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Research article

New inequalities on the Hadamard product of nonnegative matrices

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Abstract: Several novel upper bounds for the spectral radius of the Hadamard product of two nonnegative matrices are presented by leveraging spectral radius properties and the Cauchy-Schwarz inequality. The derived bounds incorporate the maximum values of the non-diagonal elements in each row of the nonnegative matrices. Furthermore, concrete examples are presented to illustrate our results are more accurate than existing relevant results.

Keywords: irreducible; Hadamard product; nonnegative matrix; spectral radius
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1. Introduction

Throughout this study, we fix a positive integer n and denote the index set $\{1, 2, \cdots, n\}$ by N. The spaces of all $n \times n$ real and complex matrices are denoted by $R^{n\times n}$ and $C^{n\times n}$, respectively.

An $n \times n$ real matrix $A = (a_{ij})$ is called a nonnegative matrix (denoted by $A \geq 0$) if $a_{ij} \geq 0$. In 1907, Perron [1] showed that every nonnegative matrix A has an eigenvalue $\rho(A)$ such that $\rho(A) \geq |\lambda|$, for every eigenvalue λ of A. The eigenvalue $\rho(A)$ is called the spectral radius of A. Since then, nonnegative matrix theory has evolved into a fundamental framework with wide-ranging applications across multiple mathematical disciplines, including the theory of stochastic processes, numerical analysis, and dynamic programming.

Let $A \in R^{n\times n}$. Then A is defined as an irreducible matrix if either

(1) $n = 1$ and $A \neq 0$; or

(2) $n \geq 2$, there is no permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix},$$

in which $A_{11} \in R^{l\times l}, A_{12} \in R^{l\times(n-l)}, A_{22} \in R^{(n-l)\times(n-l)}$, and O is an $(n-l)\times l$ zero matrix, $1 \leq l \leq n-1$. Otherwise, A is reducible.

The Hadamard product, also known as the element-wise product, is an operation that uses two matrices or vectors of the same dimensions and produces a new matrix or vector by multiplying the corresponding elements. Specifically, for two matrices A and B of the same order, their Hadamard product $A \circ B$ is defined as:

$$(A \circ B)_{ij} = (A)_{ij} \cdot (B)_{ij},$$

where $(A)_{ij}$ and $(B)_{ij}$ represent the elements of matrices A and B in the i-th row and j-th column, respectively. Unlike matrix multiplication (which involves summing products across rows and columns), the Hadamard product does not involve any summation, and its result retains the same dimension as the original matrices or vectors. For example, for two 2×2 matrices:

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix},$$

the Hadamard product of A and B is:

$$A \circ B = \begin{pmatrix} 1\times 2 & 3\times 4 \\ 5\times 6 & 7\times 8 \end{pmatrix} = \begin{pmatrix} 2 & 12 \\ 30 & 56 \end{pmatrix}.$$

The Hadamard product is commutative, and satisfies $A \circ B = B \circ A$. It is evident that if A and B are two nonnegative matrices, then the Hadamard product $A \circ B$ is also nonnegative. Moreover, the spectral radius $\rho(A \circ B)$ serves as the dominant eigenvalue, satisfying $\rho(A \circ B) \geq |\lambda|$ for all other eigenvalues λ of $A \circ B$.

Hadamard products have a wide range of applications in various fields, particularly machine learning, signal processing, and image processing. This operation is fundamental in situations where the element-wise multiplication of matrices or vectors is necessary. The Hadamard product's ability to perform element-wise multiplication efficiently makes it an invaluable tool in numerous computational tasks, especially those that require selective, position-specific adjustments or filtering of data. Its versatility and computational simplicity have ensured its continued importance in both theoretical research and practical applications. Moreover, as an important tool in matrix analysis, the Hadamard product is also widely used in statistics and econometrics [2,3]. One of the most prominent research directions in matrix theory focuses on investigating the eigenvalues of the Hadamard product. Extensive studies in this field have explored its fundamental properties and applications, including spectral radius bounds for the Hadamard product of nonnegative matrices [4–9], as well as lower bounds for the minimum eigenvalue of the Hadamard product of M-matrices [10,11]. Recent extensions to tensor operations have further advanced this area, with novel inequalities establishing the spectral radius for the Hadamard product of nonnegative tensors [12,13].

