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Research article

Local L^∞ norm estimates for the gradient solutions of variational inequalities arising from the mortgage problems

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Abstract: This paper investigates local estimates for the spatial gradient of solutions to variational inequalities within the framework of a parabolic Kirchhoff operator, which arises from mortgage problems. By utilizing the integral inequality for the gradient of the solutions derived in this study, together with the Caffarelli–Kohn–Nirenberg inequality, we establish an L^∞ norm estimate for the gradient of the solution in a local cylindrical region. This L^∞ estimate is formulated in terms of the L^p norm of the solution.

Keywords: L^∞ estimate structured by L^p norm; integral inequality for the gradient of the solution; Caffarelli–Kohn–Nirenberge inequality; mortgage
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

This paper examines a class of variational inequality problems arising from American option pricing within the framework of parabolic Kirchhoff operators, denoted as

$$\begin{cases} \max\{L\psi, \psi_0 - \psi\} = 0 \text{ in } \Omega_T, \\ \psi(\,\cdot\,, 0) = \psi_0 \text{ in } \Omega, \\ \psi = 0 \text{ in } \partial\Omega \times (0, T), \end{cases} \tag{1}$$

where the non-negative constant p satisfies conditions $p \geq 2$ and

$$L\psi = \partial_t \psi - (1 + \|\psi\|^p_{L^p(\Omega)})\mathrm{div}(|\psi|^{p-1}\psi) + f. \tag{2}$$

Here, we also define the non-negative constants α and q such that

$$f = \alpha\psi^q, \tag{3}$$

and Ω are bounded open regions in the Euclidean space R_N, with the boundary denoted by $\partial\Omega$. The initial value ψ_0 for the variational inequality problem (1) satisfies the condition

$$\psi_0 \in C^1(\Omega) \cap W_0^{1,p}(\Omega).$$

The variational inequality problem is commonly encountered in contract pricing issues in finance, particularly in the context of installment payments for real estate purchases. Suppose an investor is interested in a property but lacks sufficient funds; in this case, they might consider acquiring the property through an installment payment plan. To simplify this model, let's assume the investor makes a down payment of \$0. Consequently, the market value of the loan contract can be represented as

$$\begin{cases} \max\{HC_A, f_0 - C_A\} = 0 \,\text{in}\, R_+ \times (0, T), \\ C_A(S, 0) = 0 \,\text{in}\, R_+, \end{cases} \tag{4}$$

where

$$HC_A = \partial_t C_A + \frac{1}{2}\varepsilon^2 S^2 \partial_{SS} C_A + rS\partial_S C_A - rC_A + f_0. \tag{5}$$

Here, the parameter ε represents the volatility of the property value, while r denotes the risk-free interest rate in the market. The variable f_0 indicates the remaining repayment amount that the investor is required to pay. First, since the property price S fluctuates in response to market information, the investor is naturally concerned about the extent of the fluctuations in $\partial_S C_A$. Excessive volatility could lead the investor to a state of insolvency. Additionally, real estate transactions often incur costs such as deed tax and stamp duty. Evidence suggests that the volatility ε in formula (5) is frequently related to $\partial_S C_A$. The well-known Leland model expresses the volatility ε as

$$\varepsilon^2 = \varepsilon_0^2 \left(1 + \text{Le} \times \text{sign}(|\partial_S C_A|^p \partial_S C_A)\right), \tag{6}$$

where ε_0 represents the long-term volatility level, and the non-negative constant Le is the Leland factor determined by the ratio of trading frequency to transaction costs, which is not elaborated here. This has a structure similar to the parabolic operator (2). These factors form the motivation for the variational inequality research presented in this paper.

Model (4) carries sound financial implications. When $HC_A = 0$: The condition $C_A - f_0 \geq 0$ indicates that housing price fluctuations have driven the market value of the loan contract C_A above the originally agreed-upon remaining repayment balance f_0. This suggests that investors would benefit from early repayment to terminate the contract. When $HC_A \leq 0$: The equality $C_A = f_0$ implies that the market value of the loan contract C_A aligns with the future repayment cash flows f_0. In this scenario, retaining the loan is optimal, as early repayment offers no financial advantage and would instead forfeit liquidity.

