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Research article

A quasi-conjugate bivariate prior distribution suitable for studying dependence in reinsurance and non reinsurance models with and without a layer

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Abstract: In the collective risk model and also in a compound excess-of-loss reinsurance frameworks, it is usual to assume that the risk parameters associated with the random variables, the number of claims, and the claim size are independent for mathematical convenience. Here, we assumed Poisson and Pareto distributions for these two random variables. This paper focuses on the prior and posterior (Bayesian) net premiums of the total claims amount, assuming some degree of dependence between the two risk profiles associated with these two random variables. Here, the degree of dependence was modeled using the Sarmanov-Lee family of distributions, a special type of copula, which allows us to study the impact of this assumption on the following year's total cost of claims when prior margins are assumed to have gamma and shifted Erlang distributions. The numerical applications show that a low degree of correlation between these variables leads to collective and net Bayes premiums that can be sensitive when the hypothesis of independence is broken. The dependence hypothesis has a more significant effect in the model when no layer is considered. We illustrate the methodology proposed with some real numerical examples.

Keywords: Bayes; dependence; excess-of-loss; layer; risk parameters; Sarmanov-Lee family of distributions
Mathematics Subject Classification: 62C10, 91B05, 91G05

1. Introduction

In the preface of the book of [1], it is stated that reinsurance is a fascinating field, where several of the challenges of classical insurance are amplified, particularly when it comes to dealing with extreme situations like large claims and rare events. However, leaving this enthusiastic phrase aside, the reinsurance field is also exciting because of its practicality and ability to solve many financial problems for insurance companies. As is well known by the actuarial community, a reinsurance contract establishes an agreement or contract between the insurer and the reinsurer, whereby the former (also called the ceding company) indemnifies the latter a part of the risk subscribed by the insurer, with a third party (usually the customer) paying, in turn, a reinsurance premium. As such, it is a contract similar to the one that an insurance company signs with a client to cover damages of any kind. The difference, however, is that the reinsurer covers only a part of the compensation, called priority. This type of reinsurance is called excess-of-loss reinsurance. For a detailed study of reinsurance, in all its forms, readers are referred to the works of [1–5] and the references therein. Recent works on reinsurance include contributions such as [6–9]. In these recent works, the reinsurance technique is also used in non-life insurance, but its scope is broadened, including, for example, non-life multiline insurers under the Solvency II umbrella, risk transfers in the context of the 2007-08 global crisis, reinsurance using two layers and multiple reinsurers, and reinsurance in a game-theoretic setting, considering one insurer and two reinsurers.

As part of the collective risk model, in excess-of-loss reinsurance, it is usual to assume that the risk parameters associated with the random variables, the number of claims, and the claim size are independent. This is the main idea in [10], inspired by the ideas developed in [11]. For example, we can describe this situation by taking the Poisson frequency and the Pareto severity as appropriate distributional assumptions. The use of the Bayesian methodology facilitates the computation of predictive and posterior moments of the accumulated claims and the interpretation of the parameters involved. [12] also studied the Bayesian estimators in Paretian excess-of-loss reinsurance. Although these derivations have traditionally been built on the assumption of independence (see [13]), there has been great interest in modeling dependent risks due to the increasing complexity of the current insurance and reinsurance products.

In this paper, we will pay attention to the calculation of the prior and posterior moments of the size of the total claims, assuming some degree of dependence derived from the prior distributions of the risk parameters (profiles) involved in the Bayesian framework. This degree of dependence will be modeled using the Sarmanov-Lee family of distributions; specific details of this type of copula can be seen in [14, 15]. This model allows us to study the impact of dependence on the next year's total claims costs, showing that a low degree of correlation produces highly sensitive results. Furthermore, the case where the moments are calculated separately without assuming dependence is also considered. In recent years, some actuarial works have considered the Sarmanov family of bivariate distributions: see, for example, [16–18]. Also, [19] considered, from a Bayesian point of view, the calculation of credibility premiums for the compound loss under a bivariate prior for the risk parameters of the frequency and severity distributions.

The impact of assuming the dependence feature is a matter of concern. At first sight, it would seem that the effect of including dependence has a low impact in the first and second moments; in fact, the reinsurer should be prudent, since ignoring the hypothesis of dependence might lead to budget imbalance, especially when there is a change of scale in the monetary units used. This work assumes the dependence hypothesis between the risk profiles instead of the claims number and the size of the corresponding claims. The former approach not only retains the main features of the latter case but also reduces the mathematical burden of the independence assumption between the number and the severity.

The rest of the paper has the following structure: The basic collective risk model considered in this paper is introduced in Section 2. Here, the bivariate prior distribution assumed for the risk profile (dependent on two parameters) is also presented and studied. In the original paper of [10], a classical gamma distribution was assumed, but this seems inadequate considering this risk parameter's support. To make the paper self-contained, we recall the main results of [10] in Section 3. Here, we also give closed-form expressions for the risk, collective, and Bayes net premiums in the excess-of-loss reinsurance model. An extension of the basic collective risk model and the reinsurance model, which includes some degree of dependence, is introduced in Section 4.2. Here, we also provide properties of this extended model together with the expressions of the collective and Bayesian net premiums, which are written in closed-form expressions. The cases with and without a layer are both studied. Numerical applications are provided in Section 5, and final comments are given in the last section.

2. The specific basic collective risk model

Let $N(t)$ be the random variable denoting the number of claims reported to the insurer during $[0,t]$ and let $Y_1, Y_2, \ldots, Y_{N(t)}$ be the corresponding individual claim amounts up to time t. Assume the following: (i) $\{N(t),\, t \geq 0\}$ is a time homogeneous Poisson process with intensity $\lambda > 0$, i.e., $p_n(t) = \exp(-\lambda t)(\lambda t)^n/n!$ and (ii) $\{Y_i\}_{i=1,2,\ldots}$ are stochastically independent and independent of $N(t)$ with common survival function $\bar{F}(y|c,\psi) = (c/y)^{\psi}$, $y \geq c$, i.e., the classical Pareto distribution with scale parameter $c > 0$ and shape parameter $\psi > 0$. Its mean is given by $\mathbb{E}(Y) = c\psi/(\psi-1)$, $\psi > 1$. The compound process described above has a cumulative distribution function (see, for example, [20]) given by

$$G(x) = \sum_{n=0}^{\infty} p_n(t) F^{n*}(x), \tag{2.1}$$

where $F^{n*}(\cdot)$ represents the nth-fold convolution of $F(\cdot|c,\psi)$. In practice, it is difficult to derive a closed-form expression for the distribution function given in (2.1). Therefore, numerical approximations have been proposed based on the normal and gamma distribution and fast Fourier transform. Other approximations that have been considered are based on Edgeworth expansions and the Gram-Charlier series. Furthermore, an exact recursive expression for (2.1) can be found by using Panjer's recursion formula (see [21] and [22, Chapter 4], among others). A closed-form expression for the Pareto type III (Lomax) distribution has been derived for the convolution above in [23], and the asymptotic distribution of an infinite sum of i.i.d. classical Pareto type I random variables is provided in [24].

Nevertheless, in the expression (2.1), it is not necessary to calculate premiums based on the moments of the distribution. We will consider that the parameter c is known, that the risk profiles λ and ψ are unknown, and that they take values from the random variables Λ and Ψ, respectively. As it is well-known, a premium calculation principle ([25, 26]; among others) assigns to each vector of risk profiles $\Theta = (\lambda,\psi)$ a premium within the set $P \in \mathbb{R}^+$, the action space. Using the net premium principle, it is easy to see that, in this case, it gives the unknown risk premium

$$P(\Theta) = \mathbb{E}[N(t)]\mathbb{E}[Y] = \frac{c\lambda t\psi}{\psi-1}, \quad \psi > 1.$$

We have followed the idea of [10] in order to define the net premium. That is, the net premium is defined as the sum of losses instead of the sum of discounted losses, which leads to the exclusion of the time value of the monetary losses.

