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Research article

A dual delays epidemic model for TB with adaptive mobility behavior

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Abstract: Tuberculosis is a chronic infectious disease caused by Mycobacterium tuberculosis, primarily transmitted through the air. It has a long incubation period and complex infection dynamics, with its transmission characteristics significantly influenced by population mobility and infection risk-responsive behavior. According to reports from the World Health Organization (WHO), tuberculosis remains a severe global public health issue, particularly in developing countries, where its high transmission rate and mortality pose substantial challenges to socioeconomic development and healthcare resources. Therefore, investigating the key factors driving tuberculosis transmission is of great significance for global public health management. This paper investigates a dual delays SEIRm epidemic model for tuberculosis that incorporates mobility-adaptive behavior and a nonlinear transmission rate. The stability of these equilibria and the existence of Hopf bifurcations are analyzed. Finally, numerical simulations are performed to examine the impact of population mobility and time-delay factors on disease transmission. The simulation results indicate that properly controlled population movement can significantly reduce the spread of the epidemic. Although mobility responsiveness does not affect the basic number of reproductions, it can mitigate the peak of infections, thereby protecting a larger proportion of the susceptible population.

Keywords: mobility-adaptive behavior; tuberculosis; nonlinear transmission rate; time delays; stability; Hopf bifurcation
Mathematics Subject Classification: 34C23, 93A30

1. Introduction

Tuberculosis (TB) is a chronic zoonotic infectious disease caused by Mycobacterium tuberculosis. It is characterized by the formation of tuberculous granulomas within affected organs, followed by caseous necrosis or calcification at the granuloma center [1]. While tuberculosis primarily affects the lungs, it can also infect other parts of the body. Most infected individuals remain asymptomatic, a condition known as latent tuberculosis infection. The majority of individuals who develop tuberculosis experience a latent period before the onset of active disease. Classic symptoms of tuberculosis include chronic cough, hemoptysis, fever, night sweats, and weight loss. Infection in other organs may lead to additional symptoms. Tuberculosis is an airborne disease, meaning that the pathogen is transmitted via respiratory droplets expelled by individuals with active pulmonary TB when coughing, sneezing, or speaking. In contrast, patients with latent tuberculosis do not transmit the disease. Active tuberculosis is more frequently observed in individuals with plague or HIV/AIDS and is diagnosed through chest radiography, microscopic examination, and culture of bodily fluids. Latent tuberculosis can be detected using the tuberculin skin test or blood-based assays. Globally, approximately one-third of the population harbors latent tuberculosis, which is non-contagious. In 2014, there were an estimated 9.6 million cases of active tuberculosis and 1.5 million TB-related deaths, with 95% of fatalities occurring in developing countries. Since 2000, although the global incidence of new tuberculosis cases has been steadily declining, up to 80% of the population in many Asian and African countries still test positive for tuberculosis infection [2].

Mathematical models of infectious disease dynamics are essential tools for understanding disease transmission mechanisms and evaluating the effectiveness of intervention measures. In recent years, significant progress has been made in the study of TB transmission dynamics. Research has shown that TB transmission is influenced not only by direct contact between individuals, but also by complex dynamic mechanisms such as exogenous reinfection, endogenous reactivation, vaccination, and treatment interventions [3–10]. Various mathematical models have been developed, including those based on fractional-order derivatives [7, 8], models incorporating different isolation groups [4], and models integrating vaccination and treatment strategies [3, 7]. Additionally, considering the latent period of TB, many studies have introduced time delays to more accurately describe disease transmission dynamics [5, 6, 9]. These studies have not only deepened our understanding of TB transmission mechanisms, but also provided theoretical support for public health policy development.

Human mobility plays a crucial role in the transmission of infectious diseases, particularly in the epidemiological study of TB, where numerous studies have investigated how mobility influences disease spread [11–25]. Research has found that individual mobility patterns, population migration, and intercity movements significantly impact the spatial diffusion of diseases [11, 17, 18, 23]. Specifically, some studies have analyzed the effects of urban mobility patterns on disease transmission using complex network methods [23], while others have simulated non-local mobility between different geographic regions using reaction-diffusion models and population transfer operators [22]. Furthermore, some studies have explored how mobility influences disease control. For instance, adaptive mobility responses have been shown to effectively reduce the infection risk for susceptible individuals [13], and mobility resilience analysis has revealed the heterogeneity of population responses to epidemics [20]. These findings indicate that human mobility not only affects the geographical spread of infectious diseases, but also plays a critical role in designing and implementing control measures.

Time delay is an essential factor in infectious disease dynamic models, as it plays a key role in describing latent periods, immunity waning, and treatment delays [24–28]. In TB transmission studies, time delays are often incorporated to model the effects of latency while considering vaccination and treatment interventions [3–5]. Recent studies have further explored the impact of delays on disease transmission stability, revealing that time delays may induce backward bifurcations, thereby influencing the long-term transmission dynamics of diseases [12, 27]. Additionally, delayed models cannot only determine key parameters affecting disease transmission, but also aid in designing optimal control strategies to mitigate disease spread [26, 27]. For example, studies using delayed optimal control methods have demonstrated how control strategies can effectively suppress disease outbreaks [26]. These findings suggest that incorporating time delays can significantly enhance the descriptive power of infectious disease models, enabling more accurate predictions of disease transmission trends and the optimization of intervention measures.

Overall, significant progress has been made in tuberculosis dynamic modeling, the impact analysis of population mobility, and time-delay modeling. However, further investigation is needed to explore the interactions among these factors and their practical applications in epidemic prevention and control, aiming to enhance the predictive power of models and their policy guidance value. Therefore, this study integrates time delays and a population mobility function into the SEIR model to analyze the stability of equilibrium points, the existence of Hopf bifurcation, and the influence of the population mobility function on equilibrium states.

2. Delayed SEIRm model

Mobility-adaptive behavior refers to individuals who, when exposed to the risk of infection, reduce the risk of infection by reducing social contact, decreasing mobility, or changing behavioral patterns. This behavior is spontaneous and influenced by a combination of economic interests and health risks. In the absence of an epidemic, population mobility is usually determined by economic needs, whereas when the risk of infection rises, people go out less and avoid high-risk areas, thus affecting the kinetic properties of disease transmission. To capture the adaptive behavior of population mobility, this study introduces the parameter m, representing the intensity of population movement. Furthermore, since tuberculosis typically involves substantial delays both before individuals become infectious and during the recovery phase, the model incorporates an infection delay τ_1 and a recovery delay τ_2 to better reflect the transmission dynamics. In this section, based on the transmission mechanism of tuberculosis, we construct a dual-delay SEIRm model incorporating a population mobility function [13].

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S(t) - \dfrac{\beta m(t) I(t) S(t)}{1 + hI(t)}, \\ \dfrac{dE}{dt} = \dfrac{\beta m(t) I(t) S(t)}{1 + hI(t)} - \varepsilon E(t-\tau_1) - \mu E(t), \\ \dfrac{dI}{dt} = \varepsilon E(t-\tau_1) - \gamma I(t-\tau_2) - dI(t) - \mu I(t), \\ \dfrac{dR}{dt} = \gamma I(t-\tau_2) - \mu R(t), \\ \dfrac{dm}{dt} = m(t)\left[b - am(t) - \dfrac{\alpha I(t)}{1 + hI(t)}\right]. \end{cases} \tag{2.1}$$

The initial values are as follows:

$$S(t) = \kappa_1(t) > 0, \quad E(t) = \kappa_2(t) > 0, \quad I(t) = \kappa_3(t) > 0, \quad R(t) = \kappa_4(t) > 0, \quad m(t) = \kappa_5(t) > 0,$$

where $S(t)$, $E(t)$, $I(t)$, and $R(t)$ denote the number of susceptible, latent, infected, and recovered populations at a given moment, and $m(t)$ denotes the intensity of population movement. It is easy to conclude that $(\kappa_1(t), \kappa_2(t), \kappa_3(t), \kappa_4(t), \kappa_5(t)) \in R_+^5$ when $t \in [-\tau, 0]$.

Table 1 summarizes the definitions of the parameters used in this work.

Table 1. Definition of the parameters.

Parameters	Interpretations
Λ	Total number of births
β	Dissemination exposure rate
μ	Natural mortality
h	Infection inhibition rate
ε	Infection conversion rate
γ	Infection recovery rate
d	Disease death rate
b	Peak population movements
a	Mobility decay factor
α	Mobility impact factor
τ_1	Infection delay
τ_2	Recovery delay

3. Stability and Hopf bifurcation of disease-free equilibrium points

3.1. Disease-free equilibrium point and basic regeneration number

In this section, we prove the existence of the disease-free equilibrium points E_0 and E_1 and determine the basic reproduction number R_0 of the system (2.1). In the absence of population mobility, the semi-trivial equilibrium point is given by $E_0 = (\Lambda/\mu, 0, 0, 0)$. When population mobility is considered, the disease-free equilibrium point is $E_1 = (\Lambda/\mu, 0, 0, b/a)$.

The basic reproduction number R_0 is typically defined as the average number of secondary infections generated by a single infected individual in a fully susceptible population during their infectious period. When $R_0 > 1$, the number of cases increases exponentially, leading to an outbreak, whereas when $R_0 < 1$, the number of cases gradually declines to zero, causing the outbreak to disappear. The next-generation matrix method is employed to determine the basic reproduction number, given by $R_0 = \rho(FV^{-1})$, where $\rho(FV^{-1})$ represents the spectral radius of the matrix FV^{-1} [29]. Here, f denotes the rate of new infections, while v represents the rate at which infected individuals are removed from the infectious class (through recovery, death, or transition to other states).

$$f = \begin{bmatrix} 0 \\ \dfrac{\beta mIS}{1+hI} \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -\Lambda + \mu S + \dfrac{\beta mIS}{1+hI} \\ (\varepsilon+\mu)E \\ -\varepsilon E + (\gamma + d + \mu)I \\ -\gamma I + \mu R \end{bmatrix}.$$

At the disease-free equilibrium point E_1, we compute the Jacobian matrices F and V.

$$F=\begin{bmatrix}0 & \dfrac{\Lambda\beta b}{a\mu}\\ 0 & 0\end{bmatrix}.$$

$$V=\begin{bmatrix}\varepsilon+\mu & 0\\ -\varepsilon & \gamma+d+\mu\end{bmatrix}.$$

$$FV^{-1}=\begin{bmatrix}\dfrac{\varepsilon\Lambda\beta b}{a\mu\left(\varepsilon+\mu\right)\left(\gamma+d+\mu\right)} & \dfrac{\Lambda\beta b}{a\mu\left(\gamma+d+\mu\right)}\\ 0 & 0\end{bmatrix}.$$

$$R_0=\rho\left(FV^{-1}\right)=\frac{\varepsilon\Lambda\beta b}{a\mu\left(\varepsilon+\mu\right)\left(\gamma+d+\mu\right)}.$$

3.2. Stability of disease-free equilibrium point and Hopf bifurcation

In this paper, the system (2.1) is simplified as follows:

$$\begin{cases}\dfrac{dS}{dt}=\Lambda-\mu S\left(t\right)-\dfrac{\beta m(t)I(t)S\left(t\right)}{1+hI(t)},\\ \dfrac{dE}{dt}=\dfrac{\beta m(t)I(t)S\left(t\right)}{1+hI(t)}-\varepsilon E(t-\tau_1)-\mu E(t),\\ \dfrac{dI}{dt}=\varepsilon E(t-\tau_1)-\gamma I(t-\tau_2)-dI(t)-\mu I(t),\\ \dfrac{dm}{dt}=m(t)\left[b-am(t)-\dfrac{\alpha I(t)}{1+hI(t)}\right].\end{cases}\tag{3.1}$$

We apply the Routh–Hurwitz criterion to analyze the stability of the system and use the transversality condition to examine the existence of a Hopf bifurcation.

$$J_{E_1}=\begin{bmatrix}-\mu-\dfrac{\beta m(t)I(t)}{1+hI(t)} & 0 & -\dfrac{\beta m(t)S\left(t\right)}{[1+hI(t)]^2} & -\dfrac{\beta m(t)I(t)S\left(t\right)}{1+hI(t)}\\ \dfrac{\beta m(t)I(t)}{1+hI(t)} & -\varepsilon\dfrac{dE(t-\tau_1)}{dE(t)}-\mu & \dfrac{\beta m(t)S\left(t\right)}{[1+hI(t)]^2} & \dfrac{\beta m(t)I(t)S\left(t\right)}{1+hI(t)}\\ 0 & \varepsilon\dfrac{dE(t-\tau_1)}{dE(t)} & -\gamma\dfrac{dI(t-\tau_2)}{dI(t)}-d-\mu & 0\\ 0 & 0 & -m(t)\dfrac{\alpha}{[1+hI(t)]^2} & -am(t)\end{bmatrix}.$$

$$det\left(\lambda E-J_E\right)=\left(\lambda+\mu\right)\begin{vmatrix}\lambda+\varepsilon e^{-\lambda\tau_1}+\mu & -\dfrac{\lambda\beta b}{a\mu} & 0\\ -\varepsilon e^{-\lambda\tau_1} & \lambda+\gamma e^{-\lambda\tau_2}+\epsilon+\mu & 0\\ 0 & \dfrac{\alpha b}{a} & \lambda+b\end{vmatrix}.$$

$$\varphi_1(\lambda,\tau_1,\tau_2)=\lambda^2+(d+2\mu)\lambda+\mu(d+\mu)+\varepsilon(\lambda+d+\mu-\frac{\Lambda\beta b}{a\mu})e^{-\lambda\tau_1}+\gamma(\lambda+\mu)e^{-\lambda\tau_2}+\gamma\varepsilon e^{-\lambda(\tau_1+\tau_2)}.$$

Case 1. *When* $\tau_1 = \tau_2 = 0$*,*

$$\varphi_1(\lambda, 0, 0) = \lambda^2 + c_{01}\lambda + c_{02} = 0,$$

where

$$\begin{aligned} c_{01} &= d + 2\mu + \varepsilon + \gamma, \\ c_{02} &= (1 - R_0)(\mu + \varepsilon)(d + \mu + \gamma). \end{aligned}$$

It is evident that $c_{01} > 0$*, and when* $R_0 < 1$*, we have* $c_{02} > 0$*.*

$$\Delta = \begin{vmatrix} c_{01} & 0 \\ 1 & c_{02} \end{vmatrix} = c_{01}c_{02} > 0.$$

According to the Routh-Hurwitz criterion, we derive the following theorem.

Theorem 3.1. *When* $\tau_1 = \tau_2 = 0$*:*

(1) If $R_0 < 1$*, the system is locally asymptotically stable at the disease-free equilibrium point* E_1*.*
(2) If $R_0 > 1$*, the system is unstable at the disease-free equilibrium point* E_1*.*

Case 2. *When* $\tau_1 = 0, \tau_2 \neq 0$*,*

$$\varphi_1(\lambda, 0, \tau_2) = \lambda^2 + (d + 2\mu + \varepsilon)\lambda + c_0 + \gamma(\lambda + \mu + \varepsilon)e^{-\lambda\tau_2} = 0. \tag{3.2}$$

Let $\lambda = i\omega$ be a pure imaginary root of $\varphi_1(\lambda, 0, \tau_2)$. Substituting it into Eq (3.2), we obtain

$$\varphi_1(i\omega, 0, \tau_2) = -\omega^2 + (d + 2\mu + \varepsilon)i\omega + \mu(d + \mu) + \varepsilon(d + \mu - \frac{\Lambda\beta b}{a\mu}) + \gamma(i\omega + \mu + \varepsilon)e^{-i\omega\tau_2}.$$

From $\varphi_1(i\omega, 0, \tau_2) = 0$, it is clear that the real and imaginary parts of the above equation are both equal to zero, which gives

$$\begin{cases} \omega^2 - c_0 = \gamma\omega\sin\omega\tau_2 + \gamma(\mu + \varepsilon)\cos\omega\tau_2, \\ (d + 2\mu + \varepsilon)\omega = \gamma(\mu + \varepsilon)\sin\omega\tau_2 - \gamma\omega\cos\omega\tau_2. \end{cases} \tag{3.3}$$

By squaring and adding the two equations in Eq (3.3), we obtain the following:

$$\phi_1(\omega) = \omega^4 + c_{11}\omega^2 + c_{12} = 0,$$

where

$$\begin{aligned} c_{11} &= -2c_0 + (d + 2\mu + \varepsilon)^2 - \gamma^2, \\ c_{12} &= (c_0)^2 - \gamma^2(\varepsilon + \mu)^2, \\ c_0 &= c_0^1 - c_0^2, \\ c_0^1 &= \mu(d + \mu), \\ c_0^2 &= \varepsilon(d + \mu - \frac{\Lambda\beta b}{a\mu}). \end{aligned}$$

When $R_0 < 1$, it follows that $c_{12} < 0$. Based on the median theorem, $\lim_{\omega\to 0}\phi_1(\omega) < 0$, and $\lim_{\omega\to\infty}\phi_1(\omega) > 0$, we can deduce that there exists at least one positive root $\dot{\omega} \in (0, +\infty)$ such that $\phi_1(\dot{\omega}) = 0$.