The existence of solutions to variational inequalities is a common area of study and forms the foundation for many analytical works. Peng et al. [3] explored the existence of a nonlinear evolutionary variational-hemivariational inequality, where the parabolic operator in the variational inequality involves both convex subdifferentials and Clarke subdifferentials, which are related to the time derivative of the unknown function. By considering the differentiability and generalized convexity assumptions of some multiple integral functionals, Treanta et al. [4] introduced the existence of solutions to vector-type variational control inequalities, which depend on certain uncertainty parameters. Based on nonlinear elastic constitutive equations, Zhang et al. [5] introduced the corresponding system of partial differential equations and variational inequalities, and within the framework of variational inequalities, proved and analyzed the existence and uniqueness of solutions to such models, as well as the approximation properties of finite element numerical solutions. Wu et al. investigated the existence and stability of solutions to a class of fuzzy fractional differential variational inequalities, which involve coupled modeling through variational inequalities and fuzzy fractional derivatives [6]. By introducing a two-parameter problem in the involved mappings and constraints, they established existence results for the parameterized fuzzy fractional differential variational inequality (PFFDVI). It is important to note that the inequalities in variational inequalities often impede the study of solution existence. Research on the existence of solutions is commonly found in non-degenerate parabolic equations [7,8] and systems of parabolic equations [9,10].

Norm estimates for the gradient of solutions are also commonly studied in initial boundary value problems for parabolic equations and systems of parabolic equations [11–16]. Li [11] investigated the near-boundary $W^{2,\delta}$ regularity of the solution set for fully nonlinear parabolic inequalities in bounded open regions, generalizing the results from [12]. In contrast, Zhang and Dong [10] focused solely on the interior estimates of the solution set for fully nonlinear parabolic inequalities. The work [13] established estimates for the weighted mixed norm and endpoint regularity of the maximum regularity for discrete parabolic equations under initial boundary value problems. Meanwhile, Das [14] demonstrated local Hölder regularity for weak solutions of mixed local-nonlocal parabolic equations, specifically regarding the existence and estimation of the L^∞ norm of the gradient. Wang [15] derived new gradient estimates for positive solutions of the weighted p-Laplace heat equation under bounded m-Bakry-Émery curvature using a regularization process. Currently, there is relatively limited literature on gradient estimates for solutions of variational inequalities, with a few references available for readers [16].

The parabolic Kirchhoff operator is a class of differential operators that combines parabolic equations with Kirchhoff-type nonlinear terms. Its framework incorporates an energy function of solutions, which is commonly used to describe dynamic processes with nonlocal effects [6]. The existence of solutions to initial-boundary value problems for parabolic Kirchhoff operators has been investigated by Chen and Zhou [17], while the existence of solutions to corresponding variational inequalities has been analyzed by Wu et al. [6]. Numerous additional studies have been conducted on parabolic equations and variational inequalities; for further reading, interested readers may refer to [18,19].

This paper investigates the L^∞ norm estimate for the gradient of solutions to a class of variational inequality problems. These problems arise from American option pricing and are framed within the parabolic Kirchhoff operator architecture. By applying the energy inequality for the gradient of the solution together with the Caffarelli–Kohn–Nirenberg (C-K-N) inequality, we derive a recursive energy estimate for the gradient. This allows us to construct an upper bound for the L^∞-norm of the gradient, which is expressed in terms of the analytical norm of the solutions L^p-norm.

2. Preliminary knowledge and main results

Before presenting the main results of this paper, we first introduce the Banach spaces

$$L^p(\Omega) = \{f|f \text{ is measurable in } \Omega,\ \|f\|_{L^p(\Omega)} = \int_\Omega |f|^p \mathrm{d}x < \infty\},$$

$$W^{1,p}(\Omega) = \{f | f \text{ is measurable in } \Omega,\ \|\nabla f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} < \infty\}.$$

If $f \in W^{1,p}(\Omega)$ and $f = 0$ on $\partial\Omega$ hold, we say that $f \in W_0^{1,p}(\Omega)$. These results can be found in [3,4]. Furthermore, we will need two additional useful lemmas. Lemma 2.1 can be found in [15,16], and we use it to analyze the recursive inequalities for the gradient structure of the solution space of the variational inequality (1), thereby obtaining the boundedness of the corresponding L^∞ norm of the solution's gradient. Lemma 2.2 is found in [14], and we use it to construct the recursive inequalities for the gradient structure of the solution space mentioned earlier.

Lemma 2.1. Assume that a certain sequence $\{Z_n, n = 0, 1, 2, \cdots\}$ satisfies

$$Z_{n+1} \le C b^n Z_n^{1+\alpha}.$$

Then $Z_n \to 0$ as $n \to \infty$ holds if and only if $Z_0 \le C^{-1/\alpha} b^{-1/\alpha^2}$, where C, b, and α are all non-negative constants, and $\alpha \in (0,\ 1)$.