In the absence of experience, the actuary computes the collective (prior) premium, $\mathbb{E}_\pi[P(\Theta)]$, where π is the prior distribution assigned to the risk profile Θ. On the other hand, if experience is available, the actuary takes a sample

$$\aleph = \{N_c(T) = n, \quad Y_{c,t} = y_{ti}, \quad i = 1, 2, \ldots, n\}, \tag{2.2}$$

and uses this information to estimate the unknown risk premium $P(\Theta)$ through the Bayes premium P^*, obtained by computing the posterior expectation $\mathbb{E}_{\pi^*}[P(\Theta)]$. Here, π^* is the posterior distribution of the risk parameter, Θ, given the sample information provided in (2.2). For that purpose, we need also the likelihood function, which is given by

$$\begin{aligned} \ell(\lambda, \psi | \aleph = n, y_{t1}, \ldots, y_{tn}) &= \frac{(T\,\lambda)^n}{n!} \exp(-T\,\lambda) \prod_{i=1}^{n} \frac{\psi}{c} \left(\frac{y_{ti}}{c}\right)^{-(\psi+1)}, \\ &\propto \lambda^n \exp(-T\,\lambda)\, \psi^n \, \exp(-z\,\psi), \end{aligned} \tag{2.3}$$

where $z = \sum_{i=1}^{n} \log(y_t/c)$.

We will assume that the distribution of Λ is gamma with shape parameter $\nu > 0$ and rate parameter $\tau > 0$ and that the distribution of Ψ is Erlang with shape parameter $\gamma \in \mathbb{N}_+ = \{1, 2, \ldots\}$ and rate parameter $\xi > 0$. That is, it is assumed that $\Lambda \sim \mathcal{G}(\nu, \tau)$ and $\Psi \sim \mathcal{E}(\gamma, \xi, k)$, a $k > 0$ translated Erlang variate, with probability density functions (pdfs) given by,*

$$\pi_1(\lambda|\nu, \tau) = \frac{\tau^\nu}{\Gamma(\nu)} \lambda^{\nu-1} \exp(-\tau\lambda), \quad \lambda > 0, \tag{2.4}$$

$$\pi_2(\psi|\gamma, \xi, k) = \frac{\xi^\gamma}{\Gamma(\gamma)} (\psi - k)^{\gamma-1} \exp\left[-\xi(\psi - k)\right], \quad \psi > k, \tag{2.5}$$

where $\Gamma(\cdot)$ is the Euler gamma function and $\Gamma(\gamma) = (\gamma - 1)!$. The choice of the prior distributions, in addition to mathematical convenience (calculations are provided), is done in this way following Hesselager's work [10]. The fundamental objective of this work is to study the variation of the premiums assuming a certain degree of dependence between the risk profiles. Mean and variance of these two distributions are ν/τ, ν/τ^2, $k + \gamma/\xi$, and γ/ξ^2, respectively.

Finally, the joint pdf of $\Theta = (\Lambda, \Psi)$ is assumed to be the product of these densities. Thus,

$$\pi(\lambda, \psi|\nu, \tau, \gamma, \xi, k) = \pi_1(\lambda|\nu, \tau)\pi_2(\psi|\gamma, \xi, k), \tag{2.6}$$

i.e., both random variables are assumed to be independent. The next result provides the posterior distribution of (2.6) given the sample information provided in (2.3). As we will see, the posterior distribution is conjugate for λ and almost conjugate (quasi-conjugate) for the ψ risk profile.

Proposition 1. *Let* $\{X(t)\}_{t\geq 0}$ *be the compound Poisson process defined above. Suppose that* (Λ, Ψ) *follows the prior distribution with pdf given in* (2.6)*. Then, the posterior distribution of* (Λ, Ψ) *given*

* [10] assumes a gamma prior for the parameter ψ that seems inappropriate since from Proposition 3, we have that $\psi > k$. A translated gamma or Erlang distribution should be more accurate in the case considered here.

the sample information $\aleph = \{N_c(T) = n, \;\; Y_{c,t} = y_{ti}, \;\; i = 1, 2, \ldots, n\}$, *assumed to be independent and identically distributed, is given by,*

$$\pi^*(\lambda, \psi|\nu^*, \tau^*, \gamma^*, \xi^*, k) = \pi_1(\lambda|\nu^*, \tau^*)\pi_2^*(\psi|\gamma^*, \xi^*, k),$$

where

$$\pi_2^*(\psi|\gamma^*, \xi^*, k) = \kappa^{-1}\psi^n(\psi - k)^{\gamma-1}\exp(-\xi^*\psi), \quad \psi > k,$$

and the updated parameters are given by

$$\nu^* = \nu + n, \tag{2.7}$$
$$\tau^* = \tau + T, \tag{2.8}$$
$$\gamma^* = \gamma + n, \tag{2.9}$$
$$\xi^* = \xi + z, \tag{2.10}$$

being $\kappa \equiv \kappa(\gamma, \xi^*, k)$ *the normalization constant given by*

$$\kappa \equiv \kappa(\gamma, \xi^*, k) = \exp(-\xi^* k)\sum_{j=0}^{n}\binom{n}{j}\frac{k^{n-j}\Gamma(\gamma + j)}{(\xi^*)^{\gamma+j}}. \tag{2.11}$$

Proof. The posterior pdf $\pi_1^*(\lambda|\nu^*, \tau^*)$ is straightforwardly obtained from (2.3), (2.4), and Bayes' theorem. Now, $\pi_2^*(\psi|\gamma^*, \xi^*, k)$ is proportional to

$$\pi_2^*(\psi|\gamma^*, \xi^*, k) \propto \psi^n \exp(-z\psi)\pi_2(\psi|\gamma, \xi, k),$$

where the constant of proportionality is given by

$$\begin{aligned}
\kappa(\gamma, \xi^*, k) &= \int_k^\infty \psi^n(\psi - k)^{\gamma-1}\exp(-\xi^*\psi)\, d\psi \\
&= \exp(-\xi^* k)\sum_{j=0}^{n}\binom{n}{j}k^{n-j}\int_0^\infty \psi^{j+\gamma-1}\exp(-\xi^*\psi)\, d\psi \\
&= \exp(-\xi^* k)\sum_{j=0}^{n}\binom{n}{j}\frac{k^{n-j}\Gamma(\gamma + j)}{(\xi^*)^{\gamma+j}}.
\end{aligned}$$

Hence the result. □

As we can see in the next result, bivariate, prior, and posterior distributions can now be used to get the collective (prior) and posterior (Bayes) premiums of the aggregated model described above based on the first moment (the net premium).

Proposition 2. *The collective (prior) and Bayes (posterior) net premiums, i.e.,* $k = 1$, *for the compound aggregate model described above are given by,*

$$P(\nu, \tau, \gamma, \xi) = \frac{c\nu}{\tau}\frac{\xi + \gamma - 1}{\gamma - 1}, \quad \gamma > 1, \tag{2.12}$$
$$P^*(\nu^*, \tau^*, \gamma^*, \xi^*) = \frac{c\nu^*}{\tau^*}\frac{\kappa^*(\gamma, \xi^*, 1)}{\kappa(\gamma, \xi^*, 1)}, \quad \gamma \geq 2, \tag{2.13}$$

respectively.

