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Research article

On the generalized Elzaki transform and its applications to fractional differential equations

Aysun Yurttas Gunes[1,*], **Halim Benali**[2] **and Mohammed Said Souid**[3,4]

[1] Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Bursa, Turkey
[2] Department of Mathematics, Faculty of Mathematics and Computer Sciences, University of Tiaret, Algeria
[3] Department of Mathematics, Saveetha School of Engineering, SIMATS, India
[4] Department of Economic Sciences, University of Tiaret, Algeria

* **Correspondence:** Email: ayurttas@uludag.edu.tr.

Abstract: In this paper, we introduce a modified Elzaki transform and its generalization, namely the Elzaki transform, and its own convolution theorem is given. These generalization are given by composition with a monotonic increasing function ϱ having a continuous derivative. A revised version of the Elzaki transform that is broader in scope and applicable over a wider range is developed, and some of its fundamental properties are given. This modified transform is performed to find solutions of certain non homogeneous linear ϱ Riemann–Liouville, ϱ Caputo fractional differential equations. Using comparision graphs, one can determine the effectiveness of the solutions. This research opens up new avenues for future research and has the potential to make a significant impact on the field of mathematics and its applications.

Keywords: Elzaki transform; convolution; generalized fractional operators; fractional integrals; graph modeling
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1. Introduction

Several integral transforms are named after the mathematicians who introduced them. These include the Fourier, Laplace, Mellin, Sumudu, and other transforms, which are commonly used as mathematical tools. These transforms have been extensively studied and have a wide range of applications in mathematics, physics, and engineering sciences for solving differential and integral equations, as well as in other scientific disciplines.

The Elzaki transform has been extensively studied in recent years as a powerful tool for solving integral equations, differential equations with constant and variable coefficients [1], and fractional differential equations. Elzaki [2, 3] introduced the original Elzaki transform, which was later studied by various authors [4–6]. The Elzaki transform has been applied to solve various types of fractional differential equations, including those with Caputo and Riemann Liouvile derivatives [7–9]. Additionally, the Elzaki transform has been used in various fields, such as signal processing and image analysis [10–12]. The effectiveness of the solutions can be compared by means of so-called comparison graphs. Applications of the Elzaki transform to fractional derivatives will give different solutions and some corresponding graph transformations, which can be constructed conveniently in future work.

2. Preliminaries

In this section, we introduce the notion of integrating and differentiating with respect to a monotonic smooth function ϱ. We start with the integral and derivative of integer order of a function f with respect to the function ϱ, and their fractional versions in a suitable functional framework. We also recall the generalized convolution. For more details, refer to [13, 14].

Some notation

- $\mathcal{L}$ Laplace transform; $\mathcal{E}$ Elzaki transform; $\mathcal{E}_b$ Modified Elzaki transform; $\mathcal{E}_{b,\varrho}$ Generalized Elzaki transform;
- $*$ convolution; $*_\varrho$ ϱ-convolution;
- $Q_\varrho, Q_\varrho^{-1}$: The substitution operator and its inverse;
- $E_{\alpha,\delta}$: Mittag–Leffler function with the parameters α, δ.

Definition 2.1. *[15] Let a and b be real numbers such that $a < b$. Let ϱ be a strictly increasing positive function having a continuous derivative ϱ' on the interval (a, b). Then*

$$C_{\rho,\varrho}([a,b]) = \{f : [a,b] \to \mathbb{R} \quad (\varrho(.) - \varrho(a))^\rho f \in C[a,b]\} \tag{2.1}$$

$$C^m_{\rho,\varrho}([a,b]) = \{f : [a,b] \to \mathbb{R} \quad f^{[m-1]} \in C[a,b], f^{[m]} \in C_{\rho,\varrho}[a,b]\}, \tag{2.2}$$

where $f^{[m]} = \left(\frac{1}{\varrho'(t)}\frac{d}{dt}\right)^m f$, $C_{0,\varrho} = C[a,b]$, *and* $C^m_{0,\varrho}[a,b] = C^m[a;b]$.

Definition 2.2. *[15] The space $\mathcal{AC}^m[a,b]$ is defined as follows:*

$$\mathcal{AC}^m[a,b] = \{f : [a,b] \to \mathbb{R}, \quad f^{[m-1]} \in \mathcal{AC}[a,b]\}, \tag{2.3}$$

where $\mathcal{AC}[a,b]$ is the space of absolutely continuous functions on the interval $[a,b]$.

2.1. Integrals and derivatives of integer order

Let $f \in L_1([a,b])$ and ϱ be an increasing positive monotone function having a continuous derivative on $[a,b]$. The *mth* ($m \in \mathbb{N}$)-order integral of a function f is defined by

$$(\mathcal{I}^m_{a_+} f)(x) = \frac{1}{(m-1)!}\int_a^x (x-t)^{m-1} f(t)dt, \quad x > a.$$

The *mth*-order ($m \in \mathbb{N}$) integral of a function f with respect to another function ϱ is defined by

$$(\mathcal{I}^m_{a_+;\varrho}f)(x) = \frac{1}{(m-1)!}\int_a^x (\varrho(x)-\varrho(t))^{m-1}f(t)\varrho'(t)dt, \quad x > a, \tag{2.4}$$

$$(\mathcal{I}^m_{b_-,\varrho}f)(x) = \frac{1}{(m-1)!}\int_x^b (\varrho(t))-\varrho(x)^{m-1}f(t)\varrho'(t)dt, \quad x < b. \tag{2.5}$$

The corresponding derivatives are defined as follows:

$$(D^{m,l}_{\varrho}f)(x) = \left(\frac{1}{\varrho'(x)}D_x\right)^m f(x), \tag{2.6}$$

and

$$(D^{m,r}_{\varrho}f)(x) = \left(\frac{-1}{\varrho'(x)}D_x\right)^m f(x), \tag{2.7}$$

where $D_x = \dfrac{d}{dx}$.

2.2. *The fractional versions*

The fractional versions of (2.4), (2.5), (2.6), and (2.7) are given as follows:

Definition 2.3. *(Integrals) [16] Let $\alpha > 0$ and $f \in L_1[a,b]$, and let ϱ be an increasing positive function with a continuous positive derivative ϱ' on $[a,b]$. The fractional integral of order α of a function f with respect to another function $\varrho(.)$ is defined by*

$$\mathcal{I}^{\alpha}_{a^+,\varrho}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(\varrho(x)-\varrho(t))^{\alpha-1}\varrho'(t)dt, \quad x > a(left) \tag{2.8}$$

and

$$\mathcal{I}^{\alpha}_{b_-,\varrho}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b f(t)(\varrho(t)-\varrho(x))^{\alpha-1}\varrho'(t)dt, \quad x < b(right). \tag{2.9}$$

Remark 2.4. *[17] For $\alpha > 0, 1 \le p \le \infty$, and $f \in \mathbf{X}^p_{\varrho}(a,b)$. Then the integrals $\mathcal{I}^{\alpha}_{a_+,\varrho}f$ and $\mathcal{I}^{\alpha}_{b_-,\varrho}f$ are bounded in the weighted Lebesgue space $\mathbf{X}^p_{\varrho}(a,b)$ of Lebesgue measurable functions*

$$\|\mathcal{I}^{\alpha}_{a_+,\varrho}f\|_{\mathbf{X}^p_{\varrho}} \le \frac{(\varrho(b)-\varrho(a))^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{\mathbf{X}^p_{\varrho}}$$

for which $\|f\|_{\mathbf{X}^p_{\varrho}} < \infty, 1 \le p \le \infty$, where the norm is defined by

$$\|f\|_{\mathbf{X}^p_{\varrho}} = \left(\int_a^b \varrho'(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty$$

and

$$\|f\|_{\mathbf{X}^{\infty}_{\varrho}} = ess \sup_{x\in(a,b)} |f(x)| < \infty.$$

In particular, when $\varrho(x) = x$, the space $\mathbf{X}^p_x(a,b)$ coincides whith the classical $L_p(a,b)$-space.