Let $\omega_2^{(j)}$ $(j = 1, 2)$ be the positive root of $\phi_1(\omega)$.

From Eq (3.3), we derive

$$\begin{cases} \sin\omega\tau_2 = \dfrac{\omega^2 - c_0 - \gamma(\mu+\varepsilon)\cos\omega\tau_2}{\gamma\omega}, \\ \cos\omega\tau_2 = \dfrac{\gamma(\mu+\varepsilon)\sin\omega\tau_2 - (d+2\mu+\varepsilon)\omega}{\gamma\omega}. \end{cases}$$

Following that, it is available that

$$\cos\omega\tau_2 = \frac{(\mu+\varepsilon)(\omega^2 - c_0) - \omega^2(d+2\mu+\varepsilon)}{\gamma\left[\omega^2 + (\mu+\varepsilon)^2\right]},$$

$$\tau_2^{j,k} = \frac{1}{\omega_2^{(j)}}\left(\cos^{-1}\left[\frac{(\mu+\varepsilon)(\omega^2 - c_0) - \omega^2(d+2\mu+\varepsilon)}{\gamma\left[\omega^2 + (\mu+\varepsilon)^2\right]}\right] + 2k\pi\right),$$

where $j = 1, 2, k \in N$.

$$\frac{d\varphi_1(\lambda,0,\tau_2)}{d\tau_2} = 2\lambda\frac{d\lambda}{d\tau_2} + (d+2\mu+\varepsilon)\frac{d\lambda}{d\tau_2} + \gamma e^{-\lambda\tau_2}\frac{d\lambda}{d\tau_2} + \gamma(\lambda+\mu+\varepsilon)e^{-\lambda\tau_2}(-\lambda - \tau_2\frac{d\lambda}{d\tau_2}).$$

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{1}{\lambda(\lambda+\mu+\varepsilon)} - \frac{\tau_2}{\lambda} - \frac{2\lambda + d + 2\mu + \varepsilon}{\lambda\left[\lambda^2 + (d+2\mu+\varepsilon)\lambda + c_0\right]}. \tag{3.4}$$

Let $\tau_2^0 = \min\{\tau_2^{j,k}\}$ and $\omega_2^0 = \arg\min\{\tau_2^{j,k}\}$. When $\tau_2 = \tau_2^0$, it follows that $\psi_1(\tau_2^0) = 0$ and $\omega_1(\tau_2^0) = \omega_2^0$. Substituting $\lambda_1(\tau_2^0) = \psi_1(\tau_2^0) + i\omega_1(\tau_2^0)$ into Eq (3.4), we obtain

$$\left(\frac{d\lambda_1}{d\tau_2}\right)^{-1} = \frac{1}{i\omega_2^0\left(i\omega_2^0 + \mu + \varepsilon\right)} - \frac{\tau_2^0}{i\omega_2^0} - \frac{2i\omega_2^0 + d + 2\mu + \varepsilon}{i\omega_2^0\left[\left(i\omega_2^0\right)^2 + (d+2\mu+\varepsilon)\,i\omega_2^0 + c_0\right]}.$$

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda_1}{d\tau_2}\right)^{-1} =& \frac{\left[(\mu+\varepsilon)^2 + (\omega_2^0)^2\right]\left[(d+2\mu+\varepsilon)^2 - 2c_0 + 2\omega_2^0\right]}{\left[(\mu+\varepsilon)^2 + (\omega_2^0)^2\right]\left\{\left[c_0 - (\omega_2^0)^2\right]^2 + (d+2\mu+\varepsilon)^2(\omega_2^0)^2\right\}} \\ &- \frac{\left[c_0 - (\omega_2^0)^2\right]^2 + (d+2\mu+\varepsilon)^2(\omega_2^0)^2}{\left[(\mu+\varepsilon)^2 + (\omega_2^0)^2\right]\left\{\left[c_0 - (\omega_2^0)^2\right]^2 + (d+2\mu+\varepsilon)^2(\omega_2^0)^2\right\}}.\end{aligned}$$

$$\left[(\mu+\varepsilon)^2 + (\omega_2^0)^2\right]\left[(d+2\mu+\varepsilon)^2 - 2c_0 + 2\omega_2^0\right] - \left\{\left[c_0 - (\omega_2^0)^2\right]^2 + (d+2\mu+\varepsilon)^2(\omega_2^0)^2\right\} \neq 0. \tag{3.5}$$

It follows that if Eq (3.5) is satisfied, then

$$\mathrm{Re}\left(\frac{d\lambda_1}{d\tau_2}\right)^{-1} \neq 0.$$

Theorem 3.2. *When $\tau_1 = 0, \tau_2 \neq 0$, it suffices to satisfy Eq (3.5) to obtain the following conclusion:*

(1) If $\tau_2 \in (0, \tau_2^0)$, the system is locally asymptotically stable at the disease-free equilibrium point E_1.
(2) If $\tau_2 > \tau_2^0$, the system is unstable at the disease-free equilibrium point E_1.

(3) If $\tau_2 = \tau_2^0$, the system has a Hopf bifurcation at the disease-free equilibrium point E_1.

Case 3. *When $\tau_1 \neq 0, \tau_2 = 0$,*

$$\varphi_1(\lambda, \tau_1, 0) = \lambda^2 + c_1^1\lambda + c_1^2 + \varepsilon(\lambda + c_1^3)e^{-\lambda\tau_1} = 0. \tag{3.6}$$

Let $\lambda = i\omega$ be a pure imaginary root of $\varphi_1(\lambda, \tau_1, 0)$. Substituting it into Eq (3.6), we obtain

$$\varphi_1(i\omega, \tau_1, 0) = -\omega^2 + c_1^1 i\omega + c_1^2 + \varepsilon(i\omega + c_1^3)e^{-i\omega\tau_1}.$$

From $\varphi_1(i\omega, \tau_1, 0) = 0$, it follows that both the real and imaginary components of the equation must be equal to zero, yielding

$$\begin{cases} \omega^2 - c_1^2 = \varepsilon\omega \sin\omega\tau_1 + \varepsilon(\lambda + c_1^3)\cos\omega\tau_1, \\ c_1^1\omega = -\varepsilon\omega\cos\omega\tau_1 + \varepsilon(\lambda + c_1^3)\sin\omega\tau_1. \end{cases} \tag{3.7}$$

By squaring and adding the two equations in Eq (3.7), we obtain the following

$$\phi_2(\omega) = \omega^4 + c_{21}\omega^2 + c_{22},$$

where

$$\begin{aligned} c_{21} &= (c_1^1)^2 - 2c_1^2 - \varepsilon^2, \\ c_{22} &= (c_1^2)^2 - \varepsilon^2(c_1^3)^2, \\ c_1^1 &= d + 2\mu + \gamma, \\ c_1^2 &= \mu(d + \mu + \gamma), \\ c_1^3 &= d + \mu + \gamma - \frac{\Lambda\beta b}{a\mu}. \end{aligned}$$

When $R_0 < 1$, it directly follows that $c_{22} < 0$. Based on the median theorem, we deduce that there exists at least one positive root $\dot{\omega} \in (0, +\infty)$ such that $\phi_2(\dot{\omega}) = 0$.

Let $\omega_1^{(j)} (j = 1, 2)$ be the positive root of $\phi_2(\omega)$.

From Eq (3.7), we can obtain

$$\begin{cases} \sin\omega\tau_1 = \dfrac{\omega^2 - c_1^2 - \varepsilon(\lambda + c_1^3)\cos\omega\tau_1}{\varepsilon\omega}, \\ \cos\omega\tau_1 = \dfrac{-c_1^1\omega + \varepsilon(\lambda + c_1^3)\sin\omega\tau_1}{\varepsilon\omega}. \end{cases}$$

Then, we can get

$$\cos\omega\tau_1 = \frac{-c_1^1\omega^2 + c_1^3(\omega^2 - c_1^2)}{\varepsilon\omega^2 + \varepsilon(\lambda + c_1^3)^2},$$

$$\tau_1^{j,k} = \frac{1}{\omega_1^{(j)}}\left(\cos^{-1}\left[\frac{-c_1^1\omega^2 + c_1^3(\omega^2 - c_1^2)}{\varepsilon\omega^2 + \varepsilon(\lambda + c_1^3)^2}\right] + 2k\pi\right),$$

where $j = 1, 2, k \in N$.

$$\frac{d\varphi_2(\lambda, \tau_1, 0)}{d\tau_1} = 2\lambda\frac{d\lambda}{d\tau_1} + \varepsilon(\lambda + c_1^3)e^{-\lambda\tau_1}(-\lambda - \tau_1\frac{d\lambda}{d\tau_1}) + c_1^1\frac{d\lambda}{d\tau_1} + \varepsilon e^{-\lambda\tau_1}\frac{d\lambda}{d\tau_1}.$$

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{\tau_1}{\lambda} + \frac{1}{\lambda(\lambda + c_1^3)} - \frac{2\lambda + c_1^1}{\lambda\left[\lambda^2 + c_1^1\lambda + c_1^2\right]}. \tag{3.8}$$

Let $\tau_1^0 = \min\{\tau_1^{j,k}\}$ and $\omega_1^0 = \arg\min\{\tau_1^{j,k}\}$. When $\tau_1 = \tau_1^0$, it follows that $\psi_2(\tau_1^0) = 0$ and $\omega_2(\tau_1^0) = \omega_1^0$. Substituting $\lambda_2(\tau_1^0) = \psi_2(\tau_1^0) + i\omega_2(\tau_1^0)$ into Eq (3.8), we obtain

$$\left(\frac{d\lambda_2}{d\tau_1}\right)^{-1} = -\frac{\tau_1^0}{\mathrm{i}\omega_1^0} + \frac{1}{\mathrm{i}\omega_1^0(\mathrm{i}\omega_1^0 + c_1^3)} - \frac{2\mathrm{i}\omega_1^0 + c_1^1}{\mathrm{i}\omega_1^0\left[-(\omega_1^0)^2 + c_1^1\mathrm{i}\omega_1^0 + c_1^2\right]}.$$

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda_2}{d\tau_1}\right)^{-1} &= \frac{\left\{2\left[(\omega_1^0)^2 - c_1^2\right] + (c_1^1)^2\right\} \times \left[(\omega_1^0)^2 + (c_1^3)^2\right]}{\left[(\omega_1^0)^2 + (c_1^3)^2\right] \times \left\{\left[c_1^2 - (\omega_1^0)^2\right]^2 + (c_1^1\omega_1^0)^2\right\}} \\ &\quad - \frac{\left[c_1^2 - (\omega_1^0)^2\right]^2 - (c_1^1\omega_1^0)^2}{\left[(\omega_1^0)^2 + (c_1^3)^2\right] \times \left\{\left[c_1^2 - (\omega_1^0)^2\right]^2 + (c_1^1\omega_1^0)^2\right\}}.\end{aligned}$$

$$\left\{2\left[(\omega_1^0)^2 - c_1^2\right] + \left(c_1^1\right)^2\right\} \times \left[\left(\omega_1^0\right)^2 + \left(c_1^3\right)^2\right] - \left[c_1^2 - \left(\omega_1^0\right)^2\right]^2 - \left(c_1^1\omega_1^0\right)^2 \neq 0. \tag{3.9}$$

It follows that if Eq (3.9) is satisfied, then

$$\mathrm{Re}\left(\frac{d\lambda_2}{d\tau_1}\right)^{-1} \neq 0.$$

Theorem 3.3. *When $\tau_1 \neq 0, \tau_2 = 0$, as long as Eq (3.9) is satisfied, the following conclusions hold:*

(1) If $\tau_1 \in (0, \tau_1^0)$, the system is locally asymptotically stable at the disease-free equilibrium point E_1.
(2) If $\tau_1 > \tau_1^0$, the system is unstable at the disease-free equilibrium point E_1.
(3) If $\tau_1 = \tau_1^0$, the system has a Hopf bifurcation at the disease-free equilibrium point E_1.

Case 4. *When $\tau_1 \neq 0$ and $\tau_2 \neq 0$, analyzing the Hopf bifurcation of a model with two time lags simultaneously becomes challenging. In this case, it may be useful to treat τ_1 as the independent variable and set $\tau_2 \in (0, \tau_2^0)$ as a fixed parameter.*

$$\begin{aligned}\varphi_1(\lambda, \tau_1, \tau_2) = \lambda^2 + (d + 2\mu)\lambda + \mu(d + \mu) + \varepsilon(\lambda + d + \mu - \frac{\Lambda\beta b}{a\mu})e^{-\lambda\tau_1} \\ + \gamma(\lambda + \mu)e^{-\lambda\tau_2} + \gamma\varepsilon e^{-\lambda(\tau_1+\tau_2)} = 0.\end{aligned} \tag{3.10}$$

Let $\lambda + i\omega$ be the pure imaginary root of $\varphi_1(\lambda, \tau_1, \tau_2)$. Substituting it into Eq (3.10), we can get

$$\begin{aligned}\varphi_1(i\omega, \tau_1, \tau_2) = -\omega^2 + (d + 2\mu)i\omega + \mu(d + \mu) + \gamma\varepsilon e^{-i\omega(\tau_1+\tau_2)} \\ + \varepsilon(i\omega + d + \mu - \frac{\Lambda\beta b}{a\mu})e^{-i\omega\tau_1} + \gamma(i\omega + \mu)e^{-i\omega\tau_2}.\end{aligned}$$

From $\varphi_1(\lambda, \tau_1, \tau_2) = 0$, it follows that both the real and imaginary parts of the equation must be equal to zero, yielding

$$\begin{cases}-\omega^2+\mu(d+\mu)+\varepsilon(d+\mu-\dfrac{\Lambda\beta b}{a\mu})\cos\omega\tau_1+\varepsilon\omega\sin\omega\tau_1\\ \quad+\gamma\mu\cos\omega\tau_2+\gamma\omega\sin\omega\tau_2+\gamma\varepsilon\cos\omega(\tau_1+\tau_2)=0,\\ (d+2\mu)\omega+\varepsilon\omega\cos\omega\tau_1-\varepsilon(d+\mu-\dfrac{\Lambda\beta b}{a\mu})\sin\omega\tau_1\\ \quad+\gamma\omega\cos\omega\tau_2-\gamma\mu\sin\omega\tau_2-\gamma\varepsilon\sin\omega(\tau_1+\tau_2)=0.\end{cases}$$

After simplification, we obtain

$$\begin{cases}c_3^1\cos\omega\tau_1+c_3^2\sin\omega\tau_1=\omega^2-\mu(d+\mu)-\gamma\mu\cos\omega\tau_2-\gamma\omega\sin\omega\tau_2,\\ c_3^2\cos\omega\tau_1-c_3^1\sin\omega\tau_1=-(d+2\mu)\omega-\gamma\omega\cos\omega\tau_2+\gamma\mu\sin\omega\tau_2.\end{cases}\tag{3.11}$$

where

$$c_3^1=\varepsilon(d+\mu-\frac{\Lambda\beta b}{a\mu})+\gamma\varepsilon\cos\omega\tau_2,$$
$$c_3^2=\varepsilon\omega-\gamma\varepsilon\sin\omega\tau_2.$$

By squaring and adding the two equations in Eq (3.11), we obtain the following:

$$\phi_3(\omega)=\omega^4+c_{31}\omega^3+c_{32}\omega^2+c_{33}\omega+c_{34},$$

where

$$\begin{aligned}c_{31}&=-2\gamma\sin\omega\tau_2,\\ c_{32}&=d^2+\gamma^2-\varepsilon^2+2(d+\mu)(\mu+\gamma\cos\omega\tau_2),\\ c_{33}&=2\gamma(\varepsilon^2-\mu^2)\sin\omega\tau_2,\\ c_{34}&=\mu c_0^1(d+\mu+2\gamma\cos\omega\tau_2)+\gamma^2(\mu^2-\varepsilon^2)-c_0^2(c_0^2+2\gamma\varepsilon).\end{aligned}$$

If $c_{34}<0$, then $\lim\limits_{\omega\to 0}\phi_3(\omega)<0$ and $\lim\limits_{\omega\to\infty}\phi_3(\omega)>0$, which implies that there exists at least one positive root $\dot{\omega}\in(0,+\infty)$ such that $\phi_3(\dot{\omega})=0$.