Proof. Expression (2.12) is obtained as follows,

$$\begin{aligned} P(\nu,\tau,\gamma,\xi) &= \int_1^\infty \int_0^\infty P(\lambda,\psi)\pi_1(\lambda|\nu,\tau)\pi_2(\psi|\gamma,\xi,1)\,d\lambda\,d\psi \\ &= \frac{c\nu}{\tau}\frac{\xi^\gamma}{\Gamma(\gamma)}\int_1^\infty \psi(\psi-1)^{\gamma-2}\exp[-\xi(\psi-1)]\,d\psi \\ &= \frac{c\nu}{\tau}\frac{\xi^\gamma}{\Gamma(\gamma)}\frac{\Gamma(\gamma-1)}{\xi^{\gamma-1}}\left(1+\frac{\gamma-1}{\xi}\right), \end{aligned}$$

from which we get the result. Observe that we have used the fact that the mean of the random variable Ψ following the pdf $\pi_2(\psi|\gamma,\xi)$ is $1+(\gamma-1)/\xi$.

To get (2.13), we need to compute

$$\begin{aligned} \kappa^* &= \int_1^\infty \frac{\psi}{\psi-1}\pi_2^*(\psi|\gamma^*,\xi^*,1)\,d\psi = \kappa^{-1}\int_1^\infty \psi^{n+1}(\psi-1)^{\gamma-2}\exp(-\xi^*\psi)\,d\psi \\ &= \kappa^{-1}\sum_{j=0}^{\gamma-2}\binom{\gamma-1}{j}(-1)^j\int_1^\infty \psi^{\gamma^*-j-1}\exp(-\xi^*\psi)\,d\psi. \end{aligned}$$

Now, taking into account (2.7) and (2.8), the result is obtained directly. □

3. The reinsurance model under independence

[10] proposed the aggregate model described in the previous section to calculate reinsurance premiums in the collective risk model. For that, let $N_u(t) = \sum_{i=1}^{N(t)} I(Y_i > u)$ be the number of claims exceeding the level u during $[0,t]$. The latter expression is again a Poisson process with intensity $\lambda_u = \lambda\Pr(Y_i > u) = \lambda(\sigma/u)^\psi$, for $u \geq \sigma$.

Consider an excess-of-loss cover for the layer $b-a$ in excess of a. For a claim Y_a, exceeding the priority or deductible a, the reinsurer will cover the amount $Z_i = \min\{Y,b\} - a$. Our interest now lies on the random variable, the reinsurer's total claim amount during period $[0,t]$, $X(t) = \sum_{i=1}^{N_a(t)} Z_i$. It is clear that $\{X(t)\}_{t\geq 0}$ is a compound Poisson process with intensity λ_a and distribution function given by

$$H(z|\psi) = \begin{cases} F(a+z|a,\psi), & 0 \leq z < b-a, \\ 1, & z \geq b-a. \end{cases}$$

Now, at time T, the practitioner wishes to predict the reinsurer's total claims cost for the next year, i.e. $X = X(T+1) - X(T)$. Therefore, we must first compute the so-called risk premium of order k, $k = 1,2,\ldots$, i.e., premiums based on non-central moments. Observe that the first moment is the so-called net risk premium (see, for instance, [25]). Then, we need

$$\mathbb{E}\left[\Lambda\left(\frac{\sigma}{a}\right)^\Psi \mu_k(\psi)\right], \tag{3.1}$$

where $\Theta = (\Lambda,\Psi)$ is the vector of risk parameters and $\mu_k(\psi) = \mathbb{E}(Z_i^k|\psi)$.

Then, it is necessary to compute $\mu_k(\psi) = \mathbb{E}(Z_i^k|\psi)$. A closed-form expression for this kth moment is provided in [10] giving

$$\mu_k(\psi) = ka^k \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j \frac{1-(a/b)^{\psi-k+j}}{\psi-k+j}. \tag{3.2}$$

The latter result can be written as a closed-form expression regarding the incomplete Beta function, as shown in the following proposition.

Proposition 3. *It is verified that the non-central moments* $\mu_k(\psi) = \mathbb{E}(Z_i^k|\psi)$ *can be written as*

$$\mu_k(\psi) = \left(\frac{a}{b}\right)^{\psi} (b-a)^k + \psi\, a^k\, I_{1-a/b}(k+1, \psi - k), \tag{3.3}$$

with $\psi > k$*, where* $I_z(r,s)$ *is the incomplete Beta function, given by*

$$I_z(r,s) = \int_0^z t^{r-1}(1-t)^{s-1}\, dt, \quad 0 \le z \le 1.$$

Proof. We have that

$$\begin{aligned}\mu_k(\psi) &= \mathbb{E}(Z^k|\psi) = \left(\frac{a}{b}\right)^{\psi} (b-a)^k + \int_a^b (y-a)^k \psi a^{\psi} y^{-(1+\psi)}\, dy \\ &= \left(\frac{a}{b}\right)^{\psi} (b-a)^k + \psi a^{\psi} \int_a^b \left(1 - \frac{a}{y}\right)^k y^{k-(1+\psi)}\, dy.\end{aligned}$$

Let now the change of variable be $z = 1 - a/y$, and we get that the last integral can be rewritten as

$$\psi a^k \int_0^{1-a/b} z^{(k+1)-1}(1-z)^{\psi-k-1}\, dz = \psi a^k I_{1-a/b}(k+1, \psi-k),$$

from which we have the result. □

The main idea of Hesselager's model (see [10]) is to consider that the parameters λ and ψ (often termed as risk profiles) are unknown and will be regarded as outcomes of random parameters Λ and Ψ, respectively, and to treat the problem in a Bayesian framework. For that reason, the prior distributions of Λ and Ψ are required, and the premium provided in (3.4) is called risk premium, net risk premium when $k = 1$.

Proposition 4. *The risk, collective (prior), and Bayes (posterior) net premiums, i.e.,* $k = 1$*, are given by,*

$$\begin{aligned} P(\lambda, \psi) &= \frac{\lambda}{\psi - 1}\left[a - b\left(\frac{a}{b}\right)^{\psi}\right]\left(\frac{c}{a}\right)^{\psi}, \quad \psi > 1, && (3.4)\\ P(\nu, \tau, \gamma, \xi) &= \frac{\nu}{\tau}\frac{c\xi^{\gamma}}{\gamma - 1}\left\{\frac{1}{[\xi + \log(a/c)]^{\gamma-1}} - \frac{1}{[\xi + \log(b/c)]^{\gamma-1}}\right\}, \quad \gamma > 1, && (3.5)\\ P^*(\nu^*, \tau^*, \gamma^*, \xi^*) &= \frac{\nu^*}{\tau^*}\frac{a\,\kappa(\gamma^*, \xi^* + \log(a/c), 1) - b\,\kappa(\gamma^*, \xi^* + \log(b/c), 1)}{\kappa(\gamma^*, \xi^*, 1)}, && (3.6)\end{aligned}$$

respectively and where $a \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} c.$

Proof. By taking $t = 1$, we have that $P(\lambda, \psi) = \mathbb{E}(N(t))\mathbb{E}(Y) = \lambda\mu_1(\psi)$. Now, $\mu_1(\psi)$ is obtained directly from (3.2) or (3.3) by taking $k = 1$, from which we easily get (3.4). To get (3.5), we have to compute $P(\nu, \tau, \gamma, \xi) = \mathbb{E}_p[P(\Lambda, \Psi)]$, where the expectation is taken with respect to $p(\lambda, \psi|\nu, \tau, \gamma, \xi, 1)$. Now, due to the hypothesis of independence between the risk profiles, we have that

$$P(\nu, \tau, \gamma, \xi) = \mathbb{E}_{\pi_1}(\Lambda)\mathbb{E}_{\pi_2}\left\{\frac{1}{\Psi - 1}\left[a - b\left(\frac{a}{b}\right)^{\Psi}\right]\right\},$$