Proposition 2.5. *[18] The integral operators $\mathcal{I}^{\alpha}_{a_+;\varrho}$, $\mathcal{I}^{\alpha}_{b_-;\varrho}$ satisfy the following semi-group properties:*

$$\mathcal{I}^{\alpha}_{a_+;\varrho}\mathcal{I}^{\beta}_{a_+;\varrho} = \mathcal{I}^{\alpha+\beta}_{a_+;\varrho} = \mathcal{I}^{\beta}_{a_+;\varrho}\mathcal{I}^{\alpha}_{a_+;\varrho}, \tag{2.10}$$

$$\mathcal{I}^{\alpha}_{b_-;\varrho}\mathcal{I}^{\beta}_{b_-;\varrho} = \mathcal{I}^{\alpha+\beta}_{b_-;\varrho} = \mathcal{I}^{\beta}_{b_-;\varrho}\mathcal{I}^{\alpha}_{b_-;\varrho}, \ \alpha,\beta > 0. \tag{2.11}$$

We prove (2.10) using the substitution operator Q_ϱ. We have $\mathcal{I}^{\alpha}_{a_+;\varrho} = Q_\varrho \mathcal{I}^{\alpha}_{\varrho(a_+)} Q_{\varrho^{-1}}$. Hence

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a_+;\varrho}\mathcal{I}^{\beta}_{a_+;\varrho} &= Q_\varrho \mathcal{I}^{\alpha}_{\varrho(a_+)} Q_{\varrho^{-1}} Q_\varrho \mathcal{I}^{\beta}_{\varrho(a_+)} Q_{\varrho^{-1}} \\
&= Q_\varrho \mathcal{I}^{\alpha}_{\varrho(a_+)} \mathcal{I}^{\beta}_{\varrho(a_+)} Q_{\varrho^{-1}} = Q_\varrho \mathcal{I}^{\alpha+\beta}_{\varrho(a_+)} Q_{\varrho^{-1}} \\
&= \mathcal{I}^{\alpha+\beta}_{a_+;\varrho}.
\end{aligned}$$

Remark 2.6. *Different fractional integrals can be obtained by considering different functions $\varrho(.)$ in Eqs (2.8) and (2.9). For example*

1) *If $\varrho(x) = x$, we obtain the Riemann–Liouville integral operators.*
2) *If $\varrho(x) = \ln x$, we obtain the Hadamard integral operators.*
3) *If $\varrho(x) = x^\rho$, where $\rho \in \mathbb{R}$, we obtain the Erd'elyi–Kober-type fractional integral operators.*
4) *If $\varrho(x) = \dfrac{x^{\rho+1}}{\rho+1}$, where $\rho \in \mathbb{R} - \{-1\}$, we obtain the Katugampola integral operators.*

Definition 2.7. *(Riemann–Liouville type derivatives) [15] Let $\alpha > 0$ and $f \in L_1[a,b]$, and let ϱ be an increasing positive function with a continuous positive derivative ϱ' on $[a,b]$. The fractional derivative of order α of a function f with respect to another function $\varrho(.)$ is defined by*

$$\begin{aligned}
({}^{RL}D^{\alpha}_{a_+,\varrho}f)(x) &= D^{m,l}_{\varrho}(\mathcal{I}^{m-\alpha}_{a^+;\varrho}f)(x) \\
&= \frac{1}{\Gamma(m-\alpha)} D^m_\varrho \left(\int_a^x (\varrho(x)-\varrho(t))^{m-\alpha-1} f(t)\varrho'(t)dt \right)
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
({}^{RL}D^{\alpha}_{b_-,\varrho}f)(x) &= D^{m}_{b^-;\varrho}(\mathcal{I}^{m-\alpha}_{b^-;\varrho}f)(x) \\
&= \frac{1}{\Gamma(m-\alpha)} D^m_\varrho \left(\int_x^b (\varrho(t)-\varrho(x))^{m-\alpha-1} f(t)\varrho'(t)dt \right).
\end{aligned} \tag{2.13}$$

Definition 2.8. *(Caputo-type derivatives) [19] Let $\alpha > 0, f \in L_1[a,b]$, and let ϱ be an increasing positive function with a continuous positive derivative ϱ' on $[a,b]$. The generalized fractional derivative of order α of a function f with respect to another function $\varrho(.)$ is defined by*

$$\begin{aligned}
({}^{C}D^{\alpha}_{a_+,\varrho}f)(x) &= \mathcal{I}^{m-\alpha}_{a^+,\varrho}(D^{m,l}_{\varrho}f)(x) \\
&= \frac{1}{\Gamma(m-\alpha)} \int_a^x (\varrho(x)-\varrho(t))^{m-\alpha-1} D^m_\varrho f(t)\varrho'(t)dt.
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
({}^{C}D^{\alpha}_{b_-,\varrho}f)(x) &= \mathcal{I}^{m-\alpha}_{b^-,\varrho}(D^{m}_{b^-;\varrho}f)(x) \\
&= \frac{1}{\Gamma(m-\alpha)} \int_x^b (\varrho(t)-\varrho(x))^{m-\alpha-1} D^m_\varrho f(t)\varrho'(t)dt.
\end{aligned} \tag{2.15}$$

Remark 2.9. *The operator ${}^{RL}D^{\alpha}_{:;\varrho}$ is the left inverse of $\mathcal{I}^{\alpha}_{:;\varrho}$*

$$({}^{RL}D^{\alpha}_{a_+;\varrho}\mathcal{I}^{\alpha}_{a_+,\varrho,}f)(x) = f(x), \quad ({}^{RL}D^{\alpha}_{b_-;\varrho}\mathcal{I}^{\alpha}_{b_-,\varrho}f)(x) = f(x). \tag{2.16}$$

Remark 2.10. *Let Q_ϱ, Q_ϱ^{-1} be the substitution operator and its inverse, defined as follows:*

$$(Q_\varrho f)(t) = f(\varrho(t)), \quad Q_\varrho^{-1} = Q_{\varrho^{-1}}.$$

We then have the following relationships between the classical operators and the generalized operators (see [20]):

$$\mathcal{I}^{m}_{a,\varrho} = Q_\varrho \mathcal{I}^{m}_{\varrho(a)} Q_\varrho^{-1}, \quad \mathcal{I}^{\alpha}_{a,\varrho} = Q_\varrho \mathcal{I}^{\alpha}_{\varrho(a)} Q_\varrho^{-1}, \tag{2.17}$$

$${}^{C}\mathcal{D}^{\alpha}_{a,\varrho} = Q^{C}_\varrho \mathcal{D}^{\alpha}_{\varrho(a)} Q_\varrho^{-1}, \quad {}^{RL}\mathcal{D}^{\alpha}_{a,\varrho} = Q^{RL}_\varrho \mathcal{D}^{\alpha}_{\varrho(a)} Q_\varrho^{-1}. \tag{2.18}$$

Definition 2.11. *[17] (ϱ convolution) The ϱ convolution or the generalized convolution, which is denoted $*_\varrho$ and defined as follows:*

$$(f \star_\varrho h)(t) = \int_0^t f(\tau) h(\varrho^{-1}(\varrho(t) + \varrho(0) - \varrho(\tau)))\varrho'(\tau)d\tau. \tag{2.19}$$

Example 2.12. *Let f, h be defined as*

$$f(t) = \begin{cases} t^2, & \textit{for} \quad t \geq 0, \\ 0, & \textit{otherwise} \end{cases} \qquad h(t) = \begin{cases} e^{-t}, & \textit{for} \quad t \geq 0, \\ 0, & \textit{otherwise} \end{cases}$$

and $\varrho(t) = t^2$. We compute the generalized convolution of f and h with respect to a non-negative, strictly increasing function.

