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Research article

Equivalence of constacyclic codes over finite non-chain ring and quantum codes

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Abstract: Let $\Re_l = \mathbb{F}_q[u]/\langle u^l - u\rangle$, where $q = p^m$, p is a prime, and $m \in \mathbf{N}^+$, $(l-1) \mid (p-1)$. First, we study isometry and equivalence between constacyclic codes over $\Re_l$, and we present the equivalence conditions for $(e_1a_1 + e_2a_2 + \cdots + e_la_l)$-constacyclic code and $(e_1b_1 + e_2b_2 + \cdots + e_lb_l)$-constacyclic code. Further, based on the equivalence conditions, we classify constacyclic codes over $\Re_l$. Finally, based on the equivalence of constacyclic codes over $\Re_l$ and CSS construction, we construct some new quantum codes that are better than the existing codes in some recent references.

Keywords: quantum codes; isometric equivalence; monomial equivalence; constacyclic codes; CSS construction
Mathematics Subject Classification: 94B05, 94B15

1. Introduction

Constacyclic codes over finite non-chain rings exhibit lower encoding and decoding complexity compared to general linear codes due to their well-defined algebraic structures. Additionally, they possess unique properties absent in linear codes and encompass numerous optimal codes. Gao and Wang [1] studied self-dual constacyclic codes over the finite non-chain ring $\mathbb{F}_q[v]/\langle v^m - v\rangle$ and constructed some good self-dual codes. Tian et al. [2] studied hulls of constacyclic codes over finite non-chain rings and constructed some new quantum codes with good parameters. Castillo-Guillén et al. [3] studied constacyclic codes over finite local Frobenius non-chain rings with nilpotency index 3. Shi et al. [4] studied the $Z_pZ_{p^k}$-additive codes and their duality. Shi et al. [5] studied two new infinite families of two-weight codes over the ring $\mathbb{F}_p + u\mathbb{F}_p$, with $u^2 = u$. The concept of equivalence for constacyclic codes has been recognized and studied extensively for many years. Shi et al. [6] studied the equivalence and duality of polycyclic codes associated with trinomials over finite fields. Chen et al. [7] investigated the equivalence of constacyclic codes over finite fields and classified such codes by introducing an equivalence relation termed isometry. Chen et al. [8] introduced an equivalence relation termed n-equivalence and established a classification of constacyclic codes of length n over finite fields. Chibloun et al. [9] generalized the notions of n-isometry and n-equivalence from finite fields to finite chain rings and classified constacyclic codes over finite chain rings. Dastbasteh et al. [10] investigated the equivalence of a-constacyclic code and b-constacyclic code under the condition that a and b have distinct multiplicative orders. As an application of these results, they employ the findings to systematically search for new linear codes. Bogart et al. [11] established the criteria for equivalence between two codes over finite fields. Bierbrauer [12] established the criteria for monomial equivalence between constacyclic codes and cyclic codes over finite fields. Quantum error correction represents a pivotal challenge in achieving efficient and reliable quantum communication and quantum information processing. The error-correcting capability of quantum codes serves as an essential prerequisite for the practical implementation of these technologies. Kong and Zheng obtained some new quantum codes by using constacyclic codes over finite non-chain rings and the CSS construction [13,14]. Gowdhaman et al. [15] obtained quantum codes by studying the the structure of cyclic and λ-constacyclic codes over $\frac{\mathbb{F}_p[u,v]}{\langle v^3-v,u^3-u,uv-vu\rangle}$ and applying the CSS construction. Fu and Liu [16] extended constacyclic codes to obtain Galois self-dual codes. Huang et al. [17] obtained three classes of quantum codes with optimal parameters by using Hermitian construction. Islam and Prakash [18] obtained quantum codes from cyclic codes over a finite non-chain ring $\mathbb{F}_q[u,v]/\langle u^2-\alpha u, v^2-1, uv-vu\rangle$ by using the CSS construction. Ashraf et al. [19] obtained quantum codes from cyclic codes over ring $\mathbb{F}_q\Re_1\Re_2$.

This paper extends existing results on the equivalence of constacyclic codes over finite fields and systematically constructs quantum codes through rigorous exploitation of these equivalence relations. The structure of this paper is organized as follows. In Section 2, we provide the necessary notations, definitions, and some known results. In Section 3, we establish the equivalence conditions for $(e_1a_1 + e_2a_2 + \cdots + e_la_l)$-constacyclic code and $(e_1b_1 + e_2b_2 + \cdots + e_lb_l)$-constacyclic code over finite non-chain ring $\Re_l = \mathbb{F}_q[u]/\langle u^l - u\rangle$, as well as the equivalence conditions between constacyclic codes and cyclic codes. Based on these equivalences, we classify constacyclic codes over $\Re_l$. In Section 4, leveraging these equivalence results and the CSS construction method, we construct some new quantum codes which better than the existing codes that appeared in some papers.

2. Preliminaries

Let $\Re_l = \mathbb{F}_q[u]/\langle u^l - u\rangle$, where $q = p^m$, p is a prime, and $m \in \mathbf{N}^+$, $(l-1) \mid (p-1)$. Clearly, $\Re_l$ is a commutative non-chain ring containing q^l elements. Since $(l-1) \mid (p-1)$, then

$$u^l - u = (u-\alpha_1)(u-\alpha_2)\cdots(u-\alpha_l),$$

where $\alpha_i \in \mathbb{F}_q$ for $i = 1, 2, \cdots, l$.

Let

$$e_i = \frac{(u-\alpha_1)\cdots(u-\alpha_{i-1})(u-\alpha_{i+1})\cdots(u-\alpha_l)}{(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_l)},$$

for $i = 1, 2, \cdots, l$.

We can get that $e_i^2 = e_i$ and $e_ie_j = 0$, $1 = e_1 + e_2 + \cdots + e_l$, where $i \neq j$ and $i, j = 1, 2, \cdots, l$. By the Chinese Remainder Theorem, it follows that

$$\Re_l = \bigoplus_{i=1}^{l} e_i\Re_l = \bigoplus_{i=1}^{l} e_i\mathbb{F}_q.$$

For any $r \in \Re_l$, then r can be uniquely decomposed as

$$r = r_1e_1 + r_2e_2 + \cdots + r_le_l,$$

where $r_i \in \mathbb{F}_q$ for $i = 1, 2, \cdots, l$.

Let a be a unit in $\Re_l$ and any $\mathbf{r} = (r_0, r_1, \cdots, r_{n-1}) \in \Re_l^n$, the a-constacyclic shift σ_a of $\mathbf{r}$ is defined as $\sigma_a(\mathbf{r}) = (ar_{n-1}, r_0, \cdots, r_{n-2})$. A linear code C is called an a-constacyclic code if $\sigma_a(C) = C$. Especially if $a = 1$, then C is a cyclic code over $\Re_l$. While $a = -1$, C is a negacyclic code over $\Re_l$. Each codeword $(r_0, r_1, \cdots, r_{n-1}) \in \Re_l^n$ corresponds bijectively to a polynomial $r_0 + r_1x + \cdots + r_{n-1}x^{n-1} \in \Re_l[x]/\langle x^n - a\rangle$.

In polynomial representation, an a-constacyclic code of length n over $\Re_l$ is defined as an ideal of $\Re_l[x]/\langle x^n - a\rangle$.