where $\mathbb{E}_{\pi_1}(\Lambda) = \nu/\tau$. On the other hand, we have that

$$\begin{aligned}
\mathbb{E}_{\pi_2}\left\{\frac{1}{\Psi - 1}\left[a - b\left(\frac{a}{b}\right)^{\Psi}\right]\right\} &= \frac{c\xi^{\gamma}}{\Gamma(\gamma)}\int_1^{\infty}(\psi - 1)^{\gamma-1-1}\exp\left[-\left(\xi + \log\left(\frac{a}{c}\right)\right)(\psi - 1)\right] d\psi \\
&\quad -\frac{c\xi^{\gamma}}{\Gamma(\gamma)}\int_1^{\infty}(\psi - 1)^{\gamma-1-1}\exp\left[-\left(\xi + \log\left(\frac{c}{b}\right)\right)(\psi - 1)\right] d\psi \\
&= \frac{c\xi^{\gamma}}{\Gamma(\gamma)}\left[\frac{\Gamma(\gamma - 1)}{\left[\xi + \log\left(\frac{a}{c}\right)\right]^{\gamma-1}} - \frac{\Gamma(\gamma - 1)}{\left[\xi + \log\left(\frac{b}{c}\right)\right]^{\gamma-1}}\right],
\end{aligned}$$

from which we get (3.5) after some simple algebra. Expression (3.6), which is the posterior expectation $P^*(\nu^*, \tau^*, \gamma^*, \xi^*, k) = \mathbb{E}_{\pi^*}[P(\Lambda, \Psi)]$, is obtained from (3.4) using the updated parameters provided in (2.7)–(2.10) and arranging the constant of normalization. □

As a special case of the model studied before, we now consider that $b \to \infty$. This is the basic excess-of-loss reinsurance model considered in [27, p. 11] and [1, p. 49]. In this case, an unlimited excess-of-loss treaty exists with retention $a > 0$, and the mean excess amount gives the expected amount to be paid by the reinsurer. Thus, $z > 0$ and the risk premium can be obtained from (3.4) as

$$\lim_{b\to\infty} P(\lambda, \psi) = \frac{\lambda a}{\psi - 1}\left(\frac{c}{a}\right)^{\psi}.$$

Furthermore, from (3.5) and (3.6) and assuming that $b \to \infty$, the collective and Bayes premiums are given by,

$$P(\nu, \tau, \gamma, \xi) = \frac{\nu}{\tau}\frac{c\xi^{\gamma}}{(\gamma - 1)\left[\xi + \log(a/c)\right]^{\gamma-1}}, \tag{3.7}$$

$$P^*(\nu^*, \tau^*, \gamma^*, \xi^*) = \frac{\nu^*}{\tau^*}\frac{a\,\kappa(\gamma^*, \xi^* + \log(a/c), 1)}{\kappa(\gamma^*, \xi^*, 1)}. \tag{3.8}$$

An anonymous reviewer has pointed out this important detail: suppose the deductible a is sufficiently large. The excess loss should be roughly distributed as the generalized Pareto distribution (Pickands-Balkema-de Haan theorem), which might be of more interest because the underlying distribution of individual severity can be assumed arbitrarily. See, for instance [28].

4. Breaking the independence hypotheses

We focus here on the collective and Bayes net premiums for the aggregate amount of claims under a compound model, assuming some degree of dependence between the random variables Λ and Ψ. In our case, it is challenging to implement this assumption in the first stage of the model, involving Poisson and Pareto random variables. As pointed out by [29], an attractive way to model the dependence between risks is via the dependence between the risk profiles. Thus, we translate the non-dependence to the second stage (the prior distributions of the risk profiles).

In the statistical literature, there exists a lot of research papers dealing with the construction of bivariate distributions (see, for example, [30–32]). Most of them are based on the idea of building a bivariate distribution with given marginal distributions, and the Sarmanov-Lee family of distributions [14, 15] follows this idea. Specifically, assume $f_1(x_1)$ and $f_2(x_2)$ are univariate probability density functions for the continuous case or probability mass functions for the discrete one, with supports defined on subsets of $\mathbb{R}$ (which can be the total real line). Let $\phi_i(t)$, $i = 1, 2$, be bounded nonconstant functions such that $\int_{-\infty}^{\infty} \phi_i(t) f_i(t)\, dt = 0$ (they are usually named mixing functions). Then, provided that ω is a real number that satisfies the condition $1 + \omega\, \phi_1(x_1)\phi_2(x_2) \geq 0$ for all x_1 and x_2, the function given by

$$h(x_1, x_2) = f_1(x_1) f_2(x_2) \left[1 + \omega\, \phi_1(x_1)\phi_2(x_2)\right] \tag{4.1}$$

which is a genuine bivariate joint density (or a probability mass function in the discrete case) with specified marginals $f_1(x_1)$ and $f_2(x_2)$. It is well-known that the family given in (4.1) is a special case of the construction provided in [33] and includes some of the well-known Farlie-Gumbel-Morgenstern distributions as special cases. For details, see [34–36]. Observe that the dependence is captured through the mixing functions and the ω parameter.

Let us assume that a certain degree of dependence between the random variables Λ and Ψ exist, and specify a marginal prior gamma distribution for the first random variable and a translated Erlang distribution for the latter one, which, in both cases, are the conjugate a priori distribution of the Poisson and classical Pareto distributions. See the work of [37], where conjugate analysis for the Pareto distribution is studied.

We first need the following Lemma to obtain the prior distribution after assuming some degree of dependence.

Lemma 1. *Let us consider the following functions*

$$\begin{aligned}
\varphi_1(\lambda|\nu, \tau) &= \exp(-\lambda) - \delta_1(\nu, \tau), \quad \lambda > 0,\ \nu > 0,\ \tau > 0,\\
\varphi_2(\psi|\gamma, \xi, k) &= \exp(-\psi) - \delta_2(\gamma, \xi, k), \quad \psi > k,\ \gamma > 0,\ \xi > 0,\\
\delta_1(\nu, \tau) &= (1 + \tau^{-1})^{-\nu},\\
\delta_2(\gamma, \xi, k) &= \left(1 + \xi^{-1}\right)^{-\gamma} \exp(-k).
\end{aligned}$$

Then, it is verified that

$$H(\nu, \tau, \gamma, \xi, k, \omega) = 1 + \omega\, \varphi_1(\lambda|\nu, \tau)\, \varphi_2(\psi|\gamma, \xi, k) \geq 0, \tag{4.2}$$

if $\omega_1 \leq \omega \leq \omega_2$, where

$$\omega_1 = (-\max\{r_1, r_2\})^{-1} < 0, \tag{4.3}$$

$$\omega_2 = (\max\{r_3, r_4\})^{-1} > 0, \tag{4.4}$$

being

$$\begin{aligned} r_1 &= \delta_1(\nu,\tau)\,\delta_2(\gamma,\xi,k),\\ r_2 &= (1-\delta_1(\nu,\tau))\,(\exp(-k)-\delta_2(\gamma,\xi,k)),\\ r_3 &= (1-\delta_1(\nu,\tau))\,\delta_2(\gamma,\xi,k),\\ r_4 &= \delta_1(\nu,\tau)\,(\exp(-k)-\delta_2(\gamma,\xi,k)). \end{aligned}$$

Proof. To see this, observe that because $d(\varphi_1(\lambda|\nu,\tau))/d\lambda < 0$, we have that $\varphi_1(\lambda|\nu,\tau)$ is a decreasing function on λ; also, since the support of λ is in the interval $(0,\infty)$, we have that the range of variation of $\varphi_1(\lambda|\nu,\tau)$ is $(-\delta_1(\nu,\tau), 1-\delta_1(\nu,\tau))$. Using the same argument, we get that the range of variation of $\varphi_2(\psi|\gamma,\xi,k)$ is given by $(-\delta_2(\gamma,\xi,k), \exp(-k)-\delta_2(\gamma,\xi,k))$. Now we have that the expression $H(\nu,\tau,\gamma,\xi,\omega)\geq 0$ if

$$\begin{aligned} \omega &\geq \frac{-1}{\varphi_1(\lambda|\nu,\tau)\,\varphi_2(\psi|\gamma,\xi,k)}, \quad \text{for } \varphi_1(\lambda|\nu,\tau)\,\varphi_2(\psi|\gamma,\xi,k) > 0,\\ \omega &\leq \frac{-1}{\varphi_1(\lambda|\nu,\tau)\,\varphi_2(\psi|\gamma,\xi,k)}, \quad \text{for } \varphi_1(\lambda|\nu,\tau)\,\varphi_2(\psi|\gamma,\xi,k) < 0, \end{aligned}$$

from which, taking into account that $0<\delta_1(\nu,\tau)<1$ and $0<\delta_2(\gamma,\xi,k)<1$, it is a simple exercise to see that the range of ω is given by (ω_1,ω_2) provided above. □

The following result provides the prior distribution under the Sarmanov-Lee family of distributions.