Lemma 2.2. (Caffarelli–Kohn–Nirenberge inequality) For any $\psi \in L^p(\Omega_T) \cap L^q(\Omega)$, there exists a non-negative constant C_{C-K-N}, which depends only on N and p, such that

$$\int\int_{\Omega_T} |\psi|^{p\frac{(N+q)}{N}} \mathrm{d}x\mathrm{d}t \le C_{C-K-N} \left(\int\int_{\Omega_T} |\nabla\psi|^p \mathrm{d}x\mathrm{d}t \right) \left(\operatorname*{esssup}_{t\in(0,T)} \int_\Omega |\nabla\psi|^q \mathrm{d}x \right)^{\frac{p}{N}}.$$

The existence of a generalized solution to problem (1) has been extensively studied in the literature [16], and will not be repeated here. Suppose $(x_0, t_0) \in \Omega_T$. This paper investigates the L^∞ norm estimate for the gradient of the solution to the variational inequality (1) in the local cylindrical region

$$O(\kappa, \delta) = O(\kappa, \theta | (x_0, t_0)) = \Theta_\kappa \times \Xi_\delta = \{x \,|\, |x - x_0| < \kappa\} \times (t_0 - \delta, t_0). \tag{7}$$

In many traditional studies, the energy inequality for ψ is used (see Lemma 3.1), and after discarding certain non-negative terms, we obtain

$$\psi \in L^p(\Omega_T). \tag{8}$$

By utilizing the comparison principle, the variational inequality (1) also satisfies [16]

$$\psi \le \|\psi_0\|_{L^\infty(\Omega)}, \forall (x, t) \in \Omega_T. \tag{9}$$

Additionally, based on (16) and applying Hölder's inequality, along with Lemmas 2.1 and 2.2, we obtain the following result regarding the boundedness of the L^∞ norm for [20], which will be used in the subsequent analysis.

Lemma 2.3. For any $O(\kappa, \delta) \subset \Omega_T$, there exists a non-negative constant C that depends solely on p, N, and $\|\psi_0\|_\infty$, such that the solution ψ of the variational inequality (1) satisfies

$$\|\nabla\psi\|_{L^\infty(O(\kappa,\delta))} \le C.$$

Throughout the paper, we present the following two main results.

Theorem 2.4. Assuming $\sigma \in (0, 1)$, for any fixed non-negative constants $\varepsilon > 0$ and $p \ge 2$, the solution to the variational inequality (1) satisfies

$$\|D\psi\|_{L^\infty(O(\sigma\kappa,\sigma\delta))} \le 64^\varepsilon C_{C-K-N}^{\varepsilon/p} \left(p^2 / \kappa^2\right)^{\varepsilon/p} \left(\sup_{O(\kappa,\delta)} \psi \right)^{2\varepsilon} \|\psi\|_{L^p(O(\kappa,\delta))}.$$

Assuming $\sigma \in (0,1)$, by using formula (9) to scale $\sup\limits_{O(\kappa,\delta)} \psi$ up to $\|\psi_0\|_\infty$, we can conclude from Theorem 2.4 that for any $O(\kappa,\delta) \subset \Omega_T$, there exists a non-negative constant C that depends solely on p, N, C_{C-K-N}, and $\|\psi_0\|_\infty$ such that

$$\|D\psi\|_{L^\infty(O(\sigma\kappa,\sigma\delta))} \leq C\|\psi\|_{L^p(O(\kappa,\delta))}. \tag{10}$$

Note $p \geq 2$, so that when $\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t \geq 1$,

$$\left(\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}} \leq \left(\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}. \tag{11}$$

When $\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t \leq 1$, by choosing parameters $2/p$ and $(p-2)/p$, it is easy to obtain

$$\left(\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}} \leq \frac{2}{p}\left(\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} + \frac{p-2}{p}. \tag{12}$$

using Young's inequality. Therefore, based on the above analysis, we can obtain the following result.

Corollary 2.5. There exists a non-negative constant C that depends only on N, p, α, C_{C-K-N},, and $\|\psi_0\|_\infty$ such that

$$\|\nabla\psi\|_{L^\infty(O(\sigma\kappa,\sigma\delta))} \leq C\left(\int\int_{O(\kappa,\delta)} |\psi|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}.$$

Continuing to examine the variational inequality (4) for American option pricing, we aim to validate the main results of this paper. The American option, modeled by variational inequality (4), allows investors to exercise the option at any point in time during the option's lifespan $[0,T]$ to realize a profit. In contrast, a European option only permits investors to decide whether to exercise the option at a single point in time, specifically at T. According to reference [2], the value of the European option satisfies

$$\begin{cases} HC_E = 0 \,\text{in}\, \mathrm{R}_+ \times (0,T), \\ C_E(S,0) = (S-K)_+ \,\text{in}\, \mathrm{R}_+. \end{cases} \tag{13}$$