We define a Gray map ϕ_l as

$$\begin{aligned} \phi_l : \Re_l &\to \mathbb{F}_q^l, \\ r = \sum_{i=1}^{l} r_ie_i &\mapsto (r_1, r_2, \cdots, r_l). \end{aligned}$$

We extend ϕ_l as

$$\begin{aligned} &\phi_l : \Re_l^n \to \mathbb{F}_q^{ln}, \\ &(r_0, r_1, \cdots, r_{n-1}) \mapsto (r_{1,0}, \cdots, r_{1,n-1}, r_{2,0}, \cdots, r_{2,n-1}, \cdots, r_{l,0}, \cdots, r_{l,n-1}), \end{aligned}$$

where $r_i = r_{1,i}e_1 + r_{2,i}e_2 + \cdots + r_{l,i}e_l \in \Re_l$ for $i = 0, 1, \cdots, n-1$.

For any element $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in \Re_l^n$, the Hamming distance of $\mathbf{x}, \mathbf{y}$ is defined as $d_H(\mathbf{x}, \mathbf{y}) = w_H(\mathbf{x}-\mathbf{y})$, and the Hamming weight of $\mathbf{x}-\mathbf{y}$ is defined as $w_H(\mathbf{x}-\mathbf{y}) = \sum_{i=1}^{n} w_H(x_i - y_i)$, the Gray distance of $\mathbf{x}, \mathbf{y}$ is defined as $d_G(\mathbf{x}, \mathbf{y}) = w_G(\mathbf{x} - \mathbf{y})$. Let C be a linear code of length n over $\Re_l$, the Hamming distance of C is defined as $d_H(C) = \min\{d_H(\mathbf{x}, \mathbf{y}) | \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}$, the Gray distance of C is defined as $d_G(C) = \min\{d_G(\mathbf{x}, \mathbf{y}) | \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}$. For any element $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Re_l^n$, the Gray weight of $\mathbf{x}$ is defined as $w_G(\mathbf{x}) = w_H(\phi_l(\mathbf{x}))$.

Let C be a linear code of length n over $\Re_l$, and

$$\begin{aligned} C_1 &= \{c_1 \in \mathbb{F}_q^n \mid e_1c_1 + e_2c_2 + \cdots + e_lc_l \in C, \exists c_2, c_3, \cdots, c_l \in \mathbb{F}_q^n\}, \\ C_2 &= \{c_2 \in \mathbb{F}_q^n \mid e_1c_1 + e_2c_2 + \cdots + e_lc_l \in C, \exists c_1, c_3, \cdots, c_l \in \mathbb{F}_q^n\}, \\ &\cdots, \\ C_l &= \{c_l \in \mathbb{F}_q^n \mid e_1c_1 + e_2c_2 + \cdots + e_lc_l \in C, \exists c_1, c_2, \cdots, c_{l-1} \in \mathbb{F}_q^n\}. \end{aligned}$$

Clearly, C_i is a linear code of length n over $\mathbb{F}_q$ for $i = 1, 2, \cdots, l$. Moreover

$$C = e_1C_1 \oplus e_2C_2 \oplus \cdots \oplus e_lC_l.$$

Lemma 2.1. *([13], Lemma 3.1) An element of the form $e_1a_1 + e_2a_2 + \cdots + e_la_l$ in the ring $\Re_l$ is a unit if and only if each coefficient a_i is a unit in $\mathbb{F}_q$ for $i = 1, 2, \cdots, l$.*

Lemma 2.2. *([14], Theorem 1) A linear code $C = \bigoplus_{i=1}^{l} e_iC_i$ is an $(a_1e_1 + a_2e_2 + \cdots + a_le_l)$-constacyclic code of length n over the ring $\Re_l$ if and only if C_i is an a_i-constacyclic code of length n over $\mathbb{F}_q$ for $i = 1, 2, \cdots, l$.*

Theorem 2.1. *([14], Theorem 2) Let* $C = \bigoplus_{i=1}^{l} e_i C_i$ *be an* $(a_1e_1 + a_2e_2 + \cdots + a_le_l)$*-constacyclic code of length n over* $\Re_l$*; then* $C = \langle e_1g_1(x) + e_2g_2(x) + \cdots + e_lg_l(x)\rangle$*, where* g_i *is the generator polynomial of* C_i *for* $i = 1, 2, \cdots, l$.

Theorem 2.2. *([14], Theorem 3) Let* $C = \bigoplus_{i=1}^{l} e_i C_i$ *be a linear code of length n over* $\Re_l$*; let* $C_i^{\perp}$ *be the dual code of* C_i*; then* $C^{\perp} = \bigoplus_{i=1}^{l} e_i C_i^{\perp}$*, where* $i = 1, 2, \cdots, l$.

Lemma 2.3. *([14], Theorem 8) Let C be an a-constacyclic code of length n over* $\mathbb{F}_q$*, whose generator polynomial is* $g(x)$*. Then, C is a dual-containing code if and only if* $x^n - a$ *is the divisor of* $f^*(x)f(x)$*, and* $f^*(x)$ *is the generator polynomial of* $C^{\perp}$*, where* $f^*(x)$ *is the reciprocal polynomial of* $f(x)$*,* $f(x)g(x) = (x^n - a)$*, and* $a^{-1} = a$.

Theorem 2.3. *([14], Theorem 5) Let C be a linear code of length n over* $\Re_l$ *with* $|C| = q^k$ *and minimum distance d, then* $\phi_l(C)$ *is a linear code* $[ln, k, d]$ *and* $\phi_l(C)^{\perp} = \phi_l(C^{\perp})$.

3. Isometry and equivalence between constacyclic codes over $\Re_l$

In this section, let $a = \sum_{i=1}^{l} a_i e_i$ be a unit in $\Re_l$; then $a^{-1} = \sum_{i=1}^{l} a_i^{-1} e_i$. $\mathbb{F}_q^*$ is the multiplicative group of units of $\mathbb{F}_q$, $\Re_l^*$ is the multiplicative group of units of $\Re_l$. The multiplicative order of a in $\mathbb{F}_q^*$ is denoted $\mathrm{ord}(a_i)$, where $a_i \in \mathbb{F}_q^*$ for $i = 1, 2, \cdots, l$.

Definition 3.1. *([9], Definition 3.1) Let a and b be units in* $\Re_l$*. If the polynomial* $ax^n - b$ *of* $\Re_l[x]$ *has a root in* $\Re_l$*, we say that a and b are n-equivalent and denote* $a \sim_n b$.

Lemma 3.1. "$\sim_n$" *is an equivalence relation on* $\Re_l^*$.

Proof. Reflexivity: For any $a \in \Re_l^*$, $ax^n - a$ of $\Re_l[x]$ has a root 1 in $\Re_l$, so, $a \sim_n a$.

Symmetry: For any $a, b \in \Re_l^*$, if $a \sim_n b$, let $r \in \Re_l$ be a root of $ax^n - b$, such that $ar^n = b$. Then $r^n = a^{-1}b$, we have $r \in \Re_l^*$. It follows that r^{-1} is a root of $bx^n - a$, so $b \sim_n a$.

