Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 13825–13835.
DOI: 10.3934/math.2025622
Received: 17 February 2025
Revised: 16 May 2025
Accepted: 20 May 2025
Published: 16 June 2025

Research article

Oscillation conditions of nonlinear neutral differential equations with several delays

Maryam AlKandari*

Department of Mathematics, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

* **Correspondence:** Email: Maryam.Alkandari@Ku.edu.kw.

Abstract: In this paper, we examine whether oscillatory solutions to an even-order differential equation with multiple delays exist. We create new oscillation criteria using the comparison method. When comparing the results obtained in this paper with some of the results in the literature, we find that the results we obtained give better values for the oscillation of the studied equation, and thus we find that the results we obtained expand some of the results in the literature. To further emphasize the relevance of our proposed criteria, we offer some instances to support and exemplify our results.

Keywords: oscillatory solutions; even order neutral differential equations; several delays
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this manuscript we discuss even-order equations and the oscillation conditions for their solutions in the following form:

$$\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{p-1}\right)' + \sum_{i=1}^{r} b_i(\theta) z^{p-1}(\varphi_i(\theta)) = 0, \theta \geq \theta_0, \tag{1.1}$$

under the following assumptions:

(H$_1$) $m \in C[\theta_0, \infty)$, $m(\theta) > 0$, $m'(\theta) \geq 0$, $a, b_i \in C[\theta_0, \infty)$, $b_i(\theta) > 0$, $0 \leq a(\theta) < a_0 < \infty$, $\int_{\theta_0}^{\infty} m^{-1/(p-1)}(s)\, \mathrm{d}s = \infty$,

(H$_2$) $\emptyset \in C^1[\theta_0, \infty), \varphi_i \in C[\theta_0, \infty)$, $\emptyset'(\theta) > 0$, $\emptyset(\theta) \leq \theta$, $\lim_{\theta\to\infty} \emptyset(\theta) = \lim_{\theta\to\infty} \varphi_i(\theta) = \infty, i = 1, 2, 3, ..., r$,

(H$_3$) $j \geq 4$ is an even natural number, $(1 < p < \infty)$, $p-$Laplace type operator and $\varpi(\theta) := z(\theta) + a(\theta) z(\emptyset(\theta))$.

The solution to (1.1) is said to be oscillatory if it is ultimately neither positive nor negative. This solution is considered non-oscillatory otherwise. Therefore, we say that this. The Eq (1.1) is oscillatory if it has oscillatory solutions.

The differential equations (DEs) are a mathematical depiction of many physical wonders [1]. Electronic, chemical, biological, and transportation architectures are just a few of the many domains where these equations are used. DEs with temporal delays are required when simulating real-world problems [2, 3].

Variable delay DEs are useful for modeling a variety of systems, including electrical circuits, kinetics, physiological systems, human body control systems, ship and aviation control systems, and transmissible diseases [4, 5]. For example, in dynamical models, nonlinear diffusion and/or external sources are frequently used to design delay and oscillation situations, which disrupt the natural evolution of connected systems; see, for example, [6, 7].

Fite authored a seminal paper in the first quarter of the 20th century that established the oscillation theory of DEs with diverging reasons. Since then, a great deal of research has been done on the oscillation of solutions of many kinds of differential and functional DEs. The interest in this subject is evident from the large number of references in monographs [8, 9]. The papers [10, 11] and the references cited within them are also referred to by the reader.

A neutral DE is one in which the highest-order derivative of the unknown function appears both immediately and without delay. Neutral DE oscillation has received a lot of interest lately in research; see, for instance, [12, 13]. This is due to the fact that identical equations are utilized in a wide range of applications, including population dynamics, automatic control, mixing liquids, vibrating masses coupled to an elastic bar, and more (see [14]).

Neutral delay of the second order DEs are used to represent systems in which prior states have an impact on the current state in addition to the current inputs and state. Because of this, they can be used to represent biological systems that have feedback and delayed reactions [15, 16].