Let $\omega_{1*}^{(j)},(j=1,2,3,4)$ denote the positive roots of $\phi_3(\omega)$.

If Eq (3.12) is satisfied, then $c_{34}<0$ holds.

$$c_0^1\left[c_0^1+2\gamma\mu\cos\omega\tau_2\right]+\gamma^2\mu^2<\gamma^2\varepsilon^2+c_0^2\left[c_0^2+2\gamma\varepsilon\right].\tag{3.12}$$

By deflating and transforming the inequality, we can easily compute the following equation:

$$c_0^1\left(c_0^1+2\gamma\mu\cos\omega\tau_2\right)+\gamma^2\mu^2\le(c_1^2)^2,$$

$$\gamma^2\varepsilon^2+c_0^2\left(c_0^2+2\gamma\varepsilon\right)=\varepsilon^2(c_1^3)^2.$$

Clearly, the inequality $(c_1^2)^2<\varepsilon^2(c_1^3)^2$ naturally holds when $R_0<1$, and hence Eq (3.12) holds.

From Eq (3.11), we obtain the following:

$$\begin{cases}\sin\omega\tau_1=\dfrac{C_1^1-B_1^1\cos\omega\tau_1}{c_3^2},\\ \cos\omega\tau_1=\dfrac{A_1^1+B_1^1\sin\omega\tau_1}{c_3^2}.\end{cases}$$

Then, we can get

$$\cos\omega\tau_1 = \frac{A_1^1c_3^2 + B_1^1C_1^1}{(c_3^2)^2 + (B_1^1)^2},$$

$$\tau_{1^*}^{j,k} = \frac{1}{\omega_{1^*}^{(j)}}\left(\cos^{-1}\left[\frac{A_1^1c_3^2 + B_1^1C_1^1}{(c_3^2)^2 + (B_1^1)^2}\right] + 2k\pi\right),$$

where $j = 1, 2, k \in N$, and

$$\begin{aligned} A_1^1 &= -(d + 2\mu)\omega - \gamma\omega\cos\omega\tau_2 + \gamma\mu\sin\omega\tau_2. \\ B_1^1 &= c_0^2 + \gamma\varepsilon\cos\omega\tau_2. \\ C_1^1 &= \omega^2 - c_0^1 - \gamma\mu\cos\omega\tau_2 - \gamma\omega\sin\omega\tau_2. \end{aligned}$$

$$\begin{aligned} \frac{d\varphi_1(\lambda,\tau_1,\tau_2)}{d\tau_1} &= 2\lambda\frac{d\lambda}{d\tau_1} + (c_1^1 - \gamma)\frac{d\lambda}{d\tau_1} + (\varepsilon\lambda + c_0^2)e^{-\lambda\tau_1}(-\lambda - \tau_1\frac{d\lambda}{d\tau_1}) \\ &+ \varepsilon e^{-\lambda\tau_1}\frac{d\lambda}{d\tau_1} + \gamma(\lambda + \mu)e^{-\lambda\tau_2}(-\tau_2\frac{d\lambda}{d\tau_1}) + \gamma e^{-\lambda\tau_2}\frac{d\lambda}{d\tau_1} \\ &+ \gamma\varepsilon e^{-\lambda(\tau_1+\tau_2)}(-\lambda - (\tau_1 + \tau_2)\frac{d\lambda}{d\tau_1}). \end{aligned}$$

$$\begin{aligned} \left(\frac{d\lambda}{d\tau_1}\right)^{-1} &= -\frac{\tau_1}{\lambda} + \frac{1 - \gamma\tau_2(\lambda + \mu)e^{-\lambda\tau_2}}{\lambda(\lambda + c_1^3 - \gamma) + \gamma\lambda e^{-\lambda\tau_2}} \\ &- \frac{2\lambda + c_1^1 - \gamma + \gamma e^{-\lambda\tau_2} - \gamma\tau_2(\lambda + \mu)e^{-\lambda\tau_2}}{\lambda\left[\lambda^2 + (c_1^1 - \gamma)\lambda + c_0^1 + \gamma(\lambda + \mu)e^{-\lambda\tau_2}\right]}. \end{aligned} \tag{3.13}$$

Let $\dot{\tau}_1^0 = \min\{\dot{\tau}_1^{j,k}\}$ and $\dot{\omega}_1^0 = \arg\min\{\dot{\tau}_1^{j,k}\}$. When $\tau_1 = \dot{\tau}_1^0$, it follows that $\dot{\psi}_1(\dot{\tau}_1^0) = 0$ and $\dot{\omega}_1(\dot{\tau}_1^0) = \dot{\omega}_1^0$. Substituting $\dot{\lambda}_1(\dot{\tau}_1^0) = \dot{\psi}_1(\dot{\tau}_1^0) + i\dot{\omega}_1(\dot{\tau}_1^0)$ into Eq (3.13), we obtain the following:

$$\left(\frac{d\dot{\lambda}_1}{d\tau_1}\right)^{-1} = -\frac{\dot{\tau}_1^0}{i\dot{\omega}_1^0} + \frac{Q_1^1 i + W_1^1}{T_1^1 i + Y_1^1} + \frac{\gamma\tau_2 i\sin\dot{\omega}_1^0\tau_2 + 1 - \gamma\tau_2\cos\dot{\omega}_1^0\tau_2}{U_1^1 i + O_1^1},$$

$$\begin{aligned} \mathrm{Re}\left(\frac{d\dot{\lambda}_1}{d\tau_1}\right)^{-1} &= \frac{\left[Q_1^1T_1^1 + W_1^1Y_1^1\right]\left[(U_1^1)^2 + (O_1^1)^2\right]}{\left[(T_1^1)^2 + (Y_1^1)^2\right]\left[(U_1^1)^2 + (O_1^1)^2\right]} \\ &+ \frac{\left[U_1^1\gamma\tau_2\sin\dot{\omega}_1^0\tau_2 + (1 - \gamma\tau_2\cos\dot{\omega}_1^0\tau_2)O_1^1\right]\left[(T_1^1)^2 + (O_1^1)^2\right]}{\left[(T_1^1)^2 + (Y_1^1)^2\right]\left[(U_1^1)^2 + (O_1^1)^2\right]}, \end{aligned}$$

where

$$\begin{aligned} Q_1^1 &= 2\omega - \gamma\tau_2\cos\omega\tau_2 + \gamma(\tau_2\mu - 1)\sin\omega\tau_2, \\ W_1^1 &= c_1^1 - \gamma + \gamma(1 - \tau_2\mu)\cos\omega\tau_2 - \gamma\tau_2\mu\omega\sin\omega\tau_2, \\ T_1^1 &= \omega^3 - c_0^1 + \gamma\tau_2\mu\omega\cos\omega\tau_2 + \gamma\tau_2\omega^2\sin\omega\tau_2, \\ Y_1^1 &= (c_1^1 - \gamma)\omega^2 - \gamma\tau_2\omega^2\cos\omega\tau_2 + \gamma\tau_2\mu\omega\sin\omega\tau_2, \\ U_1^1 &= \frac{c_0^2}{\varepsilon}\omega + \gamma\omega\cos\omega\tau_2, \end{aligned}$$

$$O_1^1 = \gamma\omega\sin\omega\tau_2 - \omega^2,$$

$$\left[U_1^1\gamma\tau_2\sin\omega\tau_2 + (1-\gamma\tau_2\cos\omega\tau_2)O_1^1\right]\left[(T_1^1)^2 + (Y_1^1)^2\right] + \left[Q_1^1T_1^1 + W_1^1Y_1^1\right]\left[(U_1^1)^2 + (O_1^1)^2\right] \neq 0. \quad (3.14)$$

It follows that if Eq (3.14) is satisfied, then

$$\mathrm{Re}\left(\frac{d\lambda_1}{d\tau_1}\right)^{-1} \neq 0.$$

Theorem 3.4. *When $\tau_1 \neq 0, \tau_2 \in (0, \tau_2^0)$, as long as Eq (3.14) is satisfied, the following conclusions hold:*

(1) If $\tau_1 \in (0, \dot{\tau}_1^0)$, the system is locally asymptotically stable at the disease-free equilibrium point E_1.
(2) If $\tau_1 > \dot{\tau}_1^0$, the system is unstable in the disease-free equilibrium point E_1.
(3) If $\tau_1 = \dot{\tau}_1^0$, the system has a Hopf bifurcation at the disease-free equilibrium point E_1.

4. The existence, stability, and Hopf bifurcation of the endemic equilibrium point

4.1. The existence of the endemic equilibrium point

It follows from the fact that the equations in system (3.1) are equal to zero:

$$\begin{cases} \Lambda - (\mu + \dfrac{\beta m I}{1+hI})S = 0, \\ \dfrac{\beta m I}{1+hI}S - (\varepsilon+\mu)E = 0, \\ \varepsilon E - (\gamma + d + \mu)I = 0, \\ b - am - \dfrac{\alpha I}{1+hI} = 0. \end{cases} \quad (4.1)$$

By solving system (4.1), we obtain

$$\xi(I) = c_4^1 I^2 + c_4^2 I + c_4^3 = 0, \quad (4.2)$$

where

$$\begin{aligned} c_4^1 &= -(\varepsilon+\mu)(d+\gamma+\mu)\left[(bh-\alpha)\beta + a\mu h^2\right], \\ c_4^2 &= -(\varepsilon+\mu)(d+\gamma+\mu)(2a\mu h + b\beta) + \Lambda\varepsilon\beta(bh-\alpha), \\ c_4^3 &= \Lambda\varepsilon\beta b - a\mu(\varepsilon+\mu)(d+\gamma+\mu). \end{aligned}$$

We consider two cases for bh and α:
(I) $bh \geq \alpha$;
(II) $bh < \alpha$.

According to [13] , it can be seen that when $bh \geq \alpha$, people prioritize economic activities and daily life, and population mobility is less sensitive to the risk of infection. When $bh < \alpha$, it indicates that people are more likely to take aggressive actions to avoid the risk of infection.

When case (I) holds, we have $c_4^1 < 0$. Given that $R_0 > 1$, it follows that $c_4^3 > 0$. From the discriminant of the root, $(c_4^2)^2 - 4c_4^1c_4^3 > 0$, it can be seen that Eq (4.2) has two solutions. By Vieta's Theorem, these two roots must have opposite signs. Consequently, Eq (4.2) has a unique positive root.

When condition (II) holds, given that $R_0 > 1$, we have $c_4^3 > 0$. From the fourth equation of system (4.1), we obtain $0 < I < \dfrac{b}{\alpha - bh}$. At this point, two possible cases arise:

(1) $c_4^1 \leq 0$;

(2) $c_4^1 > 0$.

When (1) holds, there is $bh < \alpha \leq \left(b\beta h + a\mu h^2\right)/\beta$. When (2) holds, there is $\alpha > \left(b\beta h + a\mu h^2\right)/\beta$.

Substituting $I = 0$ and $I = b/(\alpha - bh)$ into Eq (4.2), respectively, yields the following:

$$\xi(0) = c_4^3 > 0.$$

$$\xi(\frac{b}{\alpha - bh}) = c_4^1(\frac{b}{\alpha - bh})^2 + c_4^2\frac{b}{\alpha - bh} + c_4^3 < 0.$$

Based on the discriminant of the roots, Vieta's theorem, and the properties of quadratic equations, when $I \in (0, b/(\alpha - bh))$, if case (1) or case (2) hold, then $\xi(I)$ has a unique positive root, denoted as $E_2 = (S^*, E^*, I^*, R^*, m^*)$.

4.2. Stability and Hopf bifurcation of endemic equilibrium point

$$J_{E_2} = \begin{bmatrix} -\mu - \dfrac{\beta m^*(b - am^*)}{\alpha} & 0 & 0 & -\dfrac{\beta S(t)(b - 2am^*)}{\alpha} \\ \dfrac{\beta m^*(b - am^*)}{\alpha} & -\varepsilon e^{-\lambda\tau_1} - \mu & 0 & \dfrac{\beta S(t)(b - 2am^*)}{\alpha} \\ 0 & \varepsilon e^{-\lambda\tau_1} & -\gamma e^{-\lambda\tau_2} - d - \mu & 0 \\ 0 & 0 & -\dfrac{\alpha m^*}{(1 + hI^*)^2} & -\alpha m^* \end{bmatrix}.$$

$$\det(\lambda E - J_{E_2}) = \begin{vmatrix} \lambda + \mu + \dfrac{\beta m^*(b - am^*)}{\alpha} & 0 & 0 & \dfrac{\beta S(t)(b - 2am^*)}{\alpha} \\ -\dfrac{\beta m^*(b - am^*)}{\alpha} & \lambda + \varepsilon e^{-\lambda\tau_1} + \mu & 0 & -\dfrac{\beta S(t)(b - 2am^*)}{\alpha} \\ 0 & -\varepsilon e^{-\lambda\tau_1} & \lambda + \gamma e^{-\lambda\tau_2} + d + \mu & 0 \\ 0 & 0 & \dfrac{\alpha m^*}{(1 + hI^*)^2} & \lambda + \alpha m^* \end{vmatrix}.$$

$$\begin{aligned} \varphi_2(\lambda, \tau_1, \tau_2) =& \lambda^4 + \left[\varepsilon e^{-\lambda\tau_1} + \gamma e^{-\lambda\tau_2} + d + 3\mu + am^* + \beta m^*\frac{b - am^*}{\alpha}\right]\lambda^3 \\ &+ \Big[(\varepsilon e^{-\lambda\tau_1} + \mu)(\gamma e^{-\lambda\tau_2} + d + \mu) + am^*(\varepsilon e^{-\lambda\tau_1} + \gamma e^{-\lambda\tau_2} + d + 2\mu) \\ &+ (\mu + \frac{\beta m^*(b - am^*)}{\alpha})(\varepsilon e^{-\lambda\tau_1} + \gamma e^{-\lambda\tau_2} + d + 2\mu + am^*)\Big]\lambda^2 \end{aligned}$$

$$
\begin{aligned}
&+\Big[(\mu+\frac{\beta m^*(b-am^*)}{\alpha}+am^*)(\varepsilon e^{-\lambda\tau_1}+\mu)(\gamma e^{-\lambda\tau_2}+d+\mu) \\
&+am^*(\mu+\frac{\beta m^*(b-am^*)}{\alpha})(\varepsilon e^{-\lambda\tau_1}+\gamma e^{-\lambda\tau_2}+d+2\mu) \\
&+\beta\varepsilon e^{-\lambda\tau_1}S^*m^*\frac{b-2am^*}{(1+hI^*)^2}\Big]\lambda+\beta\varepsilon\mu e^{-\lambda\tau_1}S^*m^*\frac{b-2am^*}{(1+hI^*)^2} \\
&+am^*(\mu+\frac{\beta m^*(b-am^*)}{\alpha})(\varepsilon e^{-\lambda\tau_1}+\mu)(\gamma e^{-\lambda\tau_2}+d+\mu)=0.
\end{aligned}
\tag{4.3}
$$

Case 5. *When $\tau_1=\tau_2=0$,*

$$
\varphi_2(\lambda,0,0)=\lambda^4+A_2^0\lambda^3+B_2^0\lambda^2+C_2^0\lambda+D_2^0=0,
\tag{4.4}
$$

where

$$
\begin{aligned}
A_2^0=&\varepsilon+\gamma+d+3\mu+am^*+\frac{\beta m^*(b-am^*)}{\alpha}, \\
B_2^0=&(\varepsilon+\mu)(\gamma+d+\mu)+am^*(\varepsilon+\gamma+d+2\mu)+\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+\gamma+d+2\mu+am^*), \\
C_2^0=&\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}+am^*\right](\varepsilon+\mu)(\gamma+d+\mu)+\beta\varepsilon S^*m^*\frac{b-2am^*}{(1+hI^*)^2} \\
&+am^*\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+\gamma+d+2\mu), \\
D_2^0=&am^*\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+\mu)(\gamma+d+\mu)+\beta\varepsilon\mu m^*\frac{b-2am^*}{(1+hI^*)^2}.
\end{aligned}
$$

Theorem 4.1. *When $\tau_1=\tau_2=0$, the system is locally asymptotically stable at the endemic equilibrium point E_2 if and only if all the coefficients in Eq* (4.4) *satisfy the following conditions:*

(1) $H_1>0$.

