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Research article

Optimal polynomial stability of the Timoshenko system with single fractional boundary dissipation

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Abstract: This study investigates Timoshenko systems with a single boundary condition involving fractional dissipation. Utilizing semigroup theory, we establish the existence and uniqueness of solutions. Our findings indicate that although the system demonstrates strong stability, it does not attain uniform stability. As a result, we derive the optimal polynomial decay rate of the system.

Keywords: fractional derivatives; differential equations; fractional differential equations; Timoshenko systems; semi-group
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1. Introduction

In this research, we examine the stabilization and well-posedness of a one-dimensional Timoshenko system expressed in the following form:

$$\begin{cases} \rho_1\varphi_{tt} - d_1(\varphi_x + \psi)_x = 0, & (x,t) \in (0,\ell) \times (0,+\infty), \\ \rho_2\psi_{tt} - d_2\psi_{xx} + d_1(\varphi_x + \psi) = 0, & (x,t) \in (0,\ell) \times (0,+\infty). \end{cases} \tag{1.1}$$

The initial conditions are

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,\ell), \\ \psi(x,0) &= \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,\ell), \end{aligned} \tag{1.2}$$

and the boundary conditions are given as follows:

$$\begin{aligned} &\varphi(0,t) = \psi(0,t) = \psi(\ell,t) = 0, \quad \text{in } (0,+\infty), \\ &(\varphi_x + \psi)(\ell,t) = -\gamma\partial_t^{\alpha,\nu}\varphi(\ell,t), \quad \text{in } (0,+\infty), \end{aligned} \tag{1.3}$$

where $\varphi = \varphi(x,t)$ denotes the rotation angle of beam cross-sections, $\psi = \psi(x,t)$ represents the transverse displacement of the beam, and the system parameters are defined as follows:

- ρ_1, ρ_2: mass densities per unit length,
- d_1, d_2: stiffness coefficients,
- γ: damping coefficient,
- ν: non-negative tempering parameter,
- ℓ: length of the beam ($x \in (0,\ell)$),
- $\alpha \in (0,1)$: fractional order of differentiation.

The notation $\partial_t^{\alpha,\nu}$ denotes the Caputo tempered fractional derivative of order α (where $0 < \alpha < 1$) with respect to time t. It is defined as:

$$\partial_t^{\alpha,\nu}\vartheta(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} e^{-\nu(t-s)} \frac{d\vartheta}{ds}(s)\, ds, \quad \nu \geq 0.$$

Compared to standard fractional derivatives ($\nu = 0$), the tempered version offers two key advantages for boundary dissipation. First, exponential tempering ($\nu > 0$) provides better physical realizability by matching observed damping behaviors in viscoelastic materials that exhibit power-law decay with an exponential cutoff. Second, the tempering ensures mathematical flexibility by guaranteeing that the dissipation operator generates an analytic semigroup, which enables our spectral analysis approach while maintaining causality. These properties make tempered derivatives suitable for boundary control problems where standard fractional derivatives face modeling and analytical limitations.

This study introduces a novel extension of the classical Timoshenko beam model—a foundational framework in structural dynamics—by integrating fractional-order derivatives into its boundary conditions. This adaptation resolves critical constraints of classical integer-order calculus approaches, which frequently struggle to capture the behavior of systems influenced by time-dependent historical interactions and non-uniform energy attenuation mechanisms. Fractional calculus, by contrast, inherently encodes historical state interactions and delayed energy transfer mechanisms, thereby enabling a more precise characterization of memory-driven phenomena. These include viscoelastic relaxation in composite materials, hysteresis in microstructured alloys, and anomalous stress propagation in metamaterials—scenarios where traditional models oversimplify time-dependent hereditary properties. Integrating fractional operators into the boundary conditions of the Timoshenko system is regarded as a methodological advancement since earlier implementations of fractional calculus in structural dynamics had been primarily limited to governing equations, with dissipative boundary effects remaining unexplored. This approach allows for an in-depth analysis of how non-local dissipation affects beam-like structures' stability and vibrational characteristics under dynamic loading conditions. By bridging gaps between fractional calculus theory and structural mechanics, this research advances predictive modeling capabilities for complex engineering systems characterized by memory effects and multiscale dissipative behavior. Fractional calculus has evolved into a well-established mathematical framework with a solid theoretical base. Its practical applications have attracted considerable attention in numerous fields of research, including signal processing, chemical processes, electrical circuits, bioengineering, viscoelasticity, and control systems (see [12]). Fractional-order control holds significance in both theoretical and practical contexts. It refines conventional integer-order control theory, allowing for more precise modeling and improving control performance. Experimental studies indicate that standard Newtonian mechanics cannot adequately describe many natural events. For example, viscoelastic materials showcase a response dictated by their microstructural composition, demonstrating characteristics of both elastic solids and viscous fluids.

Prior studies (refer to [11]) have demonstrated that the fractional derivative ∂_t^α leads to dissipation in the system, ensuring convergence towards an equilibrium state. As a result, when utilized at the boundaries, fractional derivatives can act as regulators to reduce or eliminate undesirable vibrations. In [10], B. Mbodje investigated the asymptotic behavior of solutions for the system:

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = 0, \\ \partial_x u(1,t) = -k\partial_t^{\alpha,\eta} u(1,t), \quad \alpha \in (0,1), \eta \geq 0,\ k > 0, \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = v_0(x). \end{cases}$$

He established strong asymptotic stability for the solutions when $\eta = 0$ and demonstrated a polynomial decay rate of t^{-1} as time tends to infinity when $\eta \neq 0$. This polynomial decay rate was derived by using the energy method.

Kim and Renardy [8] studied (1.1) with two boundary controls given in the following form:

$$K(\varphi_x + \psi)(L) = -\gamma\partial_t^\alpha \varphi(L), \quad b\psi_x(L) = -\gamma_2 \partial_t^\alpha \psi(L), \quad \text{for, } t \in (0,+\infty).$$

They utilized multiplier techniques to establish an exponential decay result for the natural energy of (1.1). Furthermore, Yan [14] proved a polynomial decay rate when analyzing two boundary frictional damping terms exhibiting polynomial growth near the origin.

Benaissa and Benazzouz [6] explored the stabilization of a Timoshenko system incorporating two dynamic boundary control conditions that involve fractional derivatives:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x + \psi)_x &= 0, \quad (x,t) \in (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) &= 0, \quad (x,t) \in (0,L) \times (0,+\infty). \end{cases}$$

The system is governed by the following boundary conditions:

$$\begin{cases} m_1\phi_{tt}(L,t) + K(\phi_x + \psi)(L,t) &= -\gamma_1 \partial_t^{\alpha,\eta}\phi(L,t) \text{ for } t \in (0,+\infty), \\ m_2\psi_{tt}(L,t) + b\psi_x(L,t) &= -\gamma_2 \partial_t^{\alpha,\eta}\psi(L,t) \text{ for } t \in (0,+\infty), \\ \phi(0,t) = 0, \quad \psi(0,t) &= 0 \text{ for } t \in (0,+\infty), \end{cases}$$

where m_1 and m_2 are positive constants. Using the spectrum method, they showed that the system (1) does not exhibit uniform stability. However, polynomial stability was proven through semigroup theory and by applying a result from Borichev and Tomilov.

M. Akil et al. [2] analyzed the Timoshenko system incorporating a single fractional derivative, which is described by:

$$\begin{cases} au_{tt} - (u_x + y)_x &= 0, \quad (x,t) \in (0,1) \times (0,+\infty), \\ by_{tt} - y_{xx} + c(u_x + y) &= 0, \quad (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.4}$$

where a, b, and c are positive constants. The system is governed by the following boundary conditions:

$$\begin{cases} u_x(1,t) + y(1,t) + \gamma\partial_t^{\alpha,\eta} u(1,t) &= 0, \quad t \in \mathbb{R}^+, \\ u(0,t) = y_x(0,t) = y_x(1,t) &= 0. \end{cases} \tag{1.5}$$

They established that the energy of the system (1.4)-(1.5) exhibits polynomial decay over time.

Our work advances prior results in several key directions: While [6] proved polynomial stability for a Timoshenko system with two dynamic fractional boundary controls, and [10] obtained a t^{-1} decay rate for the wave equation under single boundary control ($\eta \neq 0$), we demonstrate optimal $t^{-1/3}$ decay with minimal dissipation (single boundary condition) and fully characterize stability through conditions (C1)–(C3). Unlike [2]'s framework requiring restrictive Neumann null conditions, our tempered fractional approach eliminates such constraints. These developments build upon extensive research in fractional differential equations, including contributions from [1, 5, 13], which collectively establish the mathematical foundations for analyzing energy decay rates in dissipative systems. The key technical innovations of this work include (1) development of a new semigroup framework for the tempered fractional boundary condition, (2) proof of optimal polynomial decay rates through spectral analysis, and (3) identification of critical stability conditions (C1)–(C3) that precisely characterize the dissipation mechanism.

This article is structured as follows: Section 2 focuses on establishing the well-posedness of the system (1.1) with boundary conditions (1.3) using semigroup theory. Section 3 is concerned with analyzing the asymptotic behavior of the system (1.1)–(1.3) and establishing its strong stability. This is achieved by examining the spectrum of the associated C_o-semigroup and applying the Arendt–Batty criterion to demonstrate the decay of energy over time. In the last section, we focus on proving the polynomial stability of the Timoshenko system.

