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Research article

A class of rough generalized Marcinkiewicz integrals along twisted surfaces

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Abstract: In this paper, we investigated the mapping properties of generalized Marcinkiewicz integral operators associated with twisted surfaces. Under certain conditions on these surfaces, we established suitable L^p estimates for these operators, assuming the kernel functions belong to $L^q\left(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}\right)$. By combining these estimates with Yano's extrapolation technique, we further established the boundedness of these operators from the homogeneous Triebel-Lizorkin space $\dot{F}_p^{0,\tau}(\mathbb{R}^j\times\mathbb{R}^k)$ to the space $L^p(\mathbb{R}^j\times\mathbb{R}^k)$ under significantly weaker assumptions on the kernels. Our results extended and improved many previously known results.

Keywords: boundedness; rough kernels; generalized Marcinkiewicz integrals; twisted surfaces
Mathematics Subject Classification: 42B20, 42B25

1. Introduction

Throughout this work, we let $\mathbb{R}^\eta$ ($\eta = j$ or k) be the Euclidean space of dimension $\eta \geq 2$ and $\mathbb{U}^{\eta-1}$ be the unit sphere in $\mathbb{R}^\eta$, which is equipped with the normalized spherical measure $d\rho_\eta(\cdot)$. Also, we let $u' = u/|u|$ for $u \in \mathbb{R}^\eta \setminus \{0\}$.

Let $\Theta \in L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})$ satisfy

$$\Theta(\alpha v,\beta w) = \Theta(v,w), \quad \forall\, \alpha,\beta > 0, \tag{1.1}$$

and

$$\int_{\mathbb{U}^{j-1}} \Theta(v',w)d\rho_j(v') = \int_{\mathbb{U}^{k-1}} \Theta(v,w')d\rho_k(w') = 0. \tag{1.2}$$

For a measurable mapping $g \in L^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+)$, an integrable mapping Θ satisfying (1.1), (1.2) and an appropriate function $\Omega : \mathbb{R}^j \times \mathbb{R}^k \to \mathbb{R}^j \times \mathbb{R}^k$, we consider the generalized Marcinkiewicz integral operator $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ along the surface $\Lambda_\Omega(x,y) = \Omega(x,y)$ defined by

$$\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)(x,y) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left| V_{\alpha,\beta}(F)(x,y) \right|^{\tau} \frac{d\alpha d\beta}{\alpha\beta} \right)^{1/\tau}, \tag{1.3}$$

where $F \in C_0^{\infty}(\mathbb{R}^j \times \mathbb{R}^k)$, $\tau > 1$, and

$$V_{\alpha,\beta}(F)(x,y) = \frac{1}{\alpha\beta} \int_{|w|\le\beta} \int_{|v|\le\alpha} \frac{\Theta(v,w) g(|v|,|w|)}{|v|^{j-1}\,|w|^{k-1}} F\left((x,y) - \Omega(v,w)\right) dv dw.$$

When $\tau = 2$ and $g \equiv 1$, we denote $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ by $\mathcal{H}_{\Theta,\Omega}$ and when $\Omega(v,w) \equiv \mathrm{I}(v,w) = (v,w)$, we denote $\mathcal{H}_{\Theta,\Omega}$ by $\mathcal{M}_\Theta$, which is the classical Marcinkiewicz operator on product spaces, which was introduced in [1]. In [1] the author proved that $\mathcal{M}_\Theta$ is bounded on $L^2(\mathbb{R}^j \times \mathbb{R}^k)$ under the weak condition $\Theta \in L(\log L)^2(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$. Subsequently, numerous researchers have expanded and refined this result. For example, the L^p boundedness of $\mathcal{M}_\Theta$ was established in [2] for all $p \in (1,\infty)$ provided that $\Theta \in B_q^{(0,0)}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$ which is the optimal condition in the sense that $\mathcal{M}_\Theta$ will miss its L^2 boundedness whenever we replace this condition by any weaker condition $\Theta \in B_q^{(0,r)}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$ with $r \in (-1,0)$. Here $B_q^{(0,0)}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$ indicates to the block space that introduced in [3]. In [4] the authors proved that $\mathcal{M}_\Theta$ is bounded on L^p $(1 < p < \infty)$ provided $\Theta \in L(\log L)(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$, and they further showed that this condition is optimal. For comprehensive coverage of this topic, we refer readers to the foundational works in [5–7], the theoretical and applied developments in [8–10], contemporary studies in [11, 12], and recent advances in [13, 14].

Motivated by the extensive work on the L^p-boundedness of $\mathcal{M}_\Theta$ under various conditions on Θ, we initiate the study of $\mathcal{H}^{(2)}_{\Theta,\Omega,g}$ for the case where Ω has the standard form $\Omega(v,w) = \mathbf{S}(v,w) = (\varphi_1(v), \varphi_2(w))$. Here, $\varphi_1 : \mathbb{R}^j \to \mathbb{R}^j$ and $\varphi_2 : \mathbb{R}^k \to \mathbb{R}^k$ are assumed to satisfy certain conditions. For a survey of previous results concerning this operator, we refer the reader to [15–17] and the references therein.

Even many problems concerning the boundedness of $\mathcal{H}^{(2)}_{\Theta,\Omega,g}$ are still open, the discussion of the generalized Marcinkiewicz operator $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ in the standard form of Ω has been considered by many authors. Historically, the operator $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ was introduced in [18] in which the authors showed that if $\Theta \in L(\log L)^{2/\tau}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$, then $\mathcal{H}^{(\tau)}_{\Theta,\mathrm{I},1}$ $(\tau > 1)$ is of type (p,p) for all $1 < p < \infty$. Later, the authors of [19] established that

$$\left\| \mathcal{H}^{(\tau)}_{\Theta,\mathrm{I},1}(F) \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \le C_p \left\| F \right\|_{\dot{F}_p^{\tau,\vec{0}}(\mathbb{R}^j\times\mathbb{R}^k)},$$

for all $p, \tau \in (1,\infty)$ provided that $\Omega \equiv \mathrm{I}$, $\Theta \in B_q^{(0,\frac{2}{\tau}-1)}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1}) \cup L(\log L)^{2/\tau}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$. This result was subsequently refined in [20], where the authors established the same conclusion while weakening the assumption $g \equiv 1$ to $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\lambda > 1$. Here, $\Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ $(\lambda > 1)$ denotes the class of measurable functions g, satisfying

$$\|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)} = \sup_{m,n\in\mathbb{Z}} \left(\int_{2^n}^{2^{n+1}} \int_{2^m}^{2^{m+1}} |g(\alpha,\beta)|^{\lambda} \frac{d\alpha d\beta}{\alpha\beta} \right)^{1/\lambda} < \infty.$$

Subsequently, the operator $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ has been extensively studied under diverse conditions imposed on Θ, g, and $\Omega = \mathbf{S}$ [20–22].

