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Research article

Timelike acc-slant curves in Minkowski 3-space

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Abstract: In this work, we introduced the concept of a timelike acceleration-slant curve in three-dimensional Minkowski space, characterized by the property that the scalar product of a fixed unit direction and its acceleration vector remained constant. Moreover, we provided a classification of timelike acceleration-slant curves with a non-null axis based on their curvature properties. Later on, we explored the relationships between timelike acceleration-slant curves and various notable curves, including timelike helices, timelike slant helices, timelike Lorentzian spherical curve, and timelike Salkowski curves. To further illustrate and validate the theoretical results, we utilized Mathematica to generate visual representations.

Keywords: acceleration; helix; slant helix; Salkowski curve; Minkowski 3-space
Mathematics Subject Classification: 53A04, 53A35

1. Introduction

The object's path in space appears as a curve, and its position at any given time t can be described by the curve's position vector at parameter t. It is a well-known fact that the first derivative of a curve represents the velocity vector at a specific time t. Similarly, the second derivative corresponds to the object's acceleration vector, while the third derivative represents the object's jerk. Curve theory is utilized in various branches of physics, including general relativity and the study of relativistic particles. In general relativity, the curvature of spacetime determines the paths that objects follow under the influence of gravity. Research on relativistic particles has led to significant advancements in understanding the fundamental properties of matter and the universe [1–4].

In the study of motion, particularly in classical mechanics and kinematics, physical vector quantities play a crucial role. One such quantity is velocity, which refers to both the magnitude and direction of an object's motion. Another important vector quantity is acceleration, which measures the rate of change of velocity, and plays a critical role in dynamics, stability, and control. The determination of the direction of acceleration relies on the net force acting on an object. Sir Isaac Newton, the renowned

17th-century English mathematician and physicist, articulated this relationship in his Second Law of Motion, which establishes the connection between force and acceleration. Furthermore, the concept of jerk is defined as the rate of change of acceleration, and it influences smoothness and mechanical stress in motion systems [5–7].

In three-dimensional Euclidean space, a general helix is characterized by the property that its tangent vector field maintains a constant angle with a specified direction, referred to as the axis of the helix [8]. If a curve's principal normal vector maintains a constant angle with a specified direction, referred to as the axis of the curve, then the curve is called a slant helix, and this property is first described in [9]. A regular curve is named after the German mathematician Salkowski, who introduced it in [10], it has a constant curvature κ_1 and a non-constant torsion κ_2.

In three-dimensional Minkowski space, similar definitions apply: a curve is called a slant helix or a helix based on whether the scalar product of its principal normal or tangent or with a fixed direction is constant, respectively [11, 12]. Ali has also modified the definition of timelike Salkowski curves with an explicit parametrization in this space [13]. These specialized curves have been studied in various ambient spaces by several authors [14–20]. Additionally, the ac-slant curve have been investigated in [21, 22]. Although the definition of an acceleration slant curve in Euclidean space and that of a timelike acceleration slant curve in Minkowski space both rely on the condition that the scalar product between the acceleration vector and a fixed direction is constant, the differences in the underlying metric structures lead to significant variations in their geometric behavior and characterizations. In particular, the metric of Minkowski space introduces causal properties of vectors—such as timelike, spacelike, and lightlike—which play a central role in the classification and interpretation of such curves.

The organization of this study is outlined as follows:

In part 2, we provide a brief overview of the basic theory concerning curves in 3 dimensional Minkowski space. After that, in part 3, we introduce a new type of curve called a timelike acc-slant curve, which is defined by the property that the scalar product of a non-null fixed direction and its acceleration vector is a constant. Later on, we obtain a characterization of timelike acc-slant curves based on their curvature and torsion functions. Subsequently, it is concluded that when the timelike acceleration-slant curve takes the form of a helix, either the magnitude of the curve's velocity vector is a linear function, or the acceleration vector is perpendicular to its axis. Additionally, we establish that a unit speed curve with constant magnitude acceleration is a timelike acceleration-slant curve if, and only if, it is a slant helix. Finally, we demonstrate that a unit speed curve qualifies as a timelike acceleration-slant curve if, and only if, it represents a Salkowski curve, provided that the magnitude of acceleration is equal to 1 (i.e., when $\kappa_1 = 1$).

2. Materials and methods

Now, in this part, we give fundamental information about Minkowski 3-space, while additional context and in-depth analysis can be found in the following resources [11, 23, 24].