Zafer [17] constructed oscillation criteria by deducing new properties of positive solutions of the DE

$$\varpi^{(j)}(\theta) + b(\theta)\, z(\varphi(\theta)) = 0. \tag{1.2}$$

He obtained this new criterion

$$\liminf_{\theta\to\infty} \int_{\varphi(\theta)}^{\theta} B(s)\,\mathrm{d}s > \frac{(j-1)\,2^{(j-1)(j-2)}}{\mathrm{e}}, \tag{1.3}$$

or

$$\limsup_{\theta\to\infty} \int_{\varphi(\theta)}^{\theta} B(s)\,\mathrm{d}s > (j-1)\,2^{(j-1)(j-2)}, \quad \varphi'(\theta) \geq 0,$$

where $B(\theta) := \varphi^{j-1}(\theta)(1 - a(\varphi(\theta)))\, b(\theta)$. Their results extend, complete, and simplify some results in previous studies.

Zhang and Yan [18] studied the oscillatory behavior of (1.2). They presented new oscillation conditions

$$\lim\inf_{\theta\to\infty} \int_{\varphi(\theta)}^{\theta} B(s)\,\mathrm{d}s > \frac{(j-1)!}{\mathrm{e}}, \tag{1.4}$$

or

$$\limsup_{\theta\to\infty} \int_{\varphi(\theta)}^{\theta} B(s)\,\mathrm{d}s > (j-1)!, \quad \varphi(\theta) \geq 0,$$

In work [19], derived oscillation criteria of delay DEs and proved that the conditions

$$\left(\varphi^{-1}(\theta)\right)' \geq \varphi_0 > 0,\ \emptyset'(\theta) \geq \emptyset_0 > 0,\ \emptyset^{-1}(\varphi(\theta))\,\theta,$$

and

$$\liminf_{\theta\to\infty} \int_{\emptyset^{-1}(\varphi(\theta))}^{\theta} \frac{\widehat{b}(s)}{m(s)} \left(s^{j-1}\right)^{(p-1)} \mathrm{d}s > \left(\frac{1}{\varphi_0} + \frac{a_0^{(p-1)}}{\varphi_0\emptyset_0}\right) \frac{((j-1)!)^{(p-1)}}{\mathrm{e}}, \tag{1.5}$$

where $\widehat{b}(\theta) := \min\left\{b\left(\varphi^{-1}(\theta)\right), b\left(\varphi^{-1}(\emptyset(\theta))\right)\right\}$ makes the Eq (1.1) oscillatory.

Recently, some of the oscillation results for the equation

$$\left(m(\theta)\left|\varpi^{(j-1)}(\theta)\right|^{p-2} \varpi^{(j-1)}(\theta)\right)' + b(\theta)\left|z(\varphi(\theta))\right|^{p-2} z(\varphi(\theta)) = 0, \theta \geq \theta_0,$$

where $\int_{s_0}^{\infty-} m^{1/(p-1)}(\upsilon)\,\mathrm{d}\upsilon = \infty$, were improved and extended by the authors in [20, 21] by establishing new circumstances.

Liu et al. [22] investigated the oscillatory behavior of the DE

$$\left(m(\theta)\left|\varpi^{(n)}(\theta)\right|^{p-2} \varpi^{(n)}(\theta)\right)' + r(\theta)\left|\varpi^{(n)}(s)\right|^{p-2} \varpi^{(n)}(\theta) + b(\theta)\left|z(\varphi(\theta))\right|^{p-2} z(\varphi(\theta)) = 0,$$

where

$$\varpi(\theta) := z(\theta) + a(\theta)\, z(\emptyset(\theta)).$$

They were initially interested in introducing some new monotonic properties for the solutions of this equation, and then used these properties to obtain new oscillatory conditions which ensure that all solutions of this equation are oscillatory. Their results complement and extend some related results in the literature.

From the above, we note that there are many studies that have been interested in studying the even-order DEs in different forms, whether in the canonical case or in the non-canonical case. We also know that there are few studies that have been interested in studying the DE (1.1), and most of them were interested in the non-canonical case. As a result, the aim of this paper was to study the DE (1.1) in the canonical case and to find new criteria that expand some of the previous studies. We discussed some examples to illustrate the effectiveness of our main criteria.