In this paper, $\dot{F}_p^{k,\overrightarrow{c}}(\mathbb{R}^j \times \mathbb{R}^k)$ refers to the space of homogeneous Triebel-Lizorkin mappings. Let us recall the definition of this space. For $\tau, p \in (1,\infty)$, and $\overrightarrow{c} = (c_1, c_2) \in \mathbb{R}\times\mathbb{R}$, the space $\dot{F}_p^{\tau,\overrightarrow{c}}(\mathbb{R}^j \times \mathbb{R}^k)$ is defined to be the class of tempered distributions F on $\mathbb{R}^j \times \mathbb{R}^k$, such that

$$\|F\|_{\dot{F}_p^{\tau,\overrightarrow{c}}(\mathbb{R}^j\times\mathbb{R}^k)} = \left\| \left(\sum_{m,n\in\mathbb{Z}} \left|(\psi_m^1 \otimes \psi_n^2) * F\right|^\tau \; 2^{(c_1 m + c_2 n)\tau} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} < \infty,$$

where $\widehat{\psi_m^1}(v) = \frac{\psi_1(2^{-m}v)}{2^{mj}}$, $\widehat{\psi_n^2}(w) = \frac{\psi_2(2^{-n}w)}{2^{nk}}$ and $\psi_1 \in C_0^\infty(\mathbb{R}^j)$, $\psi_2 \in C_0^\infty(\mathbb{R}^k)$ are radial functions satisfying:

(i) $0 \le \psi_1, \psi_2 \le 1$;

(ii) $supp\,(\psi_1) \subseteq \left\{v : |v| \in [\frac{1}{2}, 2]\right\}$, $\quad supp\,(\psi_2) \subseteq \left\{w : |w| \in [\frac{1}{2}, 2]\right\}$;

(iii) For all $|v|, |w| \in [\frac{3}{5}, \frac{5}{3}]$, there exists a constant $N > 0$ such that $\psi_1(v) \ge N$ and $\psi_2(w) \ge N$;

(iv) $\sum_{m\in\mathbb{Z}} \psi_1(v/2^m) = 1$ with $v \ne 0$ and $\sum_{n\in\mathbb{Z}} \psi_2(w/2^n) = 1$ with $w \ne 0$.

The following properties can be found in [18].

(1) The Schwartz space $\mathcal{S}(\mathbb{R}^j \times \mathbb{R}^k)$ is dense in the space $\dot{F}_p^{\tau,\overrightarrow{c}}(\mathbb{R}^j \times \mathbb{R}^k)$;

(2) For all $p \in (1,\infty)$, we have $\dot{F}_p^{2,\overrightarrow{0}}(\mathbb{R}^j \times \mathbb{R}^k) = L^p(\mathbb{R}^j \times \mathbb{R}^k)$;

(3) For $\tau_1, \tau_2 \in (1,\infty)$ with $1 < \tau_1 \le \tau_2$, we have $\dot{F}_p^{\tau_1,\overrightarrow{c}}(\mathbb{R}^j \times \mathbb{R}^k) \subseteq \dot{F}_p^{\tau_2,\overrightarrow{c}}(\mathbb{R}^j \times \mathbb{R}^k)$.

The Marcinkiewicz operator $\mathcal{H}^{(2)}_{\Theta,\Omega,g}$ along twisted surfaces $\Omega(v,w) = (f_1(v,w), f_2(v,w))$ was introduced in [23], where f_1 and f_2 satisfy certain conditions $g \equiv 1$.

Motivated by the boundedness results for $\mathcal{H}^{(2)}_{\Theta,\Omega,g}$ on twisted surfaces with $g \equiv 1$ [23], and those for $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ on standard-form surfaces $\Omega \equiv \mathbf{S}$ [19–21], we naturally ask:

Can the generalized Marcinkiewicz operator $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ be extended to twisted surfaces, replacing the conditions $\tau = 2$ by $\tau > 1$ and $g \equiv 1$ by $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\lambda > 1$?

The purpose of this paper is to answer the above question in the affirmative. In this work, we let $\mathcal{G}_d$ be the class of smooth functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$, such that

$$\varphi(0) = 0, \quad |\varphi(t)| \le C_1 t^d \quad \text{and} \quad |\varphi''(t)| \ge C_2 t^{d-2}, \tag{1.4}$$

for some $d \ne 0$, $t \in \mathbb{R}_+$, and C_1, C_2 are positive constants independent of t.

The main results of this paper are the following:

Theorem 1.1. *Let $\Theta \in L^q\left(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1}\right)$ with $q \in (1,2]$ and $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\lambda \in (1,2]$. Suppose that $\Omega(v,w) = (\varphi_1(|w|)v, \varphi_2(|v|)w)$ with $\varphi_1 \in \mathcal{G}_{d_1}$ and $\varphi_2 \in \mathcal{G}_{d_2}$. Then, there is a bounded constant $C_p > 0$, such that:*
(i) If $\tau \le \lambda'$, then the estimate

$$\left\|\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \le C_p C_{\Theta,g} \left(\frac{1}{(q-1)(\lambda-1)}\right)^{2/\tau} \|F\|_{\dot{F}_p^{\overrightarrow{0},\tau}(\mathbb{R}^j\times\mathbb{R}^k)}$$

holds for all $p \in (\frac{\lambda'\tau}{\lambda'+\tau-1}, \frac{\lambda\tau'}{\tau'-\lambda})$.

(ii) If $\tau \geq \lambda'$, then the estimate

$$\left\|\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}\left(\frac{1}{(\lambda-1)(q-1)}\right)^{2/\tau} \|F\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^j\times\mathbb{R}^k)}$$

holds for all $p \in (\lambda', \infty)$, where $C_{\Theta,g} = \|\Theta\|_{L^q(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})} \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}$.