Transitivity: For any $a, b, c \in \Re_l^*$, if $a \sim_n b$, $b \sim_n c$, let $r_1, r_2 \in \Re_l$, such that $ar_1^n = b$ and $br_2^n = c$. Then $r_1^n = a^{-1}b$ and $r_2^n = b^{-1}c$, we have $r_1, r_2 \in \Re_l^*$. It follows that $a(r_1r_2)^n = c$, so $a \sim_n c$. □

Theorem 3.1. *For any* $a, b \in \Re_l^*$*, the following are equivalent.*
(1) $b \sim_n a$.
(2) There exists an element $r \in \Re_l^*$ *such that the map* $\psi_l : \Re_l[x]/\langle x^n - a\rangle \to \Re_l[x]/\langle x^n - b\rangle$ *defined by* $f(x) \mapsto f(rx)$ *is an* $\Re_l$*-algebra isomorphism.*
(3) There exists an element $s \in \Re_l^*$ *such that* $b^{-1}a = s^n$.

Proof. (1)⇒(2) Since $b \sim_n a$, we have the polynomial $bx^n - a$ of $\Re_l[x]$ has a root $r \in \Re_l$, such that $br^n = a$. Then $r^n = b^{-1}a$, we have $r \in \Re_l^*$.

Since $\psi_l(x^n - a) = (rx)^n - a = r^nx^n - a = b^{-1}ax^n - a = b^{-1}a(x^n - b) \in \langle x^n - b\rangle$, so ψ_l is well-defined.

For any $f(x) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1}$, $g(x) = \tilde{r}_0 + \tilde{r}_1x + \cdots + \tilde{r}_{n-1}x^{n-1} \in \Re_l[x]/\langle x^n - a\rangle$, then

$$\begin{aligned}\psi_l(f(x) + g(x)) &= (r_0 + \tilde{r}_0) + (r_1 + \tilde{r}_1)rx + \cdots + (r_{n-1} + \tilde{r}_{n-1})(rx)^{n-1} \\ &= r_0 + r_1rx + \cdots + r_{n-1}(rx)^{n-1} + \tilde{r}_0 + \tilde{r}_1rx + \cdots + \tilde{r}_{n-1}(rx)^{n-1} \\ &= \psi_l(f(x)) + \psi_l(g(x))\end{aligned}$$

and

$$\psi_l(f(x)g(x)) = \sum_{m=0}^{n-1}\left(\sum_{\substack{i+j=m\\0\le i,j\le n-1}} r_i\tilde{r}_j + a\sum_{\substack{i+j=m+n\\0\le i,j\le n-1}} r_i\tilde{r}_j\right)(rx)^m = \psi_l(f(x))\psi_l(g(x)).$$

So ψ_l is an $\Re_l$-algebra homomorphism.

Suppose $\psi_l(f(x)) = \psi_l(g(x))$, we have $f(rx) = g(rx)(\mathrm{mod}(x^n-b))$, which implies that $f(rx)-g(rx) = 0(\mathrm{mod}(x^n-b))$, which implies that $f(rx)-g(rx) = h(x)(x^n-b)$, which implies that $f(x)-g(x) = h(r^{-1}x)((r^{-1}x)^n-b) = h(r^{-1}x)(a^{-1}bx^n-b) = h(r^{-1}x)a^{-1}b(x^n-a) = 0(\mathrm{mod}(x^n-a))$, so ψ_l is an injection.

Let $g(x) \in \Re_l[x]/\langle x^n-b\rangle$; then $f(x) = g(r^{-1}x) \in \Re_l[x]/\langle x^n-a\rangle$ and $\psi_l(f(x)) = g(x)$, so ψ_l is a surjection. Thus ψ_l is an $\Re_l$-algebra isomorphism.

(2)⇒(3) Since there exists an element $r \in \Re_l^*$ such that ψ_l is an $\Re_l$-algebra isomorphism, which implies that in $\Re_l[x]/\langle x^n-b\rangle$, then

$$a = \psi_l(a) = \psi_l(x^n) = \psi_l(x)^n = (rx)^n = r^nx^n = r^nb,$$

so $b^{-1}a = r^n$. Let $s = r$; then $s \in \Re_l^*$ and $b^{-1}a = s^n$.

(3)⇒(1) Since there exists an element $s \in \Re_l^*$ such that $b^{-1}a = s^n$, which implies that $bs^n = a$, we have the polynomial $bx^n - a$ of $\Re_l[x]$ has a root s in $\Re_l$, so $b \sim_n a$. □

Corollary 3.1. *Let $\Re_l^{n*} = \{r^n | r \in \Re_l^*\}$, for any $a \in \Re_l^*$, the n-equivalence class of a is $a\Re_l^{n*} = \{ar^n | r \in \Re_l^*\}$ and $|a\Re_l^{n*}| = |\Re_l^{n*}|$.*

Proof. Let $b \in \Re_l^*$ and $a \sim_n b$; by Theorem 3.1, there exists an element $s \in \Re_l^*$ such that $a^{-1}b = s^n$. Which implies that $b = as^n \in a\Re_l^{n*}$, for any $r \in \Re_l^*$, then $a^{-1}ar^n = r^n \in \Re_l^{n*}$, by Theorem 3.1, $a \sim_n ar^n$, so the n-equivalence class of a is $a\Re_l^{n*} = \{ar^n | r \in \Re_l^*\}$. Since a is a unit element, the elements of $a\Re_l^{n*}$ correspond one-to-one with the elements of $\Re_l^{n*}$, so $|a\Re_l^{n*}| = |\Re_l^{n*}|$. □

Corollary 3.2. *Let $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ be units in $\Re_l$. Then $a \sim_n b$ over $\Re_l$ if and only if $a_i \sim_n b_i$ over $\mathbb{F}_q$ for $i = 1, 2, \cdots, l$.*

Proof. By Theorem 3.1, $a \sim_n b$ over $\Re_l$ if and only if there exists $s = e_1s_1 + e_2s_2 + \cdots + e_ls_l \in \Re_l^*$ such that $a^{-1}b = s^n$, if and only if $b = as^n$, if and only if $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l = (e_1a_1 + e_2a_2 + \cdots + e_la_l)(e_1s_1^n + e_2s_2^n + \cdots + e_ls_l^n) = e_1a_1s_1^n + e_2a_2s_2^n + \cdots + e_la_ls_l^n$, if and only if $b_i = a_is_i^n$, if and only if $a_i \sim_n b_i$ over $\mathbb{F}_q$, where $i = 1, 2, \cdots, l$. □

Definition 3.2. *([7], Definition 3.1) Let $a, b \in \Re_l^*$ such that $\psi_l : \Re_l[x]/\langle x^n-a\rangle \to \Re_l[x]/\langle x^n-b\rangle$ by $f(x) \mapsto f(rx)$ is an $\Re_l$-algebra isomorphism. If for any $c, c' \in \Re_l[x]/\langle x^n-a\rangle$, the Hamming distance $d_H(\psi_l(c), \psi_l(c')) = d_H(c, c')$ and ψ_l is an isometry, we say a and b are n-isometric, and denote $a \cong_n b$.*

Lemma 3.2. " $\cong_n$ " *is an equivalence relation on $\Re_l^*$.*

Proof. Reflexivity: For any $a \in \Re_l^*$, the identity map of $\Re_l[x]/\langle x^n-a\rangle$ is an isometry, so $a \cong_n a$.