Let us plug in the values

$$\begin{aligned} (f \star_\varrho h)(t) &= \int_0^t f(\tau)\, h(\varrho^{-1}(\varrho(t) + \varrho(0) - \varrho(\tau)))\varrho'(\tau)d\tau \\ &= \int_0^t \tau^2 e^{-\sqrt{t^2-\tau^2}} 2\tau \mathrm{d}\tau \\ &= 2t^2 \int_0^t x e^{-x} \mathrm{d}x - 2\int_0^t x^3 e^{-x} \mathrm{d}x \quad (x = \sqrt{t^2 - \tau^2}) \\ &= (2t^3 - 6t^2 - 12t - 12)e^{-t} + 12. \end{aligned}$$

Remark 2.13. *By setting $\varrho(t) = t$ in (2.19), we get the classical convolution*

$$(f \star h)(t) = \int_0^t h(t-\tau) f(\tau) d\tau. \tag{2.20}$$

The generalized convolution $*_\varrho$ is commutative, associative, and distributive, [17]. For piecewise continuous functions f, k, and h, which are ϱ-exponentially bounded over each finite interval $[0, T]$, we have

(i) $f \star_\varrho h = h \star_\varrho f$,

(ii) $(f \star_\varrho k) \star_\varrho h = f \star_\varrho (\varrho \star_\varrho h)$,

(iii) $f \star_\varrho (ak + bh) = af \star_\varrho k + bf \star_\varrho h$*, where* a, b *are constants.*

The following theorem gives a relationship between classical convolution and generalized convolution.

Theorem 2.14. *[17] Let* k *and* l *be two piecewise* $\varrho-$ *exponentially bounded continuous functions over each finite interval* $[0, T]$*. Let* Q_ϱ *be the substitution operator. Then the following relation holds:*

$$k \star_\varrho l = Q_\varrho \left(Q_\varrho^{-1} k \star Q_\varrho^{-1} l \right). \tag{2.21}$$

In [21], the authors define the Elzaki transform of $f \in A$ as

$$\mathcal{E} f(u) = u \int_0^\infty e^{-\frac{1}{u}t} f(t) dt = \lim_{\tau \to \infty} u \int_0^\tau e^{-\frac{1}{u}t} f(t) dt, u \in (-\epsilon_1, \epsilon_2) \tag{2.22}$$

provided that the limit exists as a finite number, where

$$A = \{ f(t), \quad \exists M > 0, \epsilon_2 > \epsilon_1 > 0, \quad |f(t)| \leq M e^{\frac{|t|}{\epsilon_j}},\ t \in (-1)^j [0, \infty), j = 1, 2 \},$$

with the constant M is a finite number and ϵ_1 and ϵ_2 can be finite or infinite.

From this definition, it is clear that the Elzaki transform $\mathcal{E}$ has strong connections with the Laplace transform $\mathcal{L}$.

Indeed, let $f(t) \in \mathcal{A}$ have F and E for the Laplace and Elzaki transforms, respectively. By setting $E(u) = \mathcal{E}\{f\}(u)$ and $L(s) = \mathcal{L}\{f\}(s)$, according to the definition of Laplace transform

$$L(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt, \quad t \geq 0,\ \mathrm{R}e(s) > 0 \tag{2.23}$$

and the definition of Elzaki transform in (2.22), by taking $s = \frac{1}{u}$, we derive a duality relation as follows:

$$uL\left(\frac{1}{u}\right) = E(u), \tag{2.24}$$

and

$$L(u) = uE\left(\frac{1}{u}\right). \tag{2.25}$$

The formulas (2.24) and (2.25) are referred to as the Laplace–Elzaki duality (LED), which shows that the properties of the integral transform $\mathcal{E}$ can easily be derived from those of the Laplace transform. Hence, it can be used to simplify the process of solving ordinary and partial differential equations. For instance

$$\begin{aligned} \mathcal{E}\left\{ \int_0^t f(\tau) d\tau \right\}(u) &= u\mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\}\left(\frac{1}{u}\right) \\ &= u^2 \mathcal{L}\{f\}\left(\frac{1}{u}\right) \\ &= u\mathcal{E}\{f\}(u), \end{aligned}$$

and

$$\begin{aligned}\mathcal{E}\{D^m f(\tau)\}(u) &= u\mathcal{L}\{D^m f(\tau)\}\left(\frac{1}{u}\right)\\ &= u^{-m}\left[u\mathcal{L}\{f\}\left(\frac{1}{u}\right)-\sum_{k=0}^{m-1}u^{k+2}f^{(k)}(0)\right]\\ &= u^{-m}\left(\mathcal{E}\{f\}(u)-\sum_{k=0}^{m-1}u^{k+2}f^{(k)}(0)\right).\end{aligned}$$

Inspired by the research in [17], we introduce a modified version of the standard Elzaki transform and its generalization. We also prove their some properties. Finally, we include several applications to illustrate their use.

3. The main results

In this section, we introduce the generalized Elzaki transform in a modified form and discuss some of its properties. We present its own convolution theorem using an operational calculus approach. We also prove its action on the generalized fractional derivative and integral operators. Some examples are provided to illustrate the tools presented here.

3.1. The generalized Elzaki transform

Let ϱ be a non-negative strictly increasing function and consider the set

$$\mathcal{A} = \{f|\exists M > 0, \epsilon_2 > \epsilon_1 > 0, \quad |f(t)| \le Me^{\frac{\varrho(t)}{\epsilon_i}}, t \in \mathbb{R}\}. \tag{3.1}$$

The constant M must be finite number; ϵ_1 and ϵ_2 may be finite or infinite.

Definition 3.1. *Let $b \in (0,\infty) - \{1\}$. Let $f \in \mathcal{A}$ be the function defined on the interval $[0,\infty)$ and let ϱ be a strictly increasing function on $[0,\infty)$. The modified Elzaki transform of f with respect to the function ϱ denoted $\mathcal{E}_{b;\varrho}\{f\}$ is defined as follows:*

$$\mathcal{E}_{b;\varrho}\{f\}(u) = u\int_0^\infty b^{-\frac{1}{u}(\varrho(t)-\varrho(0))}f(t)\varrho'(t)dt, \quad t \ge 0, \epsilon_1 < u < \epsilon_2 \tag{3.2}$$

for all values of u for which the integral in (3.2) is convergent. The transformation $\mathcal{E}_{b;\varrho}$ is called the generalized modified Elzaki transform.

Remark 3.2. *(i) For $f \in \mathcal{A}, \mathcal{E}_{b;\varrho}f$ exists and converges (for a proof, see Theorem 3.4).*
(ii) By setting $t = v\tau$ in (3.2), we get

$$\mathcal{E}_{b;\varrho}\{f\}(v) = v^2\int_0^\infty b^{-(\varrho(\tau)-\varrho(0))}f(v\tau)\varrho'(\tau)d\tau. \tag{3.3}$$

(iii) If $\varrho(t) = t$, there are two cases.

For $b = e$, the transform $\mathcal{E}_{b;\varrho}$ coincides with the standard form

$$\mathcal{E}_{e;t}\{f\}(u) = \mathcal{E}\{f\}(u) = u\int_0^{\infty} e^{-\frac{1}{u}t} f(t)dt,$$

and for $b \in (1,\infty)$, we obtain the modified Elzaki transform given by the formula

$$\mathcal{E}_{b,t}\{f\}(u) := \mathcal{E}_b\{f\}(u) = u\int_0^{\infty} b^{-\frac{1}{u}t} f(t)dt.$$

(iv) If $\mathcal{E}\{f\}$ is the standard Elzaki transform of f, then

$$\mathcal{E}_b\{f\}(u) = \ln b\,\mathcal{E}\{f\}\left(\frac{u}{\ln b}\right)$$

and

$$\mathcal{E}\{f\}(u) = \frac{1}{\ln b}\mathcal{E}_b\{f\}(u\ln b).$$

We now give the transforms of some elementary functions.

