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Theory article

A Quasi-Newton reproducing kernel method for nonlinear high-order boundary value problems

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Abstract: This paper proposes a novel Quasi-Newton reproducing kernel method (QNRKM) for efficiently solving nonlinear fifth-order two-point boundary value problems (BVPs). The proposed scheme innovatively combines the strengths of the Quasi-Newton method (QNM) and the reproducing kernel method (RKM), forming a hybrid framework that addresses the limitations of traditional kernel-based approaches. In particular, it eliminates the need for the computationally intensive Schmidt orthogonalization process required in conventional RKM, which significantly improves numerical efficiency. A comprehensive and rigorous error analysis is performed to evaluate the performance of the proposed numerical scheme. The theoretical results confirm that the method achieves second-order convergence with respect to both the solution and its first derivative. This convergence is measured in the maximum norm. These findings demonstrate the accuracy and reliability of the proposed method for solving the target class of BVPs. Several numerical examples are also presented to validate its accuracy and convergence. The approach is compared with other methods such as the Homotopy Perturbation method and the Variational Iteration method. The results are analyzed through error tables and figures under different grid sizes. They demonstrate that the proposed method is accurate, convergent, and effective. The numerical results consistently show that it achieves superior precision while maintaining low computational cost.

Keywords: reproducing kernel method; Quasi-Newton method; nonlinear differential equations; boundary value problem; convergence
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1. Introduction

Differential equations are central to many disciplines, such as engineering [1], biology [2, 3], physics [4–6], and fluid dynamics [7, 8]. Their ability to model complex systems has made them vital across various fields in recent years. Among the types of problems they address, BVPs are particularly significant, as real-world challenges often require solving differential equations subject to specific boundary conditions. In mathematical physics, for example, essential equations like wave and Laplace equations [9] are frequently framed as BVPs. Specifically, reaction-diffusion systems [10, 11], a typical class of BVPs, are widely applied in biology, ecology, neutron diffusion theory, and geology.

Given the wide-ranging use of BVPs, much research has focused on the development of numerical methods for solving them [12–14]. Various techniques have been proposed, such as spline-based methods [15, 16], the weak Galerkin method [17], the Discontinuous Galerkin method [18], and the sinc-Galerkin approach [19]. However, higher-order BVPs introduce more complexities due to intricate boundary conditions and nonlinearities, complicating numerical solutions.

The one-dimensional fifth-order two-point BVP with nonlinear characteristics represents a class of challenging high-order differential equations. This type of model captures essential features of primary elliptic-hyperbolic operators [20]. The focus of this paper is the development of a novel numerical method to address such problems.

Several researchers have explored different approaches for solving fifth-order two-point BVPs. Caglar et al. extended this approach using sixth-degree B-splines to solve specific fifth-order two-point BVPs with first-order accuracy [21]. The Homotopy Perturbation method (HPM) is an analytical technique that combines the classical perturbation method with homotopy concepts, allowing for the efficient construction of approximate solutions to nonlinear problems without requiring small parameters. The Variational Iteration method (VIM) provides an iterative framework based on variational theory to successively refine approximations to nonlinear equations, offering flexibility and rapid convergence for a broad range of BVPs. Noor et al. [20] applied VIM using He's polynomials—essentially combining VIM with HPM—to solve fifth-order BVPs. Wang et al. [22] employed the sixth-degree B-spline residual correction method, discretizing the equation with B-splines and using residual correction to form an iterative scheme. Mohyud et al. [23] introduced a new iterative approach that yields analytical solutions as a convergent series with easily computable terms. Zhang [24] also used VIM, achieving high accuracy with a single iteration.

Despite the extensive research on BVPs, efficient numerical techniques for solving two-point BVPs remain relatively limited. While the RKM has gained considerable attention in recent years, its application to high-order nonlinear problems still requires further refinement. Numerous studies [25, 26] have contributed to its development.

Existing numerical methods face several challenges. For instance, the B-spline method [21] handles boundary conditions well but has limited accuracy and may encounter stability issues when solving nonlinear problems. The VIM [20, 24] is suitable for nonlinear problems and can be combined with the HPM to improve efficiency. Nevertheless, it relies on an initial guess, which may affect convergence, and its computational complexity increases for high-order problems. The B-spline residual correction method [22] reduces errors effectively through residual correction but has a high computational cost and remains constrained by the limitations of the B-spline approach, potentially leading to insufficient accuracy in complex nonlinear problems. Iterative methods [23] have easily computable components but may suffer from convergence issues, particularly for high-order problems, and have relatively low computational efficiency.

Notably, Ramos and Momoh [27] proposed a high-order hybrid block method based on multipoint interpolation and block structure for solving fifth-order boundary value problems, achieving a theoretical convergence order of ten. However, despite its high accuracy, the method relies on high-order derivatives and specific boundary conditions, which may limit its flexibility for more general models involving integral terms or non-standard structures. Its construction is also relatively involved, which may affect practical applicability.

This paper proposes a novel QNRKM, which combines the RKM and the QNM to provide an efficient, stable, and accurate solution strategy for nonlinear fifth-order two-point BVPs. Compared to existing methods, this approach integrates the simplified RKM and QNM, overcoming the limitations of the B-spline method while enhancing computational accuracy and efficiency. It eliminates the need for the Schmidt orthogonalization process in traditional RKM, thereby reducing computational complexity. This method also exhibits greater adaptability, making it more suitable for nonlinear high-order problems and more complex boundary conditions. Additionally, it achieves faster convergence, and numerical experiments confirm its efficiency. The incorporation of the QNM further enhances computational stability.

In summary, the main novelty of this study lies in the development of a hybrid QNRKM. This approach effectively overcomes several limitations of existing kernel-based and spline-based techniques for solving nonlinear high-order BVPs. The proposed method not only improves computational efficiency and convergence but also exhibits strong adaptability to complex nonlinearities and general boundary conditions. These advances make the method broadly applicable to a wide range of high-order differential equations in applied mathematics and engineering. The new framework thus provides both theoretical and practical benefits for future research and numerical analysis in this area.

This article is structured as follows: Section 2 introduces the nonlinear fifth-order BVP under consideration, together with the associated reproducing kernel Hilbert spaces $W_2^m[a,b]$. The section also defines the relevant inner products, reproducing kernels, and functional norms. In Section 3, a novel numerical scheme, namely the QNRKM, is developed. This method combines Quasi-Newton linearization with the reproducing kernel framework to efficiently handle high-order nonlinear differential equations. Section 4 provides a rigorous error analysis of the proposed approach, establishing uniform convergence results for both the solution and its derivatives. In Section 5, several benchmark examples are presented to assess the accuracy and convergence rate of QNRKM. The numerical results are compared against those of established methods such as HPM, VIM, and B-spline-based schemes. Finally, Section 6 summarizes the theoretical and computational contributions of this study and outlines directions for future work.

