(\mathcal{M}(s,t)(\alpha,\xi))\right\|_{L^1(d\mu)} d\mu(s,t) \\ &\leq C \int_\Omega |\Phi(s,t)| \left\|f(\mathcal{M}(s,t)(\alpha,\xi))\right\|_{H^1_{\mathrm{Riesz}}} d\mu(s,t). \end{aligned} \tag{5.19}$$

By substitution, the result is derived.

□

6. Maximal Bochner-Riesz operators on $H^p(d\mu)$

For $t > 0$ and $z > \frac{N}{2}$, we define the Bochner-Riesz mean operator $\sigma^z_{N,t}$ by: for all $u \in \mathcal{S}_*(\mathbb{R}^{N+1})$,

$$\sigma^z_{N,t}(u)(\alpha,\xi) := \int_{B_{N+1}(0,t)} \left(1 - \frac{\|(\rho,\varrho)\|^2}{t^2}\right)^z \left(1 - \frac{\|\varrho\|^2}{t^2}\right)^z \overline{\varphi_{\rho,\varrho}(\alpha,\xi)} \mathcal{F}(u)(\rho,\varrho) d\gamma(\rho,\varrho). \tag{6.1}$$

The maximal operators $\varsigma^z_N,\ z > \frac{N}{2}$, associated with the Bochner-Riesz mean operators $\sigma^z_{N,t},\ t > 0$, are defined by

$$\varsigma^z_N(u) := \sup_{t>0} |\sigma^z_{N,t}(u)|. \tag{6.2}$$

Lemma 5. *If $z > \frac{N}{2}$, then*

$$\varsigma^z_N(u) := \sup_{t>0} |\Phi^z_{N,t} * u|, \tag{6.3}$$

where

$$\Phi^z_{N,t}(\alpha,\xi) := C(N,z) t^{2N+1} j_{\frac{N+1}{2}+z}(t\alpha) j_{\frac{N}{2}+z}(t\|\xi\|), \tag{6.4}$$

and

$$C(N,z) := \frac{(\Gamma(z+1))^2}{2^{2N-1} k_N \Gamma(\frac{N}{2}) \Gamma(\frac{N+1}{2}) \Gamma(\frac{N+2}{2}+z) \Gamma(\frac{N+3}{2}+z)}. \tag{6.5}$$

Proof. Let $z > \frac{N}{2}$ and $t > 0$. Since

$$(\alpha,\xi) \to \alpha^{\frac{N}{2}+z+1} \|\xi\|^{\frac{N+1}{2}+z} j_{\frac{N+1}{2}+z}(\alpha) j_{\frac{N}{2}+z}(\|\xi\|) \quad \text{and} \quad j_{\frac{N+1}{2}+z}(t\alpha) j_{\frac{N}{2}+z}(t\|\xi\|) \tag{6.6}$$

are bounded functions on Ω, then

$$\|\Phi^z_{N,t}\|_{L^1(d\mu)} < \infty.$$

On the other hand, from [48], for $z > K - \frac{1}{2}$ we have

$$\int_0^\infty j_{K+z}(ts) j_{K-1}(sy) s^{2K-1} ds = \frac{2^{2K-1}\Gamma(K+z+1)\Gamma(K)}{t^{2K}\Gamma(z+1)} \left(1 - \frac{|y|^2}{t^2}\right)^z \chi_{[0,t]}(|y|). \tag{6.7}$$

Then by (6.4) and the Hecke identity, we derive that

$$\begin{aligned}
&\mathcal{F}(\Phi^z_{N,t})(\rho,\varrho)\\
= \;& k_N C(N,z) t^{2N+1} \left(\int_0^\infty j_{\frac{N+1}{2}+z}(t\alpha) j_{\frac{N-1}{2}}\left(\alpha\sqrt{\rho^2 + \|\varrho\|^2}\right)\alpha^N d\alpha\right)\left(\int_{\mathbb{R}^N} j_{\frac{N}{2}+z}(t\|\xi\|) e^{-i\langle \xi,\varrho\rangle} d\xi\right)\\
= \;& k_N C(N,z) t^{2N+1} \left(\int_0^\infty j_{\frac{N+1}{2}+z}(t\alpha) j_{\frac{N-1}{2}}\left(\alpha\sqrt{\rho^2 + \|\varrho\|^2}\right)\alpha^N d\alpha\right)\left(\int_0^\infty j_{\frac{N}{2}+z}(ts) j_{\frac{N}{2}-1}(s\|\varrho\|) s^{N-1} ds\right).
\end{aligned}$$