Given two matrices A and B with nonnegative elements, Zhan [14] conjectured that

$$\rho(A \circ B) \leq \rho(AB). \tag{1.1}$$

This conjecture was confirmed by Audenaert [15].

Horn and Johnson [16] established the following classic estimate for $\rho(A \circ B)$, that is

$$\rho(A \circ B) \leq \rho(A)\rho(B). \tag{1.2}$$

Clearly, in some cases, the result of (1.2) may be very weak. The following example illustrates this situation. Let I be the square identity matrix of order n and $\rho(I) = 1$. Let J be the square matrix of order n with all elements equal to 1; $\rho(J) = n$. In particular, if $B = J$, then $\rho(A \circ B) = \rho(A)$(cf. Example 1). Moreover, if $A = I$, then $\rho(A \circ B) = \rho(I) = 1$, whereas the estimate (1.2) guarantees $\rho(A \circ B) \leq \rho(I)\rho(J) = n$ only. Obviously, the result may be weak when n is very large.

Recent advances in spectral radius estimation have yielded improved bounds under certain conditions. Notably, Fang [17] established the following key result:

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\}. \tag{1.3}$$

Subsequent work in [18] further refined these bounds, offering enhanced precision as follows:

$$\begin{aligned}\rho\left(A \circ B\right) \leq \max_{i\neq j} \frac{1}{2}\Big\{&a_{ii}b_{ii} + a_{jj}b_{jj} + \Big[\left(a_{ii}b_{ii} - a_{jj}b_{jj}\right)^2 \\ &+4\left(\rho\left(A\right) - a_{ii}\right)\left(\rho\left(B\right) - b_{ii}\right)\left(\rho\left(A\right) - a_{jj}\right)\left(\rho\left(B\right) - b_{jj}\right)\Big]^{\frac{1}{2}}\Big\}.\end{aligned} \tag{1.4}$$

Building on foundational work in this field, we present new theoretical advances in spectral radius estimation. The paper is organized as follows: Section 1 reviews essential definitions and key prior results.

In Section 2, we derive novel, tighter upper bounds for the spectral radius of Hadamard products of nonnegative matrices through innovative applications of the Cauchy-Schwarz inequality and spectral radius properties. These bounds incorporate the maximum values of the non-diagonal elements in each row of the nonnegative matrices, yielding significant improvements over the existing estimates.

Section 3 offers detailed numerical validation, highlighting the improved sharpness of our bounds through a systematic comparison with previously established results.

In Section 4, we summarize the study and review the main research findings.

2. Main results

To establish the principal results, we first present several fundamental lemmas that form the theoretical foundation of this study.

Lemma 1 [1]. *Let $A \in R^{n\times n}$ be irreducible and nonnegative, then*

(1) $\rho\left(A\right)$ *is an eigenvalue of* A;

(2) $AX = \rho\left(A\right)X$ *for some vector* $X > 0$.

Lemma 2 [17]. *Given a nonnegative irreducible matrix $A \in R^{n\times n}$ and a nonzero nonnegative vector X, if $AX \leq \lambda X$, then $\rho(A) \leq \lambda$.*

Our analysis employs the fundamental Cauchy-Schwarz inequality as a key component of the proof methodology.

Lemma 3 [16]. *Let* $U = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} \geq 0, V = (v_1, v_2, \cdots, v_n)^{\mathrm{T}} \geq 0$. *Then*

$$\sum_{i=1}^{n} u_i v_i \leq \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}.$$

The main theorems established in this study are presented below.

Theorem 1. *Let* $A = \left(a_{ij}\right), B = \left(b_{ij}\right) \in R^{n\times n}$ *be nonnegative matrices. It holds that*

$$\rho(A \circ B) \leq \max_{1\leq i\leq n} \left\{a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}} \left[\rho(A) - a_{ii}\right]^{\frac{1}{2}} \left[\rho(B) - b_{ii}\right]^{\frac{1}{2}}\right\}, \tag{2.1}$$

where $\alpha_i = \max\limits_{k\neq i} \{a_{ik}\}$ *and* $\beta_i = \max\limits_{k\neq i} \{b_{ik}\}$ *for any* $i \in N$.

Proof. Obviously, this conclusion holds with equality for $n = 1$. In the following section, we suppose $n \geq 2$. To analyze this problem systematically, we examine two distinct cases.