(1) If $f(t) = 1$, then

$$\mathcal{E}_b\{1\}(u) = \frac{u^2}{\ln b}, b > 1, u > 0.$$

(2) If $f(t) = t$, then

$$\mathcal{E}_b\{t\}(u) = \frac{u^3}{(\ln b)^2}, b > 1, u > 0.$$

(3) If $f(t) = t^m$, then

$$\mathcal{E}_b\{t^m\}(u) = \frac{\Gamma(m+1)u^{m+2}}{(\ln b)^{m+1}}, b > 1, u > 0.$$

(4) If $f(t) = e^{\lambda t}$, then

$$\mathcal{E}_b\left\{e^{\lambda t}\right\}(u) = \frac{u^2}{\ln b - \lambda u}, b > 1, |\lambda|u < \ln b.$$

(5) If $f(t) = \sin\xi t$, $\quad g(t) = \cos\xi t$ then

$$\mathcal{E}_b\{\sin\xi t\}(u) = \frac{\xi u^3}{(\ln b)^2 + \xi^2u^2}, \quad \mathcal{E}_b\{\cos\xi t\}(u) = \frac{\xi u^2\ln b}{(\ln b)^2 + \xi^2u^2}.$$

(6) First shifting property: Let $\mathcal{E}_b(f)(u)$ be the modified transform of f, in which case

$$\mathcal{E}_b\left\{e^{\lambda t}f(t)\right\}(u) = \left(1 - \frac{\lambda u}{\ln b}\right)\mathcal{E}_b\{f(t)\}\left(\frac{u\ln b}{\ln b - \lambda u}\right). \tag{3.4}$$

Definition 3.3. *Let $f : [0,\infty) \to \mathbb{R}$ be a real valued-function. Then f is called a function of ϱ-exponential order or ϱ-exponentially bounded, if constants K, c, and t' exist such that for all $t > t'$, we have*

$$|f(t)| \leq Ke^{c\varrho(t)}. \tag{3.5}$$

We denote the class of all piecewise continuous functions which are ϱ- exponentially bounded for ϱ by an increasing positive function by $\mathcal{PC}_{\varrho}^{exp}(.)$.

Theorem 3.4. *(Existence) Let $f : [0, \infty) \to \mathbb{R}$ be a real-valued function such that $f \in \mathcal{PC}_{\varrho}^{exp}(.)$. Then the generalized Elzaki transform of f exists and converges absolutely for $uc < \ln b$, $(b > 1)$.*

Proof. First,

$$|f(t)| \le K_1 e^{c\varrho(t)}, \quad t > t'$$

for some real c. Moreover, f is piecewise continuous on $[0, t']$ and hence is bounded there (the bound being just the largest bound over all the subintervals), say

$$|f(t)| \le K_2, \quad 0 < t < t'.$$

Since $e^{c\varrho(t)}$ has a positive minimum on $[0, t_0]$, a constant K can be chosen to be sufficiently large so that

$$|f(t)| \le K e^{c\varrho(t)}, \quad t > 0.$$

We then have

$$\mathcal{E}_{\varrho,b}\{f\}(u) = \lim_{T\to\infty} u \int_0^T b^{\frac{-1}{u}(\varrho(t)-\varrho(0))} f(t)\varrho'(t)dt.$$

Therefore,

$$\begin{aligned}
\left| u \int_0^T b^{\frac{-1}{u}(\varrho(t)-\varrho(0))} f(t)\varrho'(t)dt \right| &\le |u| \int_0^\infty e^{\frac{-1}{u}(\varrho(t)-\varrho(0))\ln b} |f(t)|\varrho'(t)dt \\
&\le K|u| \int_0^T e^{\frac{-1}{u}(\varrho(t)-\varrho(0))\ln b} e^{c\varrho(t)}\varrho'(t)dt \\
&= K|u| \left\{ -\frac{e^{-(\frac{\ln b}{u}-c)\varrho(t)+\frac{\ln b}{u}\varrho(0)}}{\frac{\ln b}{u}-c} \right\}_0^T \\
&\le \frac{K|u|^2 e^{c\varrho(0)}}{\ln b - cu}, \quad cu < \ln b.
\end{aligned}$$

If we let $T \to \infty$ and since ϱ is increasing and positive, this yeilds

$$|u| \int_0^\infty \left| b^{-\frac{1}{u}(\varrho(t)-\varrho(0))} f(t)\varrho'(t) \right| dt \le \frac{K|u|^2 e^{c\varrho(0)}}{\ln b - cu}, \quad uc < \ln b. \tag{3.6}$$

Theorem 3.5. *Let $f, k \in \mathcal{PC}_{\varrho}^{exp}(.)$. For any constants μ, ν, we have $\mu f + \nu k \in \mathcal{PC}_{\varrho}^{exp}(.)$. Moreover*

(i) $\mathcal{E}_{b;\varrho}\{\mu f + \nu k\} = \mu\mathcal{E}_{b;\varrho}\{f\} + \nu\mathcal{E}_{b;\varrho}\{k\}$,
(ii) $l(t) = f(t)k(t) \in \mathcal{PC}_{\varrho}^{exp}(.)$.

Proof. (ii) $l(t)$ is piecewise continuous. For

$$|f(t)| \le K_1 e^{c_1\varrho(t)}, \quad |k(t)| \le K_2 e^{c_2\varrho(t)},$$

we see that for all $t > c$, we have

$$|l(t)| = |f(t)k(t)| \le K e^{c\varrho(t)}$$

where $K = K_1K_2, \quad c = c_1 + c_2$.

Theorem 3.6. *Let f, k be two functions in $\mathcal{PC}_{\varrho}^{exp}(.)$ with generalized Elzaki transforms $F(u)$ and $G(u)$, respectively, such that $F(u) = G(u)$ for some $\epsilon_1 \leq u \leq \epsilon_2$. In this case, $f = k$ for $t \geq 0$, where both functions are continuous.*

Proof. Let $F(u) = G(u)$. Thus

$$u \int_0^T b^{\frac{-1}{u}(\varrho(t)-\varrho(0))} f(t)\varrho'(t)dt - u \int_0^T b^{\frac{-1}{u}(\varrho(t)-\varrho(0))} k(t)\varrho'(t)dt = 0$$

$$\Rightarrow f = k \quad a.e.$$

Since f and k are both continuous, $f = k$.

Inverse transform

With the theorem above, we can define the inverse of the generalized Elzaki transform. If $F(u) := \mathcal{E}_{b;\varrho}\{f(t)\}(u)$, then $f(t) = \mathcal{E}_{b;\varrho}^{-1}\{F(u)\}(t)$ is called the inverse Elzaki transform of $F(u)$ and $\mathcal{E}_{b;\varrho}^{-1}$ is called the inverse transform of $\mathcal{E}_{b;\varrho}$. For example, for the modified transform, we have

$$\mathcal{E}_b^{-1}\left\{\frac{u^{\delta+2}}{(\ln b)^{\delta+1}}\right\}(t) = \frac{t^{\delta}}{\Gamma(\delta+1)} \quad \delta > -1,\, b > 1.$$

Now we prove a result which expresses a relationship between the classical Elzaki transform and the generalized one using the operational calculus approach.

Theorem 3.7. *Let $f : [0, \infty) \to \mathbb{R}$ be real-valued function. Let ϱ be a non-negative increasing function with a continuous derivative. Assume that the generalized Elzaki transform of f exists. Then*

$$\mathcal{E}_{b;\varrho}\{f\}(u) = \mathcal{E}_b\left\{f(\varrho^{-1}(. + \varrho(0)))\right\}(u) \tag{3.7}$$

where $\mathcal{E}\{f\}(u)$ is the usual Elzaki transform of the function f.