Symmetry: For any $a, b \in \Re_l^*$, if $a \cong_n b$, then there exists ψ_l from $\Re_l[x]/\langle x^n-a\rangle$ to $\Re_l[x]/\langle x^n-b\rangle$ is an isometry; it is easy to know that ψ_l^{-1} is an isometry.

Transitivity: For any $a, b, c \in \Re_l^*$ such that $a \cong_n b$ and $b \cong_n c$. Let ψ_l be an isometry from $\Re_l[x]/\langle x^n-a\rangle$ to $\Re_l[x]/\langle x^n-b\rangle$, $\tilde{\psi}_l$ be an isometry from $\Re_l[x]/\langle x^n-b\rangle$ to $\Re_l[x]/\langle x^n-c\rangle$, it is easy to know that $\tilde{\psi}_l \circ \psi_l$ is an isometry from $\Re_l[x]/\langle x^n-a\rangle$ to $\Re_l[x]/\langle x^n-c\rangle$, so $a \cong_n c$. □

Corollary 3.3. *Let $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ be units in $\Re_l$. If $a \sim_n b$ over $\Re_l$, then $a \cong_n b$.*

Proof. By Theorem 3.1, if $a \sim_n b$ over $\Re_l$, then there is $r \in \Re_l^*$ such that $\psi_l : \Re_l[x]/\langle x^n - a\rangle \to \Re_l[x]/\langle x^n - b\rangle$ by $f(x) \mapsto f(rx)$ is an R_l-algebra isomorphism. It is easy to see that ψ_l is an isometry, so $a \cong_n b$. □

A monomial matrix over $\Re_l$ is a matrix in which each row and each column has exactly one nonzero element. Let C_1 and C_2 be two linear codes of length n over $\Re_l$, the generator matrix of C_1 is G_1, if there exists a monomial matrix M such that G_1M is the generator matrix of C_2, we say C_1 and C_2 are monomially equivalent.

Definition 3.3. *([10], Definition 2.2) Let C and D be two linear codes over $\Re_l$. If there exists an $\Re_l$-isomorphism $\psi_l : C \to D$ such that for any $c, c' \in C$, the Hamming distance $d_H(\psi_l(c), \psi_l(c')) = d_H(c, c')$, we say C and D are isometrially equivalent.*

Theorem 3.2. *([11], Corollary 1) Let C_1 and C_2 be two linear codes over $\mathbb{F}_q$. Then C_1 and C_2 are isometrially equivalent if and only if C_1 and C_2 are monomially equivalent.*

Theorem 3.3. *([10], Theorem 2.4) Let $a, b \in \mathbb{F}_q^*$ such that* $\mathrm{ord}(a)|\mathrm{ord}(b)$ *and n be a positive integer such that* $\gcd(n, q) = \gcd(n, q-1) = 1$. *Then the families of a-constacyclic and b-constacyclic codes of length n over $\mathbb{F}_q$ are monomially equivalent.*

Corollary 3.4. *Let $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ be units in $\Re_l$ and $a \cong_n b$. Let $C = \bigoplus_{j=1}^{l} e_jC_j$ be the a-constacyclic code of length n over $\Re_l$. Then there exists an isometry ψ_l such that $\psi_l(C)$ is a b-constacyclic code, and C and $\psi_l(C)$ are isometrially equivalent.*

Proof. By Definition 3.2, if $a \cong_n b$ in $\Re_l$, then there exists an isometry $\psi_l : \Re_l[x]/\langle x^n - a\rangle \to \Re_l[x]/\langle x^n - b\rangle$ by $f(x) \mapsto f(rx)$.

Because C is an a-constacyclic code of length n over $\Re_l$, so C is an ideal of $\Re_l[x]/\langle x^n - a\rangle$.

Suppose $C = \langle g(x)\rangle$, we have $\psi_l(g(x)) = g(rx)$, because $g(x)|(x^n - a)$; then there exists $h(x) \in \Re_l[x]$ such that $g(x)h(x) = x^n - a$; then $g(rx)h(rx) = (rx)^n - a = 0(\mathrm{mod}(x^n - b))$, so $\psi_l(g(x))|(x^n - b)$ and $\psi_l(C)$ is a b-constacyclic code of length n over $\Re_l$.

Because ψ_l is an isometry, we have C and $\psi_l(C)$ are isometrially equivalent. □

Theorem 3.4. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ and $D = \bigoplus_{i=1}^{l} e_iD_i$ be the a-constacyclic code and b-constacyclic code of length n over $\Re_l$, respectively, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ are units in $\Re_l$. If $b_i = a_is_i^n$ for $i = 1, 2, \cdots, l$, then C and D are isometrially equivalent.*

Proof. By Corollary 3.2 and Corollaries 3.3 and 3.4, we can have the result. □

Theorem 3.5. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ and $D = \bigoplus_{i=1}^{l} e_iD_i$ be the a-constacyclic code and b-constacyclic code of length n over $\Re_l$, respectively, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ are units in $\Re_l$. Then C and D are isometric equivalent if and only if C_i and D_i are isometric equivalent for $i = 1, 2, \cdots, l$.*

Proof. If C and D are isometric equivalent, then there exists an $\Re_l$-isomorphism $\psi_l : C \to D$ such that for any $\mathbf{r} = e_1\mathbf{r}_1 + e_2\mathbf{r}_2 + \cdots + e_l\mathbf{r}_l$ and $\mathbf{r}' = e_1\mathbf{r}_1' + e_2\mathbf{r}_2' + \cdots + e_l\mathbf{r}_l' \in C$, the Hamming distance

$$d_H(\psi_l(\mathbf{r}), \psi_l(\mathbf{r}')) = d_H(\mathbf{r}, \mathbf{r}').$$

Since $\Re_l = \bigoplus_{i=1}^{l} e_i\Re_l = \bigoplus_{i=1}^{l} e_i\mathbb{F}_q$, ψ_l decomposes into component maps $\varphi_i : C_i \to D_i$ such that for any $\mathbf{r}_i \in C_i$, then

$$\psi_l(e_1\mathbf{r}_1 + e_2\mathbf{r}_2 + \cdots + e_l\mathbf{r}_l) = e_1\varphi_1(\mathbf{r}_1) + e_2\varphi_2(\mathbf{r}_2) + \cdots + e_l\varphi_l(\mathbf{r}_l).$$

Because ψ_l is an $\Re_l$-isomorphism and preserves the direct sum structure, we have each φ_i is an $\mathbb{F}_q$-isomorphism, and for any $\mathbf{r}_i, \mathbf{r}_i^{'} \in C_i$, then $e_i\mathbf{r}_i = e_1 0 + \cdots + e_{i-1}0 + e_i\mathbf{r}_i + \cdots + e_l 0 \in \Re_l, e_i\mathbf{r}_i^{'} = e_1 0 + \cdots + e_{i-1}0 + e_i\mathbf{r}_i^{'} + \cdots + e_l 0 \in \Re_l$, and

$$\begin{aligned} d_H(e_i\mathbf{r}_i, e_i\mathbf{r}_i^{'}) &= d_H(\psi_l(e_i\mathbf{r}_i), \psi_l(e_i\mathbf{r}_i^{'})) \\ &= d_H(\psi_l(e_1 0 + \cdots + e_{i-1}0 + e_i\mathbf{r}_i + \cdots + e_l 0), \psi_l(e_1 0 + \cdots + e_{i-1}0 + e_i\mathbf{r}_i^{'} + \cdots + e_l 0)) \\ &= d_H(e_i\varphi_i(\mathbf{r}_i), e_i\varphi_i(\mathbf{r}_i^{'})), \end{aligned}$$

so we have

$$d_H(\mathbf{r}_i, \mathbf{r}_i^{'}) = d_H(\varphi_i(\mathbf{r}_i), \varphi_i(\mathbf{r}_i^{'})),$$

so C_i and D_i are isometric equivalent for $i = 1, 2, \cdots, l$.