(2) $H_2=\begin{vmatrix} A_2^0 & C_2^0 \\ 1 & B_2^0 \end{vmatrix}=A_2^0B_2^0-C_2^0>0.$

(3) $H_3=\begin{vmatrix} A_2^0 & C_2^0 & 0 \\ 1 & B_2^0 & D_2^0 \\ 0 & A_2^0 & C_2^0 \end{vmatrix}=A_2^0B_2^0C_2^0-(A_2^0)^2D_2^0-(C_2^0)^2>0.$

(4) $H_4=\begin{vmatrix} A_2^0 & C_2^0 & 0 & 0 \\ 1 & B_2^0 & D_2^0 & 0 \\ 0 & A_2^0 & C_2^0 & 0 \\ 0 & 1 & B_2^0 & D_2^0 \end{vmatrix}=D_2^0\left[A_2^0B_2^0C_2^0-(A_2^0)^2D_2^0-(C_2^0)^2\right]>0.$

Case 6. *When $\tau_1=0,\tau_2\neq 0$,*

$$
\varphi_2(\lambda,0,\tau_2)=\lambda^4+F_2^1\lambda^3+A_2^1\lambda^2+G_2^1\lambda+B_2^1+\left[\lambda^3+C_2^1\lambda^2+E_2^1\lambda+D_2^1\right]\gamma e^{-\lambda\tau_2}=0,
\tag{4.5}
$$

where

$$
A_2^1=(\varepsilon+\mu)(d+\mu)+\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+d+2\mu+am^*)+am^*(\varepsilon+d+2\mu),
$$

$$
\begin{aligned}
B_2^1 =& am^*\left[\mu + \frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+\mu)(d+\mu) + S^*m^*\beta\varepsilon\frac{b-2am^*}{(1+hI^*)^2},\\
C_2^1 =& \varepsilon + 2\mu + am^* + \frac{\beta m^*(b-am^*)}{\alpha},\\
D_2^1 =& am^*\left[\mu + \frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+\mu),\\
E_2^1 =& am^*(\varepsilon+\mu) + \left[\mu + \frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+\mu+am^*),\\
F_2^1 =& \varepsilon + d + 3\mu + am^* + \frac{\beta m^*(b-am^*)}{\alpha},\\
G_2^1 =& \left[\mu + \frac{\beta m^*(b-am^*)}{\alpha} + am^*\right](\varepsilon+\mu)(d+\mu) + S^*m^*\beta\varepsilon\frac{b-2am^*}{(1+hI^*)^2}\\
&+ am^*\left[\mu + \frac{\beta m^*(b-am^*)}{\alpha}\right](\varepsilon+d+2\mu).
\end{aligned}
$$

Let $\lambda = i\omega$ be the pure imaginary root of $\varphi_2(\lambda, 0, \tau_2)$. Substituting it into Eq (4.5) we obtain

$$
\begin{aligned}
\varphi_2(i\omega, 0, \tau_2) = & \omega^4 - F_2^1 i\omega^3 - A_2^1\omega^2 + G_2^1 i\omega + B_2^1\\
& + \left[-i\omega^3 - C_2^1\omega^2 + E_2^1 i\omega + D_2^1\right]\gamma\cos\omega\tau_2\\
& + \left[-\omega^3 + C_2^1 i\omega^2 + E_2^1\omega - D_2^1 i\right]\gamma\sin\omega\tau_2.
\end{aligned}
$$

From $\varphi_2(i\omega, 0, \tau_2) = 0$, it follows that the real and imaginary parts of the equation must individually equal zero, yielding

$$
\begin{cases}
\omega^4 - A_2^1\omega^2 + B_2^1 = \gamma\left(C_2^1\omega^2 - D_2^1\right)\cos\omega\tau_2 + \gamma\left(\omega^3 - E_2^1\omega\right)\sin\omega\tau_2,\\
-F_2^1\omega^3 + G_2^1\omega = \gamma\left(\omega^3 - E_2^1\omega\right)\cos\omega\tau_2 - \gamma\left(C_2^1\omega^2 - D_2^1\right)\sin\omega\tau_2.
\end{cases} \tag{4.6}
$$

By squaring and adding the two equations in Eq (4.6), we can obtain the following:

$$
\phi_4(\omega) = \omega^8 + c_{41}\omega^6 + c_{42}\omega^4 + c_{43}\omega^2 + c_{44},
$$

where

$$
\begin{aligned}
c_{41} &= -2A_2^1 + (F_2^1)^2 - \gamma^2,\\
c_{42} &= (A_2^1)^2 + 2B_2^1 - \gamma^2(C_2^1)^2 + 2\gamma^2(E_2^1)^2 - 2F_2^1G_2^1,\\
c_{43} &= -2A_2^1B_2^1 + 2\gamma^2C_2^1D_2^1 - \gamma^2(E_2^1)^2 + (G_2^1)^2,\\
c_{44} &= (B_2^1)^2 - \gamma^2(D_2^1)^2.
\end{aligned}
$$

When $R_0 > 1$, if $c_{44} < 0$ holds, it follows from the median theorem that there exists at least one positive root $\dot{\omega} \in (0, +\infty)$ such that $\phi_4(\dot{\omega}) = 0$.

Let $\omega_{2^*}^{(j)}(j = 1, 2, 3, 4)$ be the positive root of $\phi_4(\omega)$.

From Eq (4.6), we obtain the following:

$$
\begin{cases}
\sin\omega\tau_2 = \dfrac{\omega^4 - A_2^1\omega^2 + B_2^1 - \gamma(C_2^1\omega^2 - D_2^1)\cos\omega\tau_2}{\gamma(\omega^3 - E_2^1\omega)},\\[2ex]
\cos\omega\tau_2 = \dfrac{-F_2^1\omega^3 + G_2^1\omega + \gamma(C_2^1\omega^2 - D_2^1)\sin\omega\tau_2}{\gamma(\omega^3 - E_2^1\omega)}.
\end{cases}
$$

Then, we obtain the following:

$$\cos\omega\tau_2 = \frac{(\omega^4 - A_2^1\omega^2 + B_2^1)(C_2^1\omega^2 - D_2^1) + (\omega^3 - E_2^1\omega)(-F_2^1\omega^3 + G_2^1\omega)}{\gamma(C_2^1\omega^2 - D_2^1)^2 + \gamma(\omega^3 - E_2^1\omega)^2},$$

$$\bar{\tau}_2^{j,k} = \frac{1}{\bar{\omega}_2^{(j)}}\left(\cos^{-1}\left[\frac{(\omega^4 - A_2^1\omega^2 + B_2^1)(C_2^1\omega^2 - D_2^1) + (\omega^3 - E_2^1\omega)(-F_2^1\omega^3 + G_2^1\omega)}{\gamma(C_2^1\omega^2 - D_2^1)^2 + \gamma(\omega^3 - E_2^1\omega)^2}\right] + 2k\pi\right),$$

where $j = 1, 2, 3, 4,\ k \in N$.

$$\begin{aligned}\frac{d\varphi_2(\lambda, 0, \tau_2)}{d\tau_2} &= 4\lambda^3\frac{d\lambda}{d\tau_2} + 3F_2^1\lambda^2\frac{d\lambda}{d\tau_2} + 2A_2^1\lambda\frac{d\lambda}{d\tau_2} + G_2^1\frac{d\lambda}{d\tau_2} + B_2^1 \\ &+ \left[\lambda^3 + C_2^1\lambda^2 + E_2^1\lambda + D_2^1\right](-\lambda - \tau_2\frac{d\lambda}{d\tau_2})\gamma e^{-\lambda\tau_2} \\ &+ \left[3\lambda^2 + 2C_2^1\lambda + E_2^1\right]\frac{d\lambda}{d\tau_2}\gamma e^{-\lambda\tau_2}.\end{aligned}$$

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{3\lambda^2 + 2C_2^1\lambda + E_2^1}{\lambda(\lambda^3 + C_2^1\lambda^2 + E_2^1\lambda + D_2^1)} - \frac{\tau_2}{\lambda} + \frac{4\lambda^3 + 3F_2^1\lambda^2 + 2A_2^1\lambda + G_2^1}{-\lambda(\lambda^4 + F_2^1\lambda^3 + A_2^1\lambda^2\ G_2^1\lambda + B_2^1)}. \tag{4.7}$$

Let $\tau_{2^*}^0 = \min\{\tau_{2^*}^{j,k}\}$, $\omega_{2^*}^0 = \arg\min\{\tau_{2^*}^{j,k}\}$. When $\tau_2 = \tau_{2^*}^0$, we have $\psi_{1^*}(\tau_{2^*}^0) = 0$ and $\omega_{1^*}(\tau_{2^*}^0) = \omega_{2^*}^0$. Substituting $\lambda_{1^*}(\tau_{2^*}^0) = \psi_{1^*}(\tau_{2^*}^0) + i\omega_{1^*}(\tau_{2^*}^0)$ into Eq (4.7), we obtain the following:

$$\begin{aligned}\left(\frac{d\lambda_{1^*}}{d\tau_2}\right)^{-1} &= \frac{-3\left(\omega_{2^*}^0\right)^2 + 2iC_2^1\omega_{2^*}^0 + E_2^1}{i\omega_{2^*}^0\left[-i(\omega_{2^*}^0)^3 - C_2^1(\omega_{2^*}^0)^2 + iE_2^1\omega_{2^*}^0 + D_2^1)\right]} - \frac{\tau_{2^*}^0}{i\omega_{2^*}^0} \\ &+ \frac{-4i\left(\omega_{2^*}^0\right)^3 - 3F_2^1\left(\omega_{2^*}^0\right)^2 + 2iA_2^1\omega_{2^*}^0 + G_2^1}{-\lambda\left[(\omega_{2^*}^0)^4 - iF_2^1(\omega_{2^*}^0)^3 - A_2^1(\omega_{2^*}^0)^2 + iG_2^1\omega_{2^*}^0 + B_2^1)\right]},\end{aligned}$$

$$\mathrm{Re}\left(\frac{d\lambda_{1^*}}{d\tau_2}\right)^{-1} = \frac{\left(Q_2^1W_2^1 + Y_2^1T_2^1\right)\left[\left(L_2^1\right)^2 + \left(O_2^1\right)^2\right] + \left(U_2^1O_2^1 + P_2^1L_2^1\right)\left[\left(T_2^1\right)^2 + \left(W_2^1\right)^2\right]}{\left[\left(T_2^1\right)^2 + \left(W_2^1\right)^2\right]\left[\left(L_2^1\right)^2 + \left(O_2^1\right)^2\right]},$$

where

$$\begin{aligned}Q_2^1 &= 2A_2^1\omega_{2^*}^0 - 4(\omega_{2^*}^0)^3, \\ W_2^1 &= -(\omega_{2^*}^0)^5 + A_2^1(\omega_{2^*}^0)^3 - B_2^1\omega_{2^*}^0, \\ T_2^1 &= -F_2^1(\omega_{2^*}^0)^4 + G_2^1(\omega_{2^*}^0)^2, \\ Y_2^1 &= -3F_2^1(\omega_{2^*}^0)^2 + G_2^1, \\ U_2^1 &= 2C_2^1\omega_{2^*}^0, \\ O_2^1 &= -C_2^1(\omega_{2^*}^0)^3 + D_2^1\omega_{2^*}^0, \\ P_2^1 &= E_2^1 - 3(\omega_{2^*}^0)^2, \\ L_2^1 &= (\omega_{2^*}^0)^4 - E_2^1(\omega_{2^*}^0)^2,\end{aligned}$$

$$(Q_2^1W_2^1+Y_2^1T_2^1)\left[(L_2^1)^2+(O_2^1)^2\right]+(U_2^1O_2^1+P_2^1L_2^1)\left[(T_2^1)^2+(W_2^1)^2\right]\neq 0. \tag{4.8}$$

It follows that if Eq (4.8) is satisfied, then

$$\operatorname{Re}\left(\frac{d\lambda_{1^*}}{d\tau_2}\right)^{-1}\neq 0.$$

Theorem 4.2. *When $\tau_1=0$ and $\tau_2\neq 0$, if $\omega_{2^*}^{(j)}$ exists and satisfies Eq (4.8), we can draw the following conclusion:*

(1) If $\tau_2\in(0,\tau_{2^*}^0)$, the system is locally asymptotically stable at the endemic equilibrium point E_2.
(2) If $\tau_2>\tau_{2^*}^0$, the system is unstable in the endemic equilibrium point E_2.
(3) If $\tau_2=\tau_{2^*}^0$, the system has a Hopf bifurcation at the endemic equilibrium point E_2.

Case 7. *When $\tau_1\neq 0,\tau_2=0$,*

$$\varphi_2(\lambda,\tau_1,0)=\lambda^4+F_2^2\lambda^3+A_2^2\lambda^2+G_2^2\lambda+B_2^2+\left[\lambda^3+C_2^2\lambda^2+E_2^2\lambda+D_2^2\right]\varepsilon e^{-\lambda\tau_1}=0, \tag{4.9}$$

where

$$\begin{aligned}
A_2^2=&\mu(\gamma+d+\mu)+\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\gamma+d+2\mu+am^*)+am^*(\gamma+d+2\mu),\\
B_2^2=&a\mu m^*\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\gamma+d+\mu),\\
C_2^2=&\gamma+d+2\mu+am^*+\frac{\beta m^*(b-am^*)}{\alpha},\\
D_2^2=&am^*\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\gamma+d+\mu)+\beta\mu m^*S^*\frac{b-2am^*}{(1+hI^*)^2},\\
E_2^2=&\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}+am^*\right](\gamma+d+\mu)+\beta S^*m^*\frac{b-2am^*}{(1+hI^*)^2}+am^*\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right],\\
F_2^2=&\gamma+d+3\mu+am^*+\frac{\beta m^*(b-am^*)}{\alpha},\\
G_2^2=&\mu\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}+am^*\right](\gamma+d+\mu)+am^*\left[\mu+\frac{\beta m^*(b-am^*)}{\alpha}\right](\gamma+d+2\mu).
\end{aligned}$$

Let $\lambda=i\omega$ be the pure imaginary root of $\varphi_2(\lambda,\tau_1,0)$. Substituting it into Eq (4.9), we obtain

$$\begin{aligned}
\varphi_2(i\omega,\tau_1,0)=&\omega^4-F_2^2i\omega^3-A_2^2\omega^2+G_2^2i\omega+B_2^2\\
&+\left[-i\omega^3-C_2^2\omega^2+E_2^2i\omega+D_2^2\right]\varepsilon\cos\omega\tau_1\\
&+\left[-\omega^3+C_2^2i\omega^2+E_2^2\omega-D_2^2i\right]\varepsilon\sin\omega\tau_1.
\end{aligned}$$

From $\varphi_2(i\omega,\tau_1,0)=0$, it is clear that the real and imaginary parts of the equation must both equal zero, which gives

$$\begin{cases}\omega^4-A_2^2\omega^2+B_2^2=\varepsilon(C_2^2\omega^2-D_2^2)\cos\omega\tau_1+\varepsilon(\omega^3-E_2^2\omega)\sin\omega\tau_1,\\ -F_2^2\omega^3+G_2^2\omega=\varepsilon(\omega^3-E_2^2\omega)\cos\omega\tau_1-\varepsilon(C_2^2\omega^2-D_2^2)\sin\omega\tau_1.\end{cases} \tag{4.10}$$

By squaring and adding the two equations in Eq (4.10), we can obtain the following:

$$\phi_5(\omega) = \omega^8 + c_{51}\omega^6 + c_{52}\omega^4 + c_{53}\omega^2 + c_{54},$$

where

$$\begin{aligned}
c_{51} &= -2A_2^2 + (F_2^2)^2 - \varepsilon^2,\\
c_{52} &= (A_2^2)^2 + 2B_2^2 - \varepsilon^2(C_2^2)^2 + 2\varepsilon^2(E_2^2)^2 - 2F_2^2G_2^2,\\
c_{53} &= -2A_2^2B_2^2 + 2\varepsilon^2C_2^2D_2^2 - \varepsilon^2(E_2^2)^2 + (G_2^2)^2,\\
c_{54} &= (B_2^2)^2 - \varepsilon^2(D_2^2)^2.
\end{aligned}$$

When $R_0 > 1$, if $c_{54} < 0$ holds, we can deduce from the median theorem that there exists at least one positive root $\dot{\omega} \in (0, +\infty)$ such that $\phi_5(\dot{\omega}) = 0$.