2. Augmented model and well-posedness of the system

This section aims to reformulate the model (1.1) into an augmented system. To move forward, we must first obtain the following theorem:

Theorem 2.1. *[10] Let H be the function:*

$$H(r) = |r|^{(2\alpha-1)/2}, \quad r \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Let us consider the system described by the equation

$$\partial_t \varphi(r,t) + \left(|r|^2 + \nu\right) \varphi(r,t) - \mathscr{X}(t) H(r) = 0, \quad r \in \mathbb{R},\ \nu \geq 0,\ t > 0,$$

with the initial condition

$$\varphi(r,0) = 0,$$

and the output defined as

$$\mathscr{Y}(t) = \frac{\sin(\alpha\pi)}{\pi} \int_{\mathbb{R}} H(r)\varphi(r,t)\,dr.$$

Now,

$$\mathscr{Y}(t) = I^{1-\alpha,\nu} \mathscr{X}(t) = D^{\alpha,\nu} \mathscr{X}(t),$$

represents the relationship between $\mathscr{X}$ (input) and $\mathscr{Y}$ (output). where

$$[I^{\alpha,\nu}\vartheta](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\nu(t-\tau)} \vartheta(\tau)\,d\tau.$$

By applying the previous theorem, system (1.1) can be reformulated as the following augmented model:

$$\begin{cases} \rho_1\varphi_{tt} - d_1(\varphi_x + \psi)_x = 0, \quad (x,t) \in (0,\ell) \times (0,+\infty), \\ \rho_2\psi_{tt} - d_2\psi_{xx} + d_1(\varphi_x + \psi) = 0, \quad (x,t) \in (0,\ell) \times (0,+\infty), \\ \partial_t\eta\,(r,t) + \left(r^2 + \nu\right)\eta\,(r,t) - H\,(r)\,\partial_t\varphi(\ell,t) = 0, \quad t \in (0,+\infty), \quad r \in \mathbb{R}, \\ (\varphi_x + \psi)(\ell) = -\gamma\dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_{\mathbb{R}} H\,(r)\,\eta\,(r,t)\,dr, \quad t \in (0,+\infty), \end{cases} \tag{2.1}$$

with the following initial conditions:

$$\begin{aligned} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,\ell), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,\ell), \\ \varphi\,(r,0) = 0, \quad r \in \mathbb{R}, \end{aligned}$$

and the following boundary conditions:

$$\varphi(0,t) = \psi(0,t) = \psi(\ell,t) = 0, \quad \text{in } (0,+\infty). \tag{2.2}$$

Let $U = (\varphi, \varphi_t, \psi, \psi_t, \eta)$ be a solution of (2.1), and let $E(t)$ be the energy defined as follows:

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{2.3}$$

where

$$\begin{aligned} \|U\|^2_{\mathcal{H}} &= \frac{1}{2}\int_0^\ell \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + d_1|\varphi_x + \psi|^2 + d_2|\psi_x|^2\right)dx \\ &+ \frac{\zeta}{2}\int_{\mathbb{R}} |\eta|^2 dr, \end{aligned}$$

with constant $\zeta = \gamma d_1 \frac{\sin(\alpha\pi)}{\pi}$. Solutions will be sought in $\mathcal{H}$, with precise regularity requirements for initial data specified in Theorem 2.2 via the domain $D(\mathcal{A})$ defined in (2.6).

Lemma 2.1. *Let U be a regular solution to* (2.1)*, where $U = (\varphi, \varphi_t, \psi, \psi_t, \eta)$. Then, the energy functional given in* (2.3) *satisfies the expression:*

$$\frac{d}{dt}E(t) = -\zeta\int_{\mathbb{R}} \left(r^2 + \nu\right)|\eta(r,t)|^2\,dr.$$

Proof. Multiplying Eqs $(2.1)_1$ and $(2.1)_2$ by φ_t and ψ_t respectively, performing integration by parts over $(0,\ell)$, and then adding the resulting equations, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^\ell \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + d_1|\varphi_x + \psi|^2 + d_2|\psi_x|^2\right)dx\right) - d_1(\varphi_x + \psi)(\ell)\varphi_t(\ell) = 0. \tag{2.4}$$

Multiplying Eq $(2.1)_3$ by $\zeta\eta$, integrating over $\mathbb{R}$, and summing the resulting equations yields;

$$\frac{1}{2}\frac{d}{dt}\left(\zeta\int_{\mathbb{R}} |\eta|^2 dr\right) + \zeta\int_{\mathbb{R}} (r^2 + \nu)|\eta(r,t)|^2\,dr - \zeta\varphi_t(\ell)\int_{\mathbb{R}} H(r)\eta(r,t)\,dr = 0. \tag{2.5}$$

By combining (2.4) and (2.5), we get

$$\frac{d}{dt}E(t) = -\zeta\int_{\mathbb{R}} \left(r^2 + \nu\right)|\eta(r,t)|^2\,dr.$$

This completes the proof of the lemma. □

We now examine the well-posedness of (2.1). To do so, we introduce the following Hilbert space, referred to as the energy space.

$$\mathcal{H} = H_L^1(0,\ell) \times L^2(0,\ell) \times H_0^1(0,\ell) \times L^2(0,\ell) \times L^2(\mathbb{R}),$$

where $H_L^1(0,\ell)$ is defined as:

$$H_L^1(0,\ell) = \{\varphi \in H^1(0,\ell) \mid \varphi(0) = 0\}.$$

For $U = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \eta)^T$ and $\tilde{U} = (\tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3, \tilde{\Phi}_4, \tilde{\eta})^T$, we introduce the inner product in $\mathcal{H}$ in the following manner

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} &= \int_0^\ell \left(\rho_1 \Phi_2 \overline{\tilde{\Phi}_2} + \rho_2 \Phi_4 \overline{\tilde{\Phi}_4} \right) dx + \int_0^\ell d_1 (\Phi_{1x} + \Phi_3) \overline{(\tilde{\Phi}_{1x} + \tilde{\Phi}_3)} dx \\ &+ \int_0^\ell d_2 \Phi_{3x} \overline{\tilde{\Phi}_{3x}} dx + \zeta \int_{\mathbb{R}} \eta \overline{\tilde{\eta}} \, dr. \end{aligned}$$

We transform the system given by (2.1) into a semigroup framework. By introducing the vector function $U = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \eta)^T$, we express the system (2.1) in the form

$$\begin{cases} U' = \mathcal{A} U, \quad t > 0, \\ U(0) = U_0, \end{cases}$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \eta^0)^T$.

$\mathcal{A}$ is a linear operator defined as

$$\mathcal{A} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \eta \end{pmatrix} = \begin{pmatrix} \Phi_2 \\ \frac{d_1}{\rho_1} (\Phi_{1x} + \Phi_3)_x \\ \Phi_4 \\ \frac{d_2}{\rho_2} \Phi_{3xx} - \frac{d_1}{\rho_2} (\Phi_{1x} + \Phi_3) \\ -(r^2 + \nu) \eta + \Phi_2(\ell) H(r) \end{pmatrix}.$$

The domain of $\mathcal{A}$ is expressed as

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \eta)^T \in \mathcal{H} : \Phi_1, \in H^2 \cap H_L^1, \; \Phi_3 \in H^2 \cap H_0^1, \\ r\eta \in L^2(\mathbb{R}), -\left(r^2 + \nu\right) \eta + \Phi_2(\ell) H(r) \in L^2(\mathbb{R}), \\ (\Phi_{1x} + \Phi_3)(\ell) = -\gamma \dfrac{\sin(\alpha\pi)}{\pi} \displaystyle\int_{\mathbb{R}} H(r) \, \eta(r,t) \, dr. \end{array} \right\}. \tag{2.6}$$

We present the following theorem concerning existence and uniqueness:

Theorem 2.2. *Let U_0 be the initial data for the system* (2.1). *Then:*

(1) If $U_0 \in D(\mathcal{A})$, the system (2.1) *admits a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, the system (2.1) *has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Remark 2.1. *It is important to highlight that strong solutions satisfy the differential equation pointwise and demand higher regularity, while weak solutions are defined in an integral sense with fewer regularity requirements.*

Proof. To begin with, we establish that the operator $\mathcal{A}$ is dissipative.

Whether $U = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \eta) \in D(\mathcal{A})$, we get

$$\mathcal{R}e\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \;=\; -\zeta \int_{\mathbb{R}} \left(r^2+\nu\right)|\eta(r,t)|^2 dr \le 0. \tag{2.7}$$

Therefore, $\mathcal{A}$ is dissipative.

We now aim to show that the operator $I - \mathcal{A}$ is surjective. Given $Q = (q_1, q_2, q_3, q_4, q_5) \in \mathcal{H}$ we prove that there exists $U = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \eta) \in D(\mathcal{A})$ satisfying

$$(I - \mathcal{A})\, U = Q.$$

That is,

$$\begin{cases} \Phi_1 - \Phi_2 & = & q_1, \\ \Phi_2 - \frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x & = & q_2, \\ \Phi_3 - \Phi_4 & = & q_3, \\ \Phi_4 - \frac{d_2}{\rho_2}\Phi_{3xx} + \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) & = & q_4, \\ \eta(1 + r^2 + \nu) - H(r)\Phi_2(\ell, t) & = & q_5. \end{cases} \tag{2.8}$$

Then, $(2.8)_1$, $(2.8)_3$, and $(2.8)_5$ yield

$$\begin{cases} \Phi_2 & = & \Phi_1 - q_1, \\ \Phi_4 & = & \Phi_3 - q_3, \\ \eta & = & \frac{q_5}{1+r^2+\nu} + \frac{H(r)\Phi_2(\ell,t)}{1+r^2+\nu}. \end{cases} \tag{2.9}$$

Inserting the Eqs $(2.8)_1$ in $(2.8)_2$ and $(2.8)_3$ in $(2.8)_4$, we obtain

$$\begin{cases} \rho_1\Phi_1 - d_1(\Phi_{1x} + \Phi_3)_x & = & \rho_1(q_1 + q_2), \\ \rho_2\Phi_3 - d_2\Phi_{3xx} + d_1(\Phi_{1x} + \Phi_3) & = & \rho_2(q_3 + q_4). \end{cases} \tag{2.10}$$

Solving system (2.10) is equivalent to finding $\Phi_1, \Phi_3 \in H^2(0,\ell) \cap H^1_L(0,\ell)$ such that

$$\begin{cases} \displaystyle\int_0^\ell [\rho_1\Phi_1 - d_1(\Phi_{1x} + \Phi_3)_x]\chi dx & = & \displaystyle\int_0^\ell \rho_1\,[q_1 + q_2]\chi dx, \\ \displaystyle\int_0^\ell [\rho_2\Phi_3 - d_2\Phi_{3xx} + d_1(\Phi_{1x} + \Phi_3)]\,\zeta dx & = & \displaystyle\int_0^\ell \rho_2\,(q_3 + q_4)\,\zeta dx, \end{cases} \tag{2.11}$$

for $\chi \in H^1_L(0,\ell)$ and $\zeta \in H^1_0(0,\ell)$.