Case 1. We begin by assuming that the matrix $A \circ B$ is irreducible. It is straightforward to note that both matrices A and B are also irreducible. From Lemma 1, there exist two vectors $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} > 0$,$Y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} > 0$ such that

$$AX^{(2)} = \rho(A)X^{(2)}$$

and

$$BY^{(2)} = \rho(B)Y^{(2)},$$

where $X^{(2)} = \left(x_1^2, x_2^2, \cdots, x_n^2\right)^{\mathrm{T}}$ and $Y^{(2)} = \left(y_1^2, y_2^2, \cdots, y_n^2\right)^{\mathrm{T}}$. Therefore, we have

$$a_{ii}x_i^2 + \sum_{j\neq i}^{n} a_{ij}x_j^2 = \rho(A)x_i^2,$$

and

$$b_{ii}y_i^2 + \sum_{j\neq i}^{n} b_{ij}y_j^2 = \rho(B)y_i^2, \quad i \in N.$$

We can also write equivalently as

$$\sum_{j\neq i}^{n} a_{ij}x_j^2 = [\rho(A) - a_{ii}]x_i^2, \tag{2.2}$$

and

$$\sum_{j\neq i}^{n} b_{ij}y_j^2 = [\rho(B) - b_{ii}]y_i^2, \quad i \in N. \tag{2.3}$$

Now, we let $\alpha_i = \max\limits_{k\neq i} \{a_{ik}\}$ and $\beta_i = \max\limits_{k\neq i} \{b_{ik}\}, i \in N$. We define a vector $Z = (z_1, z_2, \cdots, z_n)^{\mathrm{T}} > 0$ such that $z_i = x_iy_i$ for any $i \in N$. Denote $C = A \circ B$. By Lemma 3, in conjunction with equations (2.2) and (2.3), we derive

$$\begin{aligned}
(CZ)_i &= a_{ii}b_{ii}z_i + \sum_{j\neq i}^{n} a_{ij}b_{ij}z_j \\
&= a_{ii}b_{ii}z_i + \sum_{j\neq i}^{n}(a_{ij}x_j)(b_{ij}y_j) \\
&\leq a_{ii}b_{ii}z_i + \left(\sum_{j\neq i}^{n} a_{ij}^2x_j^2\right)^{\frac{1}{2}}\left(\sum_{j\neq i}^{n} b_{ij}^2y_j^2\right)^{\frac{1}{2}} \\
&\leq a_{ii}b_{ii}z_i + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}\left(\sum_{j\neq i}^{n} a_{ij}x_j^2\right)^{\frac{1}{2}}\left(\sum_{j\neq i}^{n} b_{ij}y_j^2\right)^{\frac{1}{2}} \\
&= a_{ii}b_{ii}z_i + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}\left[\rho(A) - a_{ii}\right]^{\frac{1}{2}} x_i \left[\rho(B) - b_{ii}\right]^{\frac{1}{2}} y_i \\
&= a_{ii}b_{ii}z_i + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}\left[\rho(A) - a_{ii}\right]^{\frac{1}{2}}\left[\rho(B) - b_{ii}\right]^{\frac{1}{2}} z_i \\
&= \left\{a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}\left[\rho(A) - a_{ii}\right]^{\frac{1}{2}}\left[\rho(B) - b_{ii}\right]^{\frac{1}{2}}\right\} z_i.
\end{aligned}$$

In terms of Lemma 2, we obtain

$$\begin{aligned}
\rho(C) &\leq a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}[\rho(A) - a_{ii}]^{\frac{1}{2}}[\rho(B) - b_{ii}]^{\frac{1}{2}} \\
&\leq \max_{1\leq i\leq n}\left\{a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}[\rho(A) - a_{ii}]^{\frac{1}{2}}[\rho(B) - b_{ii}]^{\frac{1}{2}}\right\}.
\end{aligned}$$

That is

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\left\{a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}[\rho(A) - a_{ii}]^{\frac{1}{2}}[\rho(B) - b_{ii}]^{\frac{1}{2}}\right\}.$$

Case 2. We now consider the matrix $A \circ B$ to be reducible. We define a permutation matrix $P = (p_{ij})$ of order n such that

$$p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n1} = 1,$$

the rest of the elements being zero. By continuity, there exists an $\varepsilon > 0$ sufficiently small such that the matrices $A + \varepsilon P$ and $B + \varepsilon P$ remain nonnegative and irreducible, with their Hadamard product $(A + \varepsilon P) \circ (B + \varepsilon P)$ inheriting this property. The result then follows from Case 1 via a continuity argument as $\varepsilon \to 0$, thereby completing the proof.