Let $\omega_{1^*}^{(j)} (j = 1, 2, 3, 4)$ be the positive root of $\phi_5(\omega)$.

From Eq (4.10), we can obtain the following:

$$\cos\omega\tau_1 = \frac{(\omega^4 - A_2^2\omega^2 + B_2^2)(C_2^2\omega^2 - D_2^2) + (\omega^3 - E_2^2\omega)(-F_2^2\omega^3 + G_2^2\omega)}{\varepsilon(C_2^2\omega^2 - D_2^2)^2 + \varepsilon(\omega^3 - E_2^2\omega)^2},$$

$$\tau_{1^*}^{j,k} = \frac{1}{\omega_{1^*}^{(j)}}\left(\cos^{-1}\left[\frac{(\omega^4 - A_2^2\omega^2 + B_2^2)(C_2^2\omega^2 - D_2^2) + (\omega^3 - E_2^2\omega)(-F_2^2\omega^3 + G_2^2\omega)}{\varepsilon(C_2^2\omega^2 - D_2^2)^2 + \varepsilon(\omega^3 - E_2^2\omega)^2}\right] + 2k\pi\right),$$

where $j = 1, 2, 3, 4,\ k \in N$.

$$\begin{aligned}
\frac{d\varphi_2(\lambda, \tau_1, 0)}{d\tau_1} =& 4\lambda^3\frac{d\lambda}{d\tau_1} + 3F_2^2\lambda^2\frac{d\lambda}{d\tau_1} + 2A_2^2\lambda\frac{d\lambda}{d\tau_1} + G_2^2\frac{d\lambda}{d\tau_1} + B_2^2\\
&+ \left[\lambda^3 + C_2^2\lambda^2 + E_2^2\lambda + D_2^2\right](-\lambda - \tau_1\frac{d\lambda}{d\tau_1})\varepsilon e^{-\lambda\tau_1}\\
&+ \left[3\lambda^2 + 2C_2^2\lambda + E_2^2\right]\frac{d\lambda}{d\tau_1}\varepsilon e^{-\lambda\tau_1}.
\end{aligned}$$

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{3\lambda^2 + 2C_2^2\lambda + E_2^2}{\lambda(\lambda^3 + C_2^2\lambda^2 + E_2^2\lambda + D_2^2)} - \frac{\tau_1}{\lambda} + \frac{4\lambda^3 + 3F_2^2\lambda^2 + 2A_2^2\lambda + G_2^2}{-\lambda(\lambda^4 + F_2^2\lambda^3 + A_2^2\lambda^2 + G_2^2\lambda + B_2^2)}. \tag{4.11}$$

Let $\tau_{1^*}^0 = \min\{\tau_{1^*}^{j,k}\}, \omega_{1^*}^0 = \arg\min\{\tau_{1^*}^{j,k}\}$. When $\tau_1 = \tau_{1^*}^0, \psi_{2^*}(\tau_{1^*}^0) = 0$ and $\omega_{2^*}(\tau_{1^*}^0) = \omega_{1^*}^0$. Substituting $\lambda_{2^*}(\tau_{1^*}^0) = \psi_{2^*}(\tau_{1^*}^0) + i\omega_{2^*}(\tau_{1^*}^0)$ into Eq (4.11), we can obtain the following:

$$\begin{aligned}
\left(\frac{d\lambda_{2^*}}{d\tau_1}\right)^{-1} =& \frac{-3(\omega_{1^*}^0)^2 + 2iC_2^2\omega_{1^*}^0 + E_2^2}{i\omega_{1^*}^0\left[-i(\omega_{1^*}^0)^3 - C_2^2(\omega_{1^*}^0)^2 + iE_2^2\omega_{1^*}^0 + D_2^2\right]} - \frac{\tau_{1^*}^0}{i\omega_{1^*}^0}\\
&+ \frac{-4i(\omega_{1^*}^0)^3 - 3F_2^2(\omega_{1^*}^0)^2 + 2iA_2^2\omega_{1^*}^0 + G_2^2}{-\lambda\left[(\omega_{1^*}^0)^4 - iF_2^2(\omega_{1^*}^0)^3 - A_2^2(\omega_{1^*}^0)^2 + iG_2^2\omega_{1^*}^0 + B_2^2)\right]},
\end{aligned}$$

$$\mathrm{Re}\left(\frac{d\lambda_{2^*}}{d\tau_1}\right)^{-1} = \frac{(Q_2^2W_2^2 + Y_2^2T_2^2)\left[(L_2^2)^2 + (O_2^2)^2\right] + (U_2^2O_2^2 + P_2^2L_2^2)\left[(T_2^2)^2 + (W_2^2)^2\right]}{\left[(T_2^2)^2 + (W_2^2)^2\right]\left[(L_2^2)^2 + (O_2^2)^2\right]},$$

where

$$
\begin{aligned}
Q_2^2 &= -4(\omega_{1*}^0)^3 + 2A_2^2\omega_{1*}^0, \\
W_2^2 &= -(\omega_{1*}^0)^5 + A_2^2(\omega_{1*}^0)^3 - B_2^2\omega_{1*}^0, \\
T_2^2 &= -F_2^2(\omega_{1*}^0)^4 + G_2^2(\omega_{1*}^0)^2, \\
Y_2^2 &= -3F_2^2(\omega_{1*}^0)^2 + G_2^2, U_2^2 = 2C_2^2\omega_{1*}^0, \\
O_2^2 &= -C_2^2(\omega_{1*}^0)^3 + D_2^2\omega_{1*}^0, \\
P_2^2 &= -3(\omega_{1*}^0)^2 + E_2^2, \\
L_2^2 &= (\omega_{1*}^0)^4 - E_2^2(\omega_{1*}^0)^2,
\end{aligned}
$$

$$
(Q_2^2W_2^2 + Y_2^2T_2^2)\left[(L_2^2)^2 + (O_2^2)^2\right] + (U_2^2O_2^2 + P_2^2L_2^2)\left[(T_2^2)^2 + (W_2^2)^2\right] \neq 0. \tag{4.12}
$$

It follows that if Eq (4.12) is satisfied, then

$$
\mathrm{Re}\left(\frac{d\lambda_{2^*}}{d\tau_1}\right)^{-1} \neq 0.
$$

Theorem 4.3. *When $\tau_1 \neq 0$ and $\tau_2 = 0$, if $\omega_{1^*}^{(j)}$ exists and satisfies Eq (4.12), the following conclusion can be drawn:*

(1) If $\tau_1 \in (0, \tau_{1^*}^0)$, the system is locally asymptotically stable at the endemic equilibrium point E_2.
(2) If $\tau_1 > \tau_{1^*}^0$, the system is unstable in the endemic equilibrium point E_2.
(3) If $\tau_1 = \tau_{1^*}^0$, the system has a Hopf bifurcation at the endemic equilibrium point E_2.

Case 8. *When $\tau_1 \neq 0$ and $\tau_2 \neq 0$, as discussed in Case 4 of Section 3, it is useful to consider τ_1 as the independent variable while treating $\tau_2 \in (0, \tau_{2^*}^0)$ as a fixed parameter.*

$$
\begin{aligned}
\varphi_2(\lambda, \tau_1, \tau_2) =& \lambda^4 + \left[d + \mu + D_2^3\right]\lambda^3 + A_2^3\lambda^2 + G_2^3\lambda + \mu(d+\mu)B_2^3 \\
&+ \varepsilon\left[\lambda^3 + (d + D_2^3)\lambda^2 + E_2^3\lambda + F_2^3\right]e^{-\lambda\tau_1} \\
&+ \gamma\left[\lambda^3 + D_2^3\lambda^2 + C_2^3\lambda + \mu B_2^3\right]e^{-\lambda\tau_2} \\
&+ \gamma\varepsilon\left[\lambda^2 + (D_2^3 - \mu)\lambda + B_2^3\right]e^{-\lambda(\tau_1+\tau_2)} = 0,
\end{aligned} \tag{4.13}
$$

where

$$
\begin{aligned}
A_2^3 =& \mu(d+\mu) + am^*(d+2\mu) + \left[\mu + \frac{\beta m^*(b - am^*)}{\alpha}\right](d + 2\mu + am^*), \\
B_2^3 =& am^*\left[\mu + \frac{\beta m^*(b - am^*)}{\alpha}\right], \\
C_2^3 =& a\mu m^* + \left[\mu + \frac{\beta m^*(b - am^*)}{\alpha}\right](am^* + \mu), \\
D_2^3 =& 2\mu + am^* + \frac{\beta m^*(b - am^*)}{\alpha},
\end{aligned}
$$

$$E_2^3 = \left[\mu + \frac{\beta m^*(b - am^*)}{\alpha} + am^*\right](d + \mu) + am^*\left[\mu + \frac{\beta m^*(b - am^*)}{\alpha}\right] + \beta S^* m^* \frac{b - 2am^*}{(1 + hI^*)^2},$$
$$F_2^3 = am^*\left[\mu + \frac{\beta m^*(b - am^*)}{\alpha}\right](d + \mu) + \beta\mu S^* m^* \frac{b - 2am^*}{(1 + hI^*)^2},$$
$$G_2^3 = \mu\left[\mu + \frac{\beta m^*(b - am^*)}{\alpha} + am^*\right](d + \mu) + am^*\left[\mu + \frac{\beta m^*(b - am^*)}{\alpha}\right](d + 2\mu).$$

Let $\lambda = i\omega$ be the pure imaginary root of $\varphi_2(\lambda, \tau_1, \tau_2)$, Substituting it into Eq (4.13), we obtain the following:

$$\begin{aligned}
\varphi_2(i\omega, \tau_1, \tau_2) = &\omega^4 - \left[d + \mu + D_2^3\right] i\omega^3 - A_2^3\omega^2 + G_2^3 i\omega + \mu(d + \mu)B_2^3 \\
&+ \gamma\left[-i\omega^3 - D_2^3\omega^2 + C_2^3 i\omega + \mu B_2^3\right]\cos\omega\tau_2 \\
&+ \varepsilon\left[-i\omega^3 - (d + D_2^3)\omega^2 + E_2^3 i\omega + F_2^3\right]\cos\omega\tau_1 \\
&+ \gamma\varepsilon\left[-\omega^2 + (D_2^3 - \mu)i\omega + B_2^3\right]\cos\omega\tau_2\cos\omega\tau_1 \\
&+ \gamma\varepsilon\left[i\omega^2 + (D_2^3 - \mu)\omega - B_2^3 i\right]\sin\omega\tau_2\cos\omega\tau_1 \\
&+ \gamma\left[-\omega^3 + D_2^3 i\omega^2 + C_2^3\omega - \mu B_2^3 i\right]\sin\omega\tau_2 \\
&+ \varepsilon\left[-\omega^3 + (d + D_2^3)i\omega^2 + E_2^3\omega - F_2^3 i\right]\sin\omega\tau_1 \\
&+ \gamma\varepsilon\left[i\omega^2 + (D_2^3 - \mu)\omega - B_2^3 i\right]\cos\omega\tau_2\sin\omega\tau_1 \\
&- \gamma\varepsilon\left[-\omega^2 + (D_2^3 - \mu)i\omega + B_2^3\right]\sin\omega\tau_2\sin\omega\tau_1.
\end{aligned}$$

From $\varphi_2(i\omega, \tau_1, \tau_2) = 0$, we obtain the following:

$$\begin{cases} \omega^4 - A_2^3\omega^2 + c_6^1 = c_6^2\sin\omega\tau_1 + c_6^3\cos\omega\tau_1, \\ -(d + \mu + D_2^3)\omega^3 + G_2^3\omega + c_6^4 = -c_6^3\sin\omega\tau_1 + c_6^2\cos\omega\tau_1, \end{cases} \tag{4.14}$$

where

$$\begin{aligned}
c_6^1 &= \mu(d + \mu)B_2^3 + (-D_2^3\omega^2 + \mu B_2^3)\gamma\cos\omega\tau_2 + (-\omega^3 + C_2^3\omega)\gamma\sin\omega\tau_2, \\
c_6^2 &= -\varepsilon[-\omega^3 + E_2^3\omega + \gamma(\omega^2 - B_2^3)\sin\omega\tau_2 + \gamma(D_2^3 - \mu)\omega\cos\omega\tau_2], \\
c_6^3 &= -\varepsilon[-(d + D_2^3)\omega^2 + F_2^3 + \gamma(D_2^3 - \mu)\omega\sin\omega\tau_2 + \gamma(-\omega^2 + B_2^3)\cos\omega\tau_2], \\
c_6^4 &= (-\omega^3 + C_2^3\omega)\gamma\cos\omega\tau_2 + (D_2^3\omega^2 - \mu B_2^3)\gamma\sin\omega\tau_2.
\end{aligned}$$

By squaring and adding the two equations in Eq (4.14), we can obtain the following:

$$\phi_6 = \omega^8 + c_{61}\omega^7 + c_{62}\omega^6 + c_{63}\omega^5 + c_{64}\omega^4 + c_{65}\omega^3 + c_{66}\omega^2 + c_{67}\omega^1 + c_{68},$$

where

$$\begin{aligned}
c_{61} &= -2\gamma\sin\omega\tau_2, \\
c_{62} &= -2A_2^3 - 2D_2^3 + (d + \mu + D_2^3)^2 + \gamma^2 + 2\gamma(d + \mu + D_2^3)\cos\omega\tau_2 - 1, \\
c_{63} &= 2\gamma\left[A_2^3 + C_2^3 - 1 - 2(d + \mu + D_2^3)D_2^3\right]\sin\omega\tau_2, \\
c_{64} &= 2\mu(d + \mu)B_2^3 + 2\gamma\mu B_2^3\cos\omega\tau_2 + (A_2^3)^2 + 2\gamma A_2^3 D_2^3\cos\omega\tau_2 - 2\gamma^2 C_2^3
\end{aligned}$$

$$
\begin{aligned}
&- 2(d+\mu+D_2^3)G_2^3 - 2\gamma(d+\mu+D_2^3)C_2^3\cos\omega\tau_2 - \gamma^2 - (d+D_2^3)^2 \\
&- 2\gamma G_2^3\cos\omega\tau_2 + 2E_2^3 - 2\gamma(d+\mu)\cos\omega\tau_2 + \gamma^2(D_2^3)^2, \\
c_{65} =& 2\gamma\big[-A_2^3C_2^3 + \mu B_2^3D_2^3 + D_2^3G_2^3 + (d+D_2^3)(D_2^3-\mu) - B_2^3 - E_2^3\big]\sin\omega\tau_2, \\
c_{66} =& -2\mu(d+\mu)A_2^3B_2^3 - 2\gamma\mu A_2^3B_2^3\cos\omega\tau_2 + 2\gamma B_2^3(d+D_2^3)\cos\omega\tau_2 \\
&- 2\gamma\mu(d+\mu)B_2^3D_2^3\cos\omega\tau_2 + 2C_2^3\gamma^2\sin^2\omega\tau_2 - \gamma^2(D_2^3-\mu)^2 \\
&- 2\gamma C_2^3G_2^3\cos\omega\tau_2 + (G_2^3)^2 - 2\mu B_2^3D_2^3 - (E_2^3)^2 + 2F_2^3(d+D_2^3) \\
&- 2\gamma E_2^3(D_2^3-\mu)\cos\omega\tau_2 + 2\gamma^2B_2^3 + 2\gamma F_2^3\cos\omega\tau_2, \\
c_{67} =& 2\gamma\left[\mu(d+\mu)B_2^3C_2^3 - \mu B_2^3G_2^3 - F_2^3(D_2^3-\mu) + 2B_2^3E_2^3\right]\sin\omega\tau_2, \\
c_{68} =& \mu^2(d+\mu)^2(B_2^3)^2 + 2\gamma\mu^2(d+\mu)(B_2^3)^2\cos\omega\tau_2 + \gamma^2\mu^2(B_2^3)^2 \\
&- \gamma^2(B_2^3)^2 - 2\gamma B_2^3F_2^3\cos\omega\tau_2 - (F_2^3)^2.
\end{aligned}
$$