On the contrary, if C_i and D_i are isometric equivalent for $i = 1, 2, \cdots, l$, then there exists an $\mathbb{F}_q$-isomorphism $\varphi_i : C_i \to D_i$ such that for any $\mathbf{r}_i, \mathbf{r}_i^{'} \in C_j$, the Hamming distance

$$d_H(\varphi_i(\mathbf{r}_i), \varphi_i(\mathbf{r}_i^{'})) = d_H(\mathbf{r}_i, \mathbf{r}_i^{'}),$$

for $i = 1, 2, \cdots, l$.

The map $\psi : C \to D$ is defined by $\mathbf{r} = e_1\mathbf{r}_1 + e_2\mathbf{r}_2 + \cdots + e_l\mathbf{r}_l \mapsto e_1\varphi_1(\mathbf{r}_1) + e_2\varphi_2(\mathbf{r}_2) + \cdots + e_l\varphi_l(\mathbf{r}_l)$. It is easy to know that $\psi : C \to D$ is an isometry and $d_H(\psi(\mathbf{r}), \psi(\mathbf{r}^{'})) = d_H(\mathbf{r}, \mathbf{r}^{'})$, then C and D are isometric equivalent. □

By Theorems 3.2 and 3.5 we can have the following corollaries.

Corollary 3.5. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ and $D = \bigoplus_{i=1}^{l} e_iD_i$ be the a-constacyclic code and b-constacyclic code of length n over $\Re_l$, respectively, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ are units in $\Re_l$. Then C and D are isometric equivalent if and only if C and D are monomially equivalent.*

By Theorems 3.3 and 3.5 we can have the following corollaries.

Corollary 3.6. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ and $D = \bigoplus_{i=1}^{l} e_iD_i$ be the a-constacyclic code and b-constacyclic code of length n over $\Re_l$, respectively, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$ and $b = e_1b_1 + e_2b_2 + \cdots + e_lb_l$ are units in $\Re_l$, where $a_i, b_i \in \mathbb{F}_q^*$ such that* $\mathrm{ord}(a_i)|\mathrm{ord}(b_i)$ *for $i = 1, 2, \cdots, l$ and n is a positive integer such that* $\gcd(n, q) = \gcd(n, q-1) = 1$. *Then C and D are monomially equivalent.*

Theorem 3.6. *([7], Corollary3.4) Let n be a positive integer and $a \in \mathbb{F}_q^*$. The a-constacyclic code of length n over $\mathbb{F}_q$ are isometric to the cyclic code of length n over $\mathbb{F}_q$ if and only if there exists an element u in $\mathbb{F}_q^*$ such that $u^na = 1$.*

Corollary 3.7. *Let n be a positive integer and $a \in \Re_l^*$. The a-constacyclic code $C = \bigoplus_{i=1}^{l} e_iC_i$ of length n over $\Re_l$ is isometric to the cyclic code of length n over $\Re_l$ if and only if there exists an element r in $\Re_l^*$ such that $r^na = 1$, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$.*

Proof. If there exists an element r in $\Re_l^*$ such that $r^n a = 1$, let $\psi_l : \Re_l[x]/\langle x^n - 1\rangle \to \Re_l[x]/\langle x^n - a\rangle$ be defined by $f(x) \mapsto f(rx)$, then ψ_l is an isometry, so the a-constacyclic code of length n over $\Re_l$ is isometric to the cyclic codes of length n over $\Re_l$.

Conversely, if the a-constacyclic code of length n over $\Re_l$ is isometric to the cyclic code of length n over $\Re_l$, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$. By Lemma 2.2 and Theorem 3.4, we can have C_i is an a_i-constacyclic code that is isometric to the cyclic code of length n over $\mathbb{F}_q$, where $i = 1, 2, \cdots, l$. By Theorem 3.5, there exists $r_i \in \mathbb{F}_q^*$ such that $r_i^n a_i = 1$, where $i = 1, 2, \cdots, l$. Let $r = e_1r_1 + e_2r_2 + \cdots + e_lr_l$, then $r^n a = (e_1r_1^n + e_2r_2^n + \cdots + e_lr_l^n)(e_1a_1 + e_2a_2 + \cdots + e_la_l) = e_1r_1^na_1 + e_2r_2^na_2 + \cdots + e_lr_l^na_l = e_1 + e_2 + \cdots + e_l = 1$. □

Theorem 3.7. *([12], Theorem 15) Let $a \in \mathbb{F}_q^*$ such that* $\gcd(\mathrm{ord}(a), n) = 1$ *and n be a positive integer such that* $\gcd(n, q) = 1$. *Then the families of a-constacyclic and cyclic codes of length n over $\mathbb{F}_q$ are monomially equivalent.*

By Theorems 3.5 and 3.7, we can have the following corollary.

Corollary 3.8. *Let n be a positive integer such that* $\gcd(n, q) = 1$ *and $a \in \Re_l^*$ such that* $\gcd(\mathrm{ord}(a_i), n) = 1$. *Then the a-constacyclic code $C = \bigoplus_{i=1}^{l} e_iC_i$ of length n over $\Re_l$ is isometric to the cyclic code of length n over $\Re_l$, where $a = e_1a_1 + e_2a_2 + \cdots + e_la_l$.*

Example 3.1. *Let $n = 7$ and $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, $\Re_3 = \mathbb{F}_5[u]/\langle u^3 - u\rangle$, $e_1 = \frac{u^2+u}{2}, e_2 = \frac{u^2-u}{2}, e_3 = 1 - u^2$. $\Re_3 = e_1\mathbb{F}_5 \oplus e_2\mathbb{F}_5 \oplus e_3\mathbb{F}_5$. $\mathrm{ord}(2) = 4, \mathrm{ord}(3) = 4, \mathrm{ord}(4) = 2, \gcd(7, 5) = \gcd(7, 4) = \gcd(7, 2) = 1$, and by Theorems 3.3 and 3.6, the 2-constacyclic code, 3-constacyclic code, 4-constacyclic code, and cyclic code are monomially equivalent. By Corollary 3.8, for any $e_1a_1 + e_2a_2 + e_3a_3 \in \Re_l^*$, the $(e_1a_1 + e_2a_2 + e_3a_3)$-constacyclic code and cyclic code over $\Re_3$ are monomially equivalent.*

In $\mathbb{F}_5[x]$, $x^7-1 = (x+4)(x^6+x^5+x^4+x^3+x^2+x+1)$, $x^7-2 = (x+2)(x^6+3x^5+4x^4+2x^3+x^2+3x+4)$, $x^7 - 3 = (x + 3)(x^6 + 2x^5 + 4x^4 + 3x^3 + x^2 + 2x + 4)$, $x^7 - 4 = (x + 1)(x^6 + 4x^5 + x^4 + 4x^3 + x^2 + 4x + 1)$.