Numerous studies have provided pricing results for European options, which is

$$C_E(S,t) = SN(d_1) - K\exp\{-r(T-t)\}N(d_2),$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, and

$$d_1 = \frac{\ln S - \ln K + (r + \frac{1}{2}\varepsilon^2)(T-t)}{\sqrt{T-t}}, d_2 = \frac{\ln S - \ln K + (r - \frac{1}{2}\varepsilon^2)(T-t)}{\sqrt{T-t}}.$$

Further calculations reveal that [1,2]

$$\frac{\partial C_E(S,t)}{\partial S} = N(d_1). \tag{14}$$

It is important to note that American options can be exercised at any point in time during the option's lifespan $[0,T]$, while European options allow for a decision to exercise only once at time T. This implies that $C_E(S,t)$ is more sensitive to the risk asset price S than $C_A(S,t)$, specifically:

$$\left|\frac{\partial C_A(S,t)}{\partial S}\right| \leq \left|\frac{\partial C_E(S,t)}{\partial S}\right| = |N(d_1)| \leq 1.$$

From the above expression, it is clear that Theorem 2.4 and Corollary 2.5 are indeed valid.

3. Integral inequalities involving ψ and $\nabla\psi$

We first examine the integral inequalities for the solution ψ. Choose a test function $\phi = \psi^m\rho(x)^{m+1}\eta(t)$, where the truncation functions satisfy $\rho \in C^\infty(\Theta_\kappa)$ and $\eta \in C^\infty(\Xi_\delta)$, and additionally,

$$0 \le \rho \le 1 \,\text{in}\, \Theta_\kappa, 0 \le \eta \le 1 \,\text{in}\, \Xi_\delta. \tag{15}$$

Furthermore, ψ satisfies the boundary conditions $\rho = 0 \,\text{in}\, \partial\Theta_\kappa$, and η holds on the left side of the time interval Ξ_δ satisfying $\eta(t_0 - \delta) = 0$. Using the Hölder and Young inequalities, and by analogy with the proof in [16], we can derive the following integral inequality for the solution ψ.

Lemma 3.1. Assume that ψ is the solution to the variational inequality (1). For any $t \in \Xi_\delta$, $m > 0$, and $p \ge 2$, there exists a non-negative constant C, depending only on m and p, such that

$$\sup_{t\in\Xi_\delta} \int_{\Theta_\kappa} \psi^{m+1}\rho^{m+1}\eta(t)\mathrm{d}x + \int\int_{\mathrm{O}(\kappa,\delta)} \psi^{m-1}|\nabla\psi|^p\rho^m\eta \mathrm{d}x\mathrm{d}t \le \Pi_1 + \Pi_2, \tag{16}$$

where

$$\Pi_1 = \frac{m^{1-p}}{p}(m+1)^{p+1}\int\int_{\mathrm{O}(\kappa,\delta)} \psi^{p+m-1}\rho^m\eta|\nabla\rho|^p \mathrm{d}x\mathrm{d}t,\ \Pi_2 = \int\int_{\mathrm{O}(\kappa,\delta)} \psi^{m+1}\rho^{m+1}|\partial_t\eta|\mathrm{d}x\mathrm{d}t.$$

Next, we analyze the integral inequalities involving the gradient of the solution $\nabla\psi$ in order to obtain additional energy estimates. Let $\sigma \in (0, 1)$ be an undetermined constant such that

$$\kappa_n = \sigma\kappa + \frac{1-\sigma}{2^n}\kappa,\ \delta_n = \sigma\delta + \frac{1-\sigma}{2^n}\delta. \tag{17}$$

For convenience, we also set $\Theta_n = \Theta_{\kappa_n}$, $\Xi_n = \Xi_{\delta_n}$, and $\mathrm{O}_n = \Theta_n \times \Xi_n$, and it is easy to observe that

$$\mathrm{O}_0 = \mathrm{O}(\kappa,\delta),\ \mathrm{O}_\infty = \mathrm{O}(\sigma\kappa,\sigma\delta). \tag{18}$$

Furthermore, we divide O_n into two parts:

$$\mathrm{A}_1 = \{(x,t) \in \mathrm{O}_n | L\psi < 0\} \text{ and } \mathrm{A}_2 = \{(x,t) \in \mathrm{O}_n | L\psi = 0\}.$$