2. Main results

The notations utilized in this paper are shown below:

$$A_\ell(\theta) = \frac{1}{a(\emptyset^{-1}(\theta))}\left(1 - \frac{\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)^{\ell-1}}{\left(\emptyset^{-1}(\theta)\right)^{\ell-1} a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}\right), \text{ for } \ell = 2, j,$$

$$D_0(\theta) = \left(\frac{1}{m(\theta)} \int_\theta^\infty \sum_{i=1}^{r} b_i(s)\, A_2^{(p-1)}(\varphi_i(s))\,\mathrm{d}s\right)^{1/(p-1)},$$

and

$$D_\nu(\theta) = \int_\theta^\infty D_{\nu-1}(s)\,\mathrm{d}s, \quad \nu = 1, 2, ..., j-3.$$

We state the following lemma, which we will need to prove our results later.

Lemma 1. *[23, Lemma 1 and 2] If* $w, g \geq 0$. *Then*

$$(w+g)^k \leq w^k + g^k, \text{ for } k \leq 1,$$

and

$$(w+g)^k \leq 2^{k-1}\left(w^k + g^k\right), \text{ for } k \geq 1,$$

where k is a positive real number.

Lemma 2. *[24] Let* $z \in C^j([\theta_0, \infty), (0, \infty)), z^{(j-1)}(\theta) z^{(j)}(\theta) \leq 0$, *where* $z^{(j)}(\theta)$ *is of fixed sign and not identically zero on* $[\theta_0, \infty)$. *If* $\lim_{\theta \to \infty} z(\theta) \neq 0$, *then*

$$z(\theta) \geq \frac{\varepsilon}{(j-1)!}\theta^{j-1}\left|z^{(j-1)}(\theta)\right|,$$

where $\theta \geq \theta_\varepsilon, \varepsilon \in (0,1)$.

Lemma 3. *[20] Let* $z^{(j+1)}(\theta) < 0$, *then*

$$\frac{z(\theta)}{\theta^j/j!} \geq \frac{z'(\theta)}{\theta^{j-1}/(j-1)!},$$

where $z^{(i)}(\theta) > 0$, $i = 0, 1, ..., j$.

Lemma 4. *[25, Lemma 1.2] The ultimate positive solution to (1.1) is found to be the function z. Thus, we identify two instances:*

$$\begin{array}{ll} (\mathbf{I}_1) & \varpi(\theta) > 0,\ \varpi^{(j)}(\theta) < 0,\ \varpi^{(j-1)}(\theta) > 0,\ \varpi^{(c)}(\theta) > 0, c = 1, 2, \\ (\mathbf{I}_2) & \varpi(\theta) > 0, \varpi^{(r+1)}(\theta) < 0, \varpi^{(r)}(\theta) > 0 \textit{ for } r \in \{1, 3, ..., j-3\}, \\ & \varpi^{(j-1)}(\theta) > 0, \varpi^{(j)}(\theta) < 0, \end{array}$$

for $\theta \geq \theta_1$.

Lemma 5. *Case* $(\mathbf{I}_1)$ *holds, and the function z is shown to be the ultimate positive solution to (1.1). Let*

$$\frac{\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)^{j-1}}{\left(\emptyset^{-1}(\theta)\right)^{j-1} a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)} \leq 1. \tag{2.1}$$

Then

$$\varpi(\theta) \geq \frac{\varepsilon}{(j-1)!}\theta^{j-1}\varpi^{(j-1)}(\theta). \tag{2.2}$$

Proof. Given that $z(\theta)$ constitutes the final positive solution of (1.1). If Case $(\mathbf{I}_1)$ holds.
Using

$$\varpi(\theta) := z(\theta) + a(\theta) z(\emptyset(\theta)),$$

we find

$$z(\theta) = \frac{1}{a\left(\emptyset^{-1}(\theta)\right)}\left(\varpi\left(\emptyset^{-1}(\theta)\right) - z\left(\emptyset^{-1}(\theta)\right)\right).$$