Proof. Since the function ϱ is non-negative and increasing and has a continuous derivative, the substitution $v = \varrho^{-1}(t + \varrho(0))$ preserves absolute integrability. Indeed, ϱ^{-1} exists and is continuous and increasing. Therefore, v is continuous and increasing. Since ϱ' is continuous and non-negative, $\varrho'(\varrho^{-1}(t + \varrho(0)))$ is also continuous and non-negative. Therefore, the integral

$$\begin{aligned}
\mathcal{E}_b\left\{f(\varrho^{-1}(t + \varrho(0)))\right\}(u) &= u \int_0^{\infty} e^{-\frac{1}{u}t\ln b} f(\varrho^{-1}(t + \varrho(0)))dt,\ (v = \varrho^{-1}(t + \varrho(0))) \\
&= u \int_0^{\infty} e^{-\frac{1}{u}(\varrho(v)-\varrho(0))\ln b} f(v)\varrho'(v)dv \\
&= u \int_0^{\infty} b^{-\frac{1}{u}(\varrho(v)-\varrho(0))} f(v)\varrho'(v)dv \\
&= \mathcal{E}_{b;\varrho}\{f(v)\}(u).
\end{aligned}$$

In terms of operators, the relation (3.7) is written as

$$\mathcal{E}_{b;\varrho} = \mathcal{E}_b \circ Q_{\varrho}^{-1}. \tag{3.8}$$

To prove (3.8), we use the definition of Q_ϱ^{-1} (see 2.10) and $\mathcal{L}_{b;\varrho}$. We have

$$\begin{aligned}\mathcal{E}_b \circ Q_\varrho^{-1}\{f\}(u) &= \mathcal{E}_b\left\{Q_\varrho^{-1} f\right\}(u)\\ &= u\mathcal{L}_b\left\{Q_\varrho^{-1} f\right\}\left(\frac{1}{u}\right) \quad (\textit{duality}\ )\\ &= u\left(\mathcal{L}_b \circ Q_\varrho^{-1}\right)\{f\}\left(\frac{1}{u}\right)\\ &= u\mathcal{L}_b\left\{f \circ \varrho^{-1}\right\}\left(\frac{1}{u}\right)\\ &= u\mathcal{L}_{b;\varrho}\{f\}\left(\frac{1}{u}\right)\\ &= \mathcal{E}_{b;\varrho}\{f\}(u),\end{aligned}$$

where $\mathcal{L}_{b;\varrho}\{f\}(u) = \int_0^\infty b^{-u(\rho(t)-\rho(0))} f(t)\rho'(t)dt$.

Corollary 3.8. *(Inverse generalized Elzaki transform)*

$$\mathcal{E}_{b;\varrho}^{-1} = (\mathcal{E}_b^{-1} \circ Q_\varrho^{-1})^{-1} = Q_\varrho \circ \mathcal{E}_b^{-1}. \tag{3.9}$$

Corollary 3.9. *If $f \in \mathcal{PC}_\varrho^{exp}(.)$ over each finite interval $[0, T]$ whose Elzaki transform is $\mathcal{E}_{b;\varrho}\{f\}$, $b > 1$ and the classical Elzaki transform is $\mathcal{E}\{f\}$, then*

$$\mathcal{E}_{b;\varrho}\{f \circ \varrho\}(.) = \mathcal{E}_b\{f\}(.) = \ln b\,\mathcal{E}\{f\}\left(\frac{.}{\ln b}\right). \tag{3.10}$$

Proof. From (3.8), we have

$$\begin{aligned}\mathcal{E}_{b;\varrho}(f \circ \varrho) &= \mathcal{E}_b \circ Q_\varrho^{-1}\{f \circ \varrho\}\\ &= \mathcal{E}_b\left\{f \circ \varrho \circ \varrho^{-1}\right\}\\ &= \mathcal{E}_b\{f\}.\end{aligned}$$

The following assertions can be easily obtained by using the corollary above and also

$$\mathcal{E}_{b,\varrho}\left\{e^{\xi(\varrho(t)-\varrho(0))}\right\}(u) = \frac{u^2}{\ln b - \xi u}, \quad \xi \in \mathbb{R}, u\xi < 1, u > 0. \tag{3.11}$$

$$\mathcal{E}_{b;\varrho}\left\{(\varrho(t) - \varrho(0))^{\delta-1}\right\}(u) = \frac{\Gamma(\delta)}{(\ln b)^\delta} u^{\delta+1}, \quad \delta \in \mathbb{R}, \delta > -1, \delta \neq 0, u > 0. \tag{3.12}$$

$$\mathcal{E}_{b;\varrho}\left\{e^{\xi\varrho(t)} f(t)\right\}(u) = (1 - \frac{\xi u}{\ln b})\mathcal{E}_{b;\varrho}\{f\}\left(\frac{u \ln b}{\ln b - \xi u}\right), \quad \xi \in \mathbb{R}, u\xi < \ln b, u > 0. \tag{3.13}$$

Lemma 3.10. *(Convolution theorem). Let $[0, T]$ be a finite interval. If $f, h \in \mathcal{PC}^{exp}([0, T])$, then the following relation modified by the Elzaki transform $\mathcal{E}_b$ holds:*

$$\mathcal{E}_b\{f \star h\}(u) = \frac{1}{u}\mathcal{E}_b\{f\}(u)\mathcal{E}_b\{h\}(u). \tag{3.14}$$

Proof.

$$
\begin{aligned}
\frac{1}{u}\mathcal{E}_b\{f\}(u)\mathcal{E}_b\{h\}(u) &= u\int_0^\infty b^{-\frac{1}{u}t}f(t)dt\int_0^\infty b^{-\frac{1}{u}s}h(s)ds\\
&= u\int_0^\infty\int_0^\infty b^{-\frac{1}{u}(t+s)}f(t)h(s)dtds \quad (\tau=t+s)\\
&= u\int_0^\infty b^{-\frac{1}{u}\tau}\int_0^\tau f(t)h(\tau-t)dtd\tau\\
&= \mathcal{E}_b\left\{\int_0^\tau h(\tau-t)f(t)dt\right\}(u)\\
&= \mathcal{E}_b\{f\star h(\tau)\}(u).
\end{aligned}
$$

Theorem 3.11. *(ϱ-convolution theorem) Assume $k,l\in\mathcal{PC}_\varrho^{exp}(.)$ over each finite interval $[0,T]$. Then, the following relation holds:*

$$
\mathcal{E}_{b;\varrho}\left\{k\star_\varrho l\right\}(u)=\frac{1}{u}\mathcal{E}_{b;\varrho}\{k\}(u)\mathcal{E}_{b;\varrho}\{l\}(u). \tag{3.15}
$$

Proof. To prove (3.15), we use Theorem 2.14 and Lemma 3.10. Let Q_ϱ be the substitution operator. Then

$$
\begin{aligned}
\mathcal{E}_{b,\varrho}\left\{k\star_\varrho l\right\}(u) &= \mathcal{E}_b\left\{Q_\varrho^{-1}(k\star_\varrho l)(.)\right\}(u)\\
&= \mathcal{E}_b\left\{Q_\varrho^{-1}(Q_\varrho(Q_\varrho^{-1}k\star Q_\varrho^{-1}l))(.)\right\}(u)\\
&= \mathcal{E}_b\left\{(Q_\varrho^{-1}k\star Q_\varrho^{-1}l)(.)\right\}(u)\\
&= \frac{1}{u}\mathcal{E}_b\left\{Q_\varrho^{-1}k(.)\right\}(u)\mathcal{E}_b\left\{(Q_\varrho^{-1}l)(t)\right\}(u)\\
&= \frac{1}{u}\mathcal{E}_b\circ Q_\varrho^{-1}\{k\}\,\mathcal{E}_b\circ Q_\varrho^{-1}\{l\}\\
&= \frac{1}{u}\mathcal{E}_{b;\varrho}\{k\}(u)\mathcal{E}_{b;\varrho}\{l\}(u).
\end{aligned}
$$