From (1), we know that when $(x, t) \in \mathrm{A}_1$ holds, $\psi = \psi_0$ follows. Based on the assumption ψ_0, it is easy to obtain

$$\nabla\psi \in L^\infty(\mathrm{A}_1). \tag{19}$$

The remainder of this paper considers the case A_2. Let $v = |\nabla\psi|$, multiply both sides of $\nabla L\psi = 0$ by φ_n, and integrate over O_n, yielding

$$\int\int_{\mathrm{O}_n} \partial_t\nabla\psi \times \varphi_n \mathrm{d}x\mathrm{d}t + \int\int_{\mathrm{O}_n} (1 + \|\psi\|^p_{L^p(\Omega)})\nabla(\mathrm{div}(v^{p-2}\nabla\psi)) \times \varphi_n \mathrm{d}x\mathrm{d}t = \int\int_{\mathrm{O}_n} \nabla f \times \varphi_n \mathrm{d}x\mathrm{d}t, \tag{20}$$

where

$$\varphi_n = p(v - \lambda_{n+1})_+^{p-1} \times \rho_n^p\eta_n^p,\ \zeta_n = I_{\{(x,t)\in\mathrm{O}_n | v\ge k_{n+1}\}}. \tag{21}$$

Here, $\lambda_n = \lambda - \frac{1}{2^n}\lambda$, where λ is a non-negative undetermined constant. We first analyze the second term on the left-hand side of (20). By applying integration by parts, we easily obtain

$$\begin{aligned}&\int\int_{O_n} \nabla(\operatorname{div}(v^{p-2}\nabla\psi)) \times \varphi_n \mathrm{d}x\mathrm{d}t = -\int\int_{O_n} \operatorname{div}(v^{p-2}\nabla\psi)\nabla\varphi_n \mathrm{d}x\mathrm{d}t \\ &= -p(p-1)^2 \int\int_{O_n} v^{p-2}|\Delta\psi|^2 (v-\lambda_n)_+^{p-2} \times \rho_n^p \eta_n^p \mathrm{d}x\mathrm{d}t \\ &\quad -(p-1)p^2 \int\int_{O_n} v^{p-2}|\Delta\psi|(v-\lambda_n)_+^{p-1} \times \rho_n^{p-1}\eta_n^p \nabla\rho_n \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Here, we make use of $\varphi_n|_{\Theta_n} = 0$, at which point $\int_{\Xi_n} \nabla(\operatorname{div}(v^{p-2}\nabla\psi)) \times \varphi_n|_{\Theta_n}\mathrm{d}t = 0$. By performing integration transformations on $\int\int_{O_n} \partial_t\nabla\psi \times \varphi_n \mathrm{d}x\mathrm{d}t$ and $\int\int_{O_n} \nabla(\operatorname{div}(v^{p-2}\nabla\psi)) \times \varphi_n \mathrm{d}x\mathrm{d}t$, and applying inequalities such as Hölder's and Young's inequalities, we obtain the following result.

Lemma 3.2. Assuming $v = |\nabla\psi|$, for any $n = 1, 2, 3, \cdots$, we have

$$\sup_{t\in\Xi_n} \int_{\Theta_n} (v-\lambda_{n+1})^p \times \rho_n^p\eta_n^p \mathrm{d}x + \frac{(p-1)^2}{2p}\int\int_{O_n} |\nabla(v-\lambda_{n+1})_+^{\frac{1}{2}p}|^2 \times \rho_n^p\eta_n^p \mathrm{d}x\mathrm{d}t \leq \Pi_3 + \Pi_4 + \Pi_5, \quad (22)$$

where

$$\Pi_3 = p\int\int_{O_n} (v-\lambda_{n+1})_+^p \times \rho_n^p \eta_n^{p-1}|\partial_t\eta_n|\mathrm{d}x\mathrm{d}t,\ \Pi_4 = p\int\int_{O_n} |\nabla f| \times (v-\lambda_{n+1})_+^{p-1} \times \rho_n^p\eta_n^p \mathrm{d}x\mathrm{d}t,$$

$$\Pi_5 = \frac{p^2}{2(p-1)}\int\int_{O_n} v^{3p-4} \times I\{v > \lambda_{n+1}\} \times \rho_n^{p-2}|\nabla\rho_n|^2\eta_n^p \mathrm{d}x\mathrm{d}t.$$

In Lemma 3.2, we aim to incorporate several results concerning the gradient of the truncation function. We assume that ψ_n is a truncation factor on Θ_{n+1} that not only satisfies the conditions related to ψ in (15) but also meets the requirements that ψ_n is zero on the boundary of Θ_{n+1}. Furthermore, for every $n = 0, 1, 2, \cdots$, the following holds:

$$\rho_n(x) = 1 \text{ in } \Theta_{\kappa_n}, |\nabla\rho_n| \leq \frac{2^{n+2}}{(1-\sigma)\kappa}. \quad (23)$$