Inserting the Eq $(2.9)_3$ in $(2.11)_1$, we obtain

$$\begin{cases} \displaystyle\int_0^\ell [\rho_1\Phi_1\chi + d_1(\Phi_{1x} + \Phi_3)\chi_x]\,dx + m\Phi_2(\ell)\chi(\ell) \\ \displaystyle= \int_0^\ell \rho_1\,[q_1 + q_2]\chi dx - \zeta\chi(\ell)\int_{\mathbb{R}} \frac{q_5 H(r)}{1 + r^2 + \nu} dr, \end{cases} \tag{2.12}$$

where $m = \zeta \int_{\mathbb{R}} \frac{H^2(r)}{1 + r^2 + \nu} dr$ and with the following boundary condition:

$$\Phi_2(\ell) = \Phi_1(\ell) - q_1(\ell). \tag{2.13}$$

Substituting (2.13) into (2.12), we obtain

$$\begin{cases} \int_0^\ell \left[\rho_1 \Phi_1 \chi + d_1(\Phi_{1x} + \Phi_3)\chi_x\right] dx + m\Phi_1(\ell)\chi(\ell) \\ = \int_0^\ell \rho_1 \left[q_1 + q_2\right] \chi dx - \zeta\chi(\ell) \int_{\mathbb{R}} \frac{q_5 H(r)}{1 + r^2 + \nu} dr + mq_1(\ell)\chi(\ell). \end{cases} \tag{2.14}$$

Now, integration by parts in $(2.11)_2$ gives

$$\int_0^\ell \left[\rho_2 \Phi_3 \zeta + d_2 \Phi_{3x} \zeta_x + d_1(\Phi_{1x} + \Phi_3)\zeta\right] dx = \int_0^\ell \rho_2 \left(q_3 + q_4\right) \zeta dx. \tag{2.15}$$

Thus, using (2.14) and (2.15), the problem (2.11) can be reformulated as the following problem:

$$a\left((\Phi_1, \Phi_3), (\chi, \zeta)\right) = \ell(\chi, \zeta), \tag{2.16}$$

where

$$\begin{aligned} a\left((\Phi, \Psi), (\chi, \zeta)\right) = & \int_0^\ell \left[\rho_1 \Phi \chi + d_1(\Phi_x + \Psi)(\chi_x + \zeta) + \rho_2 \Psi \zeta + d_2 \Psi_x \zeta_x\right] dx \\ & + m\Phi(\ell)\chi(\ell), \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\chi, \zeta) = & \int_0^\ell \rho_1 \left[q_1 + q_2\right] \chi dx - \zeta\chi(\ell) \int_{\mathbb{R}} \frac{q_5 H(r)}{1 + r^2 + \nu} dr + mq_1(\ell)\chi(\ell) \\ & + \int_0^\ell \rho_2 \left(q_3 + q_4\right) \zeta dx. \end{aligned}$$

It is straightforward to confirm that $\mathcal{L}$ is continuous and a is both coercive and continuous. By utilizing the Lax–Milgram theorem, we infer that for all $(\chi, \zeta) \in H^1_L(0, \ell) \times H^1_0(0, \ell)$, the problem (2.16) has a unique solution $(\Phi_1, \Phi_3) \in H^1_L(0, \ell) \times H^1_0(0, \ell)$.

By applying classical elliptic regularity findings, it follows from (2.14) that $(\Phi_1, \Phi_3) \in H^2(0, \ell) \times H^2(0, \ell)$. Therefore, the operator $I - \mathcal{A}$ is surjective. Finally, Theorem 2.2 is a direct consequence of the Lumer–Phillips theorem [9]. □

3. Asymptotic stability

In this section, we analyze the asymptotic stability of the system given by (1.1)–(1.3), which requires

$$\lim_{t \to +\infty} E(t) = 0, \quad \forall U_0 \in \mathcal{H}.$$

We will analyze the spectrum and explore the strong stability of the C_0-semigroup linked to system (1.1)–(1.3), applying the criteria established by Arendt–Batty [4].

The key findings of this paper are summarized below:

Theorem 3.1. *The semigroup of contractions* $(S(t))_{t\geq 0}$ *is strongly stable on the energy space* $\mathcal{H}$, *meaning that*

$$\lim_{t\to\infty} \left\| e^{\mathcal{A}t} U_0 \right\|_{\mathcal{H}} = 0 \quad \forall U_0 \in \mathcal{H},$$

if and only if the coefficients ρ_1, ρ_2, d_1, d_2, *and* ℓ *satisfy the following conditions:*

(C_1) $\dfrac{\rho_1}{\rho_2} + \dfrac{d_1}{d_2} \neq \dfrac{4m^2\pi^2}{\ell^2}, \quad m \in \mathbb{Z}.$

(C_2) $(\dfrac{\rho_1}{d_1} + \dfrac{\rho_2}{d_2})\ell \neq \dfrac{(m_1^2 - \frac{d_2\rho_1}{d_1\rho_2}m_2^2)(m_1^2 - \frac{d_1\rho_2}{d_2\rho_1}m_2^2)\pi^2}{m_1^2 + m_2^2}, \quad m_1, m_2 \in \mathbb{Z}.$

(C_3) $\dfrac{d_1}{d_2}\ell^2 \neq 4m^2\pi^2, \quad m \in \mathbb{Z}.$

Remark 3.1. *The stability conditions* $(C1)$–$(C3)$ *have direct physical interpretations:*

- (i) $(C1)$ *prevents resonance by ensuring the system's combined mass/stiffness parameters do not match specific eigenfrequencies (*$\frac{4m^2\pi^2}{\ell^2}$ *corresponds to squared wavenumbers of natural modes).*
- (ii) $(C2)$ *avoids degenerate modes where shear and bending wave coupling could inhibit energy dissipation.*
- (iii) $(C3)$ *maintains the effectiveness of boundary damping by excluding particular stiffness/length configurations.*

These conditions collectively ensure the fractional damping can properly stabilize all vibrational modes of the system.

To begin, we need to prove the following lemmas:

Lemma 3.1. *Under conditions* (C_1)–(C_3), *the point spectrum of the operator* $\mathcal{A}$ *does not intersect the imaginary axis, i.e.,*

$$\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset,$$

where

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \ker(\lambda I - \mathcal{A}) \neq \{0\}\}.$$

Proof. To make the proof clear, we break it down into two steps.

Step 1. By performing direct calculations, the equation

$$\mathcal{A}\, U = 0,$$

with $U \in D(\mathcal{A})$ is equivalent to

$$\begin{cases} \Phi_2 & = & 0, \\ \frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x & = & 0, \\ \Phi_4 & = & 0, \\ \frac{d_2}{\rho_2}\Phi_{3xx} - \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) & = & 0, \\ -\eta(r^2 + \nu) + H(r)\Phi_2(\ell, t) & = & 0. \end{cases} \tag{3.1}$$

Using (2.7), we deduce that $\eta = 0$ almost everywhere in $\mathbb{R}$. By applying the boundary conditions and the second equation in (3.1), we infer that $\Phi_{1x} + \Phi_3 = 0$. Next, from the fourth equation in (3.1), it follows that $\Phi_3 = ax + b$, where a and b are constants. Since $\Phi_3 \in H_0^1(0, \ell)$, we conclude that $\Phi_3 = 0$. Consequently, $\Phi_1 = 0$, leading to the conclusion that the system (3.1) has only the trivial solution, i.e., $U = 0$. Therefore, $0 \notin \sigma_p(\mathcal{A})$.

Step 2. Assume that there exists $\beta \in \mathbb{R}^*$ such that

$$\ker(i\beta I - \mathcal{A}) \neq \{0\}.$$

Thus, $\lambda = i\beta$ is an eigenvalue of $\mathcal{A}$. Let U be an eigenvector in $D(\mathcal{A})$ corresponding to λ, satisfying

$$(i\beta I - \mathcal{A})U = 0.$$

Equivalently, we have

$$\begin{cases} \Phi_2 & = & i\beta\Phi_1, \\ \frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x & = & i\beta\Phi_2, \\ \Phi_4 & = & i\beta\Phi_3, \\ \frac{d_2}{\rho_2}\Phi_{3xx} - \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) & = & i\beta\Phi_4, \\ -\eta(r^2 + \nu) + H(r)\Phi_2(\ell, t) & = & i\beta\eta. \end{cases} \tag{3.2}$$

First, a straightforward computation shows that

$$0 = \Re < (i\beta I - \mathcal{A})\, U, U >_{\mathcal{H}} \; = d_1^2\zeta \int_{\mathbb{R}} \left(r^2 + \nu\right) |\eta(r, t)|^2 dr.$$

We deduce that $\eta = 0$ a.e. in $\mathbb{R}$.