Consider a Minkowski 3-space denoted by $\mathbb{E}_1^3 = \left(\mathbb{R}^3(y_1, y_2, y_3), g\right)$, where g is the standard metric given by $g = -dy_1^2 + dy_2^2 + dy_3^2$. Here, (y_1, y_2, y_3) are the canonical coordinates in $\mathbb{R}^3$. A vector y in $\mathbb{E}_1^3$ is called timelike when $g(y, y) < 0$, spacelike when $g(y, y) > 0$ or $y = 0$, or null when $g(y, y) = 0$ and $y \neq 0$. Additionally, the norm of y is defined as $\|y\| = \sqrt{|g(y, y)|}$.

A curve $\beta(t)$ is categorized as timelike, spacelike, or null based on whether its velocity vector $\mathfrak{v}(t) = \beta'(t)$ within $\mathbb{E}^3_1$ is a timelike, spacelike, or a null vector for each parameter t, in the same order. Along the timelike curve $\beta(t)$ with the Frenet-Serret frame, it can be represented by $\{V_1, V_2, V_3\}$ in $\mathbb{E}^3_1$. These three vectors correspond to the tangent vector field, the principal normal vector field, and the binormal vector field, respectively. The Frenet-Serret formulae can be expressed with curvature κ_1 and torsion κ_2 as:

$$V_1{}' = \nu\kappa_1 V_2, \quad V_2{}' = \nu\kappa_1 V_1 + \nu\kappa_2 V_3, \quad V_3{}' = -\nu\kappa_2 V_2, \tag{2.1}$$

where $g(V_1, V_1) = -1$ and $g(V_2, V_2) = g(V_3, V_3) = 1$.

Here, the quantity $\nu(t_0) = \|\beta'(t_0)\|$ is known as the speed of the curve $\beta(t)$ at the point $t_0 \in I$. For all $t \in I$, if the speed ν is constant and equal to 1, then $\beta(t)$ is classified as a unit speed curve. A curve is named a Lorentzian spherical curve if the curve lies entirely on the Lorentzian unit sphere denoted by $S^2_1 = \left\{x \in \mathbb{E}^3_1 : g(x, x) = 1\right\}$.

From a physical perspective, the curve $\beta(t)$ is defined with respect to the parameter t, which represents time. At any given moment t along the curve $\beta(t)$, the position corresponds to the motion of particle P. It is well-established that the velocity vector, acceleration vector, and jerk vector of $\beta(t)$ denoted by $\mathfrak{v}(t)$, $\mathfrak{a}(t)$, and $\mathfrak{j}(t)$, respectively, can be computed by applying the Frenet equations (2.1) as follows:

$$\mathfrak{v} = \beta' = \nu V_1, \tag{2.2}$$
$$\mathfrak{a} = \beta'' = \nu' V_1 + \nu^2\kappa_1 V_2, \tag{2.3}$$
$$\mathfrak{j} = \beta''' = \left(\nu'' + \nu^3\kappa_1{}^2\right) V_1 + \left(3\nu'\nu\kappa_1 + \nu^2\kappa_1{}'\right) V_2 + \nu^3\kappa_1\kappa_2 V_3. \tag{2.4}$$

3. Results

In this part, we define a curve which is called a timelike acceleration-slant curve in $\mathbb{E}^3_1$. Furthermore, we provide a characterization of this specific type of curve.

Definition 1. *A timelike curve $\beta(t)$ is named a timelike acceleration-slant curve when the scalar product of its acceleration vector $\mathfrak{a}$ of $\beta(t)$, and a unit fixed direction W referred to as the axis of the timelike acceleration-slant helix, is constant in $\mathbb{E}^3_1$. This can be expressed as $g(\mathfrak{a}, W) = c$.*

For the sake of conciseness, we refer to the acceleration-slant helix simply as the acc-slant helix.