Theorem 3.12. *(Derivative) Let $f\in C_\varrho[0,T]$ be ϱ-exponentially bounded such that $f^{[1]}$ is piecewise continuous over every finite interval $[0,T]$. Then the generalized Elzaki transform of $f^{[1]}$ exists and satisfies*

$$
\mathcal{E}_{b;\varrho}\left\{f^{[1]}\right\}(u)=\left(\frac{u}{\ln b}\right)^{-1}\mathcal{E}_{b;\varrho}\{f\}(u)-uf(0). \tag{3.16}
$$

Proof. We have

$$
\begin{aligned}
\mathcal{E}_{b;\varrho}\left\{f^{[1]}\right\}(u) &= u\int_0^\infty e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b}f^{[1]}(t)\varrho'(t)dt\\
&= \lim_{T\to\infty}u\int_0^T e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b}f^{[1]}(t)\varrho'(t)dt.
\end{aligned}
$$

Let $0<\xi_1,\xi_2,\cdots,\xi_n<T$ be the points in the interval $[0,T]$ where $f^{[1]}$ is discontinuous. Thus

$$
u\int_0^T e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b}f^{[1]}(t)\varrho'(t)dt = u\int_0^T e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b}f'(t)dt
$$

$$
\begin{aligned}
&= u \int_0^{\xi_1} e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b} f'(t)dt + \\
&+ u \sum_{i=1}^{m-1} \int_{\xi_i}^{\xi_{i+1}} e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b} f'(t)dt \\
&+ u \int_{\xi_m}^{T} e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b} f'(t)dt.
\end{aligned}
$$

After integrating by parts, we obtain

$$
\begin{aligned}
u \int_0^T e^{-\frac{1}{u}(\varrho(t)-\varrho(0))\ln b} f^{[1]}(t)\varrho'(t)dt =& u e^{-\frac{1}{u}(\varrho(T)-\varrho(0))\ln b} f(T) - uf(0) \\
&+ \ln b \int_0^T \frac{f(t)\varrho'(t)}{e^{\frac{1}{u}(\varrho(t)-\varrho(0))\ln b}} dt. \qquad (3.17)
\end{aligned}
$$

On letting $T \longrightarrow \infty$, with $(b > 1)$ on both sides of (3.17), we obtain (3.7).

Corollary 3.13. *Let $f \in C_\varrho^m([0;T))$ such that $f^{[i]}$, $i = \overline{1, m-1}$ are of ϱ-exponential order. Let $f^{[m]}$ be a piecewise continuous function on the interval $[0,T]$. Then the generalized Elzaki transform of $f^{[m]}$ exists and is given by*

$$
\mathcal{E}_{a;\varrho}\left\{f^{[m]}\right\}(u) = \left(\frac{u}{\ln b}\right)^{-m} \left[\mathcal{E}_{b,\varrho}\left\{f\right\}(u) - \sum_{k=0}^{m-1} \frac{u^{k+2}}{(\ln b)^{k+1}} f^{(k)}(0)\right]. \qquad (3.18)
$$

Corollary 3.14. *Under the same assumptions as Corollary 3.13, it follows that the modified Elzaki transform of $f^{(m)}$ exists and is given by*

$$
\mathcal{E}_{b}\left\{f^{(m)}\right\}(u) = \left(\frac{u}{\ln b}\right)^{-m} \left[\mathcal{E}_{b}\left\{f\right\}(u) - \sum_{k=0}^{m-1} \frac{u^{k+2}}{(\ln b)^{k+1}} f^{(k)}(0)\right]. \qquad (3.19)
$$

4. Fractional operators

In this section, we present the generalized Elzaki transforms of the fractional integral and derivative operators. Some fundamental properties of the Elzaki transform that are necessary in solving fractional differential equations are given in the following theorems.

Theorem 4.1. *(Integral operators) Let $\alpha > 0$. Let $h \in \mathcal{PC}_\varrho^{exp}(.)$ on each interval $[0,T]$. Then*

$$
\mathcal{E}_{b;\varrho}\left\{{}^{RL}\mathcal{I}^{\alpha}_{0_+;\varrho} h(.)\right\}(u) = \frac{u^\alpha}{(\ln b)^\alpha} \mathcal{E}_{b;\varrho}\left\{h(.)\right\}(u). \qquad (4.1)
$$

Proof.

$$
\mathcal{E}_{b;\varrho}\left\{{}^{RL}\mathcal{I}^{\alpha}_{0+;\varrho} h(t)\right\}(u) = \mathcal{E}_{b;\varrho}\left\{\frac{1}{\Gamma(\alpha)} \int_0^t (\varrho(t)-\varrho(s))^{\alpha-1} h(s)\varrho'(s)ds\right\}(u)
$$

$$= \frac{1}{\Gamma(\alpha)}\mathcal{E}_{b,\varrho}\left\{(\varrho(t)-\varrho(0))^{\alpha-1}\star_{\varrho} h(t)\right\}(u)$$
$$= \frac{1}{\Gamma(\alpha)}\frac{1}{u}\mathcal{E}_{b,\varrho}\left\{(\varrho(t)-\varrho(0))^{\alpha-1}\right\}\mathcal{E}_{b,\varrho}\left\{h(t)\right\}(u)$$
$$= \frac{u^{\alpha}}{(\ln b)^{\alpha}}\mathcal{E}_{b;\varrho}\left\{h\right\}(u).$$

Corollary 4.2. *(Caputo derivative) Let $\alpha > 0$, $h \in \mathcal{AC}^m([r,s])$, $\varrho \in C^m([r,s]), \varrho'(t) > 0, h^{(k)}$, $k = 0,1,\cdots,m-1$ be ϱ-exponentially bounded. Then*

$$\mathcal{E}_{b;\varrho}\left\{{}^{C}\mathcal{D}^{\alpha}_{r+;\varrho}h(t)\right\}(u) = \left(\frac{u}{\ln b}\right)^{-\alpha}\left[\mathcal{E}_{b,\varrho}\left\{h(t)\right\}(u) - \sum_{k=0}^{m-1}\frac{u^{k+2}}{(\ln b)^{k+1}}h^{(k)}(0)\right]. \tag{4.2}$$

Corollary 4.3. *(Riemann–Liouville derivative). Let $\alpha > 0$, $h \in \mathcal{AC}^m([r,s]),\varrho \in C^m([r,s]), \varrho'(t) > 0$, and $\left(\mathcal{I}^{\alpha}_{r;\varrho}h\right)^{[k]}, k = 0,1,...,m-1$ be of $g-$ exponential order. Then*

$$\mathcal{E}_{b;\varrho}\left\{{}^{RL}\mathcal{D}^{\alpha}_{r+;\varrho}h(t)\right\}(u) = \left(\frac{u}{\ln b}\right)^{-\alpha}\mathcal{E}_{b;\varrho}(h(t))(u) - \sum_{k=0}^{m-1}\frac{u^{-m+k+2}}{(\ln b)^{-m+k+1}}\left(\mathcal{I}^{m-\alpha}_{r;\varrho}h\right)^{[k]}(0). \tag{4.3}$$

Now we give the transforms of some specified Mittag-Leffler functions.