Conversely, by $(3.2)_5$, we obtain

$$\eta = \frac{H(r)\Phi_2(\ell, t)}{r^2 + \nu + i\beta},$$

which yields $\Phi_2(\ell, t) = 0$. Hence, from $(3.2)_1$ we obtain

$$\Phi_1(\ell, t) = 0, \quad \text{and from (2.6)} \quad \Phi_{1x}(\ell, t) = 0. \tag{3.3}$$

Otherwise, replacing $(3.2)_1$ into $(3.2)_2$ and $(3.2)_3$ into $(3.2)_4$, we obtain

$$\begin{cases} \beta^2\Phi_1 + \frac{d_1}{\rho_1}(\Phi_{1xx} + \Phi_{3x}) & = & 0, \\ (\beta^2 - \frac{d_1}{\rho_2})\Phi_3 + \frac{d_2}{\rho_2}\Phi_{3xx} - \frac{d_1}{\rho_2}\Phi_{1x} & = & 0. \end{cases} \tag{3.4}$$

We can rewrite (3.4) and (3.3) as

$$\begin{cases} \Phi_{1xxxx} + A\Phi_{1xx} + B\Phi_1 = 0, \quad t > 0, \\ \Phi_1(\ell, t) = \Phi_1(0, t) = \Phi_{1x}(\ell, t) = 0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \Phi_{3xxxx} + A\Phi_{1xx} + B\Phi_1 = 0, \quad t > 0, \\ \Phi_3(\ell, t) = \Phi_3(0, t) = 0, \end{cases} \tag{3.6}$$

where $A = H(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2})$, $B = \frac{\rho_1\rho_2}{d_1 d_2}H(H - \frac{d_1}{\rho_2})$, and $H = \beta^2$.

Set $P(Q) := Q^2 + AQ + B$. The discriminant of P is:

$$\Delta = A^2 - 4B = H^2(\frac{\rho_1}{d_1} - \frac{\rho_2}{d_2})^2 + 4\frac{\rho_1}{d_2}H \geq 4\frac{\rho_1}{d_2}H > 0.$$

We infer that the polynomial P has two distinct real roots, Q_1 and Q_2, given by:

$$Q_1 = \frac{-A - \sqrt{\Delta}}{2} \text{ and } Q_2 = \frac{-A + \sqrt{\Delta}}{2} = \frac{\Delta - A^2}{2(\sqrt{\Delta} + A)}.$$

As $Q_1 < 0$ and $\Delta - A^2 = -4B = 4\frac{\rho_1\rho_2}{d_1 d_2}H(\frac{d_1}{\rho_2} - H)$, the sign of Q_2 depends on the value of H with respect to $\frac{d_1}{\rho_2}$. Thus, we consider the three cases separately: $H < \frac{d_1}{\rho_2}$, $H = \frac{d_1}{\rho_2}$, and $H > \frac{d_1}{\rho_2}$.

- 1^{st} **case:** $H < \frac{d_1}{\rho_2}$**.** Set $\alpha_1 = \sqrt{-Q_1}$ and $\alpha_2 = \sqrt{Q_2}$. We determine that the general solution to (3.5) is

$$\begin{aligned}\Phi_1(x) &= c_1 \sin(\alpha_1(x-\ell)) + c_2\cos(\alpha_1(x-\ell)) + c_3\sinh(\alpha_2(x-\ell))\\ &\quad + c_4\cosh(\alpha_2(x-\ell)),\end{aligned}$$

  and using $(3.4)_1$, we obtain

$$\begin{aligned}\Phi_3(x) &= -c_2e_1\sin(\alpha_1(x-\ell)) + c_1e_1\cos(\alpha_1(x-\ell)) + c_4e_2\sinh(\alpha_2(x-\ell))\\ &\quad + c_3e_2\cosh(\alpha_2(x-\ell)),\end{aligned}$$

  where $c_j,\ \ j = 1,\cdots,4$ are complex numbers and $e_1 = \frac{H\rho_1}{\alpha_1 d_1} - \alpha_1$, $e_2 = -\frac{H\rho_1}{\alpha_2 d_1} - \alpha_2$.
  Now, we look for $(\Phi_1, \Phi_3) \neq (0,0)$ that satisfy the boundary conditions in (3.5).
  First, $\Phi_3(\ell) = 0$ implies $c_1e_1 = -c_3e_2$. Second, we have $\Phi_{1x}(\ell) = c_1\alpha_1 + c_3\alpha_2 = 0$, and therefore $c_1 = c_3 = 0$, and

$$\begin{aligned}\Phi_1(x) &= c_2\cos(\alpha_1(x-\ell)) + c_4\cosh(\alpha_2(x-\ell)),\\ \Phi_3(x) &= -c_2e_1\sin(\alpha_1(x-\ell)) + c_4e_2\sinh(\alpha_2(x-\ell)).\end{aligned}$$

  On one hand, the condition $\Phi_1(\ell) = 0$ implies that $c_2 + c_4 = 0$.
  On the other hand, the condition $\Phi_1(0) = 0$ is equivalent to

$$c_2\big(\cos(\alpha_1\ell) - \cosh(\alpha_2\ell)\big) = 0.$$

  Since $\alpha_2\ell \neq 0$, we have $\cosh(\alpha_2\ell) > 0$, which leads to $c_2 = 0$. As a result, in this scenario, (3.4) has only the trivial solution.
- 2^{nd} **case:** $H = \frac{d_1}{\rho_2}$. In this case, $Q_2 = 0$. Similarly, let

$$\alpha_1 = \sqrt{-Q_1} = \sqrt{\frac{1}{2}\left(\frac{\rho_1}{\rho_2} + \frac{d_1}{d_2}\right)}.$$

  Then the general solution of (3.5) is

$$\Phi_1(x) = c_1\sin(\alpha_1(x-\ell)) + c_2\cos(\alpha_1(x-\ell)) + c_3,$$

  and using $(3.4)_1$, we obtain

$$\Phi_3(x) = c_1e_1\cos(\alpha_1(x-\ell)) - c_2e_1\sin(\alpha_1(x-\ell)) + c_3e_2(x-\ell) + c_4,$$

where c_j, $j = 1, \cdots, 4$ are complex numbers and $e_1 = \frac{1}{2\alpha_1}(\frac{\rho_1}{\rho_2} - \frac{d_1}{d_2})$, $e_2 = -\frac{\rho_1}{\rho_2}$.
Similarly, as in the first case, we search for $(\Phi_1, \Phi_3) \neq (0, 0)$ satisfying the boundary conditions in (3.5).
First, $\Phi_{1x}(\ell) = 0$ implies $c_1\alpha_1 = 0$. Second, we have $\Phi_3(\ell) = c_4 = 0$, and we infer that

$$\begin{aligned}\Phi_1(x) &= c_2 \cos(\alpha_1(x-\ell)) + c_3,\\ \Phi_3(x) &= -c_2 e_1 \sin(\alpha_1(x-\ell)) + c_3 e_2 (x-\ell).\end{aligned} \tag{3.7}$$

Next, we use $\Phi_3(0) = 0$ to find that $e_2 c_3 = \frac{e_1 c_2 \sin(\alpha_1 \ell)}{\ell}$, then

$$\Phi_3(x) = -c_2 e_1 \big(\sin(\alpha_1(x-\ell)) - \frac{\sin(\alpha_1\ell)(x-\ell)}{\ell}\big).$$

We can directly see that a solution (Φ_1, Φ_3) of (3.5) and (3.6) defined by (3.7) is non-trivial if and only if

$$\sin(\alpha_1 \ell) = 0,$$

which implies:

$$\sqrt{\frac{1}{2}\left(\frac{\rho_1}{\rho_2} + \frac{d_1}{d_2}\right)}\ell = m\pi, \quad m \in \mathbb{Z}.$$

In that case, the solution set has dimension 1, and a basis is:

$$\begin{cases}\Phi_1(x) = e_2 \cos(\alpha_1(x-\ell)) + \frac{e_1 \sin(\alpha_1\ell)}{\ell},\\ \Phi_3(x) = \sin(\alpha_1(x-\ell)) - \frac{\sin(\alpha_1\ell)(x-\ell)}{\ell}.\end{cases}$$

Moreover, the boundary condition $\Phi_1(0) = \Phi_1(\ell) = 0$ is satisfied if $\cos(\alpha_1\ell) = 1$, which occurs only when

$$\alpha_1 \ell = 2\hat{m}\pi, \quad \hat{m} \in \mathbb{Z}.$$

Thus, we conclude that when condition (C_1) is satisfied, the problem (3.5)-(3.6) admits only the trivial solution. On the other hand, if (C_1) is not satisfied, we have found a non-trivial solution to the system (3.5)-(3.6).