Theorem 1.2. *Let Θ, Ω be given as in Theorem 1.1, and let $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\lambda \in (2, \infty)$.*
(i) If $\tau \leq \lambda'$, then the estimate

$$\left\|\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}\left(\frac{\lambda}{q-1}\right)^{2/\tau} \|F\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^j\times\mathbb{R}^m)}$$

holds for all $p \in (1, \tau)$.
(ii) If $\tau \geq \lambda'$, then the estimate

$$\left\|\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}\left(\frac{\lambda}{q-1}\right)^{2/\tau} \|F\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^j\times\mathbb{R}^m)}$$

holds for all $p \in (\lambda', \infty)$.

The estimates in Theorems 1.1 and 1.2 along with Yano's extrapolation argument (see [24, 25]), lead to the following:

Theorem 1.3. *Let Ω and g be given as in Theorem 1.1.*
(i) If $\Theta \in L(\log L)^{2/\tau}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$, then

$$\left\|\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p\left(1 + \|\Theta\|_{L(logL)^{2/\tau}(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}\right) \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^j\times\mathbb{R}^k)},$$

for $p \in (\frac{\lambda'\tau}{\lambda'+\tau-1}, \frac{\lambda\tau'}{\tau'-\lambda})$ if $\tau \leq \lambda'$, and for $p \in (\lambda', \infty)$ if $\tau \geq \lambda'$.
(ii) If $\Omega \in B_q^{(0,\frac{2}{\tau}-1)}(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$ for some $q > 1$, then

$$\left\|\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p\left(1 + \|\Omega\|_{B_q^{(0,\frac{2}{\tau}-1)}(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}\right) \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^j\times\mathbb{R}^k)},$$

for $p \in (\frac{\lambda'\tau}{\lambda'+\tau-1}, \frac{\lambda\tau'}{\tau'-\lambda})$ if $\tau \leq \lambda'$, and for $p \in (\lambda', \infty)$ if $\tau \geq \lambda'$.

Theorem 1.4. *Let $g \in \Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\lambda \in (2, \infty)$ and $\Omega \in L(\log L)^{2/\tau}(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}) \cup B_q^{(0,\frac{2}{k}-1)}(\mathbb{U}^{j-1}\times \mathbb{U}^{k-1})$ for some $q > 1$. Then, the operator $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ is bounded on $L^p(\mathbb{R}^j \times \mathbb{R}^k)$ for $p \in (1, \tau)$ if $\tau \leq \lambda'$, and for $p \in (\lambda', \infty)$ if $\tau \geq \lambda'$.*

Remarks

(1) For the case $\tau = 2$, $g \equiv 1$ and $\Omega \equiv \mathrm{I}$, the author of [6] proved the L^p boundedness of $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ for $p \in (1, \infty)$ provided that $\Theta \in L(\log L)^2(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1}) \subsetneq L(\log L)(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$. Therefore, this result is generalized and improved by the conclusion of Theorem 1.4.
(2) The main finding in [5] comes directly form Theorem 1.4 by taking the special case $\tau = 2$, $g \equiv 1$, $\Omega \equiv \mathrm{I}$, and $\Theta \in L(\log L)(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1})$.
(3) The assumptions on Θ in Theorem 1.4 are optimal for the case $\tau = 2$, $g \equiv 1$ and $\Omega \equiv \mathrm{I}$, see [2, 4].
(4) For the special case $g \equiv 1$, we confirm the L^p boundedness of $\mathcal{H}^{(\tau)}_{\Theta,\Omega,1}$ for the full range of $p \in (1, \infty)$.
(5) The main result in [20] is obtained when we take the special case $\Omega \equiv \mathrm{I}$.
(6) A model example about the class $\mathcal{G}_d$ is $\varphi(t) = t^d$ for any $t \neq 0$.

2. Preliminary lemmas

We devote this section to introducing some notations and establishing some subsidiary results needed to prove the main results of this paper. For $\mu \geq 2$, we define the family of measures $\Upsilon_{\Theta,\Omega,g,\alpha,\beta} := \{\Upsilon_{\alpha,\beta} : \alpha, \beta \in \mathbb{R}_+\}$ and its related maximal operators Υ_g^* and M_g on $\mathbb{R}^j \times \mathbb{R}^k$ by

$$\iint_{\mathbb{R}^j\times\mathbb{R}^k} F d\Upsilon_{\alpha,\beta} = \frac{1}{\alpha\beta}\int_{\frac{1}{2}\beta\leq|w|\leq\beta}\int_{\frac{1}{2}\alpha\leq|v|\leq\alpha}\frac{\Theta(v,w)g(|v|,|w|)}{|v|^{j-1}|w|^{k-1}}F(\varphi_1(|w|)v,\varphi_2(|v|)w)\,dvdw,$$

$$\Upsilon_g^*(F) = \sup_{\alpha,\beta\in\mathbb{R}_+}\left|\left|\Upsilon_{\alpha,\beta}\right| * F\right|,$$

and

$$\mathrm{M}_g(F) = \sup_{m,n\in\mathbb{Z}}\int_{\mu^n}^{\mu^{n+1}}\int_{\mu^m}^{\mu^{m+1}}\left|\left|\Upsilon_{\alpha,\beta}\right| * F\right|\frac{d\alpha d\beta}{\alpha\beta},$$

where $\left|\Upsilon_{\alpha,\beta}\right|$ is defined as $\Upsilon_{\alpha,\beta}$ with replacing $g\Theta$ by $|g\Theta|$.

We start by the following result which plays a key part in this work.

Lemma 2.1. *Let $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\Theta \in L^q\left(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}\right)$ with $\lambda, q > 1$. Suppose that $\varphi_1 \in \mathcal{G}_{d_1}$ and $\varphi_2 \in \mathcal{G}_{d_2}$. Then for all $m, n \in \mathbb{Z}$, we have*

$$\left\|\Upsilon_{\alpha,\beta}\right\| \leq C_{\Theta,g}, \tag{2.1}$$

$$\int_{\mu^n}^{\mu^{n+1}}\int_{\mu^m}^{\mu^{m+1}}\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right|^2\frac{d\alpha d\beta}{\alpha\beta} \leq C_{\Theta,g}^2(\ln\mu)^2\left|\mu^m\mu^{nd_1}\xi\right|^{\pm\frac{1}{q'\epsilon}}\left|\mu^n\mu^{md_2}\zeta\right|^{\pm\frac{1}{q'\epsilon}}, \tag{2.2}$$

where $\epsilon = \max\{2, \lambda'\}$, $\left\|\Upsilon_{\alpha,\beta}\right\|$ is the total variation of $\Upsilon_{\alpha,\beta}$ and $|a|^{\pm b} = \min\{|a|^b, |a|^{-b}\}$.