Let the generator polynomial of C_1, C_2, C_3, C_4 be $x + 4$, $x + 2$, $x + 3$, $x + 1$, respectively. The parameters of C_1, C_2, C_3, C_4 are $[7, 6, 2]$. Let the generator polynomial of C_5, C_6, C_7, C_8 be $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, $x^6 + 3x^5 + 4x^4 + 2x^3 + x^2 + 3x + 4$, $x^6 + 2x^5 + 4x^4 + 3x^3 + x^2 + 2x + 4$, $x^6 + 4x^5 + x^4 + 4x^3 + x^2 + 4x + 1$, respectively. The parameters of C_5, C_6, C_7, C_8 are $[7, 1, 7]$. The parameters of the Gray map images for $(e_1a_1 + e_2a_2 + e_3a_3)$-constacyclic code and cyclic code over $\Re_3$ are $[21, 18, 2]$,$[21, 3, 7]$,$[21, 13, 7]$,$[21, 8, 7]$.

Example 3.2. *Let $n = 13$ and $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$, $\Re_3 = \mathbb{F}_7[u]/\langle u^3 - u\rangle$, $e_1 = \frac{u^2+u}{2}, e_2 = \frac{u^2-u}{2}, e_3 = 1 - u^2$. $\Re_3 = e_1\mathbb{F}_7 \oplus e_2\mathbb{F}_7 \oplus e_3\mathbb{F}_7$. $\mathrm{ord}(2) = 3, \mathrm{ord}(3) = 6, \mathrm{ord}(4) = 3, \mathrm{ord}(5) = 6, \mathrm{ord}(6) = 2, \gcd(13, 7) = \gcd(13, 6) = \gcd(13, 3) = \gcd(13, 2) = 1$, and by Theorems 3.3 and 3.6, the 2-constacyclic code, 3-constacyclic code, 4-constacyclic code, 5-constacyclic code, 6-constacyclic code, and cyclic code are monomially equivalent. By Corollary 3.8, $\forall e_1a_1 + e_2a_2 + e_3a_3 \in \Re_3^*$, the $e_1a_1 + e_2a_2 + e_3a_3$-constacyclic code and cyclic code over $\Re_3$ are monomially equivalent.*

In $\mathbb{F}_7[x]$, $x^{13} - 1 = (x + 6)(x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$, $x^{13} - 2 = (x+5)(x^{12}+2x^{11}+4x^{10}+x^9+2x^8+4x^7+x^6+2x^5+4x^4+x^3+2x^2+4x+1)$, $x^{13}-3 = (x+4)(x^{12}+3x^{11}+2x^{10}+6x^9+4x^8+5x^7+x^6+3x^5+2x^4+6x^3+4x^2+5x+1)$, $x^{13}-4 = (x+3)(x^{12}+4x^{11}+2x^{10}+x^9+4x^8+2x^7+x^6+4x^5+2x^4+x^3+4x^2+2x+1)$, $x^{13}-5 = (x+2)(x^{12}+5x^{11}+4x^{10}+6x^9+2x^8+3x^7+x^6+5x^5+4x^4+6x^3+2x^2+3x+1)$, $x^{13} - 6 = (x + 1)(x^{12} + 6x^{11} + x^{10} + 6x^9 + x^8 + 6x^7 + x^6 + 6x^5 + x^4 + 6x^3 + x^2 + 6x + 1)$.

Let the generator polynomial of C_1, C_2, C_3, C_4, C_5, C_6 *be* $x+6$, $x+5$, $x+4$, $x+3$, $x+2$, $x+1$ *respectively. The parameters of* C_1, C_2, C_3, C_4, C_5, C_6 *are* $[13,12,2]$. *Let the generator polynomial of* C_7, C_8, C_9, C_{10}, C_{11}, C_{12} *be* $x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$, $x^{12}+2x^{11}+4x^{10}+x^9+2x^8+4x^7+x^6+2x^5+4x^4+x^3+2x^2+4x+1$, $x^{12}+3x^{11}+2x^{10}+6x^9+4x^8+5x^7+x^6+3x^5+2x^4+6x^3+4x^2+5x+1$, $x^{12}+4x^{11}+2x^{10}+x^9+4x^8+2x^7+x^6+4x^5+2x^4+x^3+4x^2+2x+1$, $x^{12}+5x^{11}+4x^{10}+6x^9+2x^8+3x^7+x^6+5x^5+4x^4+6x^3+2x^2+3x+1$, $x^{12}+6x^{11}+x^{10}+6x^9+x^8+6x^7+x^6+6x^5+x^4+6x^3+x^2+6x+1$, *respectively. The parameters of* C_7, C_8, C_9, C_{10}, C_{11}, C_{12} *are* $[13,1,13]$. *The parameters of the Gray map images for* $(e_1a_1+e_2a_2+e_3a_3)$*-constacyclic code and cyclic code are* $[39,36,2]$, $[39,25,13]$, $[39,14,13]$, $[39,3,13]$.

Example 3.3. *Let* $n=11$ *and* $\mathbb{F}_5=\{0,1,2,3,4\}$, $\mathfrak{R}_2=\mathbb{F}_5[u]/\langle u^2-u\rangle$, $e_1=u, e_2=1-u$. $\mathfrak{R}_2=e_1\mathbb{F}_5\oplus e_2\mathbb{F}_5$. $\mathrm{ord}(2)=4, \mathrm{ord}(3)=4, \mathrm{ord}(4)=2, \gcd(11,5)=\gcd(11,4)=\gcd(11,2)=1$, *and by Theorems 3.3 and 3.6, the 2-constacyclic code, 3-constacyclic code, 4-constacyclic code, and cyclic code are monomially equivalent. By Corollary 3.8,* $\forall e_1a_1+e_2a_2\in\mathfrak{R}_l^*$, *the* $(e_1a_1+e_2a_2)$*-constacyclic code and cyclic code over* $\mathfrak{R}_2$ *are monomially equivalent.*

In $\mathbb{F}_5[x]$, $x^7-1=(x+4)(x^5+2x^4+4x^3+x^2+x+4)(x^5+4x^4+4x^3+x^2+3x+4)=f_{11}f_{12}f_{13}$, $x^7-2=(x+2)(x^5+x^4+x^3+2x^2+x+2)(x^5+2x^4+x^3+2x^2+3x+2)=f_{21}f_{22}f_{23}$, $x^7-3=(x+3)(x^5+3x^4+x^3+3x^2+3x+3)(x^5+4x^4+x^3+3x^2+x+3)=f_{31}f_{32}f_{33}$, $x^{11}-4=(x+1)(x^5+x^4+4x^3+4x^2+3x+1)(x^5+3x^4+4x^3+4x^2+x+1)=f_{41}f_{42}f_{43}$.