2. Model and function space description

In this work, we introduce the QNRKM as a solution technique for nonlinear differential equations of the form

$$\begin{cases} u^{(5)}(x) + \sum_{i=0}^{4} g_i(x)u^{(i)}(x) + \mathcal{N}(u) = f(x), \quad x \in [a,b], \\ \sum_{j_1=0}^{2} u^{(j_1)}(a) = \alpha_{j_1}, \quad \sum_{j_2=0}^{1} u^{(j_2)}(b) = \beta_{j_2}. \end{cases} \tag{1}$$

Here, $u(x)$ is the unknown function, with x as the independent variable in the interval $[a, b]$. The functions $g_i(x) \in C^1[a, b]$ and $f(x)$ are given continuous functions on $[a, b]$, and $\mathcal{N}(u)$ represents a nonlinear operator. The constants α_{j_1} and β_{j_2} correspond to the boundary conditions at $x = a$ and $x = b$, respectively, where $j_1 = 0, 1, 2$, and $j_2 = 0, 1$. The objective is to solve for $u(x)$, considering the fifth-order nonlinear equations and the associated boundary conditions. Based on the results in [28,29], we can conclude that the solution to the equation exists and is unique.

To address this model, to introduce the reproducing kernel spaces $W_2^m[a, b]$, we begin by defining these spaces, along with their inner products and norms.

Definition 1. *[30] The reproducing kernel space $W_2^m[a, b]$ is given by*

$$W_2^m[a, b] = \left\{u(x) \mid u^{(m-1)} \text{ is absolutely continuous on } [a, b],\ u^{(m)} \in L^2[a, b]\right\}.$$

The corresponding inner product and norm are defined as

$$\langle u, v\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)\, dx,$$

and

$$\|u\|_{W_2^m[a,b]} = \sqrt{\langle u, u\rangle_{W_2^m[a,b]}}, \quad u(x), v(x) \in W_2^m[a, b].$$

These spaces possess unique reproducing kernel functions, and here we define $R_t(s)$ for $W_2^6[a, b]$ and $r_t(s)$ for $W_2^1[a, b]$, as follows [30].

For $W_2^6[a, b]$, the reproducing kernel $R_t(s)$ is defined as

$$R_t(s) = R(s, t) = \begin{cases} R_1(s, t), & \text{if } s \le t, \\ R_2(s, t), & \text{if } t < s. \end{cases}$$

The explicit forms of $R_1(s, t)$ and $R_2(s, t)$ are given by

$$\begin{cases} R_1(s, t) = 1 - \dfrac{x^{11}}{39916800} + \left(x + \dfrac{x^{10}}{3628800}\right)y + \left(\dfrac{x^2}{4} - \dfrac{x^9}{725760}\right)y^2 + \dfrac{x^3(6720 + x^5)y^3}{241920} \\ \quad - \dfrac{x^4(-210 + x^3)y^4}{120960} + \dfrac{x^5(6 + x)y^5}{86400}, \\ R_2(s, t) = R_1(t, s). \end{cases}$$

For $W_2^1[a, b]$, the reproducing kernel $r_t(s)$ is defined as

$$r_t(s) = r(s, t) = \begin{cases} r_1(s, t), & \text{if } s \le t, \\ r_2(s, t), & \text{if } t < s. \end{cases}$$

The explicit forms of $r_1(s, t)$ and $r_2(s, t)$ are given by

$$\begin{cases} r_1(s, t) = 1 + s, \\ r_2(s, t) = r_1(t, s). \end{cases}$$

3. Quasi-Newton reproducing kernel method

3.1. Linearization via Quasi-Newton method

For a differential operator $\mathcal{F}(u)$, the tangent equation at u_0 is obtained by using the Fréchet derivative [31] of $\mathcal{F}$ at u_0

$$\mathcal{F}(u) - \mathcal{F}(u_0) - \mathcal{F}'(u_0)(u - u_0) = 0. \tag{2}$$

For $\mathcal{F} : C^5[a, b] \to C[a, b]$, we define

$$\mathcal{F}(u) \triangleq u^{(5)}(x) + \sum_{i=0}^{4} g_i(x)u^{(i)}(x) + \mathcal{N}(u), \tag{3}$$

where $\mathcal{N}(u)$ represents the nonlinear part of the operator.

Lemma 1. *The Fréchet derivative $\mathcal{F}'(u_0) : C^5 \to C$ is*

$$\mathcal{F}'(u_0) : u \longmapsto u^{(5)}(x) + \sum_{i=0}^{4} g_i(x)u^{(i)}(x) + \mathcal{N}'(u_0)u,$$

where $\mathcal{N}'(u_0)$ denotes the Fréchet derivative of the nonlinear operator $\mathcal{N}(u)$ at u_0.

Theorem 1. *Using the previous linearization, the Quasi-Newton iteration for solving $\mathcal{F}(u) = f(x)$ is given by*

$$\begin{cases} u_{k+1}^{(5)}(x) + \sum_{i=0}^{4} g_i(x)u_{k+1}^{(i)}(x) + \mathcal{N}'(u_k)(u_{k+1} - u_k) + \mathcal{N}(u_k) = f(x), \quad x \in [a, b], \quad k = 0, 1, \dots \\ \sum_{j_1=0}^{2} u_{k+1}^{(j_1)}(a) = \alpha_{j_1}, \quad \sum_{j_2=0}^{1} u_{k+1}^{(j_2)}(b) = \beta_{j_2}. \end{cases} \tag{4}$$