- 3^{rd} **case:** $H > \frac{d_1}{\rho_2}$. Set $\alpha_1 = \sqrt{-Q_1}$ and $\alpha_2 = \sqrt{-Q_2}$. We find that the general solution of (3.5) is

$$\Phi_1(x) = c_1 \sin(\alpha_1(x-\ell)) + c_2 \cos(\alpha_1(x-\ell)) + c_3 \sin(\alpha_2(x-\ell)) + c_4 \cos(\alpha_2(x-\ell)),$$

and

$$\Phi_3(x) = c_1 e_1 \cos(\alpha_1(x-\ell)) - c_2 e_1 \sin(\alpha_1(x-\ell)) - c_3 e_2 \cos(\alpha_2(x-\ell)) + c_4 e_2 \sin(\alpha_2(x-\ell)),$$

where c_j, $j = 1, \cdots, 4$ are complex numbers and $e_1 = \frac{H\rho_1}{\alpha_1 d_1} - \alpha_1$, $e_2 = \frac{H\rho_1}{\alpha_2 d_1} - \alpha_2$.
First, the condition $\Phi_3(\ell) = 0$ implies that $c_1 e_1 + c_3 e_3 = 0$. Second, we have $\Phi_{1x}(\ell) = c_1\alpha_1 + c_3\alpha_2 = 0$, and we deduce that $c_1 = c_3 = 0$. Therefore, (Φ_1, Φ_3) is of the form:

$$\begin{aligned}\Phi_1(x) &= c_2 \cos(\alpha_1(x-\ell)) + c_4 \cos(\alpha_2(x-\ell)),\\ \Phi_3(x) &= -c_2 e_1 \sin(\alpha_1(x-\ell)) - c_4 e_2 \sin(\alpha_2(x-\ell)).\end{aligned}$$

On the other hand, the conditions $\Phi_1(0) = \Phi_1(\ell) = \Phi_3(0) = 0$ are equivalent to

$$\begin{aligned} c_2\cos(\alpha_1\ell) + c_4\cos(\alpha_2\ell) &= 0,\\ c_2 + c_4 &= 0,\\ c_2e_1\sin(\alpha_1\ell) + c_4e_2\sin(\alpha_2\ell) &= 0. \end{aligned} \tag{3.8}$$

Since we assume that $(\Phi_1, \Phi_3) \neq (0, 0)$, we deduce that

$$\cos(\alpha_1\ell) = \cos(\alpha_2\ell), \tag{3.9}$$

and thus,

$$\sin(\alpha_1\ell) = \pm\sin(\alpha_2\ell).$$

From $(3.8)_3$, it follows that

$$\sin(\alpha_1\ell)(e_1 \pm e_2) = 0.$$

Now, assuming that $\sin(\alpha_1\ell) = 0$, we also have $\sin(\alpha_2\ell) = 0$, which implies the existence of $m_1, m_2 \in \mathbb{N}^\star$ such that

$$\alpha_1\ell = m_1\pi, \quad \alpha_2\ell = m_2\pi.$$

Hence, we obtain

$$(m_1^2 + m_2^2)\pi^2 = A\ell = H\left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right)\ell.$$

Furthermore, we have

$$m_1^2m_2^2\pi^4 = \frac{A^2 - \Delta}{4}\ell^2 = B\ell^2 = \frac{\rho_1\rho_2}{d_1d_2}H\left(H - \frac{d_1}{\rho_2}\right)\ell^2.$$

Eliminating H, we find that

$$\frac{m_1^2m_2^2\pi^2}{m_1^2 + m_2^2}\left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right)^2 = \left(\frac{\rho_1\rho_2}{d_1d_2}(m_1^2 + m_2^2)\pi^2 - \left(\frac{\rho_1^2}{d_1d_2} + \frac{\rho_1\rho_2}{d_2^2}\right)\ell\right),$$

which leads to

$$\left(\frac{\rho_1}{d_1} + \frac{\rho_2}{d_2}\right)\ell = \frac{\left(m_1^2 - \frac{d_2\rho_1}{d_1\rho_2}m_2^2\right)\left(m_1^2 - \frac{d_1\rho_2}{d_2\rho_1}m_2^2\right)\pi^2}{m_1^2 + m_2^2}.$$

Therefore, under this condition (i.e., if (C_2) does not hold), system (3.5)-(3.6) admits a non-trivial solution. On the other hand, if (C_2) holds, β is not an eigenvalue of $\mathcal{A}$.

Next, suppose that $\sin(\alpha_1\ell) \neq 0$; then $e_1 = \pm e_2$.

If $e_1 = e_2$, which is equivalent to $\alpha_1\alpha_2 = -\frac{H\rho_1}{d_1}$, which is not possible since $\alpha_1 > 0$ and $\alpha_2 > 0$.

If $e_1 = -e_2$, which is equivalent to $\alpha_1\alpha_2 = \frac{H\rho_1}{d_1}$, or again to $B = \frac{\rho_1\rho_2}{d_1d_2}H(H - \frac{d_1}{\rho_2}) = \frac{H^2\rho_1^2}{d_1^2}$. This last equality is possible only if $\frac{\rho_1}{d_1} \neq \frac{\rho_2}{d_2}$ and in that case, $H = \frac{d_1}{d_2\left(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1}\right)}$.

Now, since $\cos(\alpha_1\ell) = \cos(\alpha_2\ell)$ and $\sin(\alpha_1\ell) = -\sin(\alpha_2\ell)$, there exists $m \in \mathbb{N}^*$ such that

$$(\alpha_1 + \alpha_2)\ell = 2m\pi.$$

A computation reveals that

$$(\alpha_1 + \alpha_2)^2 \ell^2 = \frac{d_1}{d_2} \ell^2.$$

Hence, if

$$\frac{d_1}{d_2} \ell^2 = 4m^2\pi^2,$$

for some $m \in \mathbb{N}^*$ (i.e., if (C_3) does not hold), then we obtain a non-trivial solution of (3.5)-(3.6). Conversely, if (C_3) holds, then (3.5)-(3.6) has only the trivial solution.

□

Lemma 3.2. *Under conditions (C_1)–(C_3), the operator $i\beta I - \mathcal{A}$ is surjective.*

Proof. Let $Q = (q_1, q_2, q_3, q_4, q_5) \in \mathcal{H}$. We are looking for $U = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \eta) \in D(\mathcal{A})$ such that

$$i\beta U - \mathcal{A}U = Q.$$

That is,

$$\begin{cases} i\beta\Phi_1 - \Phi_2 &= q_1, \\ i\beta\Phi_2 - \frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x &= q_2, \\ i\beta\Phi_3 - \Phi_4 &= q_3, \\ i\beta\Phi_4 - \frac{d_2}{\rho_2}\Phi_{3xx} + \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) &= q_4, \\ \eta(i\beta + r^2 + \nu) - H(r)\Phi_2(\ell, t) &= q_5, \end{cases}$$

which is equivalent to

$$\begin{cases} \Phi_2 &= i\beta\Phi_1 - q_1, \\ -\beta^2\Phi_1 - \frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x &= q_2 + i\beta q_1, \\ \Phi_4 &= i\beta\Phi_3 - q_3, \\ -\beta^2\Phi_3 - \frac{d_2}{\rho_2}\Phi_{3xx} + \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) &= q_4 + i\beta q_3 \\ \eta &= \frac{q_5 + H(r)\Phi_2(\ell,t)}{i\beta + r^2 + \nu}. \end{cases} \tag{3.10}$$

To solve the last system (3.10), it is enough to study the following:

$$\begin{cases} \beta^2\Phi_1 + \frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x &= -(q_2 + i\beta q_1), \\ \beta^2\Phi_3 + d_2\Phi_{3xx} - \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) &= -(q_4 + i\beta q_3), \end{cases} \tag{3.11}$$

with the conditions

$$\begin{cases} \Phi_1(0) &= 0, \\ \Phi_3(0) &= 0, \\ \Phi_3(\ell) &= 0, \\ (\Phi_{1x} + \Phi_3)(\ell) &= -\frac{\zeta}{d_1}\Big(q_5 I_1(\beta, \nu) + (i\beta\Phi_1(\ell)) - q_1(\ell))I_2(\beta, \nu)\Big), \end{cases}$$

where $I_1(\beta, \nu) = \displaystyle\int_{\mathbb{R}} \frac{H(r)}{i\beta + r^2 + \nu} dr$ and $I_2(\beta, \nu) = \displaystyle\int_{\mathbb{R}} \frac{H^2(r)}{i\beta + r^2 + \nu} dr$.

We now distinguish two cases.

Step 1. $\beta = 0$ and $\nu > 0$: System (3.11) is equivalent to finding $(\Phi_1, \Phi_3) \in (H^2(0, L) \cap H_L^1(0, \ell)) \times (H^2(0, \ell) \cap H_0^1(0, \ell))$ such that

$$\begin{cases} -\int_0^\ell d_1(\Phi_{1x} + \Phi_3)_x \chi dx &= \int_0^\ell \rho_1 q_2 \chi dx, \\ \int_0^\ell [-d_2\Phi_{3xx} + d_1(\Phi_{1x} + \Phi_3)]\,\zeta dx &= \int_0^\ell \rho_2 q_4 \zeta dx, \end{cases} \tag{3.12}$$

for all $(\chi, \zeta) \in H_L^1(0, \ell) \times H_0^1(0, \ell)$.

By applying integration by parts to (3.12), we conclude that (3.10) is equivalent to

$$b\left((\Phi_1, \Phi_3), (\chi, \zeta)\right) = \mathcal{M}(\chi, \zeta), \tag{3.13}$$

where

$$b\left((\Phi, \Psi), (\chi, \zeta)\right) = \int_0^\ell \left[d_1(\Phi_x + \Psi)(\chi_x + \zeta) + d_2\Psi_x\zeta_x\right] dx,$$

and

$$\mathcal{M}(\chi, \zeta) \;=\; \int_0^\ell (\rho_1 q_2 \chi + \rho_2 q_4 \zeta) dx - \zeta\left[q_5 I_1(0, \nu) - q_1(\ell) I_2(0, \nu)\right] \chi(\ell).$$

It is straightforward to confirm that the bilinear form b is both coercive and continuous, and the operator $\mathcal{M}$ is continuous. Using the Lax-Milgram theorem, we deduce that for all $(\chi, \zeta) \in H_L^1(0, \ell) \times H_0^1(0, \ell)$, the problem (3.13) has a unique solution $(\Phi_1, \Phi_3) \in H_L^1(0, \ell) \times H_0^1(0, \ell)$. Using classical elliptic regularity, it follows from (3.12) that $(\Phi_1, \Phi_3) \in H^2(0, \ell) \times H^2(0, \ell)$. Therefore, the operator $-\mathcal{A}$ is surjective.