We further assume that η_n is a truncation function on Ξ_{n+1}, which not only satisfies (15) but also vanishes at $t_0 - \delta_n$. Moreover, for every $n = 0, 1, 2, \ldots$, the following holds:

$$\eta_n(x) = 1 \text{ in } \Xi_n, |\nabla\eta_n| \leq \frac{2^{n+2}}{(1-\sigma)\delta}. \quad (24)$$

Using Hölder's inequality and Young's inequality,

$$\begin{aligned}&p\int\int_{O_n} |\nabla f| \times (v-\lambda_{n+1})_+^{p-1} \times \rho_n^p\eta_n^p \mathrm{d}x\mathrm{d}t \\ &\leq p^{p-1}\int\int_{O_n} |\nabla f|^p \times I\{v > \lambda_{n+1}\} \times \rho_n^p\eta_n^p \mathrm{d}x\mathrm{d}t + \frac{p-1}{p}\int\int_{O_n} (v-\lambda_{n+1})_+^p \times \rho_n^p\eta_n^p \mathrm{d}x\mathrm{d}t.\end{aligned} \quad (25)$$

Note that when $p \geq 2$, $\frac{(p-1)^2}{2p} \geq \frac{1}{2}$ follows. Therefore, by Lemma 3.2 and (15), we can obtain

$$\int_{\Theta_n} (v-\lambda_{n+1})^p \times \rho_n^p\eta_n^p \mathrm{d}x \leq p\Upsilon, \forall t \in \Xi_n \quad (26)$$

and

$$\int\int_{O_n} |\nabla(v-\lambda_{n+1})_+^{\frac{1}{2}p}|^2 \times \rho_n^p \eta_n^p \mathrm{d}x\mathrm{d}t \leq 2\Upsilon, \tag{27}$$

where

$$\begin{cases} \Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3, \ \Upsilon_1 = p^{p-1} \int\int_{O_n} |\nabla f|^p \times I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t, \\ \Upsilon_2 = \frac{2^n p}{\delta} \int\int_{O_n} (v-\lambda_{n+1})_+^p \mathrm{d}x\mathrm{d}t, \ \Upsilon_3 = \frac{2^{2n+4}p^2}{\kappa^2} \int\int_{O_n} v^{3p-4} \times I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t. \end{cases} \tag{28}$$

At the end of this section, we examine a reverse estimate for a higher-order L^p norm. Let $\tau > 0$ be set; the choice of this parameter plays a crucial role in the results presented in this paper. By applying the Caffarelli–Kohn–Nirenberg inequality to $\int\int_{O_n} |(v-\lambda_{n+1})_+^{p/2} \zeta_n|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t$, we obtain

$$\int\int_{O_n} |(v-\lambda_{n+1})_+^{p/2} \times \rho_n^{p/2} \eta_n^{p/2}|^{2(1+\tau)} \mathrm{d}x\mathrm{d}t \leq C_{C-K-N} \times \Pi_6 \times (\Pi_7)^\tau, \tag{29}$$

where

$$\Pi_6 = \int\int_{O_n} |\nabla(v-\lambda_{n+1})_+^{p/2} \times \rho_n^{p/2}\eta_n^{p/2}|^2 \mathrm{d}x\mathrm{d}t, \ \Pi_7 = \operatorname*{esssup}_{t\in\Xi_n} \int_{\Theta_n} |(v-\lambda_{n+1})_+^{p/2} \times \rho_n^{p/2}\eta_n^{p/2}|^2 \mathrm{d}x.$$

Substituting (26) and (27) into (29), we arrive at the following result.

Lemma 3.3. For any $\tau > 0$, there exists

$$\int\int_{O_n} |(v-\lambda_{n+1})_+^{p/2} \times \rho_n^{p/2}\eta_n^{p/2}|^{2(1+\tau)} \mathrm{d}x\mathrm{d}t \leq 2C_{C-K-N}\Upsilon^{1+\tau}. \tag{30}$$

Note that the left side of the above expression is of order $p(1+\tau)$ with respect to $(v-\lambda_{n+1})_+$, while the right side is of order p with respect to $(v-\lambda_{n+1})_+$ and of order $3p-4$ with respect to v. Clearly, by choosing τ sufficiently large, one can use the lower-order norms of $(v-\lambda_{n+1})_+$ and v to estimate the higher-order energy norm $(v-\lambda_{n+1})_+$.