Let the generator polynomial of C_1, C_2, C_3, C_4 *be* f_{11}, f_{21}, f_{31}, f_{41}, *respectively. The parameters of* C_1, C_2, C_3, C_4 *are* $[11,10,2]$. *Let the generator polynomial of* C_5, C_6, C_7, C_8 *be* f_{12}, f_{22}, f_{32}, f_{42}, *respectively. The parameters of* C_5, C_6, C_7, C_8 *are* $[11,6,5]$. *Let the generator polynomial of* C_9, C_{10}, C_{11}, C_{12} *be* f_{13}, f_{23}, f_{33}, f_{43}, *respectively. The parameters of* C_9, C_{10}, C_{11}, C_{12} *are* $[11,6,5]$. *Let the generator polynomial of* C_{13}, C_{14}, C_{15}, C_{16} *be* $f_{11}f_{12}$, $f_{21}f_{22}$, $f_{31}f_{32}$, $f_{41}f_{42}$, *respectively. The parameters of* C_{13}, C_{14}, C_{15}, C_{16} *are* $[11,5,6]$. *Let the generator polynomial of* C_{17}, C_{18}, C_{19}, C_{20} *be* $f_{11}f_{13}$, $f_{21}f_{23}$, $f_{31}f_{33}$, $f_{41}f_{43}$, *respectively. The parameters of* C_{17}, C_{18}, C_{19}, C_{20} *are* $[11,5,6]$. *Let the generator polynomial of* C_{21}, C_{22}, C_{23}, C_{24} *be* $f_{12}f_{13}$, $f_{22}f_{23}$, $f_{32}f_{33}$, $f_{42}f_{43}$, *respectively. The parameters of* C_{21}, C_{22}, C_{23}, C_{24} *are* $[11,1,11]$. *The parameters of the Gray map images for* $e_1a_1+e_2a_2$*-constacyclic code and cyclic code over* $\mathfrak{R}_2$ *are* $[22,20,2]$, $[22,16,5]$, $[22,15,6]$, $[22,11,11]$, $[22,12,5]$, $[22,11,6]$, $[22,7,11]$, $[22,10,6]$, $[22,6,11]$, $[22,2,11]$.

4. Quantum codes

Theorem 4.1. *(CSS Construction) Let* C *be a linear code* $[n,k,d]$ *over* $\mathbb{F}_q$ *with* $C^\perp\subseteq C$, *then there exists a quantum code* $[[n,2k-n,d]]_q$.

By Lemma 2.3 and Theorem 2.2, we can have the following theorem.

Theorem 4.2. *Let* $C=\bigoplus_{i=1}^{l}e_iC_i$ *be an a-constacyclic code of length n over* $\mathfrak{R}_l$. *Then* $C^\perp\subseteq C$ *if and only if* x^n-a_i *is the divisor of* $f_i^*(x)f_i(x)$, *where* $a^{-1}=a$ *and* $f_i^*(x)$ *is the reciprocal polynomial of* $f_i(x)$ *and the generator polynomial of* $C_i^\perp$ *for* $i=1,2,\cdots,l$.

By Theorems 2.3 and 4.2, we can have the following corollary and theorem.

Corollary 4.1. *Let* $C=\bigoplus_{i=1}^{l}e_iC_i$ *be an a-constacyclic code of length n over* $\mathfrak{R}_l$. *Then* $C^\perp\subseteq C$ *if and only if* $C_i^\perp\subseteq C_i$ *for* $i=1,2,\cdots,l$.

Theorem 4.3. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ be an a-constacyclic code of length n over $\Re_l$; if C_i is an a_i-constacyclic code over $\mathbb{F}_q$ and $C_i^{\perp} \subseteq C_i$ for $i = 1, 2, \cdots, l$, where $a^{-1} = a$. Then $\phi_l(C)^{\perp} \subseteq \phi_l(C)$ and there exists a quantum code $[[ln, 2k - ln, d_G]]_q$, where d_G is the minimum Gray weight of code C, and k is the dimension of the linear code $\phi_l(C)$.*

Proof. Let $C_i^{\perp} \subseteq C_i$ and $a^{-1} = a$. By Corollary 4.1, we have $C^{\perp} \subseteq C$, so $\phi_l(C^{\perp}) \subseteq \phi_l(C)$. By Theorem 2.3, $\phi_l(C)^{\perp} = \phi_l(C^{\perp})$; therefore $\phi_l(C)^{\perp} \subseteq \phi_l(C)$, and by Theorem 2.3, $\phi_l(C)$ is a linear code $[ln, k, d_G]$. By Theorem 4.1, there exists a quantum code $[[ln, 2k - ln, d_G]]_q$. □

Example 4.1. *Let $n = 19$ and $\Re_3 = \mathbb{F}_5[u]/\langle u^3 - u\rangle$, $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, $e_1 = \frac{u^2+u}{2}$, $e_2 = \frac{u^2-u}{2}$,$e_3 = 1 - u^2$. $\Re_3 = e_1\mathbb{F}_5 \oplus e_2\mathbb{F}_5 \oplus e_3\mathbb{F}_5$.*

In $\mathbb{F}_5[x]$, $\mathrm{ord}(2) = 4, \mathrm{ord}(3) = 4, \mathrm{ord}(4) = 2, \gcd(19, 3) = 1, \gcd(19, 2) = \gcd(19, 4) = 1$, and by Theorems 3.3 and 3.6, the 2-constacyclic code, 3-constacyclic code, 4-constacyclic code, and cyclic code are monomially equivalent.

By Corollary 3.6, for any $e_1a_1 + e_2a_2 + e_3a_3$ in $\Re_3^*$, the $(e_1a_1 + e_2a_2 + e_3a_3)$-constacyclic code and cyclic code over $\Re_3$ are monomially equivalent. Because of the equivalence of the constancyclic code and the cyclic code, we only need to consider the cyclic code when constructing the quantum code.

In $\mathbb{F}_5[x]$, $x^{19} - 1 = (x + 4)(x^9 + 3x^7 + 2x^6 + 2x^5 + 2x^4 + 4x^3 + 2x^2 + 4x + 4)(x^9 + x^8 + 3x^7 + x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 4)$.

Let C be a cyclic code of length 19 over $\Re_3$ and $g(x) = e_1g_1+e_2g_2+e_3g_3$ be the generator polynomial of C, where $g_1(x) = g_2(x) = g_3(x) = x^9 + 3x^7 + 2x^6 + 2x^5 + 2x^4 + 4x^3 + 2x^2 + 4x + 4$; then $C_1 = \langle g_1(x)\rangle$, $C_2 = \langle g_2(x)\rangle$, and $C_3 = \langle g_3(x)\rangle$ are cyclic codes of length 19 over $\mathbb{F}_5$.

By Theorem 2.3, $\phi_3(C)$ is a linear code $[57, 30, 5]$ over $\mathbb{F}_5$. By Theorem 4.3, we have $C^{\perp} \subseteq C$, and we can obtain a quantum code $[[57, 3, 5]]_5$.

Example 4.2. *Let $n = 33$ and $\Re_3 = \mathbb{F}_5[u]/\langle u^3 - u\rangle$, $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, $e_1 = \frac{u^2+u}{2}$, $e_2 = \frac{u^2-u}{2}$, $e_3 = 1 - u^2$. $\Re_3 = e_1\mathbb{F}_5 \oplus e_2\mathbb{F}_5 \oplus e_3\mathbb{F}_5$.*

In $\mathbb{F}_5[x]$, $x^{33} - 1 = (x + 4)(x^2 + x + 1)(x^5 + 2x^4 + 4x^3 + x^2 + x + 4)(x^5 + 4x^4 + 4x^3 + x^2 + 3x + 4)(x^{10} + x^9 + 2x^8 + x^7 + 4x^6 + x^5 + 3x^4 + 4x^3 + 3x + 1)(x^{10} + 3x^9 + 4x^7 + 3x^6 + x^5 + 4x^4 + x^3 + 2x^2 + x + 1)$.