Proposition 5. *The expression given by*

$$\pi(\lambda,\psi|\nu,\tau,\gamma,\xi,k,\omega) = \pi_1(\lambda|\nu,\tau)\,\pi_2(\psi|\gamma,\xi,k)H(\nu,\tau,\gamma,\xi,k,\omega),\ \lambda>0,\ \psi>k, \tag{4.5}$$

where $H(\nu,\tau,\gamma,\xi,k,\omega)$ is given in (4.2)*, defines a genuine prior bivariate distribution of* (Λ,Ψ) *with marginal distributions given by $\pi_1(\lambda|\nu,\tau)$ and $\pi_2(\psi|\gamma,\xi,k)$, which always satisfies that $\omega_1\leq\omega\leq\omega_2$ and ω_1 and ω_2 are given by* (4.3) *and* (4.4)*, respectively.*

Proof. When $\omega_1<\omega<\omega_2$, we have that $\pi(\lambda,\psi|\nu,\tau,\gamma,\xi,\omega)>0$. Now, taking into account that

$$\begin{aligned} \int_0^\infty \exp(-\lambda)\pi_1(\lambda|\nu,\tau)\,d\lambda &= \delta_1(\nu,\tau),\\ \int_k^\infty \exp(-\psi)\pi_2(\psi|\gamma,\xi,k)\,d\psi &= \delta_2(\gamma,\xi,k), \end{aligned}$$

we get in a straightforward way that

$$\begin{aligned} \int_0^\infty\int_k^\infty \pi(\lambda,\psi|\nu,\tau,\gamma,\xi,k,\omega)\,d\lambda\,d\psi &= \int_0^\infty\int_k^\infty \pi_1(\lambda|\nu,\tau)\,\pi_2(\psi|\gamma,\xi,k)\,d\lambda\,d\psi\\ &= 1. \end{aligned}$$

Hence the result. □

As can be seen, the prior bivariate distribution (4.5) can be written as a linear combination of the product of gamma and Erlang distributions. Additionally, the special case $\omega = 0$ provides the prior bivariate distribution given by $\pi(\lambda, \psi|\nu, \tau, \gamma, \xi, k) = \pi_1(\lambda|\nu, \tau)\,\pi_2(\psi|\gamma, \xi, k)$, i.e. it represents the case in which we assume independence between Λ_c and Ψ. Figure 1 shows the pdf given in (4.5) for parameters values given by $(\tau, \nu, \gamma, \xi) = (5, 10, 10, 4)$ and different values of ω moving in the range $[-1.24, 30.19]$ obtained from (4.3)-(4.4). The discrepancies between the different graphs are best appreciated in the contour plots given in Figure 1 below.

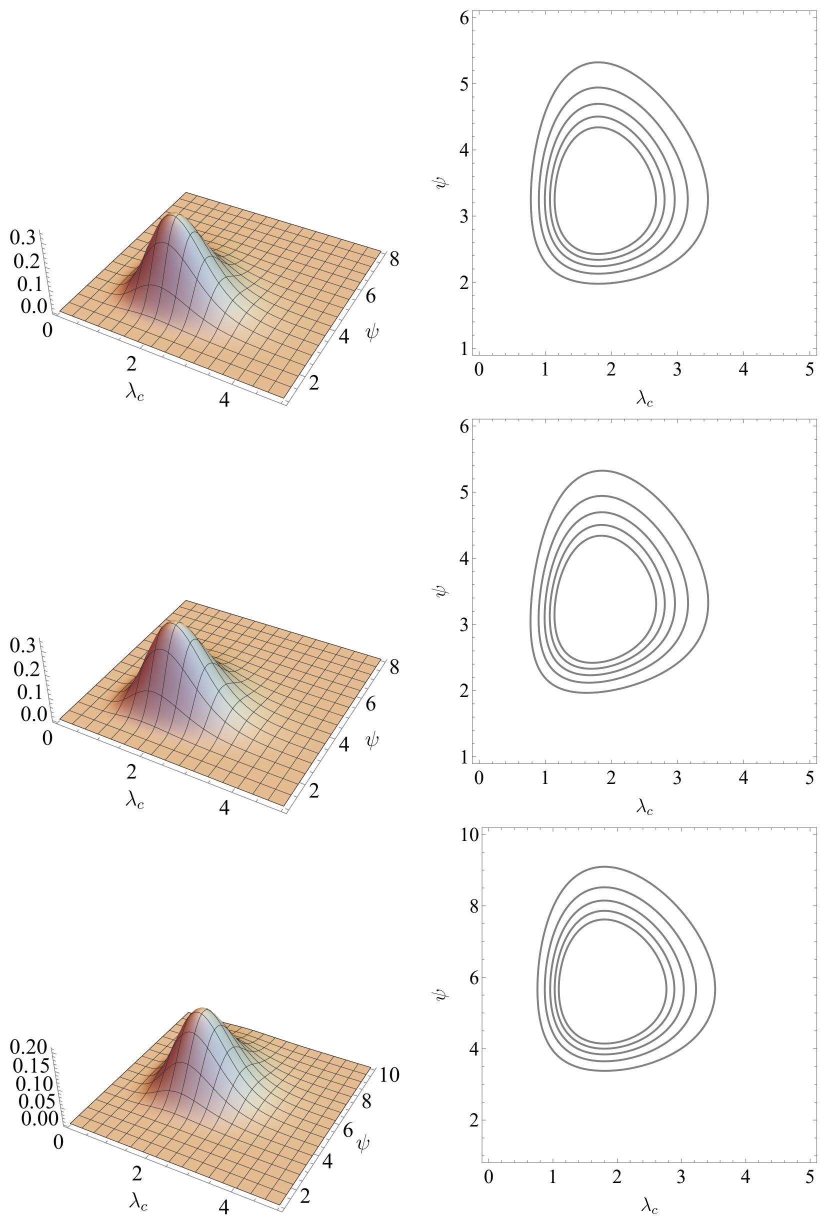

Figure 1. Bivariate densities (4.5) and corresponding contour plots for parameter values $(\tau, \nu, \gamma, \xi) = (5, 10, 10, 4)$ and ω given by 0 (top), 30 (middle), and -1.20 (bottom).

4.1. Properties of the bivariate distribution

Some properties of the bivariate distribution proposed in this section are now studied.

Lemma 2. *The following relations are verified,*

$$\mathbb{E}[\Lambda \exp(-\Lambda)] = \frac{\nu}{\tau}\delta_1(\nu+1,\tau), \tag{4.6}$$

$$\mathbb{E}[\Psi \exp(-\Psi)] = \left(k + \frac{\gamma}{\xi+1}\right)\delta_2(\xi,\gamma)\exp(-k). \tag{4.7}$$

Proof. See the Appendix. □

The next result shows the cross moment and correlation coefficient between the random variables Λ_c and Ψ when the prior distribution (4.5) is used.