When $R_0 > 1$, if $c_{68} < 0$ holds, it follows from the intermediate value theorem that there exists at least one positive root $\dot{\omega} \in (0, +\infty)$ such that $\phi_6(\dot{\omega}) = 0$. From Eq (4.14), we can get

$$
\cos\omega\tau_1 = \frac{-c_6^2(d+\mu+D_2^3)\omega^3 + c_6^2G_2^3\omega + c_6^4c_6^2 + c_6^3(\omega^4 - A_2^3\omega^2 + c_6^1)}{(c_6^2)^2 + (c_6^3)^2},
$$

$$
\dot{\tau}_{1^*}^{j,k} = \frac{1}{\dot{\omega}_{1*}^{(j)}}\left(\cos^{-1}\left[\frac{-c_6^2(d+\mu+D_2^3)\omega^3 + c_6^2G_2^3\omega + c_6^4c_6^2 + c_6^3(\omega^4 - A_2^3\omega^2 + c_6^1)}{(c_6^2)^2 + (c_6^3)^2}\right] + 2k\pi\right),
$$

where $j = 1, 2, 3, 4,\ k \in N$.

$$
\begin{aligned}
\frac{d\varphi_2(\lambda,\tau_1,\tau_2)}{d\tau_1} =& \left[4\lambda^3 + 3(d+\mu+D_2^3)\lambda^2 + 2A_2^3\lambda + G_2^3\right]\frac{d\lambda}{d\tau_1} + \varepsilon\left[3\lambda^2 + 2(d+D_2^3)\lambda + E_2^3\right]\frac{d\lambda}{d\tau_1}e^{-\lambda\tau_1} \\
&- \varepsilon\tau_1\left[\lambda^3 + (d+D_2^3)\lambda^2 + E_2^3\lambda + F_2^3\right]\frac{d\lambda}{d\tau_1}e^{-\lambda\tau_1} - \varepsilon\lambda\left[\lambda^3 + (d+D_2^3)\lambda^2 + E_2^3\lambda + F_2^3\right]e^{-\lambda\tau_1} \\
&+ \gamma(3\lambda^2 + 2D_2^3\lambda + C_2^3)\frac{d\lambda}{d\tau_1}e^{-\lambda\tau_2} - \gamma\tau_2\left[\lambda^3 + D_2^3\lambda^2 + C_2^3\lambda + \mu B_2^3\right]e^{-\lambda\tau_2}\frac{d\lambda}{d\tau_1} \\
&+ \gamma\varepsilon(2\lambda + D_2^3 - \mu)\frac{d\lambda}{d\tau_1}e^{-\lambda(\tau_1+\tau_2)} - \gamma\varepsilon(\tau_1+\tau_2)\left[\lambda^2 + (D_2^3-\mu)\lambda + B_2^3\right]\frac{d\lambda}{d\tau_1}e^{-\lambda(\tau_1+\tau_2)} \\
&- \gamma\varepsilon\lambda\left[\lambda^2 + (D_2^3-\mu)\lambda + B_2^3\right]e^{-\lambda(\tau_1+\tau_2)},
\end{aligned}
$$

$$
\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{c_{71} + c_{72} - c_{73}\tau_2}{\lambda(c_{74} + c_{73})} - \frac{\tau_1}{\lambda} + \frac{c_{75} - c_{76}\tau_2}{-\lambda(c_{77} + c_{76})}, \tag{4.15}
$$

where

$$\begin{aligned}
c_{71} &= 3\lambda^2 + 2(d + D_2^3)\lambda + E_2^3,\\
c_{72} &= \gamma(2\lambda + D_2^3 - \mu)e^{-\lambda\tau_2},\\
c_{73} &= \gamma\left[\lambda^2 + (D_2^3 - \mu)\lambda + B_2^3\right]e^{-\lambda\tau_2},\\
c_{74} &= \lambda^3 + (d + D_2^3)\lambda^2 + E_2^3\lambda + F_2^3,\\
c_{75} &= \gamma(3\lambda^2 + 2D_2^3\lambda + C_2^3)e^{-\lambda\tau_2},\\
c_{76} &= \gamma\left(\lambda^3 + D_2^3\lambda^2 + C_2^3\lambda + \mu B_2^3\right)e^{-\lambda\tau_2},\\
c_{77} &= \lambda\left[\lambda^4 + (d + \mu + D_2^3)\lambda^3 + A_2^3\lambda^2 + G_2^3\lambda + \mu(d + \mu)B_2^3\right].
\end{aligned}$$

Let $\dot{\tau}_{1^*}^0 = \min\{\dot{\tau}_{1^*}^{j,k}\}$ and $\dot{\omega}_{1^*}^0 = \arg\min\{\dot{\tau}_{1^*}^{j,k}\}$. When $\tau_1 = \dot{\tau}_{1^*}^0$, it follows that $\dot{\psi}_{1^*}(\dot{\tau}_{1^*}^0) = 0$ and $\dot{\omega}_{1^*}(\dot{\tau}_{1^*}^0) = \dot{\omega}_{1^*}^0$. Substituting $\dot{\lambda}_{1^*}(\dot{\tau}_{1^*}^0) = \dot{\psi}_{1^*}(\dot{\tau}_{1^*}^0) + i\dot{\omega}_{1^*}(\dot{\tau}_{1^*}^0)$ into Eq (4.15), we obtain the following:

$$\left(\frac{d\dot{\lambda}_{1^*}}{d\tau_1}\right)^{-1} = \frac{Y_2^3 + Q_2^3}{T_2^3 + W_2^3 i} + \frac{\tau_1 i}{\dot{\omega}_{1^*}^0} + \frac{U_2^3 + O_2^3 i}{L_2^3 + P_2^3 i},$$

$$\mathrm{Re}\left(\frac{d\dot{\lambda}_{1^*}}{d\tau_1}\right)^{-1} = \frac{\left[Q_2^3W_2^3 + Y_2^3T_2^3\right]\left[(L_2^3)^2 + (P_2^3)^2\right] + \left[O_2^3P_2^3 + U_2^3L_2^3\right]\left[(T_2^3)^2 + (W_2^3)^2\right]}{\left[(T_2^3)^2 + (W_2^3)^2\right]\left[(L_2^3)^2 + (P_2^3)^2\right]},$$

where

$$\begin{aligned}
Q_2^3 =& 2(d + D_2^3)\dot{\omega}_{1^*}^0 + \gamma\dot{\omega}_{1^*}^0(2 - D_2^3\tau_2 + \mu\tau_2)\cos\dot{\omega}_{1^*}^0\tau_2 - \gamma\left[D_2^3 - \mu - \tau_2(\dot{\omega}_{1^*}^0)^2 + B_2^3\tau_2\right]\sin\dot{\omega}_{1^*}^0\tau_2,\\
W_2^3 =& -(d + D_2^3)(\dot{\omega}_{1^*}^0)^3 + F_2^3\dot{\omega}_{1^*}^0 + \gamma\left[-(\dot{\omega}_{1^*}^0)^3 + B_2^3\dot{\omega}_{1^*}^0\right]\cos\dot{\omega}_{1^*}^0\tau_2 + \gamma(D_2^3 - \mu)(\dot{\omega}_{1^*}^0)^2\sin\dot{\omega}_{1^*}^0\tau_2,\\
T_2^3 =& (\dot{\omega}_{1^*}^0)^4 - E_2^3(\dot{\omega}_{1^*}^0)^2 - \gamma(D_2^3 - \mu)(\dot{\omega}_{1^*}^0)^2\cos\dot{\omega}_{1^*}^0\tau_2 + \gamma\left[-(\dot{\omega}_{1^*}^0)^3 + B_2^3\dot{\omega}_{1^*}^0\right]\sin\dot{\omega}_{1^*}^0\tau_2,\\
Y_2^3 =& -3(\dot{\omega}_{1^*}^0)^2 + E_2^3 + \gamma\left[D_2^3 - \mu - \tau_2(\dot{\omega}_{1^*}^0)^2 + B_2^3\tau_2\right]\cos\dot{\omega}_{1^*}^0\tau_2 + \gamma\dot{\omega}_{1^*}^0(2 - D_2^3\tau_2 + \mu\tau_2)\sin\dot{\omega}_{1^*}^0\tau_2,\\
O_2^3 =& \gamma\left\{\left[3(\dot{\omega}_{1^*}^0)^2 - C_2^3 - D_2^3\tau_2(\dot{\omega}_{1^*}^0)^2 + \mu B_2^3\tau_2\right]\sin\dot{\omega}_{1^*}^0\tau_2 + \left[\tau_2(\dot{\omega}_{1^*}^0)^3 - C_2^3\tau_2\dot{\omega}_{1^*}^0 - 2D_2^3\dot{\omega}_{1^*}^0\right]\cos\dot{\omega}_{1^*}^0\tau_2\right\},\\
P_2^3 =& -(\dot{\omega}_{1^*}^0)^5 + A_2^3(\dot{\omega}_{1^*}^0)^3 - \gamma\left[D_2^3(\dot{\omega}_{1^*}^0)^3 + \mu B_2^3\dot{\omega}_{1^*}^0\right]\cos\dot{\omega}_{1^*}^0\tau_2\\
&+ \gamma\left[(\dot{\omega}_{1^*}^0)^4 + C_2^3(\dot{\omega}_{1^*}^0)^2\right]\sin\dot{\omega}_{1^*}^0\tau_2 - \mu(d + \mu)B_2^3\dot{\omega}_{1^*}^0,\\
L_2^3 =& -(d + \mu + D_2^3)(\dot{\omega}_{1^*}^0)^4 + G_2^3(\dot{\omega}_{1^*}^0)^2 - \gamma\left[(\dot{\omega}_{1^*}^0)^4 + C_2^3(\dot{\omega}_{1^*}^0)^2\right]\cos\dot{\omega}_{1^*}^0\tau_2\\
&- \gamma\left[D_2^3(\dot{\omega}_{1^*}^0)^3 + \mu B_2^3\dot{\omega}_{1^*}^0\right]\sin\dot{\omega}_{1^*}^0\tau_2,\\
U_2^3 =& \gamma\left[-3(\dot{\omega}_{1^*}^0)^2 + C_2^3 + D_2^3\tau_2(\dot{\omega}_{1^*}^0)^2 - \mu B_2^3\tau_2\right]\cos\dot{\omega}_{1^*}^0\tau_2 + \gamma\left[\tau_2(\dot{\omega}_{1^*}^0)^3 - C_2^3\tau_2\dot{\omega}_{1^*}^0 - 2D_2^3\dot{\omega}_{1^*}^0\right]\sin\dot{\omega}_{1^*}^0\tau_2,
\end{aligned}$$

$$\left(Q_2^3W_2^3 + Y_2^3T_2^3\right)\left[(L_2^3)^2 + (P_2^3)^2\right] + \left(O_2^3P_2^3 + U_2^3L_2^3\right)\left[(T_2^3)^2 + (W_2^3)^2\right] \neq 0. \tag{4.16}$$

It follows that if Eq (4.16) is satisfied, then

$$\mathrm{Re}\left(\frac{d\dot{\lambda}_{1^*}}{d\tau_1}\right)^{-1} \neq 0.$$

Theorem 4.4. *When $\tau_1 \neq 0$ and $\tau_2 \in (0, \tau_{2*}^0)$, if $\dot{\omega}_{1*}^{(j)}$ exists and satisfies Eq (4.16), then we obtain the following conclusion:*

(1) If $\tau_1 \in (0, \dot{\tau}_{1*}^0)$, the system is locally asymptotically stable at the endemic equilibrium point E_2.
(2) If $\tau_1 > \dot{\tau}_{1*}^0$, the system is unstable in the endemic equilibrium point E_2.
(3) If $\tau_1 = \dot{\tau}_{1*}^0$, the system has a Hopf bifurcation at the endemic equilibrium point E_2.

5. Numerical simulations

In this section, we investigate the effects of two time lags in the infectious disease system and the intensity of population movement on the system's dynamical behavior. By analyzing the ranges of τ_1, τ_2, and α, we examine the dynamical behavior of the uncontrolled system. The results indicate that varying the time lags τ_1 and τ_2 can lead to three possible scenarios: local asymptotic stabilization, the emergence of Hopf bifurcations, and instability at the equilibrium point E_2. Furthermore, changes in α directly influence the value of I^*.

Throughout the numerical simulation, several key parameters were determined based on realistic epidemiological considerations. For example, according to relevant studies, the contact transmission rate in community settings generally does not exceed 0.05. Therefore, the dissemination exposure rate β was set at 0.0468. Considering that the standard treatment duration typically ranges from 6 to 9 months, a treatment period of 200 days was assumed in this study, resulting in the infection recovery rate γ being set at 0.005. The values of all parameters are summarized in Table 2.

Table 2. Value of the parameter.

Parameters	Value	Source
β	0.0468	[29]
μ	0.006	[29]
h	0.5	[12]
ε	0.000196	Fitted
γ	0.005	Fitted
d	0.013	Fitted
b	2	[12]
a	1	[12]
α	1	[12]
λ	0.00065	Fitted

The system is initialized with an initial value of $S(0) = 8000$, $E(0) = 30$, $I(0) = 5$, $R(0) = 0$, $m(0) = 1$, and numerical simulations demonstrate that it admits a unique endemic equilibrium point E_2=(6993.19, 155.404, 62.2526, 310.825). In the theoretical analysis, the double time delay τ_1 and τ_2 is categorized into four cases: (1) $\tau_1 = 0, \tau_2 = 0$; (2) $\tau_1 = 0, \tau_2 \neq 0$; (3) $\tau_1 \neq 0, \tau_2 = 0$(4) $\tau_1 \neq 0, \tau_2 \neq 0$. In this section, each of these four scenarios is numerically simulated and analyzed.

The parameter Λ in the system model represents the total number of births. While this parameter is conceptually valid within the model, directly incorporating the total number of births in numerical simulations can lead to instability in the short-term population dynamics. To address this issue, we define $\Lambda = \lambda N$ (λ being the birth rate), which ensures the stabilization of the total population in the short term.

Figure 1 illustrates the impact of the adaptive mobility response intensity parameter α on the equilibrium number of infected individuals I^* in the influenza transmission model. The horizontal axis represents α, which reflects the behavioral response intensity of the population to infection risk, while the vertical axis denotes the steady-state number of infected individuals I^*. As shown in the figure, when α gradually increases from 0, I^* initially exhibits a slight rise: I^* is approximately 500 when α is close to 0, and reaches a peak of around 600 when α is between 0.5 and 1. This suggests that during the initial stage of weak behavioral response, moderate adjustments in mobility are insufficient to effectively suppress the epidemic. On the contrary, due to economic incentives, the average mobility may slightly increase, leading to a temporary rise in the number of infections. However, as α continues to increase, the population begins to adopt more significant mobility-reducing measures, resulting in a rapid decline in infection numbers. When α exceeds approximately 2, the number of infections stabilizes at a relatively low level, indicating that sufficiently strong behavioral responses can significantly curb the epidemic. Overall, the number of infections shows a pattern of initially increasing slightly and then rapidly declining as α increases, highlighting the critical role of moderate to strong adaptive behavioral responses in epidemic control.

Table 3 presents the values of each SEI compartment at $t = 600$, corresponding to Figures 2 and 3. As shown in Table 3, when the adaptive mobility response intensity parameter α increases within the range of (0, 1), the number of susceptible individuals S also increases accordingly. This indicates that adaptive behavior can effectively protect the susceptible population. From Figure 2, it can be observed that the system is locally asymptotically stable at the equilibrium point E_2 when $\tau_1 = \tau_2 = 0, \alpha = 1$. Figure 1 reveals that, as α increases, I^* exhibits a negative correlation, means that I^* decreases as α increases. However, I^* only approaches zero as α becomes very small. Therefore, while increasing α reduces the level of infection, it does not completely halt the spread of the infectious disease.

Through numerical simulations, we determined the critical time delay thresholds under three distinct scenarios: (1) When $\tau_2 = 0$, the threshold for $\tau_1^* = 37.824$; (2) When $\tau_1 = 0$, the threshold for $\tau_2^* = 48.501$; (3) For a fixed $\tau_2 = 10.001$, the adjusted threshold for $\dot{\tau}_1^* = 35.302$.