3.2. Approximate solution via the RKM

To solve the linear differential equation derived from the nonlinear operator, we modify the RKM as follows. Consider the linear equation based on Eq (4)

$$\begin{cases} u^{(5)}(x) + \sum_{i=0}^{4} g_i(x)u^{(i)}(x) = \tilde{f}(x), \quad x \in [a, b], \\ \sum_{j_1=0}^{2} u^{(j_1)}(a) = \alpha_{j_1}, \quad \sum_{j_2=0}^{1} u^{(j_2)}(b) = \beta_{j_2}, \end{cases} \tag{5}$$

where $\tilde{f}(x) = f(x) - [\mathcal{N}'(u_k)(u_{k+1} - u_k) + \mathcal{N}(u_k)]$. We solve the given linear differential equation using the RKM, employing the reproducing kernel spaces $W_2^6[a, b]$ and $W_2^1[a, b]$. Let $\mathcal{G} : W_2^6[a, b] \to W_2^1[a, b]$ be a differential operator. Applying Eq (5) gives

$$\mathcal{G}(u) = u^{(5)}(x) + \sum_{i=0}^{4} g_i(x)u^{(i)}(x). \tag{6}$$

The equation in Eq (5) can be rewritten as the following operator equation:

$$\begin{cases} \mathcal{G}(u) = \tilde{f}(x), \quad x \in [a, b], \\ \sum_{j_1=0}^{2} u^{(j_1)}(a) = \alpha_{j_1}, \quad \sum_{j_2=0}^{1} u^{(j_2)}(b) = \beta_{j_2}. \end{cases} \tag{7}$$

Theorem 2. *The operator $\mathcal{G}$ is a bounded linear operator from $W_2^6[a, b]$ to $W_2^1[a, b]$.*

Proof. Linearity of the operator $\mathcal{G}$ is straightforward. For boundedness, we aim to show that

$$\|\mathcal{G}(u)\|^2_{W_2^1[a,b]} = \mathcal{G}(u)(a)^2 + \int_a^b \left(\mathcal{G}(u)^{(1)}(x)\right)^2 dx \le C\|u\|^2_{W_2^6[a,b]}.$$

First, we compute $\mathcal{G}(u)(a)$

$$\mathcal{G}(u)(a) = u^{(5)}(a) + \sum_{i=0}^{4} g_i(a)u^{(i)}(a).$$

Since $u \in W_2^6[a,b]$, the derivatives $u^{(i)}(a)$ for $i = 0, 1, \ldots, 5$ are bounded, and $g_i(x) \in C^1[a,b]$, it follows that $\mathcal{G}(u)(a)$ is bounded. Moreover, we can estimate $\mathcal{G}(u)(a)$ using $\|u\|_{W_2^6[a,b]}$.

Next, we compute $\mathcal{G}(u)^{(1)}(x)$

$$\mathcal{G}(u)^{(1)}(x) = u^{(6)}(x) + \sum_{i=0}^{4} \left(g_i'(x)u^{(i)}(x) + g_i(x)u^{(i+1)}(x)\right).$$

We now estimate the integral of the square of $\mathcal{G}(u)^{(1)}(x)$

$$\int_a^b \left(\mathcal{G}(u)^{(1)}(x)\right)^2 dx = \int_a^b \left(u^{(6)}(x) + \sum_{i=0}^{4} \left(g_i'(x)u^{(i)}(x) + g_i(x)u^{(i+1)}(x)\right)\right)^2 dx.$$

Expanding this and applying the triangle inequality, we obtain

$$\begin{aligned}
&\int_a^b \left(u^{(6)}(x) + \sum_{i=0}^{4} \left(g_i'(x)u^{(i)}(x) + g_i(x)u^{(i+1)}(x)\right)\right)^2 dx \\
\le &\int_a^b \left(u^{(6)}(x)\right)^2 dx + \int_a^b \left(\sum_{i=0}^{4} g_i'(x)u^{(i)}(x)\right)^2 dx + \int_a^b \left(\sum_{i=0}^{4} g_i(x)u^{(i+1)}(x)\right)^2 dx.
\end{aligned}$$

We estimate these integrals as follows:

(1) Since $u^{(6)}(x) \in L^2[a,b]$, we have

$$\int_a^b \left(u^{(6)}(x)\right)^2 dx \le C_1\|u\|^2_{W_2^6[a,b]}.$$

(2) For the second term, since $g_i'(x)$ is continuous and $u^{(i)}(x) \in L^2[a,b]$, we get

$$\int_a^b \left(\sum_{i=0}^{4} g_i'(x)u^{(i)}(x)\right)^2 dx \le C_2\|u\|^2_{W_2^6[a,b]}.$$

(3) For the third term, similarly, since $g_i(x)$ is continuous and $u^{(i+1)}(x) \in L^2[a,b]$, we have

$$\int_a^b \left(\sum_{i=0}^{4} g_i(x)u^{(i+1)}(x)\right)^2 dx \le C_3\|u\|^2_{W_2^6[a,b]}.$$

These estimates follow from the properties of Sobolev spaces; see, e.g., [32, 33]. Thus, combining all these results, we conclude that $\mathcal{G}$ is a bounded linear operator, and there exists a constant C such that

$$\|\mathcal{G}(u)\|_{W_2^1[a,b]} \leq C\|u\|_{W_2^6[a,b]}.$$

□

Remark 1. *Thanks to the existence and uniqueness of the solution to Equation* (1)*, the operator $\mathcal{G}$ is necessarily invertible.*

Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of $[a, b]$. Define

$$\xi_i(x) = \mathcal{G}^* r_y(x)\Big|_{y=x_i}, \quad i = 1, 2, \ldots,$$

where $\xi_i(x) \in W_2^6[a, b]$, and $\mathcal{G}^*$ is the adjoint operator of $\mathcal{G}$. Under this construction, the following property holds.

Theorem 3. *For each $i \in \mathbb{N}$, it holds that*

$$\xi_i(x) = \mathcal{G}R_y(x)\Big|_{y=x_i}.$$

Proof. We have $\xi_i(x) = \mathcal{G}^* r_{x_i}(x)$. By the properties of the inner product and operator $\mathcal{G}$, we can express it as

$$\xi_i(x) = \langle \mathcal{G}^* r_{x_i}, R_x \rangle_{W_2^6} = \langle r_{x_i}, \mathcal{G}R_x \rangle_{W_2^1} = \mathcal{G}R_{x_i}(x), \quad i = 1, 2, \ldots.$$

□

Theorem 4. *Let $\{x_i\}_{i=1}^{\infty}$ be a set of distinct dense points in $[a, b]$; then the set $\{\xi_i(x)\}_{i=1}^{\infty}$ is a linearly independent complete system in $W_2^6[a, b]$.*

Proof. Assume that

$$\sum_{i=1}^{\infty} c_i \xi_i(x) = 0.$$

By applying the invertibility of $\mathcal{G}$, we obtain

$$\sum_{i=1}^{\infty} c_i \mathcal{G}R_{x_i}(x) = \mathcal{G}\left(\sum_{i=1}^{\infty} c_i R_{x_i}(x)\right) = 0.$$

Since the set $\{x_i\}_{i=1}^{\infty}$ is dense in $[a, b]$, it follows that the sum $\sum_{i=1}^{\infty} c_i R_{x_i}(x)$ must be identically zero. This implies that $c_i = 0$ for all i. Therefore, for any $v(x) \in W_2^6[a, b]$, if $\langle v(x), \xi_i \rangle_{W_2^6} = 0$ for all i, then $v(x) \equiv 0$.