Proof. It is easy to check that the inequality (2.1) comes directly from the definition of $\Upsilon_{\alpha,\beta}$. Hölder's inequality along with a simple change of variables leads to

$$\begin{aligned}\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right| &\leq C\int_{\frac{1}{2}\beta}^{\beta}\int_{\frac{1}{2}\alpha}^{\alpha}|g(s,t)|\left|\iint_{\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}}e^{-i(\varphi_1(t)sv\cdot\xi+\varphi_2(s)tw\cdot\zeta)}\,\Theta(v,w)\,d\rho_j(v)\,d\rho_k(w)\right|\frac{dsdt}{st}\\ &\leq C\|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{\frac{1}{2}\beta}^{\beta}\int_{\frac{1}{2}\alpha}^{\alpha}|\mathcal{K}(s,t)|^{\lambda'}\frac{dsdt}{st}\right)^{1/\lambda'},\end{aligned}$$

where

$$\mathcal{K}(s,t) = \iint_{\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}}e^{-i(\varphi_1(t)sv\cdot\xi+\varphi_2(s)tw\cdot\zeta)}\,\Theta(v,w)\,d\rho_j(v)\,d\rho_k(w).$$

If $\lambda \in (1, 2]$, we have that

$$\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right| \leq C\|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}\|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(1-2/\lambda')}\left(\int_{\frac{1}{2}\beta}^{\beta}\int_{\frac{1}{2}\alpha}^{\alpha}|\mathcal{K}(s,t)|^2\frac{dsdt}{st}\right)^{1/\lambda'}.$$

If $\lambda > 2$ then, by using Hölder's inequality we reach that

$$\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right| \leq C\,\|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{\frac{1}{2}\beta}^{\beta}\int_{\frac{1}{2}\alpha}^{\alpha}|\mathcal{K}(s,t)|^2\,\frac{dsdt}{st}\right)^{1/2}.$$

Thus, in either case for λ,

$$\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right| \leq C\,\|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}\,\|\Theta\|^{(\epsilon-2)/\lambda'}_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}\left(\int_{\frac{1}{2}\beta}^{\beta}\int_{\frac{1}{2}\alpha}^{\alpha}|\mathcal{K}(s,t)|^2\,\frac{dsdt}{st}\right)^{1/\epsilon},$$

where $\epsilon = \max\{2,\lambda'\}$. Thus, since $\mu^m \leq \alpha \leq \mu^{m+1}$ and $\mu^n \leq \beta \leq \mu^{n+1}$, then by inequality (3.16) in [26], we conclude that

$$\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right|^2 \leq C\,\|\Theta\|^2_{L^q(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}\,\|g\|^2_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}\left|\mu^m\mu^{nd_1}\xi\right|^{\pm\frac{1}{q'\epsilon}}\left|\mu^n\mu^{md_2}\zeta\right|^{\pm\frac{1}{q'\epsilon}}.$$

Therefore,

$$\int_{\mu^n}^{\mu^{n+1}}\int_{\mu^m}^{\mu^{m+1}}\left|\hat{\Upsilon}_{\alpha,\beta}(\xi,\zeta)\right|^2\frac{d\alpha d\beta}{\alpha\beta} \leq C^2_{\Theta,g}(\ln\mu)^2\left|\mu^m\mu^{nd_1}\xi\right|^{\pm\frac{1}{q'\epsilon}}\left|\mu^n\mu^{md_2}\zeta\right|^{\pm\frac{1}{q'\epsilon}},$$

□

By Theorem 2.2 in [26] and the argument used in the proof of Lemma 3.9 in [27], we get the following:

Lemma 2.2. *Suppose that* $g \in \Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)$ *for some* $1 < \lambda \leq 2$, $\Theta \in L^q\left(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}\right)$ *for some* $1 < q \leq 2$, $\varphi_1 \in \mathcal{G}_{d_1}$, *and* $\varphi_2 \in \mathcal{G}_{d_2}$. *Then, the estimates*

$$\|\Upsilon^*_g(F)\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}(\ln\mu)^2\|F\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}, \tag{2.3}$$

and

$$\|\mathrm{M}_g(F)\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}(\ln\mu)^2\|F\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}, \tag{2.4}$$

hold for all $F \in L^p(\mathbb{R}^j\times\mathbb{R}^k)$ *with* $p \in (1,\infty)$.

Lemma 2.3. *Let* g,Θ,φ_1, *and* φ_2 *be given as in Lemma 2.2. If* $1 < \tau \leq \lambda'$ *then, for any set of functions* $\{\mathcal{A}_{m,n}(\cdot,\cdot),\ m,n\in\mathbb{Z}\}$ *on* $\mathbb{R}^j\times\mathbb{R}^k$, *we have*

$$\left\|\left(\sum_{m,n\in\mathbb{Z}}\int_{\mu^n}^{\mu^{n+1}}\int_{\mu^m}^{\mu^{m+1}}\left|\Upsilon_{\alpha,\beta}*\mathcal{A}_{m,n}\right|^\tau\frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}(\ln\mu)^{2/\tau}\left\|\left(\sum_{m,n\in\mathbb{Z}}\left|\mathcal{A}_{m,n}\right|^\tau\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}, \tag{2.5}$$

for all $\frac{\lambda'\tau}{\lambda'+\tau-1} < p < \frac{\lambda\tau'}{\tau'-\lambda}$.