Proposition 6. *For the prior bivariate distribution* (4.5)*, the cross moment and correlation coefficient between* Λ_c *and* Ψ *are given by*

$$\mathbb{E}(\Lambda\,\Psi) = \frac{\nu}{\tau\xi(1+\xi)}\left[(1+\xi)(\gamma+k\xi) + \frac{\omega\gamma}{\tau}\delta_1(\nu+1,\tau)\delta_2(\gamma,\xi,k)\right],$$

$$\rho(\Lambda,\Psi) = \frac{\omega\sqrt{\nu\gamma}\delta_1(\nu+1,\tau)\delta_2(\gamma,\xi,k)}{\tau(1+\xi)}. \tag{4.8}$$

Proof. By using Theorems 1 and 2 in [15], we have that

$$\mathbb{E}(\Lambda\,\Psi) = \mu_1\mu_2 + \omega\eta_1(\nu,\tau)\eta_2(\gamma,\xi,k),$$

$$\rho = \omega\eta_1(\nu,\tau)\eta_2(\gamma,\xi,k)/(\sigma_1\sigma_2),$$

where

$$\eta_1(\nu,\tau) = \int_0^\infty \lambda\varphi_1(\lambda|\nu,\tau)\pi_1(\lambda|\nu,\tau)\,d\lambda,$$

$$\eta_2(\gamma,\xi,k) = \int_k^\infty \psi\varphi_2(\psi|\gamma,\xi,k)\pi_2(\psi|\gamma,\xi,k)\,d\psi,$$

μ_1, μ_2, σ_1 and σ_2, the means and standard deviations of the gamma pdf (2.4) and Erlang pdf (2.5), respectively. These are given by $\mu_1 = \nu/\tau$, $\sigma_1 = \sqrt{\nu}/\tau$, $\mu_2 = k + \gamma/\xi$, and $\sigma_2 = \gamma/\xi^2$.

Now, by using the expressions (4.6) and (4.7) provided in Lemma 2 and some straightforward algebra, we get the result. □

Observe that the correlation coefficient in (4.8) is directly proportional to ω. The proportionality factor, given by $\sqrt{\nu\gamma}\delta_1(\nu+1,\tau)\delta_2(\gamma,\xi,k)/(\tau(1+\xi))$, is always positive, and the sign of the correlations will depend on the sign of ω, which allows values in the range provided by (4.3)-(4.4). A study of the linear correlation coefficient as a function of the values of the hyperparameters is carried out in the next proposition.

Proposition 7. *Assume that $\omega > 0$. Then, the following is verified:*

(1) $\rho(\Lambda, \Psi)$ *is an increasing (decreasing) function on* ν *if* $\tau > (<) \left[\exp(1/(2\nu) - 1\right]^{-1}$, *for all* $\zeta > 0$ *and* $\gamma \in \mathbb{N}$.
(2) $\rho(\Lambda, \Psi)$ *is an increasing (decreasing) function on* τ *if* $\nu > (<)\tau$, *for all* $\zeta > 0$ *and* $\gamma \in \mathbb{N}$.
(3) $\rho(\Lambda, \Psi)$ *is an increasing (decreasing) function on* γ *if* $\zeta > (<) \left[\exp(1/(2\gamma) - 1\right]^{-1}$, *for all* $\tau > 0$ *and* $\nu > 0$.
(4) $\rho(\Lambda, \Psi)$ *is an increasing (decreasing) function on* ζ *if* $\gamma > (<)\zeta$, *for all* $\tau > 0$ *and* $\nu > 0$.

Proof. See the Appendix. □

Figure 2 plots the correlation given as a function of ω. The correlation coefficients seem limited to values between -0.15 and 0.15, coherent with the correlation in most empirical cases found in actuarial statistics.

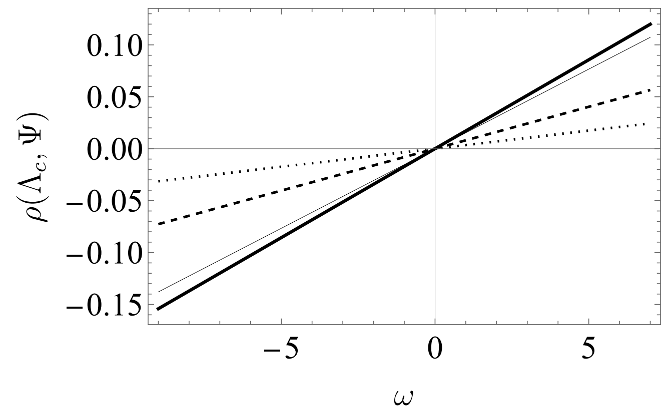

Figure 2. Correlation as a function of ω. The thick line corresponds to $(\nu, \tau, \gamma, \zeta) = (2, 1, 2, 2)$; the thin line corresponds to $(\nu, \tau, \gamma, \zeta) = (2, 2, 2, 3)$; the dashed line corresponds to $(\nu, \tau, \gamma, \zeta) = (2, 2, 2, 0.5)$; and the dotted line corresponds to $(\nu, \tau, \gamma, \zeta) = (12, 5, 10, 25)$.

The posterior distribution can be computed easily, and the next result shows that this posterior distribution can be written as a linear combination of the posterior distribution studied in Section 2.

Proposition 8. *The posterior distribution of* (Λ_c, Ψ) *given the sample information* $\aleph$ *is a linear combination of products of posterior univariate gamma and Erlang densities, given by*

$$\begin{aligned} \pi^*(\lambda, \psi | \aleph) &= \kappa_p \left\{ \left[1 + \omega \delta_1(\nu, \tau) \delta_2(\gamma, \xi, k)\right] (\tau^*)^{-n-\nu} \kappa \pi_1^*(\lambda | \nu^*, \tau^*) \pi_2^*(\psi | \gamma^*, \xi^*, k) \right. \\ &\quad + \omega (\tau^* + 1)^{-n-\nu} \kappa^* \pi_1^*(\lambda | \nu^*, \tau^* + 1) \pi_2^*(\psi | \gamma^*, \xi^* + 1, k) \\ &\quad - \omega \delta_1(\nu, \tau) (\tau^*)^{-n-\nu} \kappa^* \pi_1^*(\lambda | \nu^*, \tau^*) \pi_2^*(\psi | \gamma^*, \xi^* + 1, k) \\ &\quad \left. - \omega \delta_2(\gamma, \xi, k) (\tau^* + 1)^{-n-\nu} \kappa \pi_1^*(\lambda | \nu^*, \tau^* + 1) \pi_2^*(\psi | \gamma^*, \xi^*, k) \right\}, \end{aligned} \tag{4.9}$$

where $\kappa^* \equiv \kappa(\gamma^*, \xi^* + 1, k)$ *is given in* (2.11) *and the normalization constant,* $\kappa_p \equiv \kappa_p(\gamma^*, \xi^*, k)$, *satisfies*

$$\begin{aligned} \kappa_p^{-1} &= 1 + \omega\, \delta_1(\nu, \tau) \delta_2(\gamma, \xi) \kappa (\tau^*)^{-\nu-n} + \omega \kappa^* (\tau^* + 1)^{-\nu-n} \\ &\quad - \omega \kappa^* \delta_1(\nu, \tau) (\tau^*)^{-\nu-n} - \omega \kappa (\tau^* + 1)^{-\nu-n} \delta_2(\gamma, \xi, k). \end{aligned}$$

Proof. The result follows after some intensive algebra by using (2.3), (4.5), Bayes' Theorem and arranging parameters. □

4.2. Premiums under the dependence model

Expressions given in (4.5) and (4.9) can be used now to get the collective and Bayes net premiums under the non-independent model. These are shown in the following result.