Thus, the set $\{\xi_i(x)\}_{i=1}^{\infty}$ is both linearly independent and complete in $W_2^6[a, b]$. □

If the boundary condition is specified as $u^{(j)}(k) = \alpha_j$, we can express it as

$$\eta_j(x) = \frac{\partial^j R}{\partial y^j}\Big|_{y=\alpha_j}, \quad j = 1, 2, 3, 4, 5. \tag{8}$$

Let

$$W_{5+n} = \text{span}\{\eta_1(x), \eta_2(x), \ldots, \eta_5(x), \xi_1(x), \xi_2(x), \ldots, \xi_n(x)\}.$$

Define the projection operator $\mathcal{P}_{5+n} : W_2^6[a, b] \to W_{5+n}$. Based on this, the following theorem can be derived.

Theorem 5. *Let u be the solution to Eq (5); then the sequence $u_n = \mathcal{P}_{5+n}u$ satisfies*

$$\begin{cases} \langle u_n, \eta_j \rangle = \alpha_j, & j = 1,2,3,4,5, \\ \langle u_n, \xi_i \rangle = f(x_i), & i = 1,2,\ldots,n. \end{cases}$$

Proof. First, we compute the inner products for η_j

$$\langle \mathcal{P}_{5+n}u(x), \eta_j(x) \rangle = \langle u(x), \eta_j(x) \rangle = u^{(j)}(k) = \alpha_j, \quad j = 1,2,3,4,5.$$

Next, for ξ_i

$$\langle \mathcal{P}_{5+n}u(x), \xi_i(x) \rangle = \langle u(x), \xi_i(x) \rangle = \langle \mathcal{G}u(x), r_{x_i}(x) \rangle = \mathcal{G}u(x_i) = f(x_i), \quad i = 1,2,\ldots,n.$$

□

Thus, $(\mathcal{P}_{5+n}u)(x)$ serves as an approximation to the exact solution $u(x)$. Additionally, we present the following convergence result.

Theorem 6. $(\mathcal{P}_{5+n}u)(x)$ *converges uniformly to* $u(x)$ *over the interval* $[a,b]$*. Furthermore, for* $j = 1,2,3,4,5$*, the uniform convergence of* $(\mathcal{P}_{n+5}u)^{(j)}(x)$ *to* $u^{(j)}(x)$ *is valid on* $[a,b]$.

Proof. For $0 \le j \le 5$, we define

$$\partial_x^j R(x) = \left.\frac{\partial^j R_y(x)}{\partial y^j}\right|_{y=x}, \quad u^{(j)}(x) = \langle u(x), \partial_x^j R(x) \rangle,$$

and

$$(\mathcal{P}_{5+n}u)^{(j)}(x) = \langle \mathcal{P}_{5+n}u(x), \partial_x^j R(x) \rangle.$$

By applying the Cauchy Schwarz inequality, we derive the following result

$$|u^{(j)}(x) - (\mathcal{P}_{5+n}u)^{(j)}(x)| \le c_j \, \|u(x) - (\mathcal{P}_{5+n}u)(x)\|,$$

where c_j is a constant. Since the difference $u(x) - (\mathcal{P}_{5+n}u)(x)$ converges to zero, we conclude that

$$(\mathcal{P}_{n+5}u)^{(j)}(x) \xrightarrow{n\to\infty} u^{(j)}(x), \quad j = 1,2,3,4,5.$$

□

Let $u_n(x) = (\mathcal{P}_{5+n}u)(x)$. Then, there exist constants b_j and a_j such that

$$u_n(x) = \sum_{j=1}^{5} b_j \eta_j(x) + \sum_{i=1}^{n} a_i \xi_i(x). \tag{9}$$

The corresponding system of equations, derived from the inner products between the basis functions $\eta_j(x)$ and $\xi_i(x)$ is given by

$$\begin{cases} \sum_{j=1}^{5} b_j \langle \eta_j(x), \eta_1(x) \rangle + \sum_{i=1}^{n} a_i \langle \xi_i(x), \eta_1(x) \rangle = \gamma_1, \\ \sum_{j=1}^{5} b_j \langle \eta_j(x), \eta_2(x) \rangle + \sum_{i=1}^{n} a_i \langle \xi_i(x), \eta_2(x) \rangle = \gamma_2, \\ \vdots \\ \sum_{j=1}^{5} b_j \langle \eta_j(x), \xi_i(x) \rangle + \sum_{i=1}^{n} a_i \langle \xi_i(x), \xi_i(x) \rangle = \tilde{f}(x_i), \quad i = 1,2,\ldots,n. \end{cases} \tag{10}$$

To solve this system, we introduce the Gram matrix $\boldsymbol{G}$, which is constructed from the inner products of the basis functions. The matrix $\boldsymbol{G}$ is defined as follows:

$$\boldsymbol{G} = \begin{pmatrix} \langle \eta_1, \eta_1 \rangle & \ldots & \langle \eta_1, \eta_5 \rangle & \langle \eta_1, \xi_1 \rangle & \langle \eta_1, \xi_2 \rangle & \ldots & \langle \eta_1, \xi_n \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \eta_5, \eta_1 \rangle & \ldots & \langle \eta_5, \eta_5 \rangle & \langle \eta_5, \xi_1 \rangle & \langle \eta_5, \xi_2 \rangle & \ldots & \langle \eta_5, \xi_n \rangle \\ \langle \xi_1, \eta_1 \rangle & \ldots & \langle \xi_1, \eta_5 \rangle & \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \ldots & \langle \xi_1, \xi_n \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \xi_n, \eta_1 \rangle & \ldots & \langle \xi_n, \eta_5 \rangle & \langle \xi_n, \xi_1 \rangle & \langle \xi_n, \xi_2 \rangle & \ldots & \langle \xi_n, \xi_n \rangle \end{pmatrix}.$$

The right-hand side vector $\boldsymbol{f}$, consisting of the boundary conditions and the values of the source function, is expressed as

$$\boldsymbol{f} = \left(\gamma_1, \ldots, \gamma_5, \tilde{f}(x_1), \ldots, \tilde{f}(x_n)\right)^T. \tag{11}$$

Since $\boldsymbol{G}$ is a symmetric, positive definite Gram matrix, the solution vector $(b_1, \ldots, b_5, a_1, a_2, \ldots, a_n)^T$ is obtained by solving the linear system

$$(b_1, \ldots, b_5, a_1, a_2, \ldots, a_n)^T = \boldsymbol{G}^{-1}\boldsymbol{f},$$

which is unique due to the invertibility of $\boldsymbol{G}$, completing the required solution.