Applying the identity (6.7), we obtain

$$\mathcal{F}(\Phi^z_{N,t})(\rho,\varrho) = \left(1 - \frac{\|(\rho,\varrho)\|^2}{t^2}\right)^z \left(1 - \frac{\|\varrho\|^2}{t^2}\right)^z \chi_{B_{N+1}(0,t)}(\rho,\varrho), \tag{6.8}$$

where $\chi_{B_{N+1}(0,t)}$ is the characteristic function of the ball $B_{N+1}(0,t)$. Involving the inversion formula (2.17) and (6.8), we derive that

$$\Phi^z_{N,t} * f(\alpha,\xi) = \int_{B_{N+1}(0,t)} \left(1 - \frac{\|(\rho,\varrho)\|^2}{t^2}\right)^z \left(1 - \frac{\|\varrho\|^2}{t^2}\right)^z \overline{\varphi_{\rho,\varrho}(\alpha,\xi)} \mathcal{F}(f)(\rho,\varrho) d\gamma(\rho,\varrho). \tag{6.9}$$

Thus, by (6.1), we deduce the result. □

Theorem 13. *Let $z > \frac{N}{2}$.*

1. *The operator $\sigma^z_{N,t}$ is bounded from $L^p(d\mu)$, $1 \le p \le \infty$, onto itself.*
2. *The operator $\sigma^z_{N,t}$ is extended to a bounded operator from $H^p(d\mu)$, $0 < p \le 1$, onto $\mathcal{S}'_*(\mathbb{R}^{N+1})$.*

Proof. Let $z > \frac{N}{2}$. Then from Lemma 5 and (2.23), we obtain

$$\|\sigma^z_{N,t}(u)\|_{L^p(d\mu)} \le \|\Phi^z_{N,t}\|_{L^1(d\mu)} \|u\|_{L^p(d\mu)}. \tag{6.10}$$

On the other hand, by (3.10), if $u \in H^p(d\mu)$, then $\sigma^z_{N,t}$ belongs to $\mathcal{S}'_*(\mathbb{R}^{N+1})$ and is given by:

$$\langle \sigma^z_{N,t}(u), \phi\rangle = \int_{B_{N+1}(0,t)} \left(1 - \frac{\|(\rho,\varrho)\|^2}{t^2}\right)^z \left(1 - \frac{\|\varrho\|^2}{t^2}\right)^z \mathcal{F}(\phi)(\rho,\varrho)\mathcal{F}(u)(\rho,\varrho) d\gamma(\rho,\varrho). \tag{6.11}$$

Moreover

$$|\langle \sigma^z_{N,t}(u), \phi\rangle| \le C\|u\|_{H^p(d\mu)} \int_{\mathbb{R}^{N+1}} \|(\rho,\varrho)\|^{(2N+1)(1/p-1)} |\mathcal{F}(\phi)(\rho,\varrho)| d\gamma(\rho,\varrho). \tag{6.12}$$

Hence, $\sigma^z_{N,t}$ is bounded from $H^p(d\mu)$ onto $\mathcal{S}'_*(\mathbb{R}^{N+1})$. □

Theorem 14. *Let $\frac{2N+1}{N+2z+\frac{3}{2}} < p \le 1$. Then, the maximal Bochner-Riesz operator ς^z_N is bounded from $H^p(d\mu)$ onto $L^p(d\mu)$, provided that $\frac{N}{2} < z < \frac{N}{2} + \frac{1}{4}$.*

To prove this theorem, we need the following lemma.

Lemma 6. *Let* $z > \frac{N}{2}$.