Proposition 9. *Under the bivariate model with dependence, the prior (collective) and Bayesian (posterior) premiums are given by,*

$$\begin{aligned}\mathbb{E}_{\pi}[P(\Lambda,\Psi)] &= [1+\omega\delta_1(\mu,\tau)\delta_2(\xi,\gamma)]\,P(\nu,\tau,\gamma,\xi)\\ &\quad +\omega\delta_1(\mu,\tau)\delta_2(\xi,\gamma)\,[P(\nu,\tau+1,\gamma,\xi+1)\\ &\quad -P(\nu,\tau,\gamma,\xi+1)-P(\nu,\tau+1,\gamma,\xi)], \qquad (4.10)\\ \mathbb{E}_{\pi^*}[P(\Lambda,\Psi)|\aleph] &= \kappa_p\,\{[1+\omega\delta_1(\nu,\tau)\delta_2(\gamma,\xi,k)]\,(\tau^*)^{-n-\nu}\kappa P^*(\nu^*,\tau^*,\gamma^*,\xi^*,1),k)\\ &\quad +\omega(\tau^*+1)^{-n-\nu}\kappa^* P^*(\nu^*,\tau^*+1,\gamma^*,\xi^*+1,1)\\ &\quad -\omega\delta_1(\nu,\tau)(\tau^*)^{-n-\nu}\kappa^* P^*(\nu^*,\tau^*,\gamma^*,\xi^*+1,1)\\ &\quad -\omega\delta_2(\gamma,\xi,k)(\tau^*+1)^{-n-\nu}\kappa P^*(\nu^*,\tau^*+1,\gamma^*,\xi^*,1)\}, \qquad (4.11)\end{aligned}$$

where $P(\nu,\tau,\gamma,\xi)$ is the collective premium under the different models studied previously, the basic specific model (expression given in (2.12)*), the reinsurance model with a layer (expression given in* (3.5)*), and the reinsurance model without a layer (expression given in* (3.7)*). Alternatively, $P^*(\nu^*,\tau^*,\gamma^*,\xi^*,1)$ is the net Bayes premium under the models studied, expressions* (2.13)*,* (3.6)*, and* (3.8)*.*

Proof. See the Appendix. □

In the following corollary, we give an alternative expression for (4.10).

Corollary 1. *The collective premium given in* (4.10) *can be rewritten as*

$$\mathbb{E}_{\pi}[P(\Lambda,\Psi)] = P(\nu,\tau,\gamma,\xi)+\frac{\omega\delta_1(\mu,\tau)\delta_2(\xi,\gamma)}{\tau+1}\,[P(\nu,\tau,\gamma,\xi)-P(\nu,\tau,\gamma,\xi+1)].$$

Proof. See the Appendix. □

We get the corresponding collective and Bayes premium under the independent model by taking $\omega = 0$ in (4.10) and (4.11).

Note that the Bayesian premium has been calculated sequentially rather than in a batch, which means that the count of the number of losses may not be a homogeneous Poisson over time.

5. Numerical applications

In this section, two numerical examples will be considered. We first used the same dataset for comparison in [10, 11]. The second dataset corresponds to catastrophe data regarding the total damage caused by a storm during 1956–1977, which is displayed in Table 1. The amounts and the year of occurrence are exhibited in this table. We have only considered losses below 6 billion dollars. A comprehensive version of this dataset is shown in [38, p. 128]. For this reason, in our analysis, we only consider 10 entries.

Table 1. Hurricane data in millions of US dollars ([38, p. 128]).

Number	Loss	Year
1	2,000	1977
2	1,380	1971
3	2,000	1971
4	2,000	1964
5	2,580	1968
6	4,730	1971
7	3,700	1956
8	4,250	1961
9	4,500	1966
10	5,000	1958

The elicited hyperparameter values of the prior distributions are shown in Table 2, together with the bounds of the ω parameter and other statistics needed to obtain the collective and Bayesian net premiums. The maximum likelihood estimation method requires knowledge of the distribution obtained from (2.1), which is not simple; hence, the hyperparameters were estimated by the method of moments, where for the parameters (ν, τ), the sample mean and index of dispersion (ratio of variance to the mean assumed to be approximately 1.25) were used to calculate the estimates. Furthermore, for the parameters (ξ, γ), we used the sample mean with the coefficient of variation and the approximation of the value of γ to the next integer. The sample sizes, the values of the layers, and the corresponding time periods for each data set are also given.

Table 2. Parameter estimates and some other statistics.

	Example 1	Example 2
ν	2.56	0.3636
τ	0.8	0.8
ξ	0.972	1.197
γ	2	$2.65 \approx 3$
z	6.48165	8.54057
c	1.5	1.25
ω_1	-4.10	-12.70
ω_2	12.79	4.35
layer	5	2
T	5	21
n	16	10

The collective and Bayesian net premiums computed using (4.10) and (4.11) are shown in Table 3 for the two examples considered and the basic collective model. Observe that the case $\omega = 0$ reduces to the model where independence between the risk profiles is assumed. The correlation coefficient obtained for each value of ω obtained via (4.8) is also shown in this table. It can be observed that incorporating dependence has a low impact on both premiums, as judged by the changes in ω when this dependence parameter increases or decreases (i.e., more separated from the hypotheses of independence).

Table 3. Collective and Bayes net premiums for the basic collective model.

Example 1				Example 2			
ω	Correlation	Collective	Bayes	ω	ρ	Collective	Bayes
-4	-0.028	9.5854	8.5706	-12	-0.140	0.9920	1.8074
-3	-0.021	9.5555	8.5631	-11	-0.128	0.9850	1.7991
-2	-0.014	9.5256	8.5555	-10	-0.117	0.9780	1.7906
-1	-0.007	9.4957	8.5480	-9	-0.105	0.9710	1.7819
0	0.000	9.4658	8.5404	-8	-0.093	0.9640	1.7729
1	0.007	9.4359	8.5329	-7	-0.081	0.9570	1.7637
2	0.014	9.4060	8.5253	-6	-0.070	0.9500	1.7542
3	0.021	9.3761	8.5177	-5	-0.058	0.9431	1.7445
4	0.028	9.3462	8.5102	-4	-0.046	0.9361	1.7345
5	0.035	9.3163	8.5026	-3	-0.035	0.9291	1.7242
6	0.042	9.2864	8.4950	-2	-0.023	0.9221	1.7136
7	0.050	9.2565	8.4874	-1	-0.011	0.9151	1.7027
8	0.057	9.2266	8.4798	0	0.000	0.9081	1.6914
9	0.064	9.1967	8.4722	1	0.011	0.9011	1.6799
10	0.071	9.1668	8.4647	2	0.023	0.8941	1.6679
11	0.078	9.1369	8.4571	3	0.035	0.8871	1.6556
12	0.085	9.1070	8.4495	4	0.046	0.8801	1.6428

The collective and Bayesian net premiums, including a layer of 1.5 million in excess of the priority a for Example 1 and a layer of 2 million in excess of the priority a for Example 2, are shown in Tables 4 and 5, respectively. Again, the case $\omega = 0$ reduces to the model where independence between the risk profiles is assumed, and the correlation coefficient is also included. It can be noticed that increasing the value of the deductible a decreases the value of both premiums. In cases where the deductible is 1.5 million, the Bayes premium is higher than the collective premiums in both examples.

In addition, the collective and Bayesian net premiums without including a layer in excess of the priority a for both examples are shown in Tables 6 and 7, respectively. Again, the case $\omega = 0$ reduces to the model where independence between the risk profiles is assumed, and the correlation coefficient is also shown in these tables. It can be noticed that increasing the value of the deductible a decreases the value of both premiums. In cases where the deductible is 1.5 million, the Bayes premium is higher than the collective premiums in both examples. In Example 1, it can be observed that increasing the value of the deductible a decreases the value of both premiums. However, this is not verified in the second example, where the Bayes premiums increase for the deductible value $a = 1.5$ million and decrease again for $a = 2.2$. Also, the collective premiums increase for the deductible of $a = 2.2$ million.