Proof. We utilize an argument similar to that employed in [20]. We consider the following three cases:
Case 1. For $\tau < p < \frac{\lambda\tau'}{\tau'-\lambda}$. Since $p > \tau$, then by duality, there is a non-negative function $\mathcal{X} \in L^{(p/\tau)'}(\mathbb{R}^j\times\mathbb{R}^k)$ such that $\|\mathcal{X}\|_{L^{(p/\tau)'}(\mathbb{R}^j\times\mathbb{R}^k)} \leq 1$ and

$$\left\|\left(\sum_{m,n\in\mathbb{Z}}\int_{\mu^n}^{\mu^{n+1}}\int_{\mu^m}^{\mu^{m+1}}\left|\Upsilon_{\alpha,\beta}*\mathcal{A}_{m,n}\right|^\tau\frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|^\tau_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}$$

$$= \iint_{\mathbb{R}^j\times\mathbb{R}^k} \sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}(x,y)\right|^\tau \frac{d\alpha d\beta}{\alpha\beta} \mathcal{X}(x,y)dxdy. \tag{2.6}$$

Thanks to Hölder's inequality, we get

$$\begin{aligned} \left|\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}(x,y)\right|^\tau \leq\ & C\,\|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\tau/\tau')} \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}^{(\tau/\tau')} \int_{\beta/2}^{\beta} \int_{\alpha/2}^{\alpha} \iint_{\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}} |\Theta(v,w)| \\ & \times \left|\mathcal{A}_{m,n}(x-\varphi_1(t)sv, y-\varphi_2(s)tw)\right|^\tau d\rho_j(v)d\rho_k(w)\,|g(s,t)|^{\frac{\tau\tau'-\tau\lambda}{\tau'}} \frac{dsdt}{st}, \end{aligned} \tag{2.7}$$

which leads, along with (2.6) and Hölder's inequality, to

$$\begin{aligned} & \left\|\left(\sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}\right|^\tau \frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}^\tau \\ \leq\ & C\,\|g\|_{\Gamma_1(\mathbb{R}_+\times\mathbb{R}_+)}^{(\tau/\tau')} \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\tau/\tau')} \times \iint_{\mathbb{R}^j\times\mathbb{R}^k} \left(\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}(x,y)\right|^\tau\right) \mathrm{M}_{|g|^{\frac{\tau\tau'-\tau\lambda}{\tau'}}}(\ddot{\mathcal{X}})(-x,-y)dxdy \\ \leq\ & C\,\|g\|_{\Gamma_1(\mathbb{R}_+\times\mathbb{R}_+)}^{(\tau/\tau')} \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\tau/\tau')} \left\|\mathrm{M}_{|g|^{\frac{\tau(\tau'-\lambda)}{\tau'}}}(\ddot{\mathcal{X}})\right\|_{L^{(p/\tau)'}(\mathbb{R}^j\times\mathbb{R}^k)} \left\|\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right|^\tau\right\|_{L^{(p/\tau)}(\mathbb{R}^j\times\mathbb{R}^k)}, \end{aligned}$$

where $\ddot{\mathcal{X}}(x,y) = \mathcal{X}(-x,-y)$. Since $|g|^{\frac{\tau(\tau'-\lambda)}{\tau'}} \in \Gamma_{\frac{\tau'\lambda}{\tau(\tau'-\lambda)}}(\mathbb{R}_+\times\mathbb{R}_+)$, by the last inequality and (2.4), we directly obtain (2.5) for $\tau < p < \frac{\lambda\tau'}{\tau'-\lambda}$.

Case 2. For $p = \tau$. It is clear that Hölder's inequality and (2.7) give

$$\begin{aligned} & \left\|\left(\sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}\right|^\tau \frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}^\tau \\ \leq & C\left(\|g\|_{\Gamma_1(\mathbb{R}_+\times\mathbb{R}_+)} \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}\right)^{(\tau/\tau')} \\ & \times \sum_{m,n\in\mathbb{Z}} \iint_{\mathbb{R}^j\times\mathbb{R}^k} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \int_{\beta/2}^{\beta} \int_{\alpha/2}^{\alpha} \iint_{\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}} \left|\mathcal{A}_{m,n}(x-\varphi_1(t)sv, y-\varphi_2(s)tw)\right|^\tau \\ & \times |\Theta(v,w)|\,|g(s,t)|^{\frac{\tau(\tau'-\lambda)}{\tau'}} d\sigma_j(v)d\sigma_k(w)\frac{dsdt}{st}\frac{d\alpha d\beta}{\alpha\beta}dxdy \\ \leq & C(\ln\mu)^2\left(\|g\|_{\Gamma_1(\mathbb{R}_+\times\mathbb{R}_+)} \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}\right)^{(\tau/\tau')+1} \iint_{\mathbb{R}^j\times\mathbb{R}^k} \left(\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}(x,y)\right|^\tau\right) dxdy. \end{aligned} \tag{2.8}$$

Case 3. For $\frac{\lambda'\tau}{\lambda'+\tau-1} < p < \tau$. Let T be the linear operator on $\mathcal{A}_{m,n}$ given by

$$\mathrm{T}(\mathcal{A}_{m,n}) = \Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}.$$

So we have

$$\left\|\left\|\left\|\mathrm{T}(\mathcal{A}_{m,n})\right\|_{L^1([1,\mu)\times[1,\mu)),\frac{d\alpha d\beta}{\alpha\beta}}\right\|_{l^1(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^1(\mathbb{R}^j\times\mathbb{R}^k)} \leq C(\ln\mu)^2 \left\|\left(\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right|\right)\right\|_{L^1(\mathbb{R}^j\times\mathbb{R}^k)}. \tag{2.9}$$

On the other side, by utilizing (2.3), we conclude

$$\left\| \sup_{m,n\in\mathbb{Z}} \sup_{(\alpha,\beta)\in[1,\mu]\times[1,\mu]} \left|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\right| \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^\tau)} \leq \left\| \Upsilon_g^* \left(\sup_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right| \right) \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g} \left\| \sup_{j,k\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right| \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)},$$

for all $\lambda' < p < \infty$, which in turn implies that

$$\left\| \left\| \|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\|_{L^\infty([1,\mu]\times[1,\mu],\frac{d\alpha d\beta}{\alpha\beta})} \right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g} \left\| \left\| \mathcal{A}_{m,n} \right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}. \quad (2.10)$$

Consequently, by interpolating between (2.9) and (2.10), we prove (2.5) for all $\frac{\lambda'\tau}{\lambda'+\tau-1} < p < \tau$. □

Lemma 2.4. *Let $\Theta, \varphi_1, \varphi_2$, and $\{\mathcal{A}_{m,n}(\cdot,\cdot),\ m,n \in \mathbb{Z}\}$ be given as in Lemma 2.2. Suppose that $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $2 < \lambda < \infty$ and $1 < \tau \leq \lambda'$ Then, the estimate (2.5) holds for all $1 < p < \tau$.*