1. *For* $(s, y), (\alpha, \xi) \in \Omega$,

$$\left|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y)\right| \le Ct^{N-2z-\frac{1}{2}}\left|\, \|\xi\| - \|y\|\right|^{-(\frac{N+1}{2}+z)}|\alpha - s|^{-(\frac{N}{2}+z+1)}. \tag{6.13}$$

2. *For* $0 < \|y\| < \|\xi\|$, $0 < s < \alpha$,

$$\left|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y) - \Phi^z_{N,t}(\alpha, \xi)\right| \le Ct^{N-2z+\frac{1}{2}}(\|y\| + s)(\|\xi\| - \|y\|)^{-(\frac{N+1}{2}+z)}(\alpha - s)^{-(\frac{N}{2}+z+1)}. \tag{6.14}$$

Proof. Involving (2.19) and (6.4), we get

$$\begin{aligned}\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y) &= \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{1/2}\Gamma(N/2)}\int_0^\pi \Phi^z_{N,t}\left(\sqrt{\alpha^2 + s^2 + 2s\alpha\cos\theta}, \xi - y\right)(\sin\theta)^{N-1}d\theta \\ &= C(N, z)\frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{1/2}\Gamma(N/2)}t^{2N+1}\int_0^\pi j_{\frac{N+1}{2}+z}\left(t\sqrt{\alpha^2 + s^2 + 2s\alpha\cos\theta}\right) j_{\frac{N}{2}+z}(t\|\xi - y\|)(\sin\theta)^{N-1}d\theta.\end{aligned}$$

Using the fact that the function $k \mapsto k^{\mu+\frac{1}{2}} j_\mu(k)$ is bounded on $\mathbb{R}$, we derive that there exists a positive constant $C := \widetilde{C}(N, z)$ such that

$$\left|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y)\right| \le Ct^{N-2z-\frac{1}{2}}\int_0^\pi \left(\sqrt{\alpha^2 + s^2 + 2s\alpha\cos\theta}\right)^{-(\frac{N+2}{2}+z)}(\|\xi - y\|)^{-\frac{N+1}{2}-z}(\sin\theta)^{N-1}d\theta.$$

Finally, using the fact

$$\|\xi - y\| \ge |\|\xi\| - \|y\|| \quad \text{and } |\alpha - s| \le \sqrt{\alpha^2 + s^2 + 2s\alpha\cos\theta}, \qquad \theta \in [0, \pi],$$

we get

$$\left|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y)\right| \le C\left(\int_0^\pi (\sin\theta)^{N-1}\, d\theta\right)t^{N-2z-\frac{1}{2}}|\alpha - s|^{-(\frac{N+2}{2}+z)}\left|\, \|\xi\| - \|y\|\right|^{-\left(\frac{N+1}{2}+z\right)}.$$

On the other hand, we have

$$\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y) - \Phi^z_{N,t}(\alpha, \xi) = C_N\int_0^\pi \left(\Phi^z_{N,t}\left(\sqrt{\alpha^2 + s^2 + 2s\alpha\cos\theta}, \xi - y\right) - \Phi^z_{N,t}(\alpha, \xi)\right)(\sin\theta)^{N-1}d\theta,$$

where $C_N = \frac{\Gamma((N+1)/2)}{\pi^{1/2}\Gamma(N/2)}$. Using the mean value theorem in several variables and a similar argument as was used in the proof of (6.13), we get for $0 < \|y\| < \|\xi\|$ and $0 < s < \alpha$,

$$\left|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s, -y) - \Phi^z_{N,t}(\alpha, \xi)\right| \le Ct^{N-2z+\frac{1}{2}}(\|y\| + s)(\|\xi\| - \|y\|)^{-\left(\frac{N+1}{2}+z\right)}(\alpha - s)^{-\left(\frac{N}{2}+z+1\right)}.$$