Thus, we obtain

$$z(\theta) = \frac{\varpi\left(\emptyset^{-1}(\theta)\right)}{a\left(\emptyset^{-1}(\theta)\right)} - \frac{1}{a\left(\emptyset^{-1}(\theta)\right)}\left(\frac{\varpi\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}{a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)} - \frac{z\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}{a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}\right)$$

$$\geq \frac{\varpi\left(\emptyset^{-1}(\theta)\right)}{a\left(\emptyset^{-1}(\theta)\right)}-\frac{1}{a\left(\emptyset^{-1}(\theta)\right)}\frac{\varpi\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}{a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}. \tag{2.3}$$

Applying Lemma 3, we see

$$\varpi(\theta) \geq \frac{1}{(j-1)}\theta\varpi'(\theta),$$

so $\theta^{1-j}\varpi(\theta)$ is nonincreasing, and from $\emptyset(\theta) \leq \theta$, we see

$$\left(\emptyset^{-1}(\theta)\right)^{j-1}\varpi\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right) \leq \left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)^{j-1}\varpi\left(\emptyset^{-1}(\theta)\right). \tag{2.4}$$

Combining (2.3) and (2.4), we conclude that

$$\begin{aligned} z(\theta) &\geq \frac{1}{a\left(\emptyset^{-1}(\theta)\right)}\left(1-\frac{\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)^{j-1}}{\left(\emptyset^{-1}(\theta)\right)^{j-1}a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}\right)\varpi\left(\emptyset^{-1}(\theta)\right) \\ &= A_j(\theta)\varpi\left(\emptyset^{-1}(\theta)\right). \end{aligned} \tag{2.5}$$

By (1.1) and (2.5), we have

$$\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}\right)' + \sum_{i=1}^{r} b_i(\theta) A_j^{(p-1)}(\varphi_i(\theta))\varpi^{(p-1)}\left(\emptyset^{-1}(\varphi_i(\theta))\right) \leq 0.$$

Since $w(\theta) \leq \varphi_i(\theta)$ and $\varpi'(\theta) > 0$, we get

$$\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}\right)' \leq -\sum_{i=1}^{r} b_i(\theta) A_j^{(p-1)}(\varphi_i(\theta))\varpi^{(p-1)}\left(\emptyset^{-1}(w(\theta))\right). \tag{2.6}$$

Now, by using Lemma 2, we have

$$\varpi(\theta) \geq \frac{\varepsilon}{(j-1)!}\theta^{j-1}\varpi^{(j-1)}(\theta).$$

for some $\varepsilon \in (0,1)$. Therefore, the proof is finished. □

Lemma 6. *If (2.1) and Case* ($\mathbf{I}_2$) *holds. Then*

$$\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}\right)' \leq -\sum_{i=1}^{r} b_i(\theta) A_2^{(p-1)}(\varphi_i(\theta))\varpi^{(p-1)}\left(\emptyset^{-1}(\varrho(\theta))\right). \tag{2.7}$$

Proof. Given that $z(\theta)$ constitutes the final positive solution of (1.1). If Case ($\mathbf{I}_2$) holds. By Lemma 3, we have

$$\varpi(\theta) \geq \theta\varpi'(\theta), \tag{2.8}$$

so $\theta^{-1}\varpi(\theta)$ is nonincreasing eventually.
Since

$$\emptyset^{-1}(\theta) \leq \emptyset^{-1}\left(\emptyset^{-1}(\theta)\right),$$

thus, we find

$$\emptyset^{-1}(\theta)\,\varpi\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right) \leq \emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\varpi\left(\emptyset^{-1}(\theta)\right). \tag{2.9}$$

Combining (2.3) and (2.9), we find

$$\begin{aligned} z(\theta) &\geq \frac{1}{a\left(\emptyset^{-1}(\theta)\right)}\left(1-\frac{\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}{\left(\emptyset^{-1}(\theta)\right)a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}\right)\varpi\left(\emptyset^{-1}(\theta)\right) \\ &= A_2(\theta)\,\varpi\left(\emptyset^{-1}(\theta)\right), \end{aligned}$$

which with (1.1) yields

$$\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}\right)' + \sum_{i=1}^{r} b_i(\theta)\,A_2^{(p-1)}(\varphi_i(\theta))\,\varpi^{(p-1)}\left(\emptyset^{-1}(\varphi_i(\theta))\right) \leq 0.$$