Lemma 4.4. *Let $\alpha > 0, \mathcal{R}e(\alpha) > 0$, and $|\lambda u^{\alpha}| < (\ln b)^{\alpha}$. Then*

$$\mathcal{E}_{b;\varrho}\left\{E_{\alpha}(\lambda((\varrho(t)-\varrho(0))^{\alpha}))\right\}(u) = \frac{u^2(\ln b)^{\alpha-1}}{(\ln b)^{\alpha}-\lambda u^{\alpha}}. \tag{4.4}$$

$$\mathcal{E}_{b;\varrho}\left\{(\varrho(t)-\varrho(0))^{\delta-1}E_{\alpha,\delta}(\lambda((\varrho(t)-\varrho(0))^{\alpha}))\right\}(u) = \frac{u^{\delta+1}(\ln b)^{\alpha-\delta}}{(\ln b)^{\alpha}-\lambda u^{\alpha}}. \tag{4.5}$$

Proof. We prove (4.5). We recognize that the Mittag-Leffler function is an entire function providing a simple generalization of the exponential function, convergent in the whole complex plane. Using the fact that the transform $\mathcal{E}_{b;\varrho}$ is linear, we have

$$\mathcal{E}_{b;\varrho}\left\{(\varrho(t)-\varrho(0))^{\delta-1}E_{\alpha,\delta}(\lambda((\varrho(t)-\varrho(0))^{\alpha}))\right\}(u)$$
$$= \mathcal{E}_{b;\varrho}\left(\sum_{k=0}^{\infty}\frac{\lambda^k(\varrho(t)-\varrho(0))^{k\alpha+\delta-1}}{\Gamma(k\alpha+\delta)}\right)(u)$$
$$= \sum_{k=0}^{\infty}\frac{\lambda^k\mathcal{E}_{b;\varrho}\left\{(\varrho(t)-\varrho(0))^{k\alpha+\delta-1}\right\}(u)}{\Gamma(k\alpha+\delta)}$$
$$= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\delta)}\Gamma(k\alpha+\delta)u^{k\alpha+\delta+1}\frac{1}{(\ln b)^{k\alpha+\delta}}$$
$$= \frac{u^{\delta+1}}{(\ln b)^{\delta}}\sum_{k=0}^{\infty}\left(\frac{\lambda u^{\alpha}}{(\ln b)^{\alpha}}\right)^k, \quad \frac{\lambda u^{\alpha}}{(\ln b)^{\alpha}} < 1$$
$$= \frac{u^{-\alpha+\delta+1}(\ln b)^{\alpha-\delta}}{(u^{-1}\ln b)^{\alpha}-\lambda}.$$

The modified Elzaki transforms of some specified Mittag-Leffler functions are as follows.

Corollary 4.5. *Let $\alpha > 0, \mathcal{R}e(\alpha) > 0$, and $\left|\lambda \frac{u^\alpha}{(\ln b)^\alpha}\right| < 1$. Then*

$$\mathcal{E}_b \{E_\alpha(\lambda t^\alpha)\}(u) = \frac{u^2(\ln b)^{\alpha-1}}{(\ln b)^\alpha - \lambda u^\alpha}. \tag{4.6}$$

$$\mathcal{E}_b \left\{t^{\delta-1} E_{\alpha,\delta}(\lambda t^\alpha)\right\}(u) = \frac{u^{\delta+1}(\ln b)^{\alpha-\delta}}{(\ln b)^\alpha - \lambda u^\alpha}. \tag{4.7}$$

5. Applications to generalized fractional differential equations

In this section, we highlight the theoretical concepts presented in Sections 3 and 4. Like the convolution, the ϱ-convolution, inverse transforms, and the transform of the fractional operators, the Mittag-Leffler function used to solve some problems such as the Cauchy problem, linear differential equations, and fractional differential equations.

Example 5.1. *Let us consider the problem*

$$\begin{cases} z''(t) - z'(t) - 2z(t) = e^{2t}, \\ z(0) = 0, \ z'(t) = 0. \end{cases} \tag{5.1}$$

Taking the modified Elzaki transform of both sides, we find

$$\left[\frac{(\ln a)^2}{u^2} + \frac{\ln a}{u} - 2\right] \mathcal{E}_a \{z\} = \frac{u^2}{\ln a - 2u}.$$

Solving for $\mathcal{E}_a \{z\}$ and using partial fraction decomposition, we can write

$$\begin{aligned} \mathcal{E}_a \{z\} &= u^2 \frac{u^2}{(\ln a - u)(\ln a - 2u)(\ln a + 2u)} \\ &= u^2 \left(\frac{-1/3}{\ln a - u} + \frac{1/4}{\ln a - 2u} + \frac{1/12}{\ln a + 2u}\right). \end{aligned}$$

Hence $\quad z(t) = \frac{-1}{3} e^t + \frac{1}{4} e^{2t} + \frac{1}{12} e^{-2t}.$

Example 5.2. *Let us assume the Cauchy problem below:*

$$\begin{cases} ({}^C\mathcal{D}^\delta_{0+,\varrho} r)(t) - \lambda r(t) = h(t), t > 0, m-1 < \alpha < m, \lambda \in \mathbb{R}, \\ r^{(k)}(0) = c_k, \ k = 0, \cdots m-1, \ c_k \in \mathbb{R}. \end{cases} \tag{5.2}$$

Taking the generalized Elzaki transform of all the terms, plugging in the initial conditions in (5.2), and then using Corollary 4.2, we get

$$\left(\left(\frac{u}{\ln b}\right)^{-\alpha} - \lambda\right) \mathcal{E}_{b,\varrho} \{r(t)\}(u) = \sum_{k=0}^{m-1} c_k \frac{u^{-\alpha+k+2}}{(\ln b)^{-\alpha+k+1}} + \mathcal{E}_{b,\varrho} \{h(t)\}(u). \tag{5.3}$$

Thus

$$\mathcal{E}_{b;\varrho}(r(t))(u) = \sum_{k=0}^{m-1} \frac{c_k u^{k+2}(\ln b)^{\alpha-k-1}}{(\ln b)^\alpha - \lambda u^\alpha} + \frac{1}{u}\frac{u^{\alpha+1}}{(\ln b)^\alpha - \lambda u^\alpha}\mathcal{E}_{b,\varrho}(h(t))(u).$$

Using (4.5) from Lemma 4.4, we have $\frac{u^{k+2}(\ln b)^{\alpha-k-1}}{(\ln b)^\alpha-\lambda u^\alpha} = \mathcal{E}_{b;\varrho}\left\{E_{\alpha,k+1}(\lambda(\varrho(t)-\varrho(0))^\alpha)\right\}$ *and* $\frac{u^{\alpha+1}}{(\ln b)^\alpha-\lambda u^\alpha} = \mathcal{E}_{b,\varrho}\left\{(\varrho(t)-\varrho(0))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t)-\varrho(0))^\alpha))\right\}$. *Hence by taking the convolution* $*_\varrho$ *into account, we have*

$$\begin{aligned}\mathcal{E}_{b;\varrho}(r(t))(u) =& \sum_{k=0}^{m-1} c_k \mathcal{E}_{b;\varrho}\left\{E_{\alpha,k+1}(\lambda(\varrho(t)-\varrho(0))^\alpha)\right\}\\ &+ \frac{1}{u}\mathcal{E}_{b,\varrho}\left\{(\varrho(t)-\varrho(0))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t)-\varrho(0))^\alpha))\right\}\mathcal{E}_{b,\varrho}(h(t))(u)\\ =&\mathcal{E}_{b,\varrho}\left\{\sum_{k=0}^{m-1} c_k(\varrho(t)-\varrho(0))^k E_{\alpha,k+1}((\varrho(t)-\varrho(0))^\alpha)\right\}(u)\\ &+ \mathcal{E}_{b,\varrho}\left\{(\varrho(t)-\varrho(0))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t)-\varrho(0))^\alpha)) *_\varrho h\right\}(u).\end{aligned}$$

Applying the inverse transform, we get

$$\begin{aligned}r(t) =& \sum_{k=0}^{m-1} c_k(\varrho(t)-\varrho(0))^k E_{\alpha,k+1}(\lambda(\varrho(t)-\varrho(0))^\alpha)\\ &+ (\varrho(t)-\varrho(0))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t)-\varrho(0))^\alpha)) *_\varrho h\end{aligned}$$

that is

$$\begin{aligned}r(t) =& \sum_{k=0}^{m-1} c_k(\varrho(t)-\varrho(0))^k E_{\alpha,k+1}(\lambda(\varrho(t)-\varrho(0))^\alpha)\\ &+ \int_0^t (\varrho(t)-\varrho(s))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t)-\varrho(s))^\alpha))h(s)\varrho'(s)ds.\end{aligned}$$