Thus the result is proved. □

Proof of Theorem 14. For a (p, N)-atom a, we assume that

$$a(\alpha, \xi) = 0,\ (\alpha, \xi) \in \Omega \setminus \mathbf{Q}(0, k) \quad \text{and} \quad \|a\|_{L^\infty(d\mu)} \le k^{-2N-1/p}, \tag{6.15}$$

where

$$\mathbf{Q}(0,k) := \{(\alpha,\xi)\in\Omega \ : \ \|\xi\|\le k \quad \text{and} \quad \alpha\le k\}. \tag{6.16}$$

We choose $l\in\mathbb{Z}$ such that $2^{l-1}<k\le 2^l$, and we write

$$\int_{\Omega\setminus\mathbf{Q}(0,4k)} |\sigma^z_{N,t}(a)(\alpha,\xi)|^p d\mu(\alpha,\xi) \le I_1 + I_2, \tag{6.17}$$

where

$$I_1 := \sum_{i=1}^{\infty}\int_{\mathbf{Q}\left(0,(i+2)2^l\right)\setminus\mathbf{Q}\left(0,(i+1)2^l\right)} \sup_{t\ge\delta_i} |\sigma^z_{N,t}(a)(\alpha,\xi)|^p d\mu(\alpha,\xi) \tag{6.18}$$

and

$$I_2 := \sum_{i=1}^{\infty}\int_{\mathbf{Q}(0,(i+2)2^l)\setminus\mathbf{Q}(0,(i+1)2^l)} \sup_{t<\delta_i} |\sigma^z_{N,t}(a)(\alpha,\xi)|^p d\mu(\alpha,\xi) \tag{6.19}$$

such that $\delta_i = 2^{-l}/i^b$, where we will provide b later.

From the previous lemma, if

$$(\alpha,\xi)\in\mathbf{Q}(0,(i+2)2^l)\setminus\mathbf{Q}(0,(i+1)2^l), \ i=1,2,\ldots, \tag{6.20}$$

then

$$\begin{aligned}
|\sigma^z(a)(\alpha,\xi)| &\le \int_{\mathbf{Q}(0,k)} |a(s,y)|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s,-y)|d\mu(s,y)\\
&\le Ct^{N-2z-\frac{1}{2}}k^{-2N-1/p}\int_{\mathbf{Q}(0,2^l)} |\,\|\xi\|-\|y\|\,|^{-(\frac{N+1}{2}+z)}|\alpha-s|^{-(\frac{N}{2}+z+1)}d\mu(s,y)\\
&\le C\frac{(2^l t)^{N-2z-1/2}}{i^{N+2z+3/2}k^{2N+1)/p}}.
\end{aligned}$$

Then, using the fact that $2^{l-1}<k\le 2^l$, we obtain

$$I_1 \le C\sum_{i=1}^{\infty}\left(\frac{\delta_i^{N-2z-1/2}2^{l\{N-2z-1/2-(2N+1)/p\}}}{i^{N+2z+\frac{3}{2}}}\right)^p i^{2N}2^{l(2N+1)}, \tag{6.21}$$

and hence we conclude that

$$I_1 \le C\sum_{i=1}^{\infty} i^{2N-\{N+2z+\frac{3}{2}+(N-2z-1/2)b\}p}. \tag{6.22}$$

The series in (6.22) converges provided that: $b<((N+2z+\frac{3}{2})p-2N-1)/(p(2z-N+1/2))$.

However, since $\int_{\Omega} a\,d\mu = 0$, then from the previous lemma,

$$\begin{aligned}
&|\sigma^z_{N,t}(a)(\alpha,\xi)|\\
\le\ & \int_{\mathbf{Q}(0,t)} |a(s,y)|\left|\tau_{(\alpha,\xi)}\Phi^z_{N,t}(s,-y)-\Phi^z_{N,t}(\alpha,\xi)\right| d\mu(s,y)\\
\le\ & Ct^{N-2z+\frac{1}{2}}k^{-2N-1/p}\int_{\mathbf{Q}(0,2^l)} (\|y\|+s)\,(\|\xi\|-\|y\|)^{-(\frac{N+1}{2}+z)}(\alpha-s)^{-(\frac{N}{2}+z+1)}d\mu(s,y)
\end{aligned}$$