4. Estimates for the supremum bound of $\nabla\psi$

We continue to refine the estimates of $\int\int_{O_n} |(v-\lambda_{n+1})_+^{p/2} \times \rho_n^{p/2}\eta_n^{p/2}|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t$ in Lemma 3.3 by utilizing Lemma 4.1. To do this, we need to estimate the three non-negative terms Υ_1, Υ_2, and Υ_3 in (28). Using Lemma 4.1, it is straightforward to observe that the third term in (28) satisfies

$$\Upsilon_3 \leq \frac{2^{2n+4}p^2}{\kappa^2}\left(\sup_{O_n} v\right)^{2(p-2)} \int\int_{O_n} v^p \times I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t. \tag{31}$$

Similarly, the second term in Eq (28) satisfies

$$\Upsilon_2 \leq \frac{2^n p}{\delta} \int\int_{O_n} v^p \times I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t. \tag{32}$$

Next, we analyze the first term in Eq (28). From (9), we can obtain

$$\Upsilon_1 \leq \alpha p^{p-1} q^p \|\psi_0\|_\infty^{p(q-1)} \int\int_{O_n} v^p \times I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t. \tag{33}$$

For the sake of convenience in the discussion, we define

$$X_n = \int\int_{O_n} (v - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t. \tag{34}$$

Note that when $v > \lambda_{n+1}$, we have $(v - \lambda_{n+1})_+ > \frac{\lambda}{2^{n+1}}$ in O_n, which leads us to

$$\int\int_{O_n} I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t \leq \frac{2^{(n+1)p}}{\lambda^p}\int\int_{O_n} (v - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t = 2^{(n+1)p}\lambda^{-p}X_n. \tag{35}$$

On the other hand, by utilizing $\lambda_n = \lambda_{n+1}\frac{2^{n+1}-2}{2^{n+1}-1}$, we can similarly apply the analytical approach from [20, Eq (7.5)], which gives us

$$\int\int_{O_n} (\theta_i - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t \geq \int\int_{O_n} \theta_i{}^p \left(1 - \frac{2^{n+1}-2}{2^{n+1}-1}\right)_+^p I_{\theta_i \geq \lambda_{n+1}}\mathrm{d}x\mathrm{d}t \geq \frac{1}{2^{np}}\int\int_{O_n} \theta_i{}^p I_{\theta_i \geq \lambda_{n+1}}\mathrm{d}x\mathrm{d}t.$$

Consequently, we obtain

$$\int\int_{O_n} v^p \times I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t \leq 2^{np}X_n. \tag{36}$$

Therefore, we substitute (31)–(34), and (36) into the right-hand side of Υ, yielding

$$\Upsilon \leq \left(\alpha p^{p-1}q^p\|\psi_0\|_\infty^{p(q-1)} + \frac{2^n p}{\delta} + \frac{2^{2n+4}p^2}{\kappa^2}\left(\sup_{O_n} v\right)^{2(p-2)}\right)2^{np}X_n. \tag{37}$$

Combining this expression with Lemma 3.3, we obtain a more refined estimate for $\int\int_{O_n} |(v - k_{n+1})_+^{p/2} \times \psi_n^{p/2}\eta_n^{p/2}|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t$, which is

$$\begin{aligned}
&\int\int_{O_n} |(v - \lambda_{n+1})_+^{p/2} \times \psi_n^{p/2}\eta_n^{p/2}|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t \\
&\leq 2C_{C-K-N}16^{1+\tau}\left(\alpha p^{p-1}q^p\|\psi_0\|_\infty^{p(q-1)} + \frac{p}{\delta} + \frac{p^2}{\kappa^2}\left(\sup_{O_n} v\right)^{2(p-2)}\right)^{1+\tau} 2^{2pn(1+\alpha)}X_n{}^{1+\tau}.
\end{aligned} \tag{38}$$

Proof of Theorem 2.4. We first construct an upper bound for X_{n+1} using (38) and the previously established Lemma 3.3. Note that $O_n \supset O_{n+1}$ holds and that $\zeta_n = 1$ in O_{n+1} is satisfied, which allows us to easily obtain

$$\begin{aligned}
X_{n+1} &\leq \int\int_{O_n} |(v - \lambda_{n+1})_+^{p/2}\zeta_n|^2\mathrm{d}x\mathrm{d}t \\
&\leq \left(\int\int_{O_n} |(v - \lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{1+\tau}} \times \left(\int\int_{O_n} I_{v \geq \lambda_{n+1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{\tau}{1+\tau}}
\end{aligned} \tag{39}$$

using Hölder's inequality. Note that we constructed an upper bound for $\left(\int_{O_n} I\{v > \lambda_{n+1}\}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{1+\tau}}$ using (35). Substituting this into (39) readily reveals

$$X_{n+1} \leq \left(\int\int_{O_n} |(v - \lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{1+\tau}} \times 2^{(n+1)p}\lambda^{-p\frac{\tau}{1+\tau}}X_n{}^{\frac{\tau}{1+\tau}}. \tag{40}$$