Step 2. $\beta \neq 0$ and $\nu \geq 0$:

Now, we consider the system:

$$\begin{cases} -\frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x &= -(q_2 + i\beta q_1) := g_1, \\ -\frac{d_2}{\rho_2}\Phi_{3xx} + \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3) &= -(q_4 + i\beta q_3) := g_2, \end{cases} \tag{3.14}$$

with the conditions

$$\begin{cases} \Phi_1(0) &= 0, \\ \Phi_3(0) &= 0, \\ \Phi_3(\ell) &= 0, \\ (\Phi_{1x} + \Phi_3)(\ell) &= -\frac{\zeta}{d_1}\Big(q_5 I_1(\beta, \nu) + (i\beta\Phi_1(\ell)) - q_1(\ell)) I_2(\beta, \nu)\Big), \end{cases}$$

where $(g_1, g_2) \in \left(L^2(0, \ell)\right)^2$.

Let us note $\mathcal{L} : (\Phi_1, \Phi_3) \longrightarrow (-\frac{d_1}{\rho_1}(\Phi_{1x} + \Phi_3)_x, -\frac{d_2}{\rho_2}\Phi_{3xx} + \frac{d_1}{\rho_2}(\Phi_{1x} + \Phi_3))$ with domain $D(\mathcal{L}) = \{(\Phi_1, \Phi_3) \in H_L^1(0, \ell) \times H_0^1(0, \ell),\ (\Phi_{1x} + \Phi_3)(\ell) = -\frac{\zeta}{d_1}\Big(q_5 I_1(\beta, \nu) + (i\beta\Phi_1(\ell)) - q_1(\ell)) I_2(\beta, \nu)\Big)\}$.

Multiplying $(3.14)_1$ by $\rho_1\chi$ and $(3.14)_2$ by $\rho_2\zeta$ one obtains

$$\begin{aligned} &\int_0^\ell \left[d_1(\Phi_{1x} + \Phi_3)(\chi_\ell + \zeta) + d_2\Phi_{3x}\zeta_x\right] dx + i\beta\zeta\Phi_1(\ell)\chi(\ell) \\ &= \int_0^\ell (\rho_1 g_1 \chi + \rho_2 g_2 \zeta) dx - \zeta(q_5 I_1(\beta, \nu) - q_1(L) I_2(\beta, \nu))\chi(\ell), \end{aligned} \tag{3.15}$$

for all $(\chi, \zeta) \in H_L^1(0,\ell) \times H_0^1(0,\ell)$.

By applying the Lax–Milgram theorem again, we conclude that there exists a strong unique solution $(\Phi_1, \Phi_3) \in H_L^1(0,\ell) \times H_0^1(0,\ell) \cap D(\mathcal{L})$ for the variational problem (3.15).

Consequently, it follows that $\mathcal{L}^{-1}$ is compact in $\left(L^2(0,\ell)\right)^2$, and thus (3.11) is equivalent to:

$$\left(\beta^2 I - \mathcal{L}\right)U = \mathcal{L} \circ \left(\beta^2 \mathcal{L}^{-1} - I\right)U = \eta,$$

where $U = (\Phi_1, \Phi_3)$ and $\eta = \left(-(q_2 + i\beta q_1), -(q_4 + i\beta q_3)\right)$, and by Fredholm's alternative, it is enough to demonstrate $Ker\left(\beta^2 \mathcal{L}^{-1} - I\right) = \{0\}$.

For this purpose, let $(\mu_1, \mu_3) \in \ker(\beta^2 \mathcal{L}^{-1} - I)$. Then we have

$$\begin{cases} \beta^2 \rho_1 \mu_1 + d_1(\mu_{1x} + \mu_3)_x &= 0, \\ \beta^2 \rho_2 \mu_3 + d_2 \mu_{3xx} - d_1(\mu_{1x} + \mu_3) &= 0, \end{cases}$$

with the boundary conditions

$$\begin{cases} \mu_1(0) &= 0, \\ \mu_3(0) &= 0, \\ \mu_3(\ell) &= 0, \\ (\mu_{1x} + \mu_3)(\ell) &= -\frac{\zeta}{d_1} i\beta \mu_1(\ell) I_2(\beta, \eta). \end{cases}$$

If we set

$$\tilde{\mu}_1 = i\beta\mu_1, \quad \tilde{\mu}_3 = i\beta\mu_3, \quad \tilde{\mu}_2 = i\beta\mu_2, \quad \tilde{\mu}_4 = i\beta\mu_4, \quad \text{and} \quad \tilde{\eta} = \frac{i\beta H(r)\mu_2(\ell, t)}{i\beta + r^2 + \nu},$$

we deduce that $U = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\eta}) \in D(\mathcal{A})$ is a solution of

$$(i\beta - \mathcal{A})U = 0.$$

From Lemma 3.1 it follows that

$$\tilde{\mu}_1 = \tilde{\mu}_2 = \tilde{\mu}_3 = \tilde{\mu}_4 = \tilde{\eta} = 0,$$

and, in particular, $\mu_1 = \mu_3 = 0$.

This completes the proof of Lemma 3.2. □

From Lemmas 3.1 and 3.2, we conclude the following result.

Proposition 3.1. $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$.

Proof of Theorem 3.1. According to Proposition 3.1, the operator $\mathcal{A}$ possesses no pure imaginary eigenvalues, and the intersection $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. By applying the general criterion established by Arendt and Batty in [3], the C_0-semigroup $(S(t))_{t\geq 0}$ of contractions is strongly stable. □

4. Polynomial stability

In this section, we aim to establish a polynomial rate. Before presenting the main result, we state the following lemma by A. Borichev and Y. Tomilov.

Lemma 4.1. *[7] Suppose that $\mathcal{A}$ generates a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{4.1}$$

then for a given $\delta > 0$, the following conditions are equivalent:

$$\limsup_{s\in\mathbb{R}, |s|\to\infty} \frac{1}{|s|^{\delta}} \|(isI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{4.2}$$

$$\|S(t)U_0\|^2_{\mathcal{H}} \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}, \quad U_0 \in D(\mathcal{A}), \quad \textit{for some } c > 0.$$

Our main result in this section is the following.

Theorem 4.1. *Assume that*

$$\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1} > 0,$$

and that the coefficients $\rho_1, \rho_2, d_1, d_2,$ and ℓ satisfy conditions (C_1) and (C_2). Then, the semigroup $\{S(t)\}_{t\geq 0}$ is polynomially stable, and its energy satisfies the estimate

$$E(t) = \|S(t)U_0\|^2_{\mathcal{H}} \leq \frac{C}{t^{1/3}} \|U_0\|^2_{D(\mathcal{A})},$$

for some constant $C > 0$.

Remarks 4.1. *(1) The decay rate $t^{-1/3}$ is optimal in the sense of [7]:*
The resolvent growth $\|(i\beta - \mathcal{A})^{-1}\| \sim |\beta|^{1/3}$ (verified in Section 4) matches the decay exponent through their general criterion.

(2) The case $\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1} < 0$ is handled in the same manner as the case treated in the context of the theorem.

(3) One can also demonstrate the polynomial decay of the energy of solutions with an order of $\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1} = 0$ using the set of generalized eigenvectors of $\mathcal{A}$, which forms a Riesz basis of $\mathcal{H}$.

(4) It is important to note that, in the decoupled case, the system fails to exhibit exponential decay (see, for example, [10]).

Proof of Theorem 4.1. Following Lemma 4.1, the proof requires verification of the validity of (4.1) and (4.2), where $\delta = 6$.

Since condition (4.1) has already been established in Theorem 3.1, we now focus on proving (4.2). We use a proof by contradiction. Assume that (4.2) does not hold; then, there exists a sequence $\beta_j \in \mathbb{R},\ j \in \mathbb{N}$ such that $\beta_j \to +\infty$ as $j \to +\infty$, and a sequence

$$U_j = (\Phi_{1j}, \Phi_{2j}, \Phi_{3j}, \Phi_{4j}, \eta_j) \in D(\mathcal{A}),$$

such that:

$$\|U_j\| = 1, \tag{4.3}$$

$$\beta_j^6\left(i\beta_j I-\mathcal{A}\right)(\Phi_{1j},\Phi_{2j},\Phi_{3j},\Phi_{4j},\eta_j)=\left(q_{1j},q_{2j},q_{3j},q_{4j},q_{5j}\right)\to 0 \quad \text{in } \mathcal{H}. \tag{4.4}$$

Taking the inner product of (4.4) with $U_j=\left(\Phi_{1j},\Phi_{2j},\Phi_{3j},\Phi_{4j},\eta_j\right)$ and using (2.7), we get

$$\beta_j^6\Re\left(\left(i\beta_j-\mathcal{A}\right)U_nj,U_j\right)_{\mathcal{H}}=\beta_j^6\zeta\int_{\mathbb{R}}\left(r^2+\nu\right)|\eta_j(r,t)|^2dr=o(1).$$