Remark 1. *The result can alternatively be established through the following approach. Define two nonsingular positive diagonal matrices as follows:*

$$P_1 = \operatorname{diag}(x_1, x_2, \cdots, x_n)$$

and

$$P_2 = \operatorname{diag}(y_1, y_2, \cdots, y_n).$$

Let

$$\tilde{A} = P_1^{-1} A P_1 = \begin{pmatrix} a_{11} & \frac{a_{12}x_2}{x_1} & \cdots & \frac{a_{1n}x_n}{x_1} \\ \frac{a_{21}x_1}{x_2} & a_{22} & \cdots & \frac{a_{2n}x_n}{x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}x_1}{x_n} & \frac{a_{n2}x_2}{x_n} & \cdots & a_{nn} \end{pmatrix}$$

and

$$\tilde{B} = P_2^{-1} B P_2 = \begin{pmatrix} b_{11} & \frac{b_{12}y_2}{y_1} & \cdots & \frac{b_{1n}y_n}{y_1} \\ \frac{b_{21}y_1}{y_2} & b_{22} & \cdots & \frac{b_{2n}y_n}{y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}y_1}{y_n} & \frac{b_{n2}y_2}{y_n} & \cdots & b_{nn} \end{pmatrix}.$$

It is simple to observe that

$$\begin{aligned} \tilde{A} \circ \tilde{B} &= \begin{pmatrix} a_{11}b_{11} & \frac{a_{12}b_{12}x_2y_2}{x_1y_1} & \cdots & \frac{a_{1n}b_{1n}x_ny_n}{x_1y_1} \\ \frac{a_{21}b_{21}x_1y_1}{x_2y_2} & a_{22}b_{22} & \cdots & \frac{a_{2n}b_{2n}x_ny_n}{x_2y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}b_{n1}x_1y_1}{x_ny_n} & \frac{a_{n2}b_{n2}x_2y_2}{x_ny_n} & \cdots & a_{nn}b_{nn} \end{pmatrix} \\ &= (P_1P_2)^{-1} (A \circ B) (P_1P_2). \end{aligned}$$

Hence, we have $\rho(A \circ B) = \rho\left(\tilde{A} \circ \tilde{B}\right)$. Since $\rho\left(\tilde{A} \circ \tilde{B}\right)$ *is an eigenvalue of* $\tilde{A} \circ \tilde{B}$, *by the Gerschgorin theorem (see Theorem 1.11 in [19]), we obtain*

$$\begin{aligned} |\rho(A \circ B) - a_{ii}b_{ii}| &\le \sum_{j \ne i}^{n} \frac{a_{ij}b_{ij}x_jy_j}{x_iy_i} \\ &= \sum_{j \ne i}^{n} \frac{a_{ij}x_j}{x_i} \frac{b_{ij}y_j}{y_i} \\ &\le \left(\sum_{j \ne i}^{n} \frac{a_{ij}^2 x_j^2}{x_i^2} \right)^{\frac{1}{2}} \left(\sum_{j \ne i}^{n} \frac{b_{ij}^2 y_j^2}{y_i^2} \right)^{\frac{1}{2}} \\ &\le \alpha_i^{\frac{1}{2}} \beta_i^{\frac{1}{2}} \left(\sum_{j \ne i}^{n} \frac{a_{ij}x_j^2}{x_i^2} \right)^{\frac{1}{2}} \left(\sum_{j \ne i}^{n} \frac{b_{ij}y_j^2}{y_i^2} \right)^{\frac{1}{2}}. \end{aligned}$$

From equalities (2.2) and (2.3), we derive

$$|\rho(A \circ B) - a_{ii}b_{ii}| \le \alpha_i^{\frac{1}{2}} \beta_i^{\frac{1}{2}} \left[\rho(A) - a_{ii}\right]^{\frac{1}{2}} \left[\rho(B) - b_{ii}\right]^{\frac{1}{2}}. \tag{2.4}$$

Note that $\rho(A \circ B) \geq a_{ii}b_{ii}$, from inequality (2.4), we can obtain the desired result.