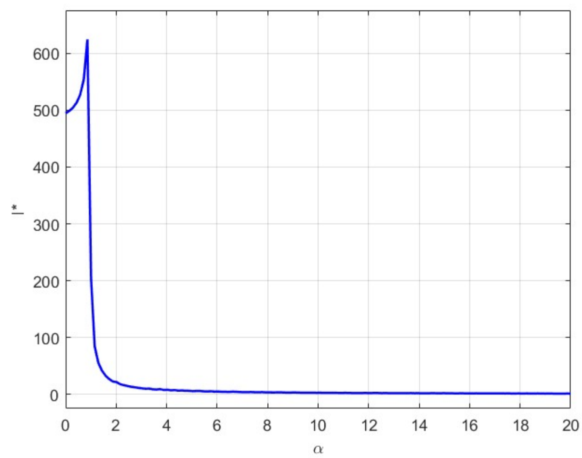


Figure 1. Plot of the variation of I^* versus α.

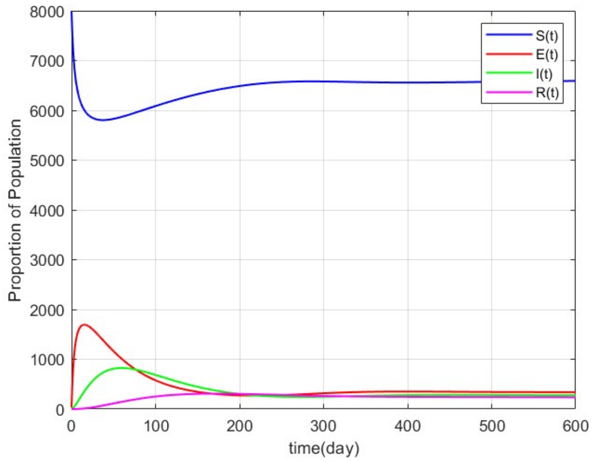
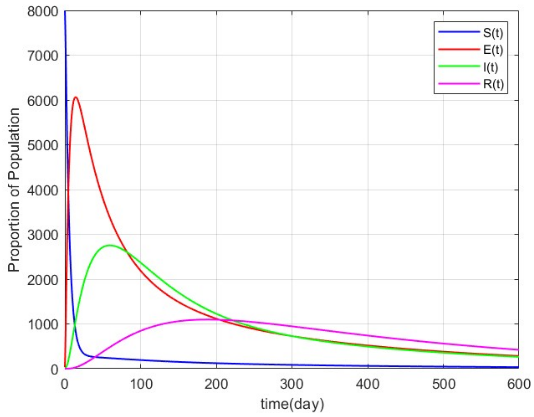
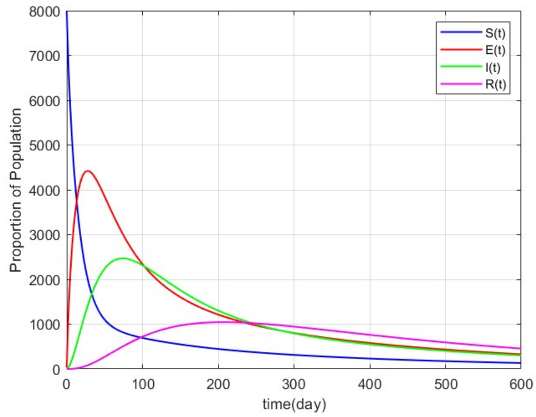


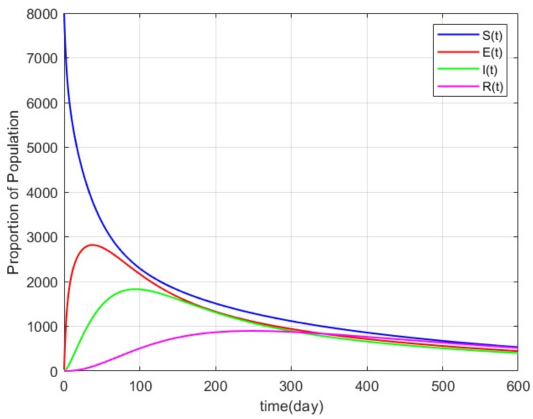
Figure 2. Phase diagram of the variation of $S(t)$,$E(t)$,$I(t)$, and $R(t)$ when $\tau_1 = \tau_2 = 0, \alpha = 1$.



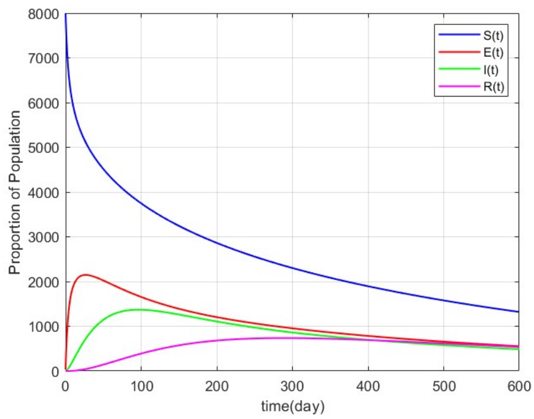
(a) $\alpha = 0$



(b) $\alpha = 0.7$



(c) $\alpha = 0.9$



(d) $\alpha = 0.95$

Figure 3. Variations in SEIR compartment dynamics under different α values.

Table 3. SEIR compartment values at t = 600 under different α values.

α	S	E	I
0	62	490	357
0.7	224	535	387
0.9	533	445	401
0.95	1320	554	487
1	6591	340	278

By systematically varying the time delay τ near the critical value τ^*, we observed dynamic changes in the system's behavior. Figures 4–6 respectively illustrate the cases where $\tau_2 = 0$ with $\tau_1 = 24.8 < \tau_1^*$, $\tau_1 = 0$ with $\tau_2 = 35 < \tau_2^*$, and $\tau_2 = 10.001$ with $\tau_1 = 26 < \dot{\tau}_1^*$. Collectively, these three figures demonstrate that when $\tau < \tau^*$, the phase space trajectories exhibit inward spiraling behavior toward the endemic equilibrium point E_2, thereby confirming its local asymptotic stability. This contraction phenomenon indicates that minor perturbations gradually decay over time, dynamically stabilizing the system.

Figures 7–9 respectively illustrate the cases where $\tau_2 = 0$ with $\tau_1 = \tau_1^* = 37.824$, $\tau_1 = 0$ with $\tau_2 = \tau_2^* = 48.501$, and $\tau_2 = 10.001$ with $\tau_1 = \dot{\tau}_1^* = 35.302$. These three figures depict a critical transition at $\tau = \tau^*$, where the trajectories merge into a closed circular orbit. This persistent oscillation signals the emergence of a periodic solution via a Hopf bifurcation, marking a qualitative shift from stability to sustained periodic behavior. The recovery delay refers to the time required for an infected individual to lose infectiousness after initiating treatment. For tuberculosis patients, effective treatment typically leads to a significant reduction in infectiousness within 1-2 months (approximately 30–60 days). In Figure 8, we obtained $\tau_2 = 48.501$ through numerical simulation, which falls within the expected range of 30–60 days, indicating that the simulation results are consistent with clinical observations.

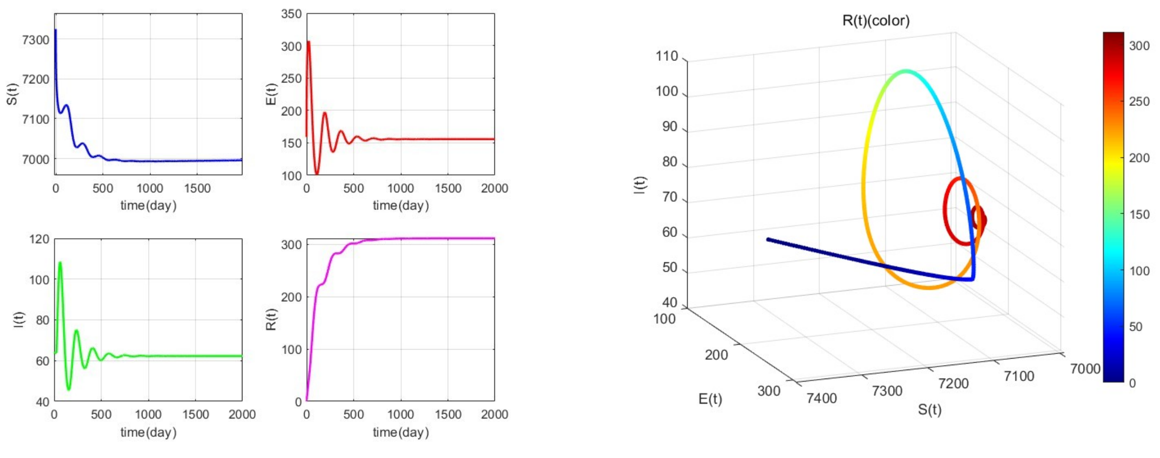


Figure 4. When $\tau_1 = 24.8, \tau_2 = 0$, the system is locally asymptotically stable at E_2.

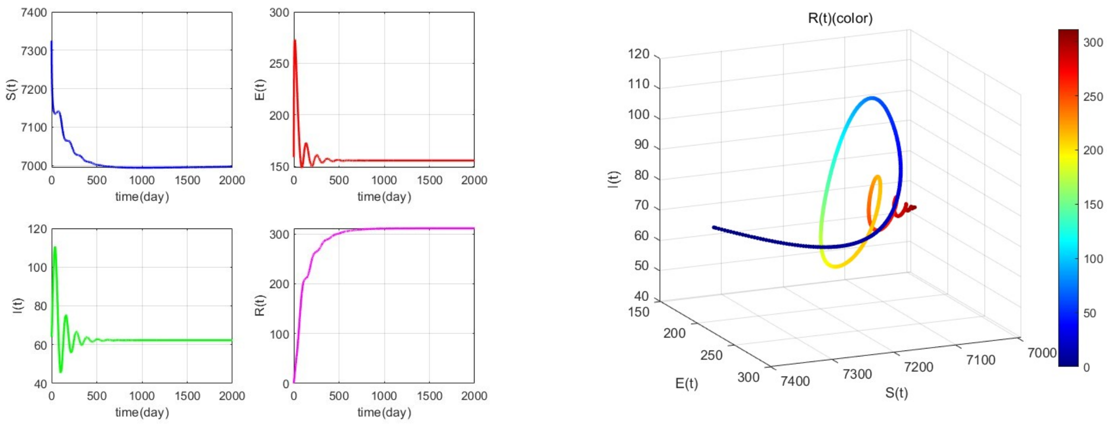


Figure 5. When $\tau_1 = 0, \tau_2 = 35$, the system is locally asymptotically stable at E_2.

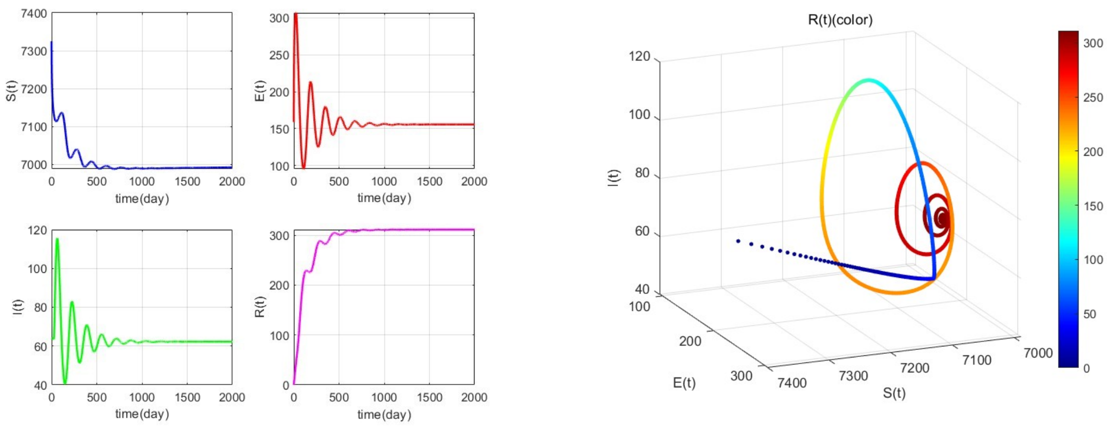


Figure 6. When $\tau_1 = 26, \tau_2 = 10.001$, the system is locally asymptotically stable at E_2.

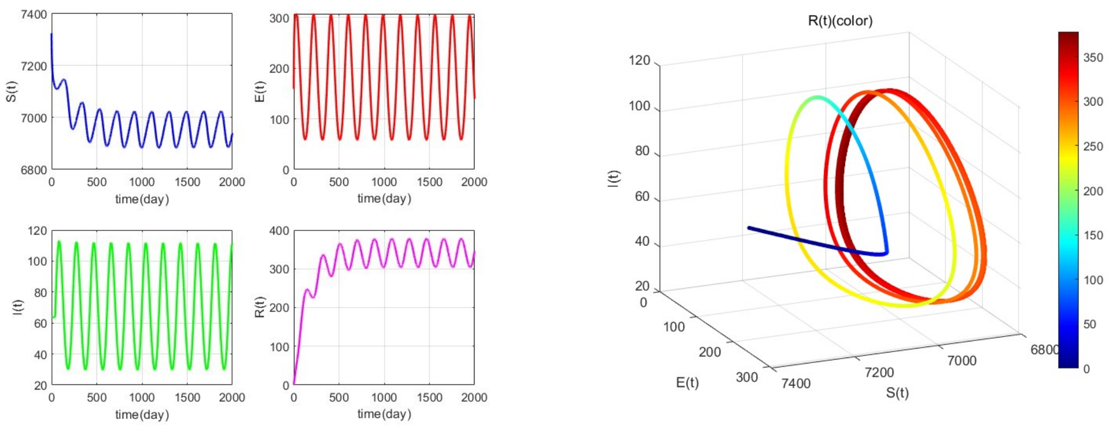


Figure 7. When $\tau_1 = 37.824, \tau_2 = 0$, the system undergoes a Hopf bifurcation at E_2.

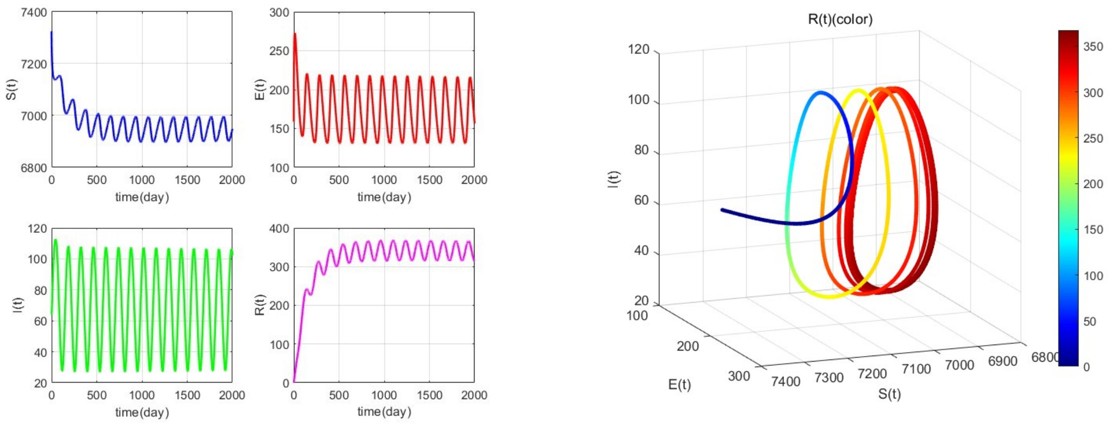


Figure 8. When $\tau_1 = 0, \tau_2 = 48.501$, the system undergoes a Hopf bifurcation at E_2.

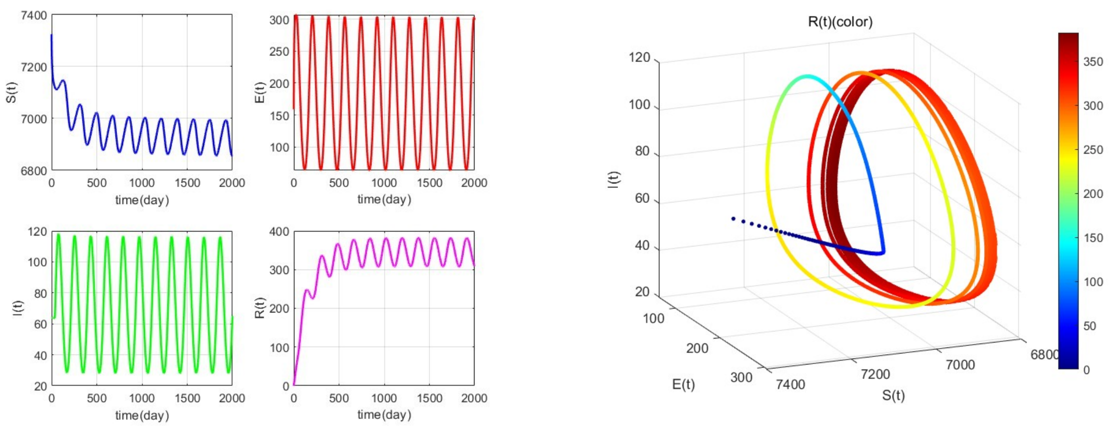


Figure 9. When $\tau_1 = 35.302, \tau_2 = 10.001$, the system undergoes a Hopf bifurcation at E_2.