4. Error analysis

In this section, we derive the error estimates for the previously introduced scheme and present several fundamental approximation results.

$$\|u^{(i)} - (I_h u)^{(i)}\|_{L_2} \leq K h^{2-i} \|u''\|_{L_2}, \quad i = 0, 1. \tag{12}$$

Here, I_h denotes the interpolation operator based on piecewise linear polynomials, with h representing the maximum mesh size.

Moreover, several key norm equivalences in reproducing kernel Hilbert spaces are derived. For a detailed exposition, see [34].

$$\|u\|_{W_2^6} \leq K \|\mathcal{P}u\|_{W_2^1}, \quad \forall u \in W_2^6[a,b], \tag{13}$$

$$\|w\|_{W_2^2} \leq K \|\mathcal{P}^* w\|_{W_2^6}, \quad \forall w \in W_2^2[a,b] \cap W_2^1[a,b], \tag{14}$$

where K is a constant.

Lemma 2. *Let $u(x) \in W_2^6[a,b]$ be the solution to Eq. (7), and let $u_n = \mathcal{P}_{5+n} u \in W_{5+n}$. Then, the following inequality holds:*

$$\|u - u_n\|_{W_2^6} \leq C \cdot \|w - w_n\|_{W_2^1},$$

where $\mathcal{G}^* w = u$, $w_n = \mathcal{P}_{\hat{H}} w$, and $\mathcal{P}_{\hat{H}}$ is the projection operator from $W_2^1[a,b]$ onto $\hat{H}$.

Proof. Since $\mathcal{G}$ is a bounded linear operator, it follows that $\mathcal{G}^*$ is also bounded and linear, with the same norm as $\mathcal{G}$. Consequently, we can deduce the following estimate:

$$\|u - u_n\|_{W_2^6} \leq \|\mathcal{G}^*(w - w_n)\|_{W_2^6} \leq \|\mathcal{G}^*\|\|w - w_n\|_{W_2^1} \leq C\|w - w_n\|_{W_2^1}.$$

□

Theorem 7. *Under the assumptions of Lemma 2, we have the following estimates:*

$$\|u - u_n\|_\infty \leq M_1 h^2, \tag{15}$$

$$\|u' - u_n'\|_\infty \leq M_2 h^2, \tag{16}$$

where M_1 and M_2 are constants independent of h.

Proof. We begin with the first term:

$$|u(x) - u_n(x)| = |\langle u - u_n, R_x\rangle_{W_2^6}| = |\langle u - u_n, R_x - \mathcal{P}_{5+n}R_x\rangle_{W_2^6}| \tag{17}$$

$$\leq \|R_x - \mathcal{P}_{5+n}R_x\|_{W_2^6} \cdot \|u - u_n\|_{W_2^6}. \tag{18}$$

For the first factor, applying reasoning similar to that in Lemma 2, we derive the estimate

$$\|R_x - \mathcal{P}_{5+n}R_x\|_{W_2^6} \leq \|\mathcal{P}^*\| \cdot \|\tilde{w} - \tilde{w}_n\|_{W_2^1}. \tag{19}$$

Since $\tilde{w}_n \in \hat{H}$ and satisfies $\tilde{w}(x_j) - \tilde{w}_n(x_j) = \langle \tilde{w} - \tilde{w}_n, r_{x_j}\rangle = 0$ at each node x_j, it follows that $\tilde{w}_n$ coincides with the piecewise linear interpolation of $\tilde{w}(x)$. By utilizing the interpolation properties from (12), we obtain

$$\|\tilde{w} - \tilde{w}_n\|_{W_2^1} \leq C_1\|\tilde{w} - \tilde{w}_n\|_{W_2^2} \leq C_2 h\|\tilde{w}\|_{W_2^2},$$

and combining this with (14), we conclude that

$$\|R_x - \mathcal{P}_{5+n}R_x\|_{W_2^6} \leq C_0 h\|\mathcal{P}^* w\|_{W_2^6} = C_0 h\|R_x\|_{W_2^6}. \tag{20}$$

For the second term, leveraging the estimates from (13)-(14), we obtain

$$\|w - w_n\|_{W_2^1} \leq C_1\|w - w_n\|_{W_2^2} \leq C_2 h\|w\|_{W_2^2} \leq C_3 h\|\mathcal{P}^* w\|_{W_2^6} \tag{21}$$

$$= C_3 h\|u\|_{W_2^6} \leq C_4 h\|\mathcal{P}u\|_{W_2^1} = C_4 h\|g(u)\|_{W_2^1}, \tag{22}$$

where $C_i, i = 0, 1, 2, 3, 4$ are constants. By combining (18), (20), and (21) and applying the Lipschitz condition on the nonlinearity, we conclude that

$$\|u - u_n\|_\infty \leq M_1 h^2.$$

Similarly, for the derivative of u, we obtain the following estimate:

$$|u' - u_n'| = |\partial_x\langle u - u_n, R_x - \mathcal{P}_{5+n}R_x\rangle| = |\langle u - u_n, \partial_x(R_x - \mathcal{P}_{5+n}R_x)\rangle| \tag{23}$$

$$\leq \|u - u_n\|_{W_2^3}\|\partial_x(R_x - \mathcal{P}_{5+n}R_x)\|_{W_2^3} \tag{24}$$

$$\leq \|u - u_n\|_{W_2^3}\|\partial_x R_x\|_{W_2^3} \tag{25}$$

$$\leq \|\mathcal{P}^*\|\|w - w_n\|_{W_2^1} \cdot D_1 h\|g(u)\|_{W_2^1} \cdot D_2 h\|\partial_x R_x\|_{W_2^3} \tag{26}$$

$$\leq M_2 h^2. \tag{27}$$

Thus, we conclude that

$$\|u' - u_n'\|_\infty \leq D_0 h^2\|g\|_{W_2^1}\|\partial_x R_x\|_{W_2^6} \leq M_2 h^2,$$

where $D_i, i = 0, 1, 2$, are constants. □

5. Numerical examples

Within this segment, numerous numerical instances are provided to showcase the precision and effectiveness of the method we have put forth. We compare the results to the exact solution and show that our approach is highly effective and practical under various scenarios. The absolute error e is computed using the following metrics:

$$\|e\|_a = \|e\|_{L^\infty} = \max_{0<x<1}|u - u_n|, \qquad \|e\|_b = \|e\|_{L^2} = \left(\int_0^1 (u-u_n)^2 \, \mathrm{d}x\right)^{1/2},$$
$$|e|_c = \left\|u' - u_n'\right\|_{L^2}, \qquad |e|_d = \left\|u'' - u_n''\right\|_{L^2}.$$

And we evaluate the convergence rate r by using the following formulas

$$r = \log_2 \frac{\|e_n\|}{\|e_{2n}\|}.$$

Table 1 represents the number of iterations of the Quasi-Newton scheme using k.