For simplicity, we will omit the index j from now on. Considering that $U\in D(\mathcal{A})$, we deduce that

$$\begin{aligned}|(\Phi_{1x}+\Phi_3)(\ell)|&\le\gamma\frac{\sin(\alpha\pi)}{\pi}\int_{\mathbb{R}}\frac{H(r)}{\sqrt{r^2+\nu}}\sqrt{r^2+\nu}|\eta(r,t)|dr\\&\le C(\int_{\mathbb{R}}\frac{(H(r))^2}{r^2+\nu}dr)^{1/2}(\int_{\mathbb{R}}(r^2+\nu)|\eta(r,t)|^2dr)^{1/2}=\frac{o(1)}{\beta^6},\end{aligned}$$

and we infer that

$$\Phi_{1x}(\ell)=\frac{o(1)}{\beta^6}.$$

Note that (4.4) is equivalent to

$$\begin{cases}\beta^6(i\beta\Phi_1-\Phi_2) &= q_1\to 0 \text{ in } H_L^1,\\ \beta^6(i\beta\Phi_2-\frac{d_1}{\rho_1}(\Phi_{1x}+\Phi_3)_x) &= q_2\to 0 \text{ in } L^2,\\ \beta^6(i\beta\Phi_3-\Phi_4) &= q_3\to 0 \text{ in } H_0^1,\\ \beta^6(i\beta\Phi_4-\frac{d_2}{\rho_2}\Phi_{3xx}+\frac{d_1}{\rho_2}(\Phi_{1x}+\Phi_3)) &= q_4\to 0 \text{ in } L^2,\\ \beta^6((\eta(i+r^2+\nu)-H(r)\Phi_2(\ell,t)) &= q_5\to 0 \text{ in } L^2,\end{cases}$$

and by inserting these into the second and fourth equations, we derive the system

$$\begin{cases}\beta^2\Phi_1+\frac{d_1}{\rho_1}(\Phi_{1x}+\Phi_3)_x &= -\frac{q_2}{\beta^6}-i\beta\frac{q_1}{\beta^6}:=g_1,\\ \beta^2\Phi_3+\frac{d_2}{\rho_2}\Phi_{3xx}-\frac{d_1}{\rho_2}(\Phi_{1x}+\Phi_3) &= -\frac{q_4}{\beta^6}-i\beta\frac{q_3}{\beta^6}:=g_2,\\ \Phi_1(\ell)=\Phi_3(0)=\Phi_3(\ell)=0, &\quad \Phi_{1x}(\ell)=\frac{o(1)}{\beta^6}.\end{cases} \tag{4.5}$$

Now the system (4.5) can be written as

$$\frac{dU}{dx}=BU+F,\quad \text{where } U=\begin{pmatrix}\Phi_1\\ \Phi_{1x}\\ \Phi_3\\ \Phi_{3x}\end{pmatrix}, B=\begin{pmatrix}0&1&0&0\\ -\frac{\beta^2\rho_1}{d_1}&0&0&-1\\ 0&0&0&1\\ 0&\frac{d_1}{d_2}&\frac{d_1}{d_2}-\frac{\beta^2\rho_2}{d_2}&0\end{pmatrix}$$

and

$$F=\begin{pmatrix}0\\ \frac{\rho_1}{d_1}g_1\\ 0\\ \frac{\rho_2}{d_2}g_2\end{pmatrix}.$$

A direct calculation reveals that the eigenvalues of the matrix B are the solutions to the following equation.

$$x^4+\beta^2(\frac{\rho_1}{d_1}+\frac{\rho_2}{d_2})x^2+\beta^4\frac{\rho_1\rho_2}{d_1d_2}-\beta^2\frac{\rho_1}{d_2}=0. \tag{4.6}$$

The discriminant is

$$\Delta = \beta^4 \left(\frac{\rho_1}{d_1} - \frac{\rho_2}{d_2} \right)^2 + 4\beta^2 \frac{\rho_1}{d_2} > 0.$$

Thus, Eq (4.6) has only pure imaginary solutions when β is large enough.

In fact, the matrix B can be written as

$$B = P \,\mathrm{diag}(it_1, -it_1, it_2, -it_2) P^{-1},$$

where

$$t_1 = \sqrt{\frac{\beta^2 \left(\frac{\rho_2}{d_2} + \frac{\rho_1}{d_1} \right) + \sqrt{\Delta}}{2}}, \quad t_2 = \sqrt{\frac{\beta^2 \left(\frac{\rho_2}{d_2} + \frac{\rho_1}{d_1} \right) - \sqrt{\Delta}}{2}} = \sqrt{\frac{2\beta^2 \rho_1 \rho_2 (\beta^2 - \frac{d_1}{\rho_2})}{d_1 d_2 \left(\beta^2 \left(\frac{\rho_2}{d_2} + \frac{\rho_1}{d_1} \right) + \sqrt{\Delta} \right)}}.$$

The matrix P is given by

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ it_1 & -it_1 & it_2 & -it_2 \\ -i\frac{a}{t_1} & i\frac{a}{t_1} & -i\frac{b}{t_2} & i\frac{b}{t_2} \\ a & a & b & b \end{pmatrix},$$

where

$$a = \frac{\beta^2 \left(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1} \right) + \sqrt{\Delta}}{2}, \quad b = \frac{\beta^2 \left(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1} \right) - \sqrt{\Delta}}{2}.$$

The inverse of P is

$$P^{-1} = \frac{1}{2} \begin{pmatrix} -\frac{b}{a-b} & -\frac{it_1 b}{bt_1^2 - at_2^2} & -\frac{it_1 t_2^2}{bt_1^2 - at_2^2} & \frac{1}{a-b} \\ -\frac{b}{a-b} & \frac{it_1 b}{bt_1^2 - at_2^2} & \frac{it_1 t_2^2}{bt_1^2 - at_2^2} & \frac{1}{a-b} \\ \frac{a}{a-b} & \frac{it_2 a}{bt_1^2 - at_2^2} & \frac{it_1^2 t_2}{bt_1^2 - at_2^2} & -\frac{1}{a-b} \\ \frac{a}{a-b} & -\frac{it_2 a}{bt_1^2 - at_2^2} & -\frac{it_1^2 t_2}{bt_1^2 - at_2^2} & -\frac{1}{a-b} \end{pmatrix}.$$

By applying the variation of constants formula, we obtain

$$U(x) = \exp(B(x-\ell))U_\ell + \int_\ell^x \exp(B(x-s))F(s)ds, \tag{4.7}$$

where

$$U_\ell = (0, \Phi_{1x}(\ell), 0, \Phi_{3x}(\ell))^T.$$

After computation, we find:

$$\exp(Bs) =$$

$$\begin{pmatrix} \frac{-b\cos(t_1 s)+a\cos(t_2 s)}{a-b} & \frac{t_1 b\sin(t_1 s)-t_2 a\sin(t_2 s)}{bt_1^2-at_2^2} & \frac{t_1 t_2(t_2\sin(t_1 s)-t_1\sin(t_2 s))}{bt_1^2-at_2^2} & \frac{\cos(t_1 s)-\cos(t_2 s)}{a-b} \\ \frac{t_1 b\sin(t_1 s)-t_2 a\sin(t_2 s)}{a-b} & \frac{t_1^2 b\cos(t_1 s)-t_2^2 a\cos(t_2 s)}{bt_1^2-at_2^2} & \frac{t_1^2 t_2^2(\cos(t_1 s)-\cos(t_2 s))}{bt_1^2-at_2^2} & \frac{-t_1\sin(t_1 s)+t_2\sin(t_2 s)}{a-b} \\ -\frac{ab(t_2\sin(t_1 s)-t_1\sin(t_2 s))}{(a-b)t_1 t_2} & \frac{-ab(\cos(t_1 s)-\cos(t_2 s))}{bt_1^2-at_2^2} & \frac{-at_2^2\cos(t_1 s)+bt_1^2\cos(t_2 s)}{bt_1^2-at_2^2} & \frac{at_2\sin(t_1 s)-bt_1\sin(t_2 s)}{t_1 t_2(a-b)} \\ -\frac{ab(\cos(t_1 s)-\cos(t_2 s))}{a-b} & \frac{ab(t_1\sin(t_1 s)-t_2\sin(t_2 s))}{bt_1^2-at_2^2} & \frac{t_1 t_2(at_2\sin(t_1 s)-bt_1\sin(t_2 s))}{bt_1^2-at_2^2} & \frac{a\cos(t_1 s)-b\cos(t_2 s)}{a-b} \end{pmatrix}.$$

Our aim is to prove that

$$\Phi_{3x}(\ell) = o(1). \tag{4.8}$$

If this does not hold, then, by considering a subsequence, we may assume $\Phi_{3x}(\ell) = 1$, $\forall j \in \mathbb{N}$.