Figures 10–12 respectively illustrate the cases where $\tau_2 = 0$ with $\tau_1 = 38 > \tau_1^*$, $\tau_1 = 0$ with $\tau_2 = 49 > \tau_2^*$, and $\tau_2 = 10.001$ with $\tau_1 = 35.4 > \dot{\tau}_1^*$. These three figures reveal a destabilizing regime when $\tau > \tau^*$: the trajectories spiral outward, diverging from E_2. This divergence confirms that surpassing the time-delay threshold destabilizes the equilibrium, rendering the system unstable despite its prior stability at lower delays.

These results highlight the system's sensitivity to time delays and emphasize the critical role of τ^* as a bifurcation parameter. The transition from stability to instability through oscillatory dynamics is consistent with classical Hopf bifurcation theory, providing a mechanistic understanding of delay-induced instability in epidemiological models.

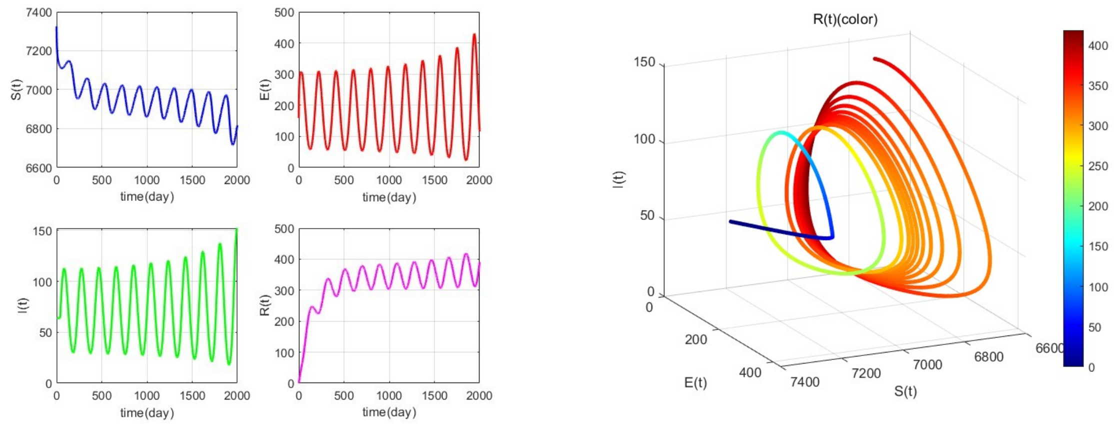


Figure 10. When $\tau_1 = 38, \tau_2 = 0$, the system is unstable at E_2.

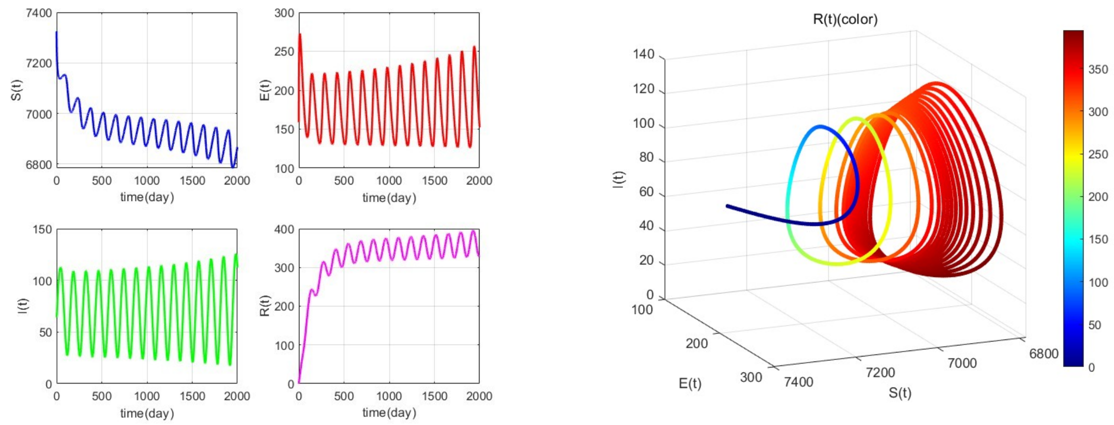


Figure 11. When $\tau_1 = 0, \tau_2 = 49$, the system is unstable at E_2.

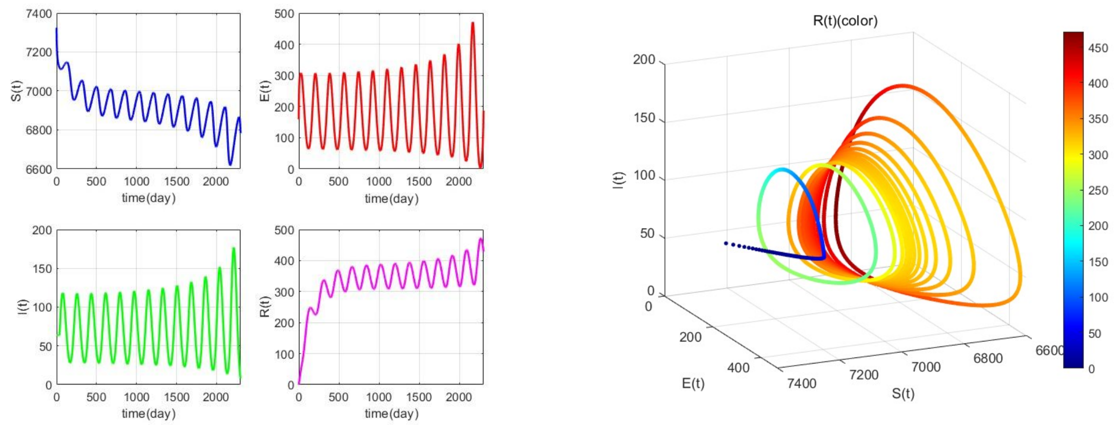


Figure 12. When $\tau_1 = 35.4, \tau_2 = 10.001$, the system is unstable at E_2.

6. Conclusions

This paper investigates a delayed SEIRm epidemic model with mobility-adaptive behavior, examining the impact of population movement on disease transmission and the effect of time delays on system stability. First, an SEIRm biological epidemic model with a population mobility function is formulated, and the existence and stability of both the disease-free equilibrium and the endemic equilibrium are analyzed. By calculating the basic reproduction number R_0, the study reveals the transmission dynamics of the epidemic under different conditions. When $R_0 < 1$, the disease tends to vanish; when $R_0 > 1$, the disease persists in the population and leads to an endemic state.

In the stability analysis of the disease-free equilibrium, this paper employs the Routh-Hurwitz criterion to prove its local asymptotic stability and further investigates the possibility of a Hopf bifurcation.

For the endemic equilibrium E_2, this paper proves its existence under the condition $R_0 > 1$ and analyzes its stability. The study reveals that the stability of the endemic equilibrium is jointly determined by the intensity of population movement, the response coefficient of infection risk, and the delay parameters. When the delay parameters are relatively small, the endemic equilibrium remains locally asymptotically stable. However, as the delay increases, the system may undergo a Hopf bifurcation, leading to periodic oscillations. If the delay continues to increase, the system may eventually become unstable. This finding indicates that disease transmission is influenced not only by the basic number of reproductions, but also by the significant effects of time delays. Specifically, delays in incubation and recovery periods may trigger subsequent outbreaks, resulting in periodic infection surges.

The numerical simulations validate the theoretical analysis and illustrate the dynamical characteristics of disease transmission under different parameter settings. The simulation results indicate that properly controlled population movement can significantly reduce the spread of the epidemic. Although mobility responsiveness does not affect the basic number of reproductions, it can mitigate the peak of infections, thereby protecting a larger proportion of the susceptible population. Using the influence of human mobility on epidemic transmission enables the design of more effective intervention measures to better control and prevent outbreaks of infectious diseases. However, excessive reliance on movement restrictions can lead to long-term economic and social problems, highlighting the need to balance mobility control with socioeconomic activities in epidemic prevention strategies.

In general, this study emphasizes the critical roles of time-delayed effects and population movement in disease transmission, revealing the key role of Hopf bifurcation in epidemic dynamics. This study also provides theoretical support for the development of infectious disease control strategies, contributing to a better understanding of periodic epidemic fluctuations in the real world. Furthermore, the findings offer scientific insight for public health policy makers to optimize intervention measures and improve the effectiveness of epidemic prevention and control.

Author contributions

Both authors contributed equally to the manuscript. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. A. E. Kovach, Rubin's pathology, clinicopathologic foundations of medicine, *Am. J. Surg. Pathol.*, **32** (2008), 1427. https://doi.org/10.1097/PAS.0B013E31816D7194
2. *Tuberculosis*, Wikipedia, 2025. Available from: `https://en.wikipedia.org/wiki/Tuberculosis`.
3. Z. Zhang, W. Zhang, K. S. Nisar, N. Gul, A. Zeb, V. Vijayakumar, Dynamical aspects of a tuberculosis transmission model incorporating vaccination and time delay, *Alex. Eng. J.*, **66** (2023), 287–300. https://doi.org/10.1016/J.AEJ.2022.11.010
4. M. M. Ojo, O. J. Peter, E. F. D. Goufo, H. S. Panigoro, F. A. Oguntolu, Mathematical model for control of tuberculosis epidemiology, *J. Appl. Math. Comput.*, **69** (2022), 69–87. https://doi.org/10.1007/S12190-022-01734-X
5. S. Georgiev, Mathematical identification analysis of a fractional-order delayed model for tuberculosis, *Fractal Fract.*, **7** (2023), 538. https://doi.org/10.3390/FRACTALFRACT7070538
6. S. Kundu, S. Kouachi, S. Kumar, N. Kumari, Pattern formation and stability analysis in a delayed epidemic model with two aware classes, *Eur. Phys. J. Plus*, **139** (2024), 1079. https://doi.org/10.1140/EPJP/S13360-024-05840-6
7. A. E. Gemeda, L. L. Obsu, E. D. Gurmu, S. T. Degefa, Mathematical modeling of tuberculosis transmission dynamics with vaccination and two lines of treatments: a caputo fractional approach, *J. Appl. Math. Comput.*, **71** (2024), 2017–2049. https://doi.org/10.1007/S12190-024-02308-9
8. R. Chinnathambi, F. A. Rihan, H. J. Alsakaji, A fractional-order model with time delay for tuberculosis with endogenous reactivation and exogenous reinfections, *Math. Methods Appl. Sci.*, **44** (2021), 8011–8025. https://doi.org/10.1002/mma.5676
9. M. Saade, S. Ghosh, M. Banerjee, V. Volpert, Delay epidemic models determined by latency, infection, and immunity duration, *Math. Biosci.*, **370** (2024), 109155. https://doi.org/10.1016/j.mbs.2024.109155

10. D. Kalyan, B. S. N. Murthy, S. A. Samad, M. H. A. Biswas, Mathematical transmission analysis of SEIR tuberculosis disease model, *Sens. Int.*, **2** (2021), 100120. https://doi.org/10.1016/j.sintl.2021.100120

11. T. S. Brown, D. A. Robinson, C. O. Buckee, B. Mathema, Connecting the dots: understanding how human mobility shapes TB epidemics, *Trends Microbiol.*, **30** (2022), 1036–1044. https://doi.org/10.1016/j.tim.2022.04.005

12. A. Jabbari, M. Lotfi, H. Kheiri, S. Khajanchi, Mathematical analysis of the dynamics of a fractional-order tuberculosis epidemic in a patchy environment under the influence of re-infection, *Math. Meth. Appl. Sci.*, **46** (2023), 17798–17817. https://doi.org/10.1002/mma.9532

13. W. Wang, Modeling adaptive behavior in influenza transmission, *Math. Model. Nat. Phenom.*, **7** (2012), 253–262. https://doi.org/10.1051/mmnp/20127315

14. Y. Cai, W. Wang, Global stability for an influenza transmission model incorporating human mobility behavior, *Int. J. Biomath.*, **10** (2017), 1750100. https://doi.org/10.1142/S1793524517501005

15. Y. Cai, J. Li, Y. Kang, K. Wang, W. Wang, The fluctuation impact of human mobility on the influenza transmission, *J. Franklin Inst.*, **357** (2020), 8899–8924. https://doi.org/10.1016/j.jfranklin.2020.07.002

16. S. Hazarie, D. Soriano-Paños, A. Arenas, J. Gómez-Gardeñes, G. Ghoshal, Interplay between population density and mobility in determining the spread of epidemics in cities, *Commun. Phys.*, **4** (2021), 191. https://doi.org/10.1038/s42005-021-00679-0

17. D. Gao, L. Cao, Vector-borne disease models with Lagrangian approach, *J. Math. Biol.*, **88** (2024), 22. https://doi.org/10.1007/s00285-023-02044-x

18. K. Castellano, R. B. Salako, On the effect of lowering population's movement to control the spread of an infectious disease, *J. Differ. Equations*, **316** (2022), 1–27. https://doi.org/10.1016/j.jde.2022.01.031

19. Y. Qin, L. Yang, Z. Gu, Dynamics behavior of a novel infectious disease model considering population mobility on complex network, *Int. J. Dynam. Control*, **12** (2024), 2295–2309. https://doi.org/10.1007/s40435-023-01371-7

20. Y. Liu, X. Wang, C. Song, J. Chen, H. Shu, M. Wu, et al., Quantifying human mobility resilience to the COVID-19 pandemic: a case study of Beijing, China, *Sustain. Cities Soc.*, **89** (2023), 104314. https://doi.org/10.1016/j.scs.2022.104314

21. W. Zhang, R. Xie, X. Dong, J. Li, P. Peng, E. D. R. S. Gonzalez, SEIR-FMi: a coronavirus disease epidemiological model based on intra-city movement, inter-city movement and medical resource investment, *Comput. Biol. Med.*, **149** (2022), 106046. https://doi.org/10.1016/j.compbiomed.2022.106046

22. M. Grave, A. Viguerie, G. F. Barros, A. Reali, R. F. S. Andrade, A. L. G. A. Coutinho, Modeling nonlocal behavior in epidemics via a reaction-diffusion system incorporating population movement along a network, *Comput. Meth. Appl. Mech. Eng.*, **401** (2022), 115541–115541. https://doi.org/10.1016/j.cma.2022.115541

23. Y. Zhang, X. Zheng, M. Helbich, N. Chen, Z. Chen, City2vec: urban knowledge discovery based on population mobile network, *Sustain. Cities Soc.*, **85** (2022), 104000. https://doi.org/10.1016/j.scs.2022.104000
24. Q. Wang, H. Wu, There exists the "smartest" movement rate to control the epidemic rather than "city lockdown", *Appl. Math. Model.*, **106** (2022), 696–714. https://doi.org/10.1016/j.apm.2022.02.018
25. X. Kang, Y. Hu, Z. Liu, S. Sarwar, Forecast and evaluation of asymptomatic COVID-19 patients spreading in China, *Results Phys.*, **34** (2022), 105195. https://doi.org/10.1016/j.rinp.2022.105195
26. C. Liu, J. Gao, J. Kanesan, Dynamics analysis and optimal control of delayed SEIR model in COVID-19 epidemic, *J. Inequalities Appl.*, **31** (2024), 66. https://doi.org/10.1186/s13660-024-03140-2
27. J. Deng, H. Xiang, How do time delays influence dynamics and controls of a generalized SEAIR model? *Commun. Nonlinear Sci. Numer. Simul.*, **137** (2024), 108156. https://doi.org/10.1016/j.cnsns.2024.108156
28. Z. Zhang, S. Kundu, J. P. Tripathi, S. Bugalia, Stability and Hopf bifurcation analysis of an SVEIR epidemic model with vaccination and multiple time delays, *Chaos Soliton. Fract.*, **131** (2020), 109483. https://doi.org/10.1016/j.chaos.2019.109483
29. P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48. https://doi.org/10.1016/S0025-5564(02)00108-6