Proof. By duality, there is a set of mappings $\{E_{m,n}(x,y,\alpha,\beta)\}$ on $\mathbb{R}^j \times \mathbb{R}^k \times \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $\left\| \left\| \|E_{m,n}\|_{L^{\tau'}([\mu^m,\mu^{m+1}]\times[\mu^n,\mu^{n+1}],\frac{d\alpha d\beta}{\alpha\beta})} \right\|_{l^{\tau'}(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^{p'}(\mathbb{R}^j\times\mathbb{R}^k)} \leq 1$ and

$$\begin{aligned}
&\left\| \left(\sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}\right|^\tau \frac{d\alpha d\beta}{\alpha\beta} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \\
&= \iint_{\mathbb{R}^j\times\mathbb{R}^k} \sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left(\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}(x,y)\right) E_{j,k}(x,y,\alpha,\beta) \frac{d\alpha d\beta}{\alpha\beta} dxdy \\
&\leq C(\ln\mu)^{2/\tau} \left\| (\mathcal{B}(E_{m,n})^{1/\tau'} \right\|_{L^{p'}(\mathbb{R}^j\times\mathbb{R}^k)} \left\| \left(\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}, \quad (2.11)
\end{aligned}$$

where

$$\mathcal{B}(E_{m,n})(x,y) = \sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * E_{m,n}(x,y,\alpha,\beta)\right|^{\tau'} \frac{d\alpha d\beta}{\alpha\beta}.$$

As $\tau \leq \lambda' < 2 < \lambda$, we obtain that

$$\begin{aligned}
\left|\Upsilon_{\alpha,\beta} * E_{m,n}(x,y)\right|^{\tau'} &\leq C \left(\|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)} \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})} \right)^{\tau/\tau'} \\
\times \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \iint_{\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}} &\left|E_{m,n}(x-\varphi_1(t)sv, y-\varphi_2(s)tw)\right|^{\tau'} |\Theta(v,w)|\, d\rho_j(v) d\rho_k(w) \tfrac{dsdt}{st}. \quad (2.12)
\end{aligned}$$

Again, as $p' > \tau'$, a function $h \in L^{(p'/\tau')'}(\mathbb{R}^j \times \mathbb{R}^k)$ exists such that

$$\left\|\mathcal{B}(E_{m,n})\right\|_{L^{(p'/\tau')}(\mathbb{R}^j\times\mathbb{R}^k)} = \sum_{m,n\in\mathbb{Z}} \iint_{\mathbb{R}^j\times\mathbb{R}^k} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * E_{m,n}(x,y,\alpha,\beta)\right|^{\tau'} \frac{d\alpha d\beta}{\alpha\beta} h(x,y) dxdy.$$

Therefore, by Lemma 2.2 and (2.12), we deduce

$$\begin{aligned}
\left\|\mathcal{B}(E_{m,n})\right\|_{L^{(p'/\tau')}(\mathbb{R}^j\times\mathbb{R}^k)} \leq& C \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\tau'/\tau)} \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}^{\tau'} \|\Upsilon^*(h)\|_{L^{(p'/\tau')'}(\mathbb{R}^j\times\mathbb{R}^k)} \\
&\times \left\| \left(\sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{j+1}} \int_{\mu^m}^{\mu^{k+1}} \left|E_{m,n}(x,y,\alpha,\beta)\right|^{\tau'} \frac{d\alpha d\beta}{\alpha\beta} \right) \right\|_{L^{(p'/\tau')}(\mathbb{R}^j\times\mathbb{R}^k)} \quad (2.13) \\
\leq& C_p \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}^{\tau'} \|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\tau'/\tau)+1} \|h\|_{L^{(p'/\tau')'}(\mathbb{R}^j\times\mathbb{R}^k)}.
\end{aligned}$$

Consequently, by (2.11) and (2.13), we prove that the estimate (2.5) holds for all $1 < p < \tau$. □

Lemma 2.5. *Let $\Theta, \varphi_1, \varphi_2$, and $\{\mathcal{A}_{m,n}(\cdot,\cdot),\ m,n \in \mathbb{Z}\}$ be given as in Lemma 2.2. Assume that $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $1 < \lambda < \infty$ and $\tau \geq \lambda'$. Then, the estimate (2.5) holds for all $\lambda' < p < \infty$.*

Proof. By the inequality (2.3), we directly get

$$\left\|\sup_{j,k\in\mathbb{Z}} \sup_{(\alpha,\beta)\in[1,\mu]\times[1,\mu]} \left|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\right|\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq \left\|\Upsilon_g^*\left(\sup_{j,k\in\mathbb{Z}}\left|\mathcal{A}_{m,n}\right|\right)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g} \left\|\sup_{j,k\in\mathbb{Z}}\left|\mathcal{A}_{m,n}\right|\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \tag{2.14}$$

for $\lambda' < p < \infty$. This means that

$$\left\|\left\|\|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\|_{L^\infty([1,\mu]\times[1,\mu],\frac{d\alpha d\beta}{\alpha\beta})}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_{p,\Theta,g} \left\|\left\|\mathcal{A}_{m,n}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}. \tag{2.15}$$

Thanks to duality, there exists $\mathcal{Y} \in L^{(p/\lambda')'}(\mathbb{R}^j \times \mathbb{R}^k)$ satisfying $\|\mathcal{Y}\|_{L^{(p/\lambda')'}(\mathbb{R}^j\times\mathbb{R}^k)} \leq 1$ and