Remark 1. *Let $\beta(t)$ be a timelike curve in Minkowski 3-space. So, $\beta(t)$ is a timelike acc-slant helix if, and only if, the axis of timelike acc-slant W is perpendicular to the jerk vector of $\beta(t)$. More specifically, $g(\mathfrak{j}, W) = 0$.*

Theorem 1. *Let $\beta(t)$ be a timelike curve equipped with Frenet apparatus $\{V_1, V_2, V_3, \kappa_1, \kappa_2\}$ $\mathbb{E}^3_1$. Then, β is a timelike acc-slant curve if, and only if,*

$$-\eta_1^2 + \eta_2^2 + \eta_3^2 = \varepsilon \tag{3.1}$$

such that

$$\eta_1 = \frac{\left(\frac{1}{\nu\kappa_2}\left(\frac{\nu'}{\nu^3\kappa_1}\right)\right)' - \frac{1}{\nu}\left(\frac{\kappa_2}{\kappa_1} - f\right) - \left(\frac{1}{\nu\kappa_2}\left(\frac{1}{\nu^2\kappa_1}\right)'\right)'}{f' + \frac{\nu'}{\nu}\left(\frac{\kappa_2}{\kappa_1} - f\right)}, \tag{3.2}$$

$$\eta_2 = \frac{1}{\nu^2\kappa_1} + \frac{\nu'}{\nu^2\kappa_1}\eta_1, \tag{3.3}$$

$$\eta_3 = f\eta_1 + \frac{1}{\nu\kappa_2}\left(\left(\frac{1}{\nu^2\kappa_1}\right)' - \frac{\nu'}{\nu^3\kappa_1}\right), \tag{3.4}$$

where c is a nonzero constant, $f = \frac{\kappa_1}{\kappa_2} + \frac{1}{\nu\kappa_2}\left(\left(\frac{\nu'}{\nu^2\kappa_1}\right)' - \frac{(\nu')^2}{\nu^3\kappa_1}\right)$, *and* $\varepsilon = \pm 1$.

Proof. Suppose that $\beta(t)$ is a timelike acc-slant curve with a non-null axis (timelike or spacelike) W. According to Definition 1, there exist differentiable functions λ_i and a constant $c = g(\mathfrak{a}, W)$ such that

$$W = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3. \tag{3.5}$$

By using Eqs (2.3) and (3.5), we derive

$$\lambda_2 = \frac{c}{\nu^2\kappa_1} + \frac{\nu'}{\nu^2\kappa_1}\lambda_1. \tag{3.6}$$

After differentiating Eq (3.5) and using Eq (3.6), the resulting system of differential equations is as follows:

$$\lambda_1' + \frac{c}{\nu} + \frac{\nu'}{\nu}\lambda_1 = 0, \tag{3.7}$$

$$\nu\kappa_1\lambda_1 + c\left(\frac{1}{\nu^2\kappa_1}\right)' + \left(\frac{\nu'}{\nu^2\kappa_1}\right)'\lambda_1 + \frac{\nu'}{\nu^2\kappa_1}\lambda_1' - \nu\kappa_2\lambda_3 = 0, \tag{3.8}$$

$$\lambda_3' + \frac{c\,\kappa_2}{\nu\kappa_1} + \frac{\nu'\kappa_2}{\nu\kappa_1}\lambda_1 = 0. \tag{3.9}$$

By substituting Eq (3.7) in Eq (3.8), we obtain

$$f\lambda_1 + \frac{c}{\nu\kappa_2}\left(\left(\frac{1}{\nu^2\kappa_1}\right)' - \frac{\nu'}{\nu^3\kappa_1}\right) - \lambda_3 = 0 \tag{3.10}$$

where

$$f = \frac{\kappa_1}{\kappa_2} + \frac{1}{\nu\kappa_2}\left(\left(\frac{\nu'}{\nu^2\kappa_1}\right)' - \frac{(\nu')^2}{\nu^3\kappa_1}\right).$$

By differentiating Eq (3.10) and utilizing Eqs (3.7) and (3.9), we get the following result.

$$\lambda_1 = \frac{\left(\frac{c}{\nu\kappa_2}\left(\frac{\nu'}{\nu^3\kappa_1}\right)\right)' - \frac{c}{\nu}\left(\frac{\kappa_2}{\kappa_1} - f\right) - \left(\frac{1}{\nu\kappa_2}\left(\frac{c}{\nu^2\kappa_1}\right)'\right)'}{f' + \frac{\nu'}{\nu}\left(\frac{\kappa_2}{\kappa_1} - f\right)}. \tag{3.11}$$

Clearly, from Eq (3.10), we obtain

$$\lambda_3 = f\lambda_1 + \frac{c}{\epsilon\nu\kappa_2}\left(\left(\frac{1}{\nu^2\kappa_1}\right)' - \frac{\nu'}{\nu^3\kappa_1}\right). \tag{3.12}$$

Therefore, employing Eqs (3.6), (3.11), and (3.12), it becomes evident that there exist differentiable functions $\eta_j = \frac{1}{c}\lambda_j$ that satisfy (3.1) for $j = 1, 2, 3$.