Theorem 2. *Let $A = \left(a_{ij}\right), B = \left(b_{ij}\right) \in R^{n\times n}$ be nonnegative matrices. It holds that*

$$\rho(A \circ B) \leq \max_{1\leq i\leq n} \{(b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A)\}, \tag{2.5}$$

where $\beta_i = \max_{k\neq i} \{b_{ik}\}$ for any $i \in N$.

Proof. Obviously, inequality (2.5) becomes equal when $n = 1$. We next assume that $n \geq 2$. To demonstrate this problem, we distinguish between the two cases.

Case 1. We begin by assuming that the matrix $A \circ B$ is irreducible. It is straightforward to note that both matrices A and B are also irreducible. From Lemma 1, there exists $U = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} > 0$ such that

$$AU = \rho(A)\, U.$$

So, we obtain

$$a_{ii}u_i + \sum_{j\neq i}^{n} a_{ij}u_j = \rho(A)u_i, \quad i \in N,$$

or equivalently

$$\sum_{j\neq i}^{n} a_{ij}u_j = [\rho(A) - a_{ii}]u_i, \quad i \in N.$$

Let $\beta_i = \max_{k\neq i} \{b_{ik}\}$ for any $i \in N$. We define $C = A \circ B$. Therefore, we have that

$$\begin{aligned}
(CU)_i &= a_{ii}b_{ii}u_i + \sum_{j\neq i}^{n} a_{ij}b_{ij}u_j \\
&\leq a_{ii}b_{ii}u_i + \beta_i \sum_{j\neq i}^{n} a_{ij}u_j \\
&= a_{ii}b_{ii}u_i + \beta_i[\rho(A) - a_{ii}]u_i \\
&= \left[(b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A)\right] u_i.
\end{aligned}$$

In terms of Lemma 2, we obtain

$$\begin{aligned}
\rho(C) &\leq (b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A) \\
&\leq \max_{1\leq i\leq n} \{(b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A)\}.
\end{aligned}$$

That is

$$\rho(A \circ B) \leq \max_{1\leq i\leq n} \{(b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A)\}.$$

Case 2. Suppose $A \circ B$ is reducible. Following the methodology in Theorem 1, we apply analogous reasoning to establish the corresponding result.

Based on the commutativity of the Hadamard product, we obtain $\rho(A \circ B) = \rho(B \circ A)$. Consequently, the following conclusion is drawn.

Theorem 3. *Let $A = \left(a_{ij}\right), B = \left(b_{ij}\right) \in R^{n\times n}$ be nonnegative matrices. It holds that*

$$\rho(A \circ B) \leq \max_{1\leq i\leq n} \{(a_{ii} - \alpha_i)\, b_{ii} + \alpha_i\rho(B)\}, \tag{2.6}$$

where $\alpha_i = \max_{k\neq i} \{a_{ik}\}$ *for any* $i \in N$.

Remark 2. *The estimation of* $\rho(A \circ B)$ *in inequalities (1.3) and (1.4) requires prior knowledge of both* $\rho(A)$ *and* $\rho(B)$. *However, inequalities (2.5) and (2.6) demonstrate that only one spectral radius (either* $\rho(A)$ *or* $\rho(B)$*) is sufficient to determine the bound for* $\rho(A \circ B)$.

The subsequent conclusion directly follows from the application of Theorem 2 in conjunction with Theorem 3.

Theorem 4. *Let* $A = \left(a_{ij}\right), B = \left(b_{ij}\right) \in R^{n\times n}$ *be nonnegative matrices. It holds that*

$$\rho(A \circ B) \leq \min\left\{\max_{1\leq i\leq n}\{(b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A)\}, \max_{1\leq i\leq n}\{(a_{ii} - \alpha_i)b_{ii} + \alpha_i\rho(B)\}\right\}, \tag{2.7}$$

where $\alpha_i = \max_{k\neq i} \{a_{ik}\}$ *and* $\beta_i = \max_{k\neq i} \{b_{ik}\}$ *for any* $i \in N$.

3. Illustrative examples

In this section, we present several representative examples demonstrating the superior precision of our novel upper bounds compared with the existing results in the literature.