Table 1. Error comparison at $n = 32$ for Example 1.

x	[21]	[23]	MHP [20]	VIM [20]	[22]	[24]	$\|e\|$ (QNRKM)
0	0.000	0.000	0.000	0.000	0.000	0.000	1.53932E-15
0.1	8.0E-3	1.8E-5	3E-11	3E-11	1.0E-8	4.0E-7	2.96505E-13
0.2	1.2E-3	1.0E-4	2E-10	2E-10	6.0E-8	3.1E-6	1.8999E-12
0.3	5.0E-3	2.4E-4	4E-10	4E-10	1.5E-7	8.8E-6	4.9989E-12
0.4	3.0E-3	3.7E-4	8E-10	8E-10	2.7E-7	1.6E-5	8.87155E-12
0.5	8.0E-3	4.2E-4	1.2E-9	1.2E-9	3.8E-7	2.3E-5	1.22689E-11
0.6	6.0E-3	3.6E-4	2.0E-9	2.0E-9	4.3E-7	2.7E-5	1.38417E-11
0.7	5.0E-3	2.3E-4	2.2E-9	2.2E-9	3.9E-7	2.6E-5	1.26192E-11
0.8	9.0E-3	1.1E-4	1.9E-9	1.9E-9	2.7E-7	1.8E-5	8.54851E-12
0.9	9.0E-3	8.7E-5	1.4E-9	1.4E-9	1.0E-7	6.9E-6	3.10766E-12
1	0.000	0.000	0.000	0.000	0.000	0.000	3.19687E-15

Example 1. *Examine the following problem [20–24, 35].*

$$u^{(5)}(x) - u(x) = -15e^x - 10xe^x, \quad 0 \le x \le 1. \tag{28}$$

The boundary conditions are specified as follows: $a = 0, b = 1, \alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 0, \beta_0 = 0, \beta_1 = -e$.

The problem considered in this paper has an exact solution $u(x) = x(1-x)e^x$. We first present the results for $n = 32$ and compare them with those derived using the Homotopy Perturbation method, Variational Iteration method, and Adomian Decomposition method, as documented in [20–24]. The results are summarized in Table 1, highlighting the enhanced accuracy of the proposed method. Subsequently, we extend the analysis to different values of n, specifically $n = 8, 16, 32, 64, 128$ and 256 and compare these outcomes with those reported in [35]. The summary of these findings is provided in

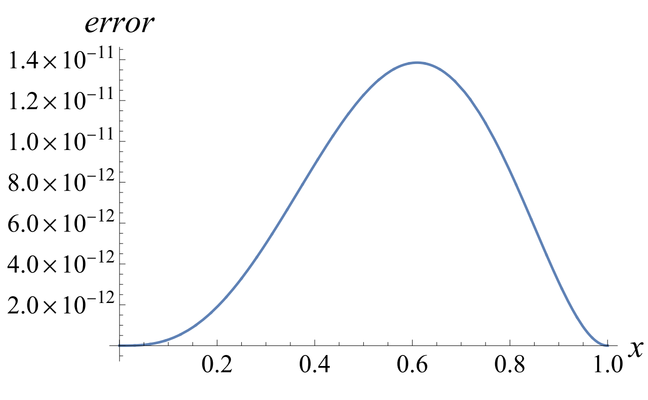
Table 2. As shown in Table 3, the proposed method yields steadily decreasing errors and maintains a high convergence rate close to or above 2, with acceptable increases in CPU time as n grows. Figure 1 shows the absolute errors for $u(x)$ and $u'(x)$ at $n = 32$. Figure 2 displays the convergence behavior of the numerical error with a logarithmic scale for Example 1. From the figure, it is clear that the convergence rate is around 2, demonstrating the favorable convergence properties of the method.

Table 2. Computational results for Example 1.

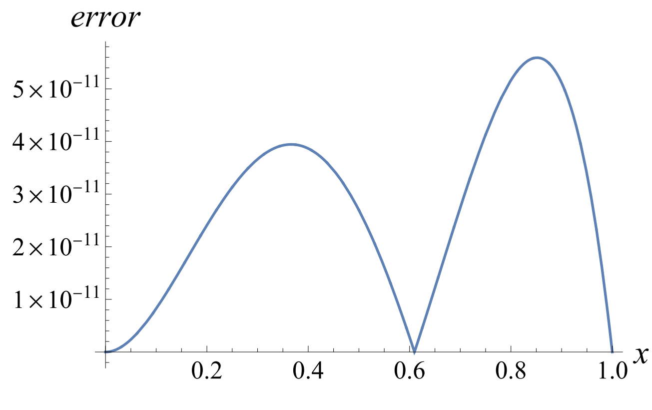
n	$\|u - u_n\|_\infty$	$\|u' - u_n'\|_\infty$	$\|u'' - u_n''\|_\infty$	$\|u''' - u_n'''\|_\infty$	$\|u'''' - u_n''''\|_\infty$	$\|u''''' - u_n'''''\|_\infty$
			QNRKM			
8	2.19202E-10	8.8676E-10	9.89132E-9	8.72268E-8	4.69438E-6	6.99747E-5
16	5.58093E-11	2.25497E-10	1.24431E-9	2.7282E-8	9.3558E-7	5.23751E-7
32	1.39995E-11	5.59071E-11	3.08005E-10	7.56532E-9	2.34355E-7	1.26264E-7
64	2.75569E-12	1.38387E-11	7.64894E-11	1.9867E-9	5.86268E-8	3.16565E-8
128	5.09872E-13	3.38622E-12	1.90497E-11	5.08672E-10	1.46598E-8	7.92509E-9
256	1.33359E-13	8.20656E-13	4.42344E-12	1.28612E-10	4.92824E-10	1.97194E-8
			Method of [35]			
8	5.59E-4	5.1E-3	2.95E-2	9.19E-2	1.63E-1	5.95E-4
16	3.41E-5	3.59E-4	2.5E-3	9.9E-3	2.55E-2	3.41E-5
32	1.59E-6	2.40E-5	2.39E-4	1.4E-3	4.9E-3	1.59E-6
64	2.47E-7	1.56E-6	3.33E-5	2.61E-4	1.1E-3	2.47E-7
128	8.90E-8	3.68E-7	6.79E-6	6.20E-5	2.83E-4	8.91E-8
256	6.65E-8	2.60E-7	3.99E-6	3.59E-5	1.9E-3	6.65E-8