Conversely, assume that $\beta(t)$ is a timelike curve with Frenet apparatus $\{V_1, V_2, V_3, \kappa_1, \kappa_2\}$. Then, there is a unit fixed direction W given by (3.5) where the differentiable functions $\eta_i = \frac{1}{c}\lambda_j$ for $i = 1, 2, 3$ are given by (3.2)–(3.4). So, it is easily seen that the scalar product of unit fixed direction W and acceleration vector $\mathfrak{a}$ as given in Eq (2.3) of $\beta(t)$ is equal to c, which is a nonzero constant. Hence, $\beta(t)$ is a timelike acc-slant curve with non-null axis W. □

Now, it is straightforward to derive the following corollaries from Theorem 1.

Corollary 2. *It is easy to see that W does not exist when $c = 0$ and v is a non-constant function.*

Corollary 3. *Let $\beta(t)$ be a non-helical unit speed timelike curve with curvatures κ_1, κ_2 in $\mathbb{E}_1^3$. Then, $\beta(t)$ is a unit speed timelike acc-slant curve if, and only if,*

$$-\left(\frac{\left(\frac{\kappa_2}{\kappa_1}\right)\varsigma}{\left(\frac{\kappa_2}{\kappa_1}\right)'}\right)^2 + \frac{1}{{\kappa_1}^2} + \left(\frac{1}{\kappa_2}\left(\frac{1}{\kappa_1}\right)' + \frac{\varsigma}{\left(\frac{\kappa_2}{\kappa_1}\right)'}\right)^2 = \pm\frac{1}{c^2}$$

where $\varsigma = -1 + \left(\frac{\kappa_2}{\kappa_1}\right)^2 + \frac{\kappa_2}{\kappa_1}\left(\frac{1}{\kappa_2}\left(\frac{1}{\kappa_1}\right)'\right)'$ and c is a nonzero constant.

Corollary 4. *Let $\beta(t)$ be a unit speed timelike acc-slant curve with curvatures $\kappa_1 = 1$ and κ_2 in $\mathbb{E}_1^3$. Then,*

$$\kappa_2(t) = \pm\frac{\sqrt{\frac{c^2}{\varepsilon - c^2}}\, t}{\sqrt{1 + \frac{c^2}{\varepsilon - c^2}t^2}}$$

where $\varepsilon = \pm 1$ and c is a nonzero constant.

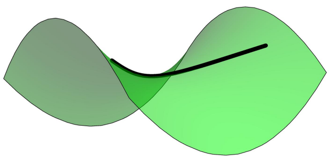
Example 1. *For $t \in (-0.7, 0.7)$, the curve*

$$\beta(t) = \left(t\cosh t, \frac{t^2}{2}, t\sinh t\right)$$

is a timelike curve in $\mathbb{E}_1^3$ for $g(\beta'(t), \beta'(t)) < 0$. Also, the curve $\beta(t)$ entirely lies on the surface $x^2 - z^2 = 2y$, and the acceleration vector of $\beta(t)$ is

$$\mathfrak{a}(t) = (2\sinh t + t\cosh t, 1, t\sinh t + 2\cosh t)$$

in $\mathbb{E}_1^3$. It is straightforward to calculate that $g(\mathfrak{a}, W) = 1$ for the spacelike axis $W = (0, 1, 0)$; therefore, β is a timelike acc-slant curve (see Figure 1).



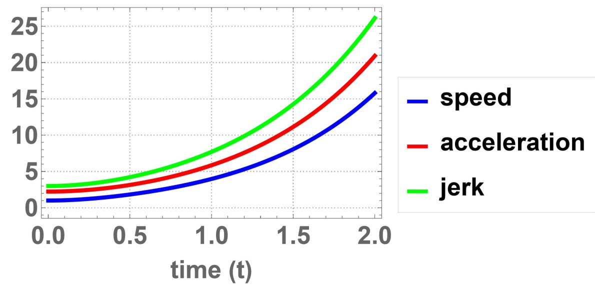


Figure 1. A timelike acc-slant curve lies on the surface $x^2 - z^2 = 2y$.