Example 1. *First, we consider the two nonnegative matrices below:*

$$A = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 0.05 & 1 & 1 \\ 0 & 1 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We compute the Hadamard product

$$A \circ B = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 0.05 & 1 & 1 \\ 0 & 1 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{pmatrix}.$$

By direct calculation with MATLAB, we have

$$\rho(A \circ B) = \rho(A) = 5.7339, \quad \rho(B) = 4.$$

By inequality (1.1), we obtain

$$\rho(A \circ B) \leq \rho(AB) = 21.0500.$$

From inequality (1.2), we obtain

$$\rho(A \circ B) \leq \rho(A)\rho(B) = 22.9336.$$

By inequality (1.3) in [17], we obtain

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} = 17.1017.$$

From inequality (1.4) in [18], we obtain

$$\rho(A \circ B) \le \max_{i \neq j} \frac{1}{2} \Big\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \Big[\big(a_{ii}b_{ii} - a_{jj}b_{jj} \big)^2$$
$$+4 (\rho(A) - a_{ii}) (\rho(B) - b_{ii}) \big(\rho(A) - a_{jj}\big) \big(\rho(B) - b_{jj}\big) \Big]^{\frac{1}{2}} \Big\} = 11.6478.$$

According to Theorem 1, we obtain

$$\rho(A \circ B) \le \max_{1 \le i \le n} \left\{ a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}} \beta_i^{\frac{1}{2}} [\rho(A) - a_{ii}]^{\frac{1}{2}} [\rho(B) - b_{ii}]^{\frac{1}{2}} \right\} = 7.2254.$$

However, if we apply Theorem 2 and Theorem 3, one can obtain

$$\rho(A \circ B) \le \max_{1 \le i \le n} \{(b_{ii} - \beta_i)a_{ii} + \beta_i \rho(A)\} = 5.7339$$

and

$$\rho(A \circ B) \le \max_{1 \le i \le n} \{(a_{ii} - \alpha_i)b_{ii} + \alpha_i \rho(B)\} = 10,$$

respectively. Finally, by Theorem 4, we obtain the optimal upper bound:

$$\rho(A \circ B) \le \min \left\{ \max_{1 \le i \le n} \{(b_{ii} - \beta_i)a_{ii} + \beta_i \rho(A)\}, \max_{1 \le i \le n} \{(a_{ii} - \alpha_i)b_{ii} + \alpha_i \rho(B)\} \right\} = 5.7339.$$

Unexpectedly, our analysis revealed that this upper bound coincides exactly with the spectral radius $\rho(A \circ B)$.

Example 2. *Now, we present the second example and examine the following two* 4×4 *nonnegative matrices:*

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 5 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

We compute the Hadamard product

$$A \circ B = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 2 & 15 & 2 & 0 \\ 0 & 2 & 16 & 3 \\ 0 & 0 & 1 & 20 \end{pmatrix}.$$

By direct calculation with MATLAB, we have

$$\rho(A \circ B) = 20.7439, \quad \rho(A) = 7.6631, \quad \rho(B) = 6.5414.$$

From inequality (1.1), we obtain

$$\rho(A \circ B) \le \rho(AB) = 44.1410.$$

By inequality (1.2), we acquire

$$\rho(A \circ B) \le \rho(A)\rho(B) = 50.1274.$$

According to inequality (1.3) in [17], we obtain

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} = 25.6463.$$

From inequality (1.4) in [18], we obtain

$$\rho(A \circ B) \leq \max_{i\neq j} \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Big[\big(a_{ii}b_{ii} - a_{jj}b_{jj}\big)^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})\big(\rho(A) - a_{jj}\big)\big(\rho(B) - b_{jj}\big)\Big]^{\frac{1}{2}}\Big\} = 25.5209.$$

According to Theorem 1, we obtain

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\Big\{a_{ii}b_{ii} + \alpha_i^{\frac{1}{2}}\beta_i^{\frac{1}{2}}[\rho(A) - a_{ii}]^{\frac{1}{2}}[\rho(B) - b_{ii}]^{\frac{1}{2}}\Big\} = 23.4737.$$

However, if we utilize Theorem 2 and Theorem 3, we will arrive at

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\{(b_{ii} - \beta_i)\, a_{ii} + \beta_i\rho(A)\} = 26.9893$$

and

$$\rho(A \circ B) \leq \max_{1\leq i\leq n}\{(a_{ii} - \alpha_i)\, b_{ii} + \alpha_i\rho(B)\} = 22.0828,$$

respectively. Finally, according to Theorem 4, we obtain the optimal upper bound:

$$\rho(A \circ B) \leq \min\Big\{\max_{1\leq i\leq n}\{(b_{ii} - \beta_i)\, a_{ii} + \beta_i\rho(A)\}, \max_{1\leq i\leq n}\{(a_{ii} - \alpha_i)\, b_{ii} + \alpha_i\rho(B)\}\Big\} = 22.0828.$$