Since $\varrho(\theta) \leq \varphi_i(\theta)$ and $\varpi'(\theta) > 0$, we have that

$$\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}\right)' \leq -\sum_{i=1}^{r} b_i(\theta)\,A_2^{(p-1)}(\varphi_i(\theta))\,\varpi^{(p-1)}\left(\emptyset^{-1}(\varrho(\theta))\right).$$

The proof is finished. □

Theorem 1. *Let positive functions $w,\ \varrho \in C^1([\theta_0,\infty),\mathbb{R})$ satisfy*

$$w(\theta) \leq \varphi_i(\theta),\ w(\theta) < \emptyset(\theta),\ \varrho(\theta) \leq \varphi_i(\theta),\ \varrho(\theta) < \emptyset(\theta),\ \varrho'(\theta) \geq 0 \text{ and } \lim_{\theta\to\infty} w(\theta) = \lim_{\theta\to\infty}\varrho(\theta) = \infty. \tag{2.10}$$

If the equations

$$\varpi'(\theta) + \left(\frac{\varepsilon\left(\emptyset^{-1}(w(\theta))\right)^{j-1}}{(j-1)!m^{1/(p-1)}\left(\emptyset^{-1}(w(\theta))\right)}\right)^{(p-1)} \sum_{i=1}^{r} b_i(\theta)\,A_j^{(p-1)}(\varphi_i(\theta))\,\varpi\left(\emptyset^{-1}(w(\theta))\right) = 0, \tag{2.11}$$

where $\varepsilon \in (0,1)$, and

$$\phi'(\theta) + \emptyset^{-1}(\varrho(\theta))\,D_{j-3}(\theta)\,\phi\left(\emptyset^{-1}(\varrho(\theta))\right) = 0, \tag{2.12}$$

are oscillatory, then (1.1) is oscillatory.

Proof. Given that $z(\theta)$ constitutes the final positive solution of (1.1). The cases ($\mathbf{I}_1$) and ($\mathbf{I}_2$) are holds from Lemma 4. Using Lemma 5, we see from (2.6) and (2.2) that, for all $\varepsilon \in (0,1)$,

$$\left(\frac{\varepsilon\left(\emptyset^{-1}(w(\theta))\right)^{j-1}}{(j-1)!}\right)^{(p-1)} \sum_{i=1}^{r} b_i(\theta)\,A_j^{(p-1)}(\varphi_i(\theta))\left(\varpi^{(j-1)}\left(\emptyset^{-1}(w(\theta))\right)\right)^{(p-1)} \leq -\left(m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}\right)'.$$

Thus, if we choose $\varpi(\theta) = m(\theta)\left(\varpi^{(j-1)}(\theta)\right)^{(p-1)}$, we have ϖ is a positive solution of

$$\varpi'(\theta) + \left(\frac{\varepsilon\left(\emptyset^{-1}(w(\theta))\right)^{j-1}}{(j-1)!m^{1/(p-1)}\left(\emptyset^{-1}(w(\theta))\right)}\right)^{(p-1)} \sum_{i=1}^{r} b_i(\theta)\,A_j^{(p-1)}(\varphi_i(\theta))\,\varpi\left(\emptyset^{-1}(w(\theta))\right) \leq 0.$$

From [26, Theorem 1], it is also clear to us that the Eq (2.11) becomes a positive solution. Therefore, it is clear that this is a contradiction.