Lemma 5. *Let $c = 0$ and ν be a constant function. Thus, $\beta(t)$ is a timelike acc-slant curve if, and only if,*

$$-\lambda_1^2 + \lambda_3^2 = \varepsilon, \quad \lambda_2 = 0, \tag{3.13}$$

where λ_1, λ_3 are nonzero constants.

Proof. Assume that $\beta(t)$ is a timelike acc-slant curve with a non-null axis W. According to Definition 1, there exist a constant $g(\mathfrak{a}, W) = c$ along with differentiable functions λ_i such that

$$W = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3. \tag{3.14}$$

Assuming that $c = 0$ and ν is a constant function, we get $\lambda_2 = 0$ from Eq (2.3). Later, differentiating Eq (3.14) and using the fact that $\lambda_2 = 0$, we obtain the following differential equations:

$$\lambda_1' = 0, \tag{3.15}$$

$$\nu\kappa_1\lambda_1 - \nu\kappa_2\lambda_3 = 0, \tag{3.16}$$

$$\lambda_3' = 0. \tag{3.17}$$

It is easy to see that Eq (3.13) holds, since W is a unit fixed direction.

Contrarily, let $\beta(t)$ be a timelike curve with Frenet apparatus $\{V_1, V_2, V_3, \kappa_1, \kappa_2\}$. Assume that there exists a unit fixed direction W which is given by Eq (3.14) where differentiable functions λ_i are given by Eq (3.13). So, it is obvious that $g(\mathfrak{a}, W) = 0$ when ν is a constant function. □

Corollary 6. *[25] Let $\beta(t)$ be a timelike Lorentzian spherical curve in 3 dimensional Minkowski space. So, the following expressions are equivalent:*

1. $\left(\frac{1}{\kappa_1}\right)^2 + \left(\frac{1}{\nu\kappa_2}\left(\frac{1}{\kappa_1}\right)'\right)^2 = r^2$
2. $-\frac{\kappa_2}{\kappa_1} = \frac{1}{\nu}\left(\frac{1}{\nu\kappa_2}\left(\frac{1}{\kappa_1}\right)'\right)'$

where $r \in \mathbb{R}^+$.

Theorem 7. *Let $\beta(t)$ be a unit speed non-helical timelike Lorentzian spherical curve in $\mathbb{S}_1^2$. So, $\beta(t)$ is a timelike acc-slant curve if, and only if,*

$$\left(\frac{1}{\left(\frac{\kappa_2}{\kappa_1}\right)'}\right)^2 \left(1 - \left(\frac{\kappa_2}{\kappa_1}\right)^2\right) - 2\frac{\frac{1}{\kappa_2}\left(\frac{1}{\kappa_1}\right)'}{\left(\frac{\kappa_2}{\kappa_1}\right)'} = \frac{\varepsilon - c^2 r^2}{c^2} \tag{3.18}$$

where $r \in \mathbb{R}^+$, c is a nonzero constant, and $\varepsilon = \pm 1$.

Proof. Suppose $\beta(t)$ is a unit speed non-helical timelike Lorentzian spherical curve in $\mathbb{S}_1^2$ that satisfies Corollary 6. If $\beta(t)$ is also a timelike acc-slant curve with a nonzero constant $c = g(\mathfrak{a}, W)$, then, by employing Corollary 6 in conjunction with Eqs (3.2)–(3.4), there exist differentiable functions λ_i such that:

$$W = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3,$$

where

$$\begin{aligned}\lambda_1 &= -c\frac{\frac{\kappa_2}{\kappa_1}}{\left(\frac{\kappa_2}{\kappa_1}\right)'},\\ \lambda_2 &= c\frac{1}{\kappa_1},\\ \lambda_3 &= c\left(\frac{1}{\kappa_2}\left(\frac{1}{\kappa_1}\right)' - \frac{1}{\left(\frac{\kappa_2}{\kappa_1}\right)'}\right).\end{aligned}$$

Since W is a unit fixed direction, and by using Corollary 6, by straightforward calculations, we obtain that Eq (3.18). Conversely, the proof is obvious. □