In $\mathbb{F}_5[x]$, $\mathrm{ord}(2) = 4, \mathrm{ord}(3) = 4, \mathrm{ord}(4) = 2, \gcd(33, 5) = \gcd(33, 4) = \gcd(33, 2) = 1$, and by Theorems 3.3 and 3.6, the 2-constacyclic code, 3-constacyclic code, 4-constacyclic code, and cyclic code are monomially equivalent.

By Corollary 3.6, $\forall e_1a_1 + e_2a_2 + e_3a_3 \in \Re_3^*$, the $(e_1a_1 + e_2a_2 + e_3a_3)$-constacyclic code and cyclic code over $\Re_3$ are monomially equivalent. Because of the equivalence of the constancyclic code and the cyclic code, we only need to consider the cyclic code when constructing the quantum code.

Let C be a cyclic code of length 33 over $\Re_3$, and $g_1(x) = x^5 + 2x^4 + 4x^3 + x^2 + x + 4 = g_2(x) = g_3(x)$, then $C_1 = \langle g_1(x)\rangle$, $C_2 = \langle g_2(x)\rangle$ and $C_3 = \langle g_3(x)\rangle$ are cyclic codes of length 33 over $\mathbb{F}_5$.

By Theorem 2.3, $\phi_3(C)$ is a linear code $[99, 84, 5]$ over $\mathbb{F}_5$. By Theorem 4.3, we have $C^{\perp} \subseteq C$, we can get a quantum code $[[99, 79, 5]]_5$.

Example 4.3. *Let $n = 55$ and $\Re_3 = \mathbb{F}_3[u]/\langle u^3 - u\rangle$.*

In $\mathbb{F}_3[x]$, $x^{55} - 1 = (x + 2)(x^4 + x^3 + x^2 + x + 1)(x^5 + 2x^3 + x^2 + 2x + 2)(x^5 + x^4 + 2x^3 + x^2 + 2)(x^{20} + x^{18} + 2x^{17} + 2x^{16} + 2x^{15} + x^{14} + 2x^{10} + x^9 + 2x^8 + 2x^7 + x^5 + x^4 + 2x^3 + 2x^2 + 2x + 1)(x^{20} + 2x^{19} + 2x^{18} + 2x^{17} + x^{16} + x^{15} + 2x^{13} + 2x^{12} + x^{11} + 2x^{10} + x^6 + 2x^5 + 2x^4 + 2x^3 + x^2 + 1)$, $x^{55} + 1 = (x + 1)(x^4 + 2x^3 + x^2 + 2x + 1)(x^5 +$

$2x^3+2x^2+2x+1)(x^5+2x^4+2x^3+2x^2+1)(x^{20}+x^{18}+x^{17}+2x^{16}+x^{15}+x^{14}+2x^{10}+2x^9+2x^8+x^7+2x^5+x^4+x^3+2x^2+x+1)(x^{20}+x^{19}+2x^{18}+x^{17}+x^{16}+2x^{15}+x^{13}+2x^{12}+2x^{11}+2x^{10}+x^6+x^5+2x^4+x^3+x^2+1)$.

In $\mathbb{F}_3[x]$, $\mathrm{ord}(2)=2, \gcd(55,3)=\gcd(55,2)=1$, *by Theorems 3.3 and 3.6, the 2-constacyclic code and cyclic code are monomially equivalent. By Corollary 3.6, for any* $e_1a_1+e_2a_2+e_3a_3$ *in* $\Re_3^*$, *the* $(e_1a_1+e_2a_2+e_3a_3)$*-constacyclic code and cyclic code over* $\Re_3$ *are monomially equivalent.*

Let C *be a* $(2e_1+e_2+e_3)$*-constacyclic code of length* 55 *over* $\Re_3$ *and* $g_1(x)=x^5+2x^3+2x^2+2x+1$, $g_2(x)=x^5+2x^3+x^2+2x+2$, $g_3(x)=1$. *By Theorem 2.3,* $\phi_3(C)$ *is a linear code* $[165,154,4]$ *over* $\mathbb{F}_3$. *By Theorem 4.3, we have* $C^\perp\subseteq C$, *so we can get a quantum code* $[[165,143,4]]_3$.

In Table 1, we provide some new quantum codes (in the seventh column) from constacyclic codes over $\Re_l$. Compared to the existing quantum codes that appeared in some recent references, the new quantum codes we obtained exhibit larger dimension, greater minimum distance, and higher code rate, enhancing their performance metrics and error correction capabilities. In the fifth column, the generator polynomials $\langle g_1(x),\cdots,g_l(x)\rangle$, where $g_i(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is denoted by $a_na_{n-1}\cdots a_1a_0$.

Table 1. New quantum codes from constacyclic codes over $\Re_l$.

q	n	l	$(a_1,\cdots,a_l)$	$\langle g_1(x),\cdots,g_l(x)\rangle$	$\phi_l(C)$	New codes	Rate	Existing codes	Rate
5	93	3	$(1,-1,1)$	$(1014,1011,1)$	$[279,272,3]$	$[[279,265,3]]_5$	0.95	$[[279,225,3]]_5$ in [15]	0.81
3	88	2	$(1,1)$	$(10102010002,11212)$	$[176,156,5]$	$[[176,136,5]]_3$	0.77	$[[176,126,5]]_3$ in [18]	0.72
3	48	2	$(1,1,1)$	$(1211011,11,1)$	$[144,137,4]$	$[[144,130,4]]_3$	0.90	$[[144,36,3]]_3$ in [15]	0.25
3	27	3	$(1,1,1)$	$(12211,11,11)$	$[81,75,4]$	$[[81,69,4]]_9$	0.85	$[[81,33,3]]_9$ in [19]	0.41
3	40	3	$(1,1,1)$	$(10202102,112,112)$	$[120,109,4]$	$[[120,98,4]]_3$	0.82	$[[120,68,4]]_3$ in [19]	0.57
3	60	3	$(1,1,1)$	$(10144,12,12)$	$[180,174,4]$	$[[180,168,4]]_3$	0.93	$[[180,136,3]]_3$ in [19]	0.76

5. Conclusions

In this article, we investigate the equivalence of constacyclic codes over finite non-chain ring $\Re_l$ and establish a classification of such codes over specific rings based on their equivalence properties. Furthermore, leveraging this classification and the CSS construction, we construct some new quantum codes and compare these codes better with the existing codes that appeared in some recent references. In the future, we will study the equivalence of constacyclic codes over $\mathbb{F}_q\mathbb{R}$, where $\mathbb{R}=\mathbb{F}_q[u_1,u_2,\cdots,u_k]/\langle f_1(u_1),f_2(u_2),\cdots,f_k(u_k),u_iu_j-u_ju_i\rangle$ and work on the construction of quantum codes from constacyclic codes over $\mathbb{F}_q\mathbb{R}$. It is also interesting to study the equivalence of polycyclic codes.

Author contributions

Jie Liu: Conceptualization, data curation, formal analysis, investigation, software, writing-original draft; Xiying Zheng: Supervision, methodology, project administration, writing-review and editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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