Example 3. *Now, we present the third example and examine the following two* 8×8 *nonnegative dense matrices:*

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 2 & 2 & 4 & 3 \\ 1 & 2 & 1 & 4 & 1 & 3 & 4 & 2 \\ 4 & 3 & 3 & 3 & 2 & 2 & 3 & 1 \\ 1 & 3 & 1 & 4 & 3 & 3 & 1 & 4 \\ 4 & 2 & 3 & 3 & 2 & 1 & 3 & 2 \\ 2 & 2 & 3 & 4 & 2 & 3 & 4 & 1 \\ 4 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 1 & 3 & 2 & 3 & 2 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 5 & 3 & 4 & 4 & 5 & 5 & 4 & 4 \\ 6 & 4 & 5 & 5 & 5 & 6 & 5 & 4 \\ 6 & 4 & 3 & 3 & 6 & 4 & 5 & 6 \\ 5 & 4 & 3 & 5 & 4 & 3 & 5 & 5 \\ 5 & 3 & 5 & 5 & 5 & 5 & 5 & 6 \\ 4 & 5 & 5 & 5 & 5 & 6 & 5 & 4 \\ 6 & 4 & 5 & 3 & 3 & 5 & 4 & 5 \end{pmatrix}.$$

By direct calculation with MATLAB, we have

$$\rho(A \circ B) = 91.0304\ , \rho(A) = 19.7998, \rho(B) = 36.5906\ .$$

From inequality (1.1), we obtain

$$\rho(A \circ B) \leq \rho(AB) = 724.7848.$$

By inequality (1.2), we acquire

$$\rho(A \circ B) \leq \rho(A)\rho(B) = 724.4877.$$

According to inequality (1.3) in [17], we obtain

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} = 603.906.$$

From inequality (1.4) in [18], we obtain

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \Big\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \Big[\big(a_{ii}b_{ii} - a_{jj}b_{jj} \big)^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})\big(\rho(A) - a_{jj}\big)\big(\rho(B) - b_{jj}\big) \Big]^{\frac{1}{2}} \Big\} = 595.9397.$$

According to Theorem 1, we obtain

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} + {\alpha_i}^{\frac{1}{2}}{\beta_i}^{\frac{1}{2}}[\rho(A) - a_{ii}]^{\frac{1}{2}}[\rho(B) - b_{ii}]^{\frac{1}{2}} \right\} = 127.8591.$$

However, if we utilize Theorem 2 and Theorem 3, we will arrive at

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{(b_{ii} - \beta_i) a_{ii} + \beta_i\rho(A)\} = 116.7988$$

and

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{(a_{ii} - \alpha_i) b_{ii} + \alpha_i\rho(B)\} = 146.3624$$

respectively. Finally, according to Theorem 4, we obtain the optimal upper bound:

$$\rho(A \circ B) \leq \min \left\{ \max_{1 \leq i \leq n} \{(b_{ii} - \beta_i) a_{ii} + \beta_i\rho(A)\}, \max_{1 \leq i \leq n} \{(a_{ii} - \alpha_i) b_{ii} + \alpha_i\rho(B)\} \right\} = 116.7988.$$

The data calculated above indicate that, in certain instances, our bounds are more accurate than the corresponding results.

4. Conclusions

In this study, we have established novel bounds for the spectral radius of the Hadamard product of two nonnegative matrices. By utilizing the properties of the spectral radius and applying the Cauchy-Schwarz inequality, we derived bounds that incorporate the maximum values of the non-diagonal elements in each row of the matrices. These newly established bounds provide a more precise characterization of the spectral radius compared to previous results in the literature.

Furthermore, through concrete examples, we demonstrated that our new bounds yield sharper estimates, thereby illustrating the effectiveness of our approach. Future research could extend these findings to explore additional types of matrix products or consider more general classes of matrices, potentially leading to even tighter bounds and broader applications in matrix theory and related fields.

Author contributions

Qin Zhong: Writing-original draft, Methodology, Conceptualization, Data curation; Ling Li: Formal analysis, Writing-review & editing; Gufang Mou: Methodology, Investigation, Data curation. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no competing interests.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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