Example 2. *For $t \in (-1.5, 1.5)$, the curve*

$$\beta(t) = \left(\sqrt{1-\frac{t^4}{16}}\sinh t, \sqrt{1-\frac{t^4}{16}}\cosh t, \frac{t^2}{4}\right)$$

is a timelike curve in $\mathbb{E}_1^3$ since $g(\beta(t), \beta(t)) < 0$. Also, the curve β entirely lies on the Lorentzian sphere and the acceleration vector of $\beta(t)$ is

$$\mathfrak{a}(t) = \begin{pmatrix} \sqrt{1-\frac{t^4}{16}}\sinh t - \dfrac{t^6\sinh t}{64\left(1-\frac{t^4}{16}\right)^{3/2}} + \dfrac{t^3\cosh t}{4\sqrt{1-\frac{t^4}{16}}} - \dfrac{3t^2\sinh t}{8\sqrt{1-\frac{t^4}{16}}}, \\ \sqrt{1-\frac{t^4}{16}}\cosh t - \dfrac{t^6\cosh t}{64\left(1-\frac{t^4}{16}\right)^{3/2}} - \dfrac{t^3\sinh t}{4\sqrt{1-\frac{t^4}{16}}} - \dfrac{3t^2\cosh t}{8\sqrt{1-\frac{t^4}{16}}}, \\ \dfrac{1}{2} \end{pmatrix}$$

in $\mathbb{E}_1^3$. It is straightforward to calculate that $g(\mathfrak{a}, W) = \frac{1}{2}$ for the spacelike axis $W = (0, 0, 1)$; therefore, β is a timelike acc-slant curve (see Figure 2).

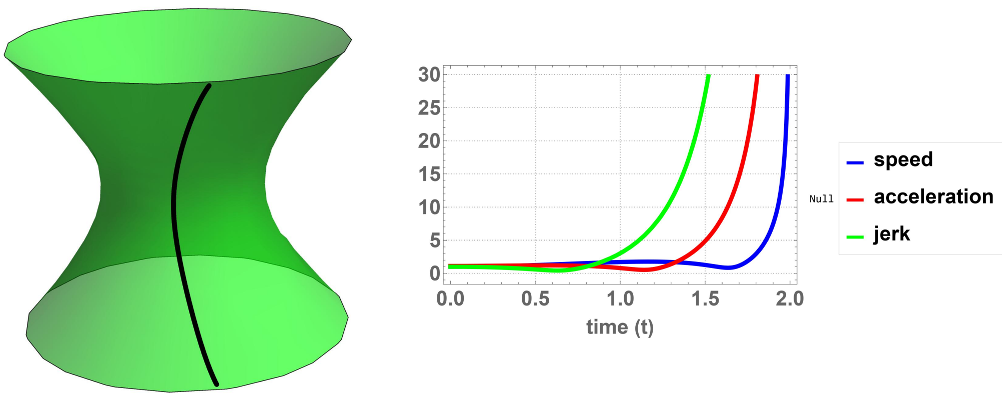


Figure 2. A Lorentzian spherical timelike acc-slant helix.

Now, we obtain easily the following corollary for timelike acc-slant curves from Theorem 1*b*.

Corollary 8. *The curve $\beta(t)$ is a timelike acc-slant curve with the spacelike or timelike axis W, which satisfies Eq* (3.13) *in Theorem 1 if, and only if, $\beta(t)$ is a helix with the same axis W.*

Proof. The proof is clear from (3.13). □

Theorem 9. *Let $\beta(t)$ be a timelike helix with non-null axis W in $\mathbb{E}_1^3$. In that case, $\beta(t)$ qualifies as a timelike acc-slant curve with the non-null axis W if, and only if, the magnitude of the velocity, $|\mathfrak{v}| = |\beta'(t)| = \nu$, is a linear function with respect to the parameter t of $\beta(t)$.*

Proof. Assume that $\beta(t)$ is a timelike helix with non-null axis W and Frenet apparatus $\{V_1, V_2, V_3, \kappa_1, \kappa_2\}$ in $\mathbb{E}_1^3$. So, there exists a constant c_1 such that

$$g\left(V_1, W\right) = c_1. \tag{3.19}$$

Differentiating equation (3.19) and using the Frenet equations (2.1), we have

$$g\left(V_2, W\right) = 0. \tag{3.20}$$

Now, we assume that $\beta(t)$ is a timelike acc-slant curve with the same axis W as given by Eq (3.5). Then, using Eqs (3.19) and (3.20), we have $\lambda_1 = -c_1$. By using Eq (3.6), we obtain

$$\lambda_2 = \frac{1}{\nu^2\kappa_1}\left(c - \nu' c_1\right) = 0. \tag{3.21}$$

Therefore, we conclude that ν' is a constant.