By Lemma 4, integrating (2.7) from θ to ∞, we obtain

$$\varpi^{(j-1)}(\theta) \geq D_0(\theta)\,\varpi\left(\emptyset^{-1}(\varrho(\theta))\right).$$

Integrating from θ to ∞ a total of $j-3$ times, we obtain

$$\varpi''(\theta) + D_{j-3}(\theta)\,\varpi\left(\emptyset^{-1}(\varrho(\theta))\right) \leq 0. \tag{2.13}$$

So, if we choose $\phi(\theta) := \varpi'(\theta)$ and apply (2.8), we find

$$\phi'(\theta) + \emptyset^{-1}(\varrho(\theta))\,D_{j-3}(\theta)\,\phi\left(\emptyset^{-1}(\varrho(\theta))\right) \leq 0, \tag{2.14}$$

where ϕ is a positive solution. Based on [26, Theorem 1], it is also clear to us that Eq (2.12) becomes a positive solution. Therefore, it is clear that this is a contradiction, so the proof is finished. □

Corollary 1. *Let (2.1) and (2.10) holds. If*

$$\liminf_{\theta\to\infty}\int_{\emptyset^{-1}(w(\theta))}^{\theta}\left(\frac{\left(\emptyset^{-1}(w(s))\right)^{j-1}}{m^{1/(p-1)}\left(\emptyset^{-1}(w(s))\right)}\right)^{(p-1)}\sum_{i=1}^{r} b_i(s)\,A_j^{(p-1)}(\varphi_i(s))\,\mathrm{d}s > \frac{((j-1)!)^{(p-1)}}{\mathrm{e}}, \tag{2.15}$$

and

$$\liminf_{\theta\to\infty}\int_{\emptyset^{-1}(\varrho(\theta))}^{\theta}\emptyset^{-1}(\varrho(s))\,D_{j-3}(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.16}$$

then (1.1) is oscillatory.

Proof. Based on ([27, Theorem 2]) we see that Conditions (2.15) and (2.16) imply oscillation of (2.11) and (2.12), respectively. □

Now, we will give some examples to illustrate the effectiveness of our results.

Example 1. *Consider the equation:*

$$\left[\theta\left(z(\theta) + \frac{1}{2}z\left(\frac{\theta}{3}\right)\right)'''\right]' + \frac{b_0}{\theta}\left(z^2+z\right)\left(\frac{\theta}{2}\right) = 0, \iota \geq 1, \tag{2.17}$$

where $b_0 > 0$ is a constant. Let $p = 2,\ m(\theta) = \theta,\ a(\theta) = 1/2,\ \emptyset(\theta) = \theta/3,\ b(\theta) = b_0/\theta$, and $\varphi(\theta) = \theta/2$.
Now, we find

$$\int_{\theta_0}^{\infty} m^{-1/(p-1)}(s)\,\mathrm{d}s$$
$$= \int_{\theta_0}^{\infty} 1/s\mathrm{d}s = \infty.$$

Let $w(\theta) = \varrho(\theta) = \beta\theta$; we see that (2.1) and (2.10) are satisfied.
Also, we obtain

$$\begin{aligned} A_\ell(\theta) &= \frac{1}{a(\emptyset^{-1}(\theta))}\left(1 - \frac{\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)^{\ell-1}}{\left(\emptyset^{-1}(\theta)\right)^{\ell-1} a\left(\emptyset^{-1}\left(\emptyset^{-1}(\theta)\right)\right)}\right) \\ &= \frac{1}{2}\ \left(1 - 2^{2(\ell-1)}3^{(1-\ell)}\right)\ \textit{for } \ell = 2, j, \end{aligned}$$

and

$$\begin{aligned} D_0(\theta) &= \left(\frac{1}{m(\theta)}\int_\theta^\infty b(s)\, A_2^{(p-1)}(\varphi_i(s))\,\mathrm{d}s\right)^{1/(p-1)} \\ &= \frac{1}{\theta}\int_\theta^\infty \frac{b_0}{2s}\left(1 - \frac{2^2}{3}\right)\mathrm{d}s \\ &= \infty \end{aligned}$$

By Corollary 1, we see (2.17) is oscillatory.