$$\begin{aligned}
&\left\|\left(\sum_{m,n\in\mathbb{Z}} \int_1^\mu \int_1^\mu \left|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\right|^{\lambda'} \frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\lambda'}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}^{\lambda'} \\
=\ & \iint_{\mathbb{R}^j\times\mathbb{R}^k} \sum_{m,n\in\mathbb{Z}} \int_1^\mu \int_1^\mu \left|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\right|^{\lambda'} \frac{d\alpha d\beta}{\alpha\beta} \mathcal{Y}(x,y)dxdy \\
\leq\ & C\,\|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\lambda'/\lambda)} \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}^{\lambda'} \iint_{\mathbb{R}^j\times\mathbb{R}^k} \left(\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}(x,y)\right|^{\lambda'}\right) \Upsilon_g^*(\ddot{\mathcal{Y}})(x,y)dxdy \\
\leq\ & C(\ln\mu)^2\,\|\Theta\|_{L^1(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})}^{(\lambda'/\lambda)} \|g\|_{\Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)}^{\lambda'} \left\|\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right|^{\lambda'}\right\|_{L^{(p/\lambda')}(\mathbb{R}^j\times\mathbb{R}^k)} \left\|\Upsilon_g^*(\ddot{\mathcal{Y}})\right\|_{L^{(p/\lambda')'}(\mathbb{R}^j\times\mathbb{R}^k)},
\end{aligned} \tag{2.16}$$

where $\ddot{\mathcal{Y}}(x,y) = \mathcal{Y}(-x,-y)$. Use the linear operator T that is defined in the proof of Lemma 2.3, and combine (2.15) with (2.16), we obtain

$$\begin{aligned}
\left\|\left(\sum_{m,n\in\mathbb{Z}} \int_{\mu^n}^{\mu^{n+1}} \int_{\mu^m}^{\mu^{m+1}} \left|\Upsilon_{\alpha,\beta} * \mathcal{A}_{m,n}\right|^\tau \frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} &\leq C \left\|\left(\sum_{m,n\in\mathbb{Z}} \int_1^\mu \int_1^\mu \left|\Upsilon_{\mu^m\alpha,\mu^n\beta} * \mathcal{A}_{m,n}\right|^\tau \frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \\
&\leq C_{p,\Theta,g}(\ln\mu)^{2/\tau} \left\|\left(\sum_{m,n\in\mathbb{Z}} \left|\mathcal{A}_{m,n}\right|^\tau\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}
\end{aligned}$$

for all $\lambda' < p < \infty$ with $\lambda' < \tau$. The proof is complete. □

3. Proof of the major results

Let us first prove Theorem 1.1. Assume that $\Theta \in L^q\left(\mathbb{U}^{j-1} \times \mathbb{U}^{k-1}\right)$ and $g \in \Gamma_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ with $q, \lambda \in (1,2]$. It is clear that Minkowski's inequality leads to

$$\mathcal{H}_{\Theta,\Omega,g}^{(\tau)}(F)(x,y) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\sum_{m,n=0}^{\infty} \frac{1}{\alpha\beta} \iint_{\Lambda_{\alpha,\beta}(v,w)} \frac{\Theta(v,w)g(|v|,|w|)}{|v|^{j-1}|w|^{k-1}}\right.\right.$$

$$\times F\left(x-\varphi_1(|w|)v,y-\varphi_2(|v|)w\right)\ dvdw\ \Bigg|^{\tau}\frac{d\alpha d\beta}{\alpha\beta}\Bigg)^{1/\tau}$$

$$\leq\ C\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\Upsilon_{\alpha,\beta}*F(x,y)\right|^{\tau}\frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}, \qquad (3.1)$$

where $\Gamma_{\alpha,\beta}(v,w)=\{(v,w)\in\mathbb{R}^j\times\mathbb{R}^k:|v|\in(2^{-m-1}\alpha,2^{-m}\alpha],\ \ |w|\in(2^{-n-1}\beta,2^{-n}\beta]\}$.

For $m,n\in\mathbb{Z}$, we let $\{f_{1,m,n}\}_{-\infty}^{\infty}$ $\{f_{2,m,n}\}_{-\infty}^{\infty}$ be sets of partition of unity in the space $C^{\infty}(\mathbb{R}_+)$, such that

$$0\leq f_{1,m,n}\leq 1,\ \ \sum_{m,n\in\mathbb{Z}}(f_{1,m,n}f_{2,m,n})^2=1,$$

$$\text{supp } f_{1,m,n}\ \subseteq\ \{t:\frac{4}{5}\mu^{-(m+1)}\mu^{-d_1(n+1)}<t<\frac{5}{4}\mu^{-(m-1)}\mu^{-d_1(n-1)}\}\equiv\mathbf{I}_1,$$

$$\text{supp } f_{2,m,n}\ \subseteq\ \{t:\frac{4}{5}\mu^{-(n+1)}\mu^{-d_2(m+1)}<t<\frac{5}{4}\mu^{-(n-1)}\mu^{-d_2(m-1)}\}\equiv\mathbf{I}_2,$$

$$\left|\frac{d^sf_{1,m,n}(t)}{dt^s}\right|\leq\frac{C_s}{t^s},\ \ and\ \ \left|\frac{d^sf_{2,m,n}(t)}{dt^s}\right|\leq\frac{C_s}{t^s},$$

where C_s does not depend on μ. Let $S_{m,n}$ be the multiplier operators defined on $\mathbb{R}^j\times\mathbb{R}^k$ by

$$(\widehat{S_{m,n}F})(\xi,\zeta)=f_{1,m,n}(|\xi|)f_{2,m,n}(|\zeta|)\hat{F}(\xi,\zeta).$$

So, for any $F\in\mathcal{S}(\mathbb{R}^j\times\mathbb{R}^k)$, we obtain that

$$\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}(F)(x,y)\leq\sum_{m,n\in\mathbb{Z}}\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)(x,y), \qquad (3.2)$$

where

$$\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)(x,y)=\left(\iint_{\mathbb{R}^+\times\mathbb{R}^+}\left|\mathcal{N}_{\Theta,\Omega,g,m,n}(x,y,\alpha,\beta)\right|^{\tau}\frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau},$$

$$\mathcal{N}_{\Theta,\Omega,g,m,n}(x,y,\alpha,\beta)=\sum_{a,b\in\mathbb{Z}}(S_{a+m,b+n}*\Upsilon_{\alpha,\beta}*F)(x,y)\chi_{[\mu^a,\mu^{a+1})\times[\mu^b,\mu^{b+1})}(\alpha,\beta).$$