Contrarily, we suppose that $\beta(t)$ be a timelike helix in $\mathbb{E}_1^3$ with non-null axis W, and ν is a linear function with respect to the parameter t of $\beta(t)$. Hence, a constant c_1 exists such that $W = c_1 V_1 + \sqrt{1 - c_1^2} V_3$. Therefore, using Eq (2.3), we have

$$g\left(\mathfrak{a}, W\right) = \nu' c_1 = const. \tag{3.22}$$

As a result, $\beta(t)$ is a timelike acc-slant curve with the non-null axis W. □

Example 3. *[26] For all $t \in \mathbb{R}$, the curve*

$$\beta(t) = \left(\sqrt{\frac{3}{2}}\left(t^2 + 1\right),\ \sqrt{6}(\sin t - t\cos t),\ \sqrt{5}(t\sin t + \cos t) \right)$$

is a timelike acc-slant helix in $\mathbb{E}_1^3$ since $g\left(\beta'(t), \beta'(t)\right) = -t^2\cos^2 t < 0$. Also, the curve β entirely lies on the surface $\left(\frac{2y}{\sqrt{5}}\right)^2 + z^2 = \frac{48x}{3\sqrt{15}}$ with the acceleration vector

$$\mathfrak{a}(t) = \left(\sqrt{6},\ \sqrt{6}(\sin t + t\cos t),\ \sqrt{5}(\cos t - t\sin t) \right)$$

in $\mathbb{E}_1^3$. It is straightforward to calculate that $g\left(\mathfrak{a}, W\right) = -\sqrt{6}$ for the timelike axis $W = (1, 0, 0)$; therefore, $\beta(t)$ is a timelike acc-slant curve (see Figure 3). Also, the tangent vector field associated with $\beta(t)$ is

$$V_1 = \left(\sqrt{6}\sec t,\ \sqrt{6}\tan t,\ \sqrt{5} \right)$$

then $\beta(t)$ *is the timelike helix since* $g(V_1,(0,0,1)) = \sqrt{5}$, *and* $\frac{\kappa_2}{\kappa_1} = -\sqrt{\frac{5}{6}}$. *Morever, Theorem 9 holds since* $|\mathfrak{v}(t)| = t\cos t$ *is a linear function.*

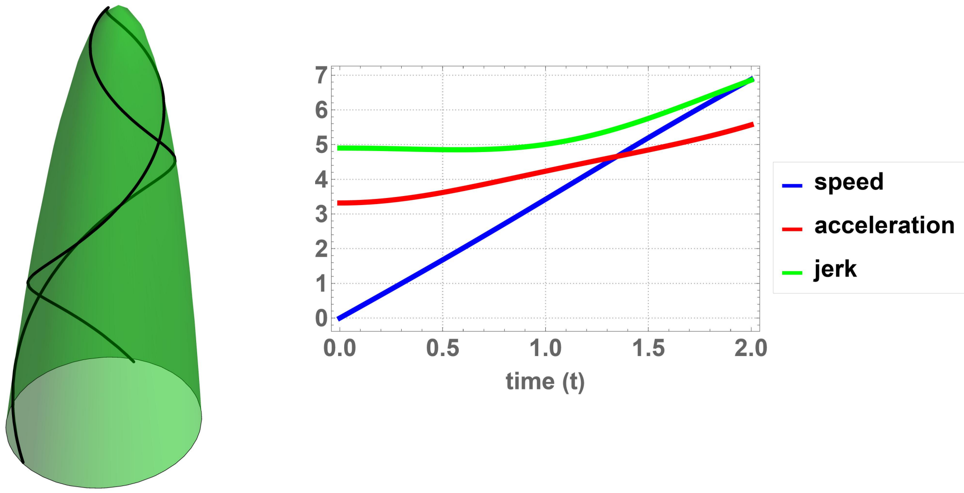


Figure 3. A timelike helix acc-slant curve in $\mathbb{E}_1^3$.

Corollary 10. *Let* $\beta(t)$ *be a unit speed timelike curve with a constant curvature and constant magnitude of acceleration in* $\mathbb{E}_1^3$. *Under this condition,* $\beta(t)$ *is a timelike acc-slant curve if, and only if,* $\beta(t)$ *is a slant helix.*

Proof. Assume that the curvature $\kappa_1 = \kappa_0$ is a nonzero constant of a unit speed curve β. So, acceleration vector $\mathfrak{a}$ is equal to $\kappa_0 V_2$. Therefore, the proof is straightforward. □

Lemma 11. *[13] Let* $\beta(t)$ *be a unit speed timelike curve with* $\kappa_1 = 1$. *Then, the normal vector field* V_2 *of* $\beta(t)$ *makes a constant hyperbolic angle* ψ, *with a fixed straight line in* $\mathbb{E}_1^3$ *if, and only if,* $\kappa_2(s) = \pm\frac{s}{\sqrt{\epsilon(s^2-\tanh^2\psi)}}$.