Table 3. Maximum error, CPU time, and convergence rate r for Example 1.

n	$\|u - u_n\|_\infty$	CPU time (s)	r
8	2.19202E−10	-	-
16	5.58093E−11	0.015625	1.97368
32	1.39995E−11	0.078125	1.99513
64	2.75569E−12	0.265625	2.34489
128	5.09872E−13	1.01563	2.43421
256	1.33359E−13	4.07813	1.93482



(a) $|u - u_n|$



(b) $|u' - u'_n|$

Figure 1. Error results for Example 1 with $n = 32$.

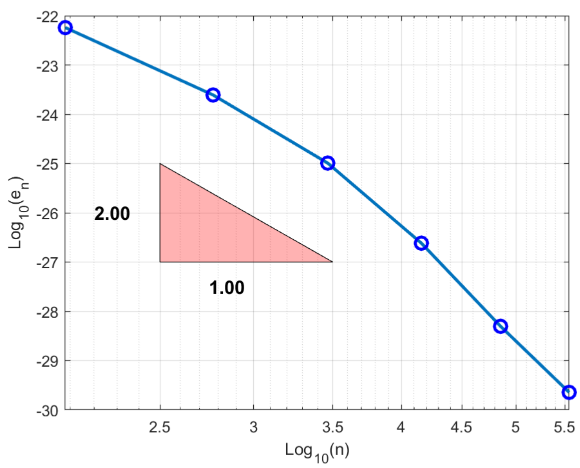


Figure 2. Convergence trend of numerical error with Log-scale variables for Example 1.

Example 2. *Examine the following problem [20, 21, 23, 24].*

$$u^{(5)}(x) - e^{-x}u^2(x) = 0, \quad 0 \le x \le 1. \tag{29}$$

The boundary conditions are outlined as follows: $a = 0, b = 1, \alpha_0 = 1, \alpha_1 = 1, \alpha_2 = 1, \beta_0 = e, \beta_1 = e.$

The problem considered in this study has an exact solution given by $u(x) = e^x$. We first solve this problem for $n = 32$ and $k = 2$ and compare the results with the errors reported in [20, 21, 23, 24]. Table 4 summarizes the findings, highlighting that our method has significantly lower maximum absolute errors than the approaches in [20, 21, 23, 24]. Next, we extend our analysis to a range of values for n, specifically $n = 8, 16, 32, 64$ and 128. To verify the convergence of our method in practical applications, we compute the orders of convergence for various values of n with $k = 2$, and present the results in Table 5. Figure 3 shows the convergence pattern of the numerical error with log-scale variables for Example 2, with $k = 2$. From the figure, it is clear that the convergence rate is approximately 2, indicating that the numerical method demonstrates strong convergence for this problem.

Table 4. Comparison of errors for Example 2 ($n = 32, k = 2$).

x	[21]	[23]	[20] MHP	[20] VIMHP	[20] VIM	[24]	$\|e\|$ (QNRKM)
0	0.000	0.000	0.000	0.000	0.000	0.000	1.39266E-14
0.1	7.0E-4	2.3E-7	1.0E-9	1.0E-9	1.0E-9	0.000	8.13592E-12
0.2	7.2E-4	1.6E-6	2.0E-9	2.0E-9	2.0E-9	1.0E-5	5.22023E-11
0.3	4.1E-4	4.6E-6	1.0E-8	1.0E-8	1.0E-8	1.0E-5	1.37116E-10
0.4	4.6E-4	8.9E-6	2.0E-8	2.0E-8	2.0E-8	1.0E-4	2.32741E-10
0.5	4.7E-4	1.3E-5	3.1E-8	3.1E-8	3.1E-8	3.2E-4	2.14657E-10
0.6	4.8E-4	1.6E-5	3.7E-8	3.7E-8	3.7E-8	3.6E-4	2.26136E-10
0.7	3.9E-4	1.6E-5	4.1E-8	4.1E-8	4.1E-8	1.4E-4	2.4008E-10
0.8	3.1E-4	1.2E-5	3.1E-8	3.1E-8	3.1E-8	3.1E-4	2.30049E-10
0.9	1.6E-4	5.1E-6	1.4E-8	1.4E-8	1.4E-8	5.8E-4	8.31103E-11
1	0.000	0.000	0.000	0.000	0.000	0.000	3.81162E-14

Table 5. Convergence rates r of various norm errors by QNRKM for Example 2 ($k = 2$).

n	$\|\|e\|\|_a$	r	$\|\|e\|\|_b$	r	$\|e\|_c$	r	$\|e\|_d$	r
8	5.12451E-9	-	3.56002E-9	-	1.37991E-8	-	1.00778E-7	-
16	1.11487E-9	2.20054	9.09464E-10	1.9688	3.51164E-9	1.97436	2.62241E-8	1.94221
32	2.40113E-10	2.21509	2.25904E-10	2.00931	8.71416E-10	2.01071	6.53243E-9	2.0052
64	5.95833E-11	2.01073	5.61505E-11	2.00834	2.16555E-10	2.00862	1.62403E-9	2.00804
128	1.52808E-11	1.96319	1.40529E-11	1.99843	5.39545E-11	2.00492	4.03635E-10	2.00845

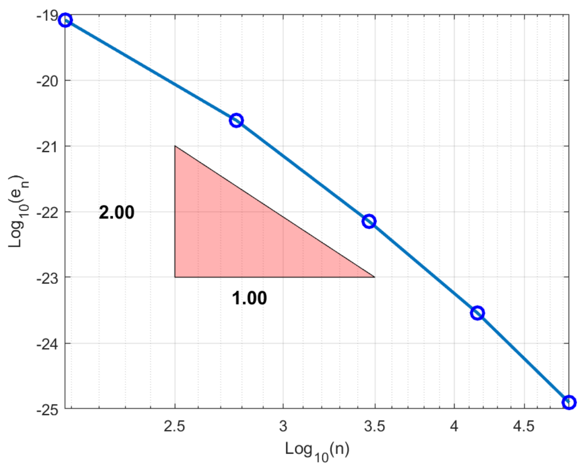


Figure 3. Convergence trend of numerical error with Log-scale variables for Example 2 ($k = 2$).