Example 2. *Consider the equation*

$$(z(\theta) + a_0 z(\pi\theta))^{(j)} + \frac{b_0}{\theta^j} z(\beta\theta) = 0, \theta \geq 1,\ b_0 > 0, \tag{2.18}$$

where $\pi \in \left(a_0^{-1/(j-1)}, 1\right)$ and $\beta \in (0, \pi)$, $m(\theta) = 1, p = 2$, $a(\theta) = a_0$, $\emptyset(\theta) = \pi\theta$, $\varphi(\theta) = \beta\theta$ and $b(\theta) = b_0/\theta^j$.
If we set $w(\theta) = \varrho(\theta) = \beta\theta$, then it's easy to obtain that (2.1) and (2.10) are satisfied.
Moreover, we see

$$\begin{aligned} A_\ell(\theta) &= \frac{1}{a_0}\left(1 - \frac{\pi^{1-\ell}}{a_0}\right), \textit{for } \ell = 2, j, \\ D_0(\theta) &= \frac{b_0}{a_0}\left(1 - \frac{1}{\pi a_0}\right)\frac{\theta^{1-j}}{(j-1)}, \end{aligned}$$

and

$$D_{j-3}(\theta) = \frac{1}{(j-3)!}\frac{b_0}{a_0}\left(1 - \frac{1}{\pi a_0}\right)\frac{1}{(j-2)(j-1)\theta^2}.$$

Thus, Conditions (2.15) and (2.16) become

$$b_0\frac{1}{a_0}\left(\frac{\beta}{\pi}\right)^{j-1}\left(1 - \frac{\pi^{1-j}}{a_0}\right)\ln\frac{\pi}{\beta} > \frac{(j-1)!}{\mathrm{e}}, \tag{2.19}$$

and

$$b_0\frac{1}{a_0}\frac{\beta}{\pi}\left(1 - \frac{1}{\pi a_0}\right)\ln\frac{\pi}{\beta} > \frac{(j-1)!}{\mathrm{e}}. \tag{2.20}$$

So, it is clear that we find (2.19) implies (2.20).
Thus, applying Corollary 1, we obtain that (2.18) is oscillatory if (2.19) holds.

3. Conclusions

During this work, we have highlighted the oscillatory behavior of solutions of DEs (1.1) in the canonical case. Using the comparison technique, we have established new criteria that are more effective than the relevant criteria in the literature. Moreover, some examples have been given to show that our results extend the results reported in [17–19]. As a further research task, it would be interesting to apply the approach in studying delay DEs $\left(m(\theta)\,\varpi^{(j-1)}(\theta)\right)' + b(\theta)\,z(\varphi(\theta)) = 0$.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author acknowledges the support of Kuwait University for their joint research project No. SM06/24, "Some new oscillation criteria for neutral differential equations of fourth-order", funded by Kuwait University.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. J. K. Hale, Theory of Functional Differential Equations, Berlin/Heidelberg: Springer, 1977. http://dx.doi.org/10.1007/978-1-4612-9892-2
2. J. Dzurina, S. R. Grace, I. Jadlovska, T. Li, Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term, *Math. Nachr.*, **293** (2020), 910–922. https://doi.org/10.1002/mana.201800196
3. S. R. Grace, Oscillation of certain neutral difference equations of mixed type, *J. Math. Anal. Appl.*, **224** (1998), 241–254. https://doi.org/10.1006/jmaa.1998.6001
4. R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations, *Appl. Math. Lett.*, **18** (2005), 1201–1207. https://doi:10.1007/978-94-017-2515-6
5. S. H. Saker, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, *J. Comput. Appl. Math.*, **187** (2006), 123–141. https://doi.org/10.1016/j.cam.2005.03.039
6. B. Batiha, N. Alshammari, F. Aldosari, F. Masood, O. Bazighifan, Asymptotic and oscillatory properties for even-order nonlinear neutral differential equations with damping term, **Symmetry**, **17** (2025), 87. https://doi.org/10.3390/sym17010087
7. C. G. Philos, Oscillation theorems for linear differential equation of second order, *Arch. Math.*, **53** (1989), 483–492. http://dx.doi.org/10.1007/BF01324723