Finally, substituting the estimate for $\int\int_{O_n} |(v-\lambda_{n+1})_+^{p/2} \times \psi_n^{p/2}\eta_n^{p/2}|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t$ from (38) into the previous expression yields

$$X_{n+1} \leq 32p^2 \frac{C_{C-K-N}}{(1-\sigma)^2}\left(\alpha p^{p-1}q^p\|\psi_0\|_\infty^{p(q-1)} + \frac{p}{\delta} + \frac{p^2}{\kappa^2}\left(\sup_{O_n} v\right)^{2(p-2)}\right)8^{pn} \times \lambda^{-p\frac{\tau}{1+\tau}} X_n{}^{1+\frac{\tau}{1+\tau}}. \tag{41}$$

To simplify the results, we define

$$\left(\sup_{O_n} v\right)^{2(p-2)} \geq \max\{\frac{\kappa^2}{\delta}, \alpha p^{p-3}q^p\|\psi_0\|_\infty^{p(q-1)}\kappa^2\}, \tag{42}$$

such that (41) can be reduced to

$$X_{n+1} \leq 64p^2 \frac{C_{C-K-N}}{(1-\sigma)^2\kappa^2}\left(\sup_{O_n} v\right)^{2(p-2)} 8^{pn} \times X_n{}^{1+\frac{\tau}{1+\tau}}. \tag{43}$$

Consequently, according to Lemma 2.1, in order to achieve $X_n \to 0$ as $n \to \infty$, it is sufficient to choose

$$X_0 \leq \left(\frac{64p^2}{(1-\sigma)^2\kappa^2}\right)^{-(1+\alpha^{-1})} C_{C-K-N}^{-(1+\tau^{-1})}\left(\sup_{O_n} v\right)^{-2(p-2)(1+\tau^{-1})} 8^{-p(1+\tau^{-1})^2}\lambda^p.$$

Note that $O_0 = O(\kappa,\delta)$ and $X_0 = \int_{O(\kappa,\delta)} v^p\mathrm{d}x\mathrm{d}t$, and when κ is sufficiently small,

$$\left(\sup_{O_n} v\right)^{2(p-2)} \geq \alpha p^{p-3}q^p\|\psi_0\|_\infty^{p(q-1)}\kappa^2$$

always holds. Therefore, we choose

$$\lambda = 8^{(1+\tau^{-1})^2}\left(\sup_{O(\kappa,\delta)} v\right)^{2(1+\tau^{-1})\frac{p-2}{p}} C_{C-K-N}^{\frac{1+\tau}{\tau p}}\left(\frac{64p^2}{(1-\sigma)^2\kappa^2}\right)^{\frac{1+\tau}{\tau p}}\left(\int_{O(\kappa,\delta)} v^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}. \tag{44}$$

At this point, $X_n \to 0$ as $n \to \infty$ holds, and by combining this with $O_\infty = O(\sigma\kappa,\sigma\delta)$, we can derive

$$\sup_{O(\sigma\kappa,\sigma\delta)} v \leq \lambda. \tag{45}$$

Thus, we have completed the proof of Theorem 2.4.

5. Conclusions

This paper examines a class of variational inequality boundary value problems governed by Kirchhoff operators, specifically the variational inequality (1), which arises from the analysis of the value of American contingent claims in finance. We attempt to estimate the infinite norm of the gradient of the solution to variational inequality (1) in local regions. This result helps financial scholars characterize the sensitivity of the value of American contingent claims with respect to the value of risky assets. Using the integral inequality (Lemma 3.2) of the gradient of the solution to variational inequality (1) in a cylindrical region O_n applying the Caffarelli–Kohn–Nirenberg inequality, we construct an estimate for

$$\int\int_{O_n} |(v-\lambda_{n+1})_+^{p/2} \times \rho_n^{p/2}\eta_n^{p/2}|^{2(1+\tau)}\mathrm{d}x\mathrm{d}t.$$

Subsequently, we use this to construct a pushforward inequality for $\int_{O_n}(v-\lambda_n)_+^p\mathrm{d}x\mathrm{d}t$. Finally, by choosing the parameter λ, we complete the infinite norm estimate of the gradient v of the solution to variational inequality (1).

This paper restricts $p \geq 2$; if $p \in (1,2)$, obtaining formula (31) becomes challenging. Similarly, formulas (11) and (12) no longer hold, as they likewise rely on the condition $p \geq 2$. Additionally, α must be non-negative; otherwise, Lemma 3.1 may become more complex in form. If this condition is omitted, there will be considerable uncertainty when handling Υ_1 in formulas (28) and (33). In future research, we will attempt to overcome these limitations.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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