Corollary 12. *Let* $\beta(t)$ *be a unit speed timelike curve with* $\kappa_1 = 1$. *In that case,* β *is a timelike acc-slant curve if, and only if,* β *is a timelike Salkowski curve.*

Proof. Suppose that β is a unit speed timelike acc-slant curve with $\kappa_1 = 1$. So, by using Corollary 10 and Lemma 11, β is a timelike Salkowski curve. Contrarily, β is a unit speed timelike Salkowski curve with $\kappa_1 = 1$. So, β is a timelike acc-slant curve by Corollary 3 and Lemma 11. □

Example 4. *For all* $t \in \mathbb{R}$, *the curve*

$$\beta(t) = \left(\frac{\left(3 - 2\sqrt{3}t\right)^{5/2}}{45\sqrt{2}}, \frac{\left(-2\sqrt{3}t - 3\right)^{5/2}}{45\sqrt{2}}, \frac{3}{8} - t^2 \right)$$

is a unit speed timelike curve in $\mathbb{E}_1^3$ *for* $g(\beta'(t), \beta'(t)) = -1 < 0$. *Also, the curve* β *entirely lies on the* *surface* $-\frac{x^2}{2}+\frac{y^2}{2}+\left(\frac{z-\frac{9}{8}}{\frac{\sqrt{15}}{2\sqrt{2}}}\right)^2=\frac{6}{25}$ *with the acceleration vector*

$$\mathfrak{a}(t)=\left(\frac{\sqrt{3-2\sqrt{3}t}}{\sqrt{2}},\frac{\sqrt{-2\sqrt{3}t-3}}{\sqrt{2}},-2\right)$$

in $\mathbb{E}_1^3$. *It is straightforward to calculate that* $g(\mathfrak{a},W)=-2$ *for the spacelike axis* $W=(0,0,1)$*; therefore,* $\beta(t)$ *is a timelike acc-slant curve (see Figure 4). Morever,* $g(V_2,W)=-2$*, so* β *is also a timelike slant helix. Besides,* β *is a Salkowski curve, and Corollary 10 is held as* $\kappa_1=1$*. Also, Corollary 4 holds* $\kappa_2(t)=-\frac{2t}{\sqrt{-\frac{2t}{\sqrt{3}}-1}\sqrt{3-2\sqrt{3}t}}$.

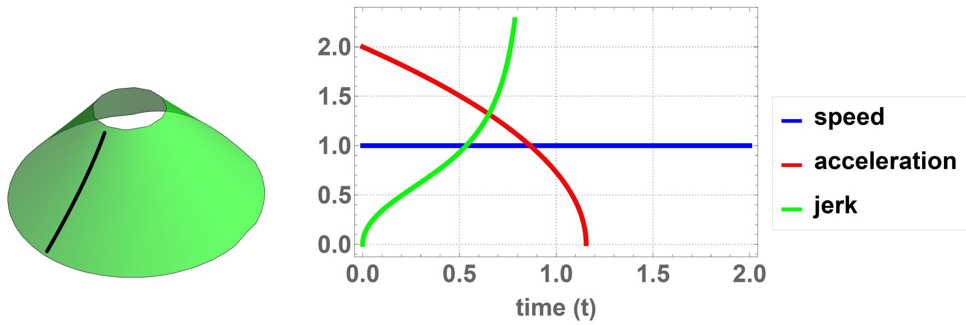


Figure 4. A timelike Salkowski acc-slant curve in $\mathbb{E}_1^3$.

4. Conclusions

In this study, we explored the concept of the timelike acc-slant helix, emphasizing its curvature, torsion, and fundamental role in differential geometry in Minkowski 3-space. Future research could focus on the extension of timelike acc-slant helices to higher-dimensional spaces. Also, curves that give a constant inner product of the jerk vector and a fixed direction vector can be studied, investigating their implications in advanced motion planning and material science in Euclidean and Minkowski spaces.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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