Let us estimate the norm of $\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)$. First, we consider the case $p=2=\tau$. Thanks to Fubini's theorem as well as Plancherel's theorem and inequality (2.1), we have

$$\left\|\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)\right\|^2_{L^2(\mathbb{R}^j\times\mathbb{R}^k)}\leq\sum_{a,b\in\mathbb{Z}}\iint_{\mathcal{D}_{m+a,n+b}}\left(\int_{\mu^a}^{\mu^{a+1}}\int_{\mu^b}^{\mu^{b+1}}\left|\hat{\Upsilon}_{\alpha,\beta}(\zeta,\xi)\right|^2\frac{d\alpha d\beta}{\alpha\beta}\right)\left|\hat{F}(\zeta,\xi)\right|^2d\zeta d\xi$$

$$\leq\ C^2_{\Theta,g}(\ln\mu)^2\sum_{a,b\in\mathbb{Z}}\iint_{\mathcal{D}_{m+a,n+b}}\left(\left|\mu^a\mu^{bd_1}\xi\right|^{\pm\frac{1}{q'\epsilon}}\left|\mu^b\mu^{ad_2}\zeta\right|^{\pm\frac{1}{q'\epsilon}}\right)\left|\hat{F}(\zeta,\xi)\right|^2d\zeta d\xi$$

$$\leq\ C^2_{\Theta,g}(\ln\mu)^2\ 2^{-\epsilon(|m|+|n|)}\sum_{m,n\in\mathbb{Z}}\iint_{\mathcal{D}_{m+a,n+b}}\left|\hat{F}(\zeta,\xi)\right|^2d\zeta d\xi$$

$$\leq\ C_pC^2_{\Theta,g}(\ln\mu)^2\ 2^{-\epsilon(|m|+|n|)}\left\|F\right\|^2_{L^2(\mathbb{R}^j\times\mathbb{R}^k)},$$

where $\mathcal{D}_{m,n} = \left\{(\zeta,\xi) \in \mathbb{R}^j \times \mathbb{R}^k : (|\zeta|, |\xi|) \in \mathbf{I}_1 \times \mathbf{I}_2\right\}$ and $\epsilon \in (0,1)$. Thus, choosing $\mu = 2^{\lambda' q'}$ leads to

$$\left\|\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)\right\|_{L^2(\mathbb{R}^j\times\mathbb{R}^k)} \leq C_p C_{\Theta,g}\left((\lambda-1)(q-1)\right)^{-1}\ 2^{-\frac{\epsilon}{2}(|m|+|n|)}\|F\|_{\dot{F}_0^{2,2}(\mathbb{R}^j\times\mathbb{R}^k)}. \tag{3.3}$$

Next, the L^p-norm of $\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)$ for the other cases of p is estimated by following the arguments employed in [7]. In fact, we utilize the Littlewood-Paley theory together with Lemmas 2.3 and 2.5, so we conclude

$$\begin{aligned}
&\left\|\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}\\
\leq\ & C\left\|\left(\sum_{m,n\in\mathbb{Z}}\int_{\mu^m}^{\mu^{m+1}}\int_{\mu^n}^{\mu^{n+1}}\left|\Upsilon_{\alpha,\beta} * S_{m+\alpha,n+\beta} * F\right|^\tau \frac{d\alpha d\beta}{\alpha\beta}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)}\\
\leq\ & \left\|\left(\sum_{m,n\in\mathbb{Z}}\left|S_{m+\alpha,n+\beta} * F\right|^\tau\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} C_{p,\Theta}(\ln\mu)^{2/\tau}\\
\leq\ & \left(\frac{1}{(\lambda-1)(q-1)}\right)^{2/\tau}\|F\|_{\dot{F}_p^{\tau,\vec{0}}(\mathbb{R}^j\times\mathbb{R}^k)} C_{p,\Theta}
\end{aligned} \tag{3.4}$$

for all $\frac{\lambda'\tau}{\lambda'+\tau-1} < p < \frac{\lambda\tau'}{\tau'-\lambda}$ with $\tau \leq \lambda'$; and also holds for all $\lambda' < p < \infty$ with $\tau \geq \lambda'$. Therefore, by interpolating between (3.3) and (3.4), we obtain that

$$\left\|\mathcal{M}^{(\tau)}_{\Theta,\Omega,g,m,n}(F)\right\|_{L^p(\mathbb{R}^j\times\mathbb{R}^k)} \leq \left(\frac{1}{(\lambda-1)(q-1)}\right)^{2/\tau}\|F\|_{\dot{F}_p^{\tau,\vec{0}}(\mathbb{R}^j\times\mathbb{R}^k)} C_{p,\Theta}\, 2^{-\frac{\epsilon}{2}(|m|+|n|)}, \tag{3.5}$$

for $\frac{\lambda'\tau}{\lambda'+\tau-1} < p < \frac{\lambda\tau'}{\tau'-\lambda}$ with $\tau \leq \lambda'$; and for $\lambda' < p < \infty$ with $\tau \geq \lambda'$, which in turn with (3.2) finish the proof of Theorem 1.1.

The proof of Theorem 1.2 can be obtained by following the same above argument with invoking Lemma 2.4 instead of Lemma 2.3.

4. Conclusions

In this paper, we introduced the generalized Marcinkiewicz integrals $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ along the twisted surfaces

$$\left\{\Omega(v,w) = (\varphi_1(|w|)v, \varphi_2(|v|)w) : (v,w) \in \mathbb{R}^j \times \mathbb{R}^k,\ \varphi_1 \in \mathcal{G}_{d_1} \textit{ and } \varphi_2 \in \mathcal{G}_{d_2}\right\}.$$

We established appropriate L^p bounds for $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ whenever $\Theta \in L^q\left(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1}\right)$ and $g \in \Gamma_\lambda(\mathbb{R}_+\times\mathbb{R}_+)$, which enables us to employ Yano's extrapolation approach to prove the L^p boundedness of $\mathcal{H}^{(\tau)}_{\Theta,\Omega,g}$ under the weaker conditions $\Theta \in L(\log L)^{2/\tau}(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})$ or $\Theta \in B_q^{(0,\frac{2}{\tau}-1)}(\mathbb{U}^{j-1}\times\mathbb{U}^{k-1})$ with $\tau, q > 1$. For the case $g \equiv 1$, we confirmed the L^p boundedness of the aforementioned operator for the full range of $p \in (1,\infty)$; and for case $\tau = 2$, the conditions on Θ are the weakest possible conditions among their respective classes. In this work, several known results are generalized and improved.

Author contributions

Hussain Al-Qassem: writing-original draft, commenting; Mohammed Ali: writing-original draft, formal analysis, commenting. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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