$$\leq \quad C\frac{(2^l t)^{N-2z+1/2}}{i^{N+2z+3/2}\alpha^{2N+1/p}}.$$

Therefore, for $(\alpha,\xi)\in\Omega\setminus\mathbf{Q}(0,4k)$, we obtain

$$I_2 \leq C\sum_{i=1}^{\infty}\left(\frac{\delta_i^{N-2z+1/2}2^{l\{N-2z+1/2-(2N+1)/p\}}}{i^{N+2z+\frac{3}{2}}}\right)^p i^{2N}2^{l(2N+1)}, \tag{6.23}$$

and thus

$$I_2 \leq C\sum_{i=1}^{\infty} i^{2N-\{N+2z+\frac{3}{2}+(N-2z+1/2)b)\}p}. \tag{6.24}$$

The series in (6.24) converges if $b > (2N+1-(N+2z+3/2)p)/(p(N-2z+1/2))$.

The real number b can be calculated provided that the two series in (6.22) and (6.24) converge if and only if $p > \frac{2N+1}{N+2z+\frac{3}{2}}$.

From (6.22) and (6.24), we deduce that there is a constant $C > 0$, such that

$$\int_{\Omega\setminus\mathbf{Q}(0,4k)} |\varsigma_N^z(a)(\alpha,\xi)|^p d\mu(\alpha,\xi) \leq C, \tag{6.25}$$

where C does not depend on a. It follows by Theorem 13, (1), and (6.25) that

$$\begin{aligned}
\|\varsigma_N^z(a)\|_{L^p(d\mu)}^p &\leq \int_{\mathbf{Q}(0,4k)} |\varsigma_N^z(a)(\alpha,\xi)|^p d\mu(\alpha,\xi) + \int_{\Omega\setminus B(0,4k)} |\varsigma_N^z(a)(\alpha,\xi)|^p d\mu(\alpha,\xi) \\
&\leq C\left(\|a\|_{L^\infty(d\mu)}^p \int_{\mathbf{Q}(0,4k)} d\mu(\alpha,\xi) + 1\right) \leq C.
\end{aligned}$$

That is, for every (p,N)-atom a,

$$\|\varsigma_N^z\|_{L^p(d\mu)} \leq C. \tag{6.26}$$

Suppose that f is now in $H^p(d\mu)$. For each $j\in\mathbb{N}$, a_j will be a (p,N)-atom and let $t_j\in\mathbb{C}$ be such that $\sum_{j=0}^{\infty}|t_j|^p < \infty$. Assume that $f = \sum_{j=0}^{\infty} t_j a_j$, where the series converges in $\mathcal{S}'_*(\mathbb{R}^{N+1})$.

From Theorem 13 (2), for $0 < p \leq 1$, we can write

$$\varsigma_N^z(f)(\alpha,\xi) = \sum_{j=0}^{\infty} t_j\sigma^z(a_j)(\alpha,\xi). \tag{6.27}$$

Hence, from (6.26), it follows that

$$\|\varsigma_N^z(a)\|_{L^p(d\mu)}^p \leq C\sum_{j=0}^{\infty}|t_j|^p. \tag{6.28}$$

Thus, we conclude the desired result. □

7. Conclusions

In this paper, we investigated the notion of Hardy spaces in the spherical mean setting. We proved Hardy-type inequalities on some functional spaces associated with spherical mean operators. We studied the boundedness of generalized Hausdorff operators associated with spherical mean operators. Finally, we introduced maximal Bochner-Riesz-type operators on generalized Hardy-type spaces.

In further work, our aim is to extend the notion of bounded mean oscillation spaces in the spherical mean setting. We will next seek to prove the duality between these spaces and Hardy-type spaces $H^1(d\mu)$, and to study the boundedness of Hausdorff-type operators on these bounded mean oscillation spaces.

Author contributions

Saifallah Ghobber: Conceptualization, validation, writing—review and editing, project administration, funding acquisition; Hatem Mejjaoli: Methodology, formal analysis, investigation, writing—original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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