Example 3. *Examine the following problem [20–24].*

$$u^{(5)}(x) + 24e^{-5u(x)} = \frac{48}{(1+x)^5}, \quad 0 \le x \le 1. \tag{30}$$

The boundary conditions are specified as follows: $a = 0$, $b = 1$, $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_0 = \ln 2$, *and* $\beta_1 = 0.5$.

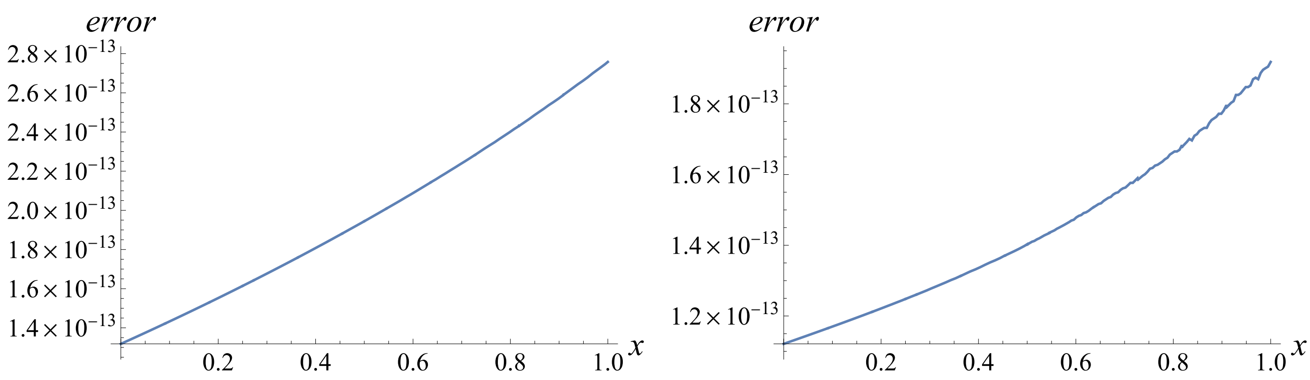
The problem addressed in this research has the exact solution $u(x) = \ln(1 + x)$. The numerical solutions for this problem are obtained for $k = 4$, and the corresponding absolute errors at various nodes are listed in Table 6. Furthermore, we determine the absolute errors for different types of error norms with $k = 2$ and 4, and the results are compiled in Table 7. The absolute error curves for both $u(x)$ and $u'(x)$ with $n = 11$ and $k = 4$ are illustrated in Figure 4.

Table 6. Computational findings and absolute error with $n = 9$ for Example 3 ($k = 4$).

x	Analytical Solution	[21]	[22]	$\|e\|$ (QNRKM)
0	0.000	0.000	0.000	1.43973E-14
0.1	0.0953102	0.000	-2E-8	7.1567E-14
0.2	0.182322	-0.015	-1.2E-7	1.33394E-13
0.3	0.262364	-0.029	-2.8E-7	1.99667E-13
0.4	0.336472	-0.028	-4.5E-7	2.69854E-13
0.5	0.405465	-0.026	-5.6E-7	3.43059E-13
0.6	0.470004	-0.024	-5.8E-7	4.17721E-13
0.7	0.530628	-0.026	-4.8E-7	4.92939E-13
0.8	0.587787	-0.033	-3.0E-7	5.65659E-13
0.9	0.641854	-0.046	-1.0E-7	6.32827E-13
1	0.693147	0.000	0.000	6.90559E-14

Table 7. Errors in various norms for Example 3.

n	3	7	11	15
		$k = 2$		
$\|\|e\|\|_\infty$	1.77206E-9	4.23682E-9	3.12699E-9	2.12205E-9
$\|\|e\|\|_l$	1.15524E-9	2.70806E-9	1.99862E-9	1.35519E-9
$\|e\|_1$	4.9558E-9	1.18424E-8	8.71688E-9	5.89739E-9
$\|e\|_2$	3.93454E-8	9.20278E-8	6.66106E-8	4.39418E-8
		$k = 4$		
$\|\|e\|\|_\infty$	1.36631E-12	8.07636E-13	2.79056E-13	5.60074E-14
$\|\|e\|\|_l$	1.08456E-12	9.02311E-14	2.01766E-13	4.02464E-14
$\|e\|_1$	9.50935E-13	6.77778E-14	1.45774E-13	6.76908E-14
$\|e\|_2$	5.531E-12	3.30245E-14	8.44705E-14	5.49586E-14



(a) $|u - u_n|$ **(b)** $|u' - u'_n|$

Figure 4. Error results for example 3 with $n = 11$.

Numerical experiments confirm that the proposed QNRKM significantly outperforms several established approaches, including the HPM, VIM [20, 24], B-spline-based techniques [21, 35], and residual correction methods [22]. QNRKM delivers higher accuracy, faster convergence, and greater stability, especially for nonlinear high-order BVPs with complex boundary conditions. By eliminating the need for Schmidt orthogonalization, QNRKM reduces computational cost and improves efficiency. These results demonstrate that QNRKM is a robust and efficient alternative to traditional kernel- and spline-based methods for challenging problems in this class.

6. Conclusions

In this work, we developed the QNRKM for solving nonlinear fifth-order two-point boundary value problems. Numerical experiments confirm that the proposed method achieves second-order convergence for both the solution and its first derivative under the maximum norm. Compared with existing techniques such as the HPM, VIM, sixth-degree B-spline approaches, and residual correction methods, it consistently delivers smaller errors and faster convergence while maintaining strong numerical stability. The main innovation lies in the hybrid integration of Quasi-Newton iteration with the reproducing kernel framework. By avoiding the Schmidt orthogonalization process, the method reduces computational cost and improves robustness when handling complex nonlinearities and intricate boundary conditions. We rigorously established the boundedness and convergence of the scheme within the reproducing kernel Hilbert space setting. Numerical results across multiple examples demonstrate that this hybrid approach outperforms traditional kernel- and spline-based solvers. Our original hypothesis that such integration yields an efficient, stable, and highly accurate algorithm for high-order nonlinear problems has been fully validated. Moreover, the method shows clear advantages in terms of accuracy, convergence rate, and computational efficiency. While currently formulated for one-dimensional problems, its extension to higher-dimensional or more complex systems remains a promising direction for future research.

Author contributions

Chaoyue Guan: Software, Validation, Visualization, Writing-original draft. **Jian Zhang**: Conceptualization, Formal analysis, Methodology, Project administration, Supervision, Writing-review and editing, Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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