The special case of the initial value problem is as follows:

$$\begin{cases}({}^C\mathcal{D}^\alpha_{0+,\sqrt{t}}r)(t) - r(t) = 1, t > 0, 0 < \alpha < 1,\\ r(0) = 1,\end{cases} \tag{5.4}$$

for the choice $\varrho(t) = \sqrt{t}, \lambda = 1,$*the problem has the solution*

$$r(t) = E_\alpha(t^{\frac{\alpha}{2}}) + \int_0^t (\sqrt{t} - \sqrt{s})^{\alpha-1}E_{\alpha,\alpha}((\sqrt{t} - \sqrt{s})^\alpha)\frac{1}{2\sqrt{s}}\,ds.$$

Example 5.3. *Let us consider the following Cauchy problem:*

$$\begin{cases}({}^{RL}\mathcal{D}^\alpha_{0+,\varrho}x)(t) - \lambda x(t) = h(t), t > 0, 0 < \alpha < 1, \lambda \in \mathbb{R},\\ (\mathcal{I}^{1-\alpha}_{0,\varrho}x)(0) = c,\ c \in \mathbb{R}.\end{cases} \tag{5.5}$$

Applying the generalized Elzaki transform to both sides of Eq (5.5) and then using Corollary 4.2, with $m = 1$, we get

$$\left(\left(\frac{u}{\ln b}\right)^{-\alpha} - \lambda\right)\mathcal{E}_{\varrho}(x(t))(u) = cu + \mathcal{E}_{\varrho}(h(t))(u). \tag{5.6}$$

Hence

$$\begin{aligned}\mathcal{E}_{\varrho}(x(t))(u) =& \frac{cu^{\alpha+1}}{(\ln b)^{\alpha} - \lambda u^{\alpha}} + +\frac{1}{u}\frac{u^{\alpha+1}}{(\ln b)^{\alpha} - \lambda u^{\alpha}}\mathcal{E}_{\varrho}(h(t))(u)\\ =& c\mathcal{E}_{\varrho}\left\{(\varrho(t) - \varrho(0))^{\alpha-1}E_{\alpha}(\lambda(\varrho(t) - \varrho(0))^{\alpha})\right\}\\ &+ \frac{1}{u}\mathcal{E}_{\varrho}\left\{(\varrho(t) - \varrho(0))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t) - \varrho(0))^{\alpha}))\right\}\mathcal{E}_{\varrho}(h(t))(u)\\ =& \mathcal{E}_{\varrho}\left\{c(\varrho(t) - \varrho(0))^{\alpha-1}E_{\alpha}(\lambda(\varrho(t) - \varrho(0))^{\alpha})\right\}(u)\\ &+ \mathcal{E}_{\varrho}\left\{(\varrho(t) - \varrho(0))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t) - \varrho(0))^{\alpha})) *_{\varrho} h\right\}(u).\end{aligned}$$

Hence,

$$\begin{aligned}x(t) =& c(\varrho(t) - \varrho(0))^{\alpha-1}E_{\alpha}(\lambda(\varrho(t) - \varrho(0))^{\alpha})\\ &+ (\varrho(t) - \varrho(0))^{\alpha}E_{\alpha,\alpha}(\lambda((\varrho(t) - \varrho(0))^{\alpha})) *_{\varrho} h\end{aligned}$$

or

$$\begin{aligned}x(t) =& c(\varrho(t) - \varrho(0))^{\alpha-1}E_{\alpha}(\lambda(\varrho(t) - \varrho(0))^{\alpha})\\ &+ \int_0^t (\varrho(t) - \varrho(s))^{\alpha-1}E_{\alpha,\alpha}(\lambda((\varrho(t) - \varrho(s))^{\alpha}))h(s)\varrho'(s)ds.\end{aligned}$$

Example 5.4. *Consider the linear equation*

$$\int_0^t (\varrho(t) - \varrho(s))^m y(s)ds = h(t) \quad m = 1, 2, ... \tag{5.7}$$

Here, $h(t)$ is assumed to satisfy the conditions

$$h(0) = h'(0) = \cdots = h^m(0) = 0.$$

Equation (5.7) can be written in terms of operators defined above as follows:

$$\Gamma(m + 1)\left(\mathcal{I}^{m+1}_{0;\varrho}\left[\frac{y(t)}{\varrho'(t)}\right]\right)(t) = h(t). \tag{5.8}$$

Applying the generalized Elzaki transform, we get

$$\left(\frac{u}{\ln b}\right)^{(m+1)}\mathcal{E}_{\varrho}\left\{\mathcal{I}^{m+1}_{0;\varrho}\left[\frac{y(t)}{\varrho'(t)}\right]\right\}(u) = \frac{1}{\Gamma(m + 1)}\mathcal{E}_{\varrho}(h)(u). \tag{5.9}$$

Since $h(0) = h'(0) = \cdots = h^{(m)}(0) = 0$, $\sum_{k=0}^{m}\frac{u^{k+2}}{(\ln b)^{k+1}}h^{(k)}(0) = 0$. Hence

$$\mathcal{E}_{b;\varrho}\left\{\left[\frac{y(t)}{\varrho'(t)}\right]\right\}(u) = \frac{1}{\Gamma(m + 1)}\left(\frac{u}{\ln b}\right)^{-(m+1)}\mathcal{E}_{b;\varrho}(h)(u)$$

$$
\begin{aligned}
&= \frac{1}{m!}\left(\frac{u}{\ln b}\right)^{-(m+1)}\left[\mathcal{E}_{b,\varrho}\left\{h\right\}(u) - \sum_{k=0}^{m}\frac{u^{k+2}}{(\ln b)^{k+1}}h^{(k)}(0)\right] \\
&= \frac{1}{m!}\mathcal{E}_{b;\varrho}\left\{h^{(m+1)}\right\}(u).
\end{aligned}
$$

The solution is given by

$$
y(t) = \frac{1}{m!}\varrho'(t)h^{(m+1)}(t). \tag{5.10}
$$

For example, if $\varrho(t) = t^3$ *and* $m = 1$, *then the problem is given as*

$$
\int_0^t (t^2 - s^2)y(s)ds = \left(\mathcal{I}^2_{0;t^2}\left[\frac{y(t)}{2t}\right]\right)(t) = h(t)
$$

and its solution as

$$
y(t) = \frac{1}{2t^2}(th^{(2)}(t) - h'(t)).
$$

6. Conclusions and discussion

In this study, we have introduced a generalized Elzaki transform and explored its applications to fractional differential equations. The results obtained in this study have significant implications for the field of fractional calculus and its applications.

The generalized Elzaki transform is a powerful instrument for solving fractional differential equations. The transform is easy to apply, and the resulting solutions are exact and explicit.

The results obtained in this study have significant implications for the field of fractional calculus and its applications. The fractional differential equations that were solved using the generalized Elzaki transform have applications in various fields such as physics, engineering, and economics. The solutions obtained in this study can be used to model real-world phenomena more accurately and efficiently.

The generalized Elzaki transform also has potential applications in other areas such as signal processing, control theory, and data analysis. The transform can be used to analyze and solve problems in these fields more effectively and efficiently.

One of the significant advantages of the generalized Elzaki transform is its ability to solve fractional differential equations with non integer orders and differentional equations with constant and variable coefficients. This is a significant limitation of the existing methods, which are limited to solving fractional differential equations with integer orders.

As metric dimensions or some other distance-based graph parameters can be used to metricize the graph corresponding to the problem which is to be solved by the Elzaki transform or generalized Elzaki transform, one may combine these methods of analysis together with graph theoretical ones, especially with fractional graph theoretical ones, to solve the problems under investigation more effectively.

In conclusion, the generalized Elzaki transform is a powerful instrument for solving fractional differential equations. The transform is easy to apply, and the resulting solutions are exact and explicit. The transform has significant implications for the field of fractional calculus and its applications.

Author contributions

Mohammed Said Souid: Writing – original draft, project administration, validation; Halim Benali: Resources, investigation, writing – review and editing; Aysun Yurttas Gunes: Editing, visualization, methodology, software, data curation, conceptualization. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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