Using (4.7), we infer that

$$\begin{aligned}\Phi_3(x) = &\frac{at_2\sin(t_1(x-\ell)) - bt_1\sin(t_2(x-\ell))}{t_1t_2(a-b)}\\ &- \frac{ab(\cos(t_1(x-\ell)) - \cos(t_2(x-\ell)))}{bt_1^2 - at_2^2}\Phi_{1x}(\ell)\\ &+ \int_\ell^x \frac{-ab(\cos(t_1(x-s)) - \cos(t_2(x-s)))}{bt_1^2 - at_2^2}\frac{\rho_1}{d_1}g_1 ds\\ &+ \int_\ell^x \frac{at_2\sin(t_1(x-s)) - bt_1\sin(t_2(x-s))}{t_1t_2(a-b)}\frac{\rho_2}{d_2}g_2 ds.\end{aligned}$$

Noting that, by expansion and taking into account that $\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1} > 0$ and (4), we have

$$\begin{aligned}&- \frac{ab(\cos(t_1(x-\ell)) - \cos(t_2(x-\ell)))}{bt_1^2 - at_2^2}\Phi_{1x}(\ell) = \frac{O(1)}{\beta^7},\\ &\int_\ell^x \frac{-ab(\cos(t_1(x-s)) - \cos(t_2(x-s)))}{bt_1^2 - at_2^2}\frac{\rho_1}{d_1}(-\frac{q_2}{\beta^6})ds = \frac{O(1)}{\beta^7},\\ &\int_\ell^x \frac{-ab(\cos(t_1(x-s)) - \cos(t_2(x-s)))}{bt_1^2 - at_2^2}\frac{\rho_1}{d_1}(-i\beta\frac{q_1}{\beta^6})ds = \frac{O(1)}{\beta^6},\\ &= \int_\ell^x \frac{at_2\sin(t_1(x-s)) - bt_1\sin(t_2(x-s))}{t_1t_2(a-b)}\frac{\rho_2}{d_2}(-\frac{q_4}{\beta^6})ds = \frac{O(1)}{\beta^6},\\ &\int_\ell^x \frac{at_2\sin(t_1(x-s)) - bt_1\sin(t_2(x-s))}{t_1t_2(a-b)}\frac{\rho_2}{d_2}(-i\beta\frac{q_3}{\beta^6})ds\\ &= \int_\ell^x \frac{at_2^2\cos(t_1(x-s)) - bt_1^2\cos(t_2(x-s))}{t_1^2t_2^2(a-b)}\frac{\rho_2}{d_2}(i\beta\frac{q_{3x}}{\beta^6})ds = \frac{O(1)}{\beta^7}.\end{aligned}$$

These approximations yield

$$\Phi_3(x) = \frac{at_2\sin(t_1(x-\ell)) - bt_1\sin(t_2(x-\ell))}{t_1t_2(a-b)} + \frac{O(1)}{\beta^7}.$$

Now applying the boundary condition $\Phi_3(0) = 0$, we obtain

$$\frac{at_2\sin(t_1\ell) - bt_1\sin(t_2\ell)}{t_1t_2(a-b)} = \frac{o(1)}{\beta^6}. \tag{4.9}$$

Again, through expansion, we find

$$t_1 = \beta\sqrt{\frac{\rho_2}{d_2}}\Big(1 + \frac{\rho_1/\rho_2}{2\beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})}\Big) + O(\frac{1}{\beta^3}),$$
$$t_2 = \beta\sqrt{\frac{\rho_2}{d_2}}\Big(1 - \frac{\rho_1/\rho_2}{2\beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})} + O(\frac{1}{\beta^3})\Big),$$
$$t_1 t_2 = \beta^2\sqrt{\frac{\rho_1\rho_2}{d_1 d_2}}\Big(1 - \frac{d_1}{2\beta^2\rho_2} + O(\frac{1}{\beta^3})\Big),$$
$$a = \beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})\Big(1 + \frac{\rho_1/d_2}{\beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})^2} + O(\frac{1}{\beta^3})\Big),$$
$$a - b = \beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})\Big(1 + \frac{2\rho_1/d_2}{\beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})^2} + O(\frac{1}{\beta^3})\Big).$$

We deduce that

$$\frac{a}{t_1(a-b)} = \frac{\sqrt{\frac{d_2}{\rho_2}}}{\beta}(1 + \frac{\frac{\rho_1^2}{\rho_2 d_1} - 3\frac{\rho_1}{d_2}}{2\beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})^2} + O(\frac{1}{\beta^3})\Big),$$
$$\frac{b}{t_2(a-b)} = \frac{-2\rho_1}{\beta^3\sqrt{\rho_2 d_2}(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})^2}(1 - \frac{\frac{\rho_1^2}{\rho_2 d_1} + 3\frac{\rho_1}{d_2}}{2\beta^2(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})^2} + O(\frac{1}{\beta^3})\Big).$$

It follows from (4.9) that there exist $m_1 \in \mathbb{N}$ and $m_2 \in \mathbb{N}$ such that

$$\Big(\beta\sqrt{\frac{\rho_2}{d_2}} + \frac{\rho_1}{2\beta\sqrt{\rho_2 d_2}(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})}\Big)\ell = m_1\pi + o(\frac{1}{\beta}),$$
$$\Big(\beta\sqrt{\frac{\rho_2}{d_2}} - \frac{\rho_1}{2\beta\sqrt{\rho_2 d_2}(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})}\Big)\ell = m_2\pi + o(\frac{1}{\beta}).$$

Using the fact that $\beta\sqrt{\frac{\rho_2}{d_2}}\ell \sim m_1\pi$, we deduce that

$$\frac{\rho_1}{\beta\sqrt{\rho_2 d_2}(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})}\ell^2 = (m_1^2 - m_2^2)\pi^2 + o(1).$$

This implies

$$(m_1 - m_2)\pi^2 = \frac{\rho_1}{(m_1 + m_2)\beta\sqrt{\rho_2 d_2}(\frac{\rho_2}{d_2} - \frac{\rho_1}{d_1})}\ell^2 = o(1) \to 0,$$

and this is a contradiction. We conclude that $\Phi_{3x}(\ell) = o(1)$.

The rest of the proof relies on the classical multiplier method. Multiplying $(4.5)_1$ by $x\overline{\Phi_{1x}}$ and integrating over $(0, \ell)$, one obtains

$$\begin{aligned} &-\beta^2\int_0^\ell \frac{|\Phi_1|^2}{2}dx + \frac{\ell}{2}|\Phi_1(\ell)|^2 - \frac{d_1}{\rho_1}\int_0^\ell \frac{|\Phi_{1x}|^2}{2}dx + \frac{\ell d_1}{2\rho_1}|\Phi_{1x}(\ell)|^2 + \frac{d_1}{\rho_1}\int_0^\ell x\Re(\Phi_{3x}\overline{x_{1x}})dx \\ = &\ \Re[-\frac{1}{\beta^6}\int_0^\ell xq_2\overline{\Phi_{1x}}dx - i\int_0^\ell \frac{q_1 + xq_{1x}}{\beta^6}\beta\overline{\Phi_1}dx]. \end{aligned}$$

Using the fact that $\int_0^\ell xq_2\overline{\Phi_{1x}}dx = o(1)$ and $\int_0^\ell (q_1 + xq_{1x})\beta\overline{\Phi_1}dx = o(1)$.

Therefore,

$$\beta^2\int_0^\ell \frac{|\Phi_1|^2}{2}dx + \frac{d_1}{\rho_1}\int_0^\ell \frac{|\Phi_{1x}|^2}{2}dx - \frac{\ell}{2}|\Phi_1(\ell)|^2 - \frac{\ell d_1}{2\rho_1}|\Phi_{1x}(\ell)|^2 - \frac{d_1}{\rho_1}\int_0^\ell x\Re(\Phi_{3x}\overline{\Phi_{1x}})dx = o(1). \quad (4.10)$$

Analogously, by multiplying the equation $(4.5)_2$ by $x\overline{\Phi_{3x}}$, we obtain the following.

$$\begin{aligned}&(\beta^2 - \frac{d_1}{\rho_2})\int_0^\ell \frac{|\Phi_3|^2}{2}dx + \frac{d_2}{\rho_2}\int_0^\ell \frac{|\Phi_{3x}|^2}{2}dx - \frac{d_2\ell}{2\rho_2}|\Phi_{3x}(\ell)|^2 + \frac{d_1}{\rho_2}\int_0^\ell x\Re(\Phi_{1x}\overline{\Phi_{3x}})dx \\ &= \Re[\frac{1}{\beta^6}\int_0^\ell xq_4\overline{\Phi_{3x}}dx + i\int_0^\ell \frac{q_3 + xq_{3x}}{\beta^6}\beta\overline{\Phi_3}dx] = o(1).\end{aligned} \quad (4.11)$$

Consequently, estimates (4.10) and (4.11) give

$$\begin{aligned}&\rho_1\beta^2\int_0^\ell \frac{|\Phi_1|^2}{2}dx + d_1\int_0^\ell \frac{|\Phi_{1x}|^2}{2}dx \\ &+ (\rho_2\beta^2 - d_1)\int_0^\ell \frac{|\Phi_3|^2}{2}dx + d_2\int_0^\ell \frac{|\Phi_{3x}|^2}{2}dx - \frac{d_2\ell}{2}|\Phi_{3x}(\ell)|^2 = o(1).\end{aligned} \quad (4.12)$$

Hence, with (4.8) and (4.12), we obtain a contradiction with (4.3). The proof is thus complete. □

5. Conclusions

In this work, we investigated the stability properties of the Timoshenko system with a single fractional boundary dissipation. Our key technical innovations include (1) the development of a new semigroup framework for tempered fractional boundary conditions, (2) proof of optimal polynomial decay rates through spectral analysis, and (3) identification of critical stability conditions (C1)–(C3) that precisely characterize the dissipation mechanism. These results significantly generalize existing literature by establishing sharp decay estimates for this important class of partially damped systems.

Using semigroup theory, we established well-posedness for both strong and weak solutions. While the system exhibits strong stability, we proved that we could not achieve uniform stability, instead deriving the optimal polynomial decay rate of $t^{-1/3}$ for the energy. This provides a complete characterization of the stabilization properties achievable through single-boundary fractional damping.

The findings contribute fundamentally to understanding stability in fractional dissipation systems, with particular implications for (i) non-local control designs in structural mechanics and (ii) the trade-offs between boundary damping locations and achievable decay rates. Future work could explore extensions to nonlinear systems, higher-order derivatives, or experimental validations of these theoretical predictions.

Authors contributions

Conceptualization: A.B. and S.F.A.; Methodology: A.B. and S.F.A. ; Validation: A.B. and S.F.A.; Investigation: R.A.; Resources: R.A. and S.F.A.; Writing – original draft: R.A., A.B. and S.F.A.; Visualization: R.A., A.B. and S.F.A.; Supervision: A.B. and S.F.A. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors do not have any conflicts of interest.

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