8. Z. Han, T. Li, S. Sun, Y. Sun, Remarks on the paper, *Appl. Math. Comput.*, **215** (2010), 3998–4007. https://doi.org/10.1016/j.amc.2009.12.006
9. B. Baculikova, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.*, **91** (2019), 68–75. https://doi.org/10.1016/j.aml.2018.11.021
10. B. Almarri, A. H. Ali, A. M. Lopes, O. Bazighifan, Nonlinear differential equations with distributed delay: Some new oscillatory solutions, *Mathematics*, **10** (2022), 995. https://doi.org/10.3390/math10060995
11. M. Aldiaiji, B. Qaraad, L. F. Iambor, E. M. Elabbasy, New oscillation theorems for second-order superlinear neutral differential equations with variable damping terms, *Symmetry*, **15** (2023), 1630. https://doi.org/10.3390/sym15091630
12. Y. G. Sun, F. W. Meng, Note on the paper of Dzurina and Stavroulakis, *Appl. Math. Comput.*, **164** (2006), 1634–1641.
13. T. Kusano, Y. Naito, Oscillation and nonoscillation criteria for second order quasilinear differential equations, *Acta Math. Hung.*, **76** (1997), 81–99. https://doi.org/10.1007/bf02907054
14. M. Bohner, S. R. Grace, I. Jadlovska, Oscillation criteria for second-order neutral delay differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **2017** (2017), 60. http://doi:10.14232/ejqtde.2017.1.60
15. B. Batiha, N. Alshammari, F. Aldosari, F. Masood, O. Bazighifan, Nonlinear neutral delay differential equations: Novel criteria for oscillation and asymptotic behavior, *Mathematics*, **13** (2025), 147. https://doi.org/10.3390/math13010147
16. S. Sun, T. Li, Z. Han, H. Li, Oscillation theorems for Second-Order quasilinear neutral functional differential equations, *Abstr. Appl. Anal.*, **2012** (2012), 819342. https://doi.org/10.1155/2012/819342
17. A. Zafer, Oscillation criteria for even order neutral differential equations, *Appl. Math. Lett.*, **11** (1998), 21–25. https://doi.org/10.1016/S0893-9659(98)00028-7
18. Q. Zhang, J. Yan, Oscillation behavior of even order neutral differential equations with variable coefficients, *Appl. Math. Lett.*, **19** (2006), 1202–1206. https://doi.org/10.1016/j.aml.2006.01.003
19. G. Xing, T. Li, C. Zhang, Oscillation of higher-order quasi linear neutral differential equations, *Adv. Difference Equ.*, **2011** (2011), 1–10. https://doi.org/10.1186/1687-1847-2011-45
20. O. Bazighifan, T. Abdeljawad, Improved approach for studying oscillatory properties of fourth-order advanced differential equations with p-Laplacian like operator, *Mathematics*, **8** (2020), 656. https://doi.org/10.3390/math8050656
21. T. Li, B. Baculikova, J. Dzurina, C. Zhang, Oscillation of fourth order neutral differential equations with p-Laplacian like operators, *Bound. Value Probl.*, **56** (2014), 41–58. https://doi.org/10.1186/1687-2770-2014-56
22. S. Liu, Q. Zhang, Y. Yu, Oscillation of even-order half-linear functional differential equations with damping, *Comput. Math. Appl.*, **61** (2011), 2191–2196. https://doi.org/10.1016/j.camwa.2010.09.011

23. B. Baculikova, J. Dzurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comput. Math. Appl.*, **62** (2011), 4472–4478. https://doi.org/10.1016/j.camwa.2011.10.024
24. R. P. Agarwal, M. Bohner, T. Li, C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay diferential equations, *Appl. Math. Comput.*, **225** (2013), 787–794. https://doi.org/10.1016/j.amc.2013.09.037
25. Ch. G. Philos, A new criterion for the oscillatory and asymptotic behavior of delay differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math.*, **39** (1981), 61–64.
26. Ch.G. Philos, On the existence of non-oscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math.* **36** (1981), 168–178. https://doi.org/10.1007/BF01223686
27. Y. Kitamura, T. Kusano, Oscillation of first-order nonlinear differential equations with deviating arguments, *Proc. Amer. Math. Soc.*, **78** (1980), 64–68. https://doi.org/10.1090/S0002-9939-1980-0548086-5