Table 4. Collective and Bayes net premiums for Example 1 with a layer of 1.5 millions in excess of the priority a million for different values of the parameter ω.

		$a = 0.8$		$a = 1.5$		$a = 2.2$	
ω	Correlation	Collective	Bayes	Collective	Bayes	Collective	Bayes
-4	-0.028	11.1805	8.0838	2.8365	2.9943	1.5891	1.6574
-3	-0.021	11.1990	8.0846	2.8288	2.9911	1.5823	1.6545
-2	-0.014	11.2175	8.0855	2.8211	2.9879	1.5755	1.6517
-1	-0.007	11.2361	8.0863	2.8135	2.9847	1.5687	1.6489
0	0.000	11.2546	8.0871	2.8058	2.9816	1.5619	1.6460
1	0.007	11.2731	8.0880	2.7981	2.9784	1.5551	1.6432
2	0.014	11.2917	8.0888	2.7905	2.9752	1.5483	1.6403
3	0.021	11.3102	8.0896	2.7828	2.9720	1.5415	1.6374
4	0.028	11.3287	8.0905	2.7751	2.9688	1.5347	1.6346
5	0.036	11.3472	8.0913	2.7675	2.9656	1.5279	1.6317
6	0.042	11.3658	8.0922	2.7598	2.9625	1.5211	1.6289
7	0.050	11.3843	8.0930	2.7521	2.9593	1.5143	1.6260
8	0.057	11.4028	8.0938	2.7445	2.9561	1.5075	1.6232
9	0.064	11.4214	8.0947	2.7368	2.9529	1.5007	1.6203
10	0.071	11.4399	8.0955	2.7291	2.9497	1.4939	1.6174
11	0.078	11.4584	8.0964	2.7214	2.9465	1.4871	1.6146
12	0.085	11.4769	8.0972	2.7138	2.9433	1.4803	1.6117

Table 5. Collective and Bayes net premiums for Example 2 with a layer of 2 millions in excess of the priority a million for different values of the parameter ω.

		$a = 0.8$		$a = 1.5$		$a = 2.2$	
ω	Correlation	Collective	Bayes	Collective	Bayes	Collective	Bayes
-12	-0.140	0.7151	0.7078	0.1829	0.3554	0.0914	0.2264
-11	-0.128	0.7174	0.7059	0.1808	0.3542	0.0899	0.2255
-10	-0.117	0.7197	0.7040	0.1787	0.3529	0.0883	0.2246
-9	-0.105	0.7220	0.7020	0.1767	0.3516	0.0868	0.2236
-8	-0.093	0.7243	0.7000	0.1746	0.3503	0.0852	0.222
-7	-0.081	0.7266	0.6979	0.1725	0.3489	0.0837	0.2217
-6	-0.070	0.7290	0.6957	0.1704	0.3475	0.0822	0.2207
-5	-0.058	0.7313	0.6935	0.1683	0.3461	0.0806	0.2196
-4	-0.046	0.7336	0.6913	0.1662	0.3446	0.0791	0.2185
-3	-0.035	0.7359	0.6889	0.1641	0.3431	0.0775	0.2174
-2	-0.023	0.7382	0.6865	0.1620	0.3415	0.0760	0.2163
-1	-0.011	0.7405	0.6840	0.1599	0.3398	0.0744	0.2151
0	0.000	0.7420	0.6815	0.1578	0.3382	0.0729	0.2139
1	0.011	0.7452	0.6789	0.1557	0.3364	0.0714	0.2126
2	0.023	0.7475	0.6762	0.1536	0.3347	0.0698	0.2113
3	0.035	0.7498	0.6734	0.1516	0.3328	0.0683	0.2100
4	0.046	0.7521	0.6705	0.1495	0.3309	0.0667	0.2086

Table 6. Collective and Bayes net premiums for Example 1 without a layer in excess of the priority a million for different values of the parameter ω.

		a = 0.8		a = 1.5		a = 2.2	
ω	Correlation	Collective	Bayes	Collective	Bayes	Collective	Bayes
-4	-0.028	13.2230	8.9452	4.7854	3.7701	3.4611	2.3652
-3	-0.021	13.2187	8.9416	4.7555	3.7627	3.4326	2.3584
-2	-0.014	13.2144	8.9379	4.7256	3.7553	3.4041	2.3515
-1	-0.007	13.2101	8.9343	4.6957	3.7478	3.3756	2.3446
0	0.000	13.2058	8.9306	4.6658	3.7404	3.3470	2.3377
1	0.007	13.2015	8.9270	4.6359	3.7330	3.3185	2.3308
2	0.014	13.1972	8.9233	4.6060	3.7256	3.2900	2.3239
3	0.021	13.1929	8.9197	4.5761	3.7181	3.2615	2.3170
4	0.028	13.1886	8.9160	4.5462	3.7107	3.2330	2.3101
5	0.036	13.1843	8.9123	4.5163	3.7032	3.2044	2.3032
6	0.042	13.1800	8.9087	4.4864	3.6958	3.1759	2.2963
7	0.050	13.1758	8.9050	4.4565	3.6883	3.1474	2.2894
8	0.057	13.1715	8.9013	4.4266	3.6809	3.1189	2.2825
9	0.064	13.1672	8.8977	4.3967	3.6734	3.0903	2.2756
10	0.071	13.1629	8.8940	4.3668	3.6660	3.0618	2.2687
11	0.078	13.1586	8.8903	4.3369	3.6585	3.0333	2.2617
12	0.085	13.1543	8.8866	4.3070	3.6510	3.0048	2.2548

Table 7. Collective and Bayes net premiums for Example 2 without a layer in excess of the priority a million for different values of the parameter ω.

		a = 0.8		a = 1.5		a = 2.2	
ω	Correlation	Collective	Bayes	Collective	Bayes	Collective	Bayes
-12	-0.140	0.8994	0.3378	0.2273	1.5146	1.0860	0.9017
-11	-0.128	0.8965	0.3310	0.2214	1.5072	1.0796	0.8959
-10	-0.117	0.8935	0.3242	0.2156	1.4996	1.0729	0.8898
-9	-0.105	0.8906	0.3174	0.2097	1.4918	1.0661	0.8836
-8	-0.093	0.8877	0.3106	0.2038	1.4838	1.0592	0.8773
-7	-0.081	0.8848	0.3038	0.1980	1.4755	1.0520	0.8707
-6	-0.070	0.8819	0.2970	0.1921	1.4671	1.0446	0.8640
-5	-0.058	0.8789	0.2902	0.1862	1.4584	1.0370	0.8571
-4	-0.046	0.8760	0.2834	0.1804	1.4494	1.0292	0.8500
-3	-0.035	0.8731	0.2766	0.1745	1.4402	1.0212	0.8427
-2	-0.023	0.8702	0.2698	0.1687	1.4307	1.0129	0.8352
-1	-0.011	0.8673	0.2630	0.1628	1.4210	1.0044	0.8275
0	0.000	0.8643	0.2562	0.1569	1.4109	0.9957	0.8195
1	0.011	0.8614	0.2494	0.1511	1.4006	0.9866	0.8113
2	0.023	0.8585	0.2426	0.1452	1.3899	0.9773	0.8028
3	0.035	0.8556	0.2358	0.1394	1.3789	0.9677	0.7940
4	0.046	0.8527	0.2290	0.1335	1.3675	0.9578	0.7850

Even though, for all the tables above, the predicted correlation coefficients seem to be minor, this is not a trivial issue since it could have a significant impact when changes of scales in the monetary units are considered; therefore, the analysis we have carried out in this work shows that although the correlation is usually low, the impact of assuming independence might have potential financial consequences. It is important to note that the dependence hypothesis is established between the risk profiles and not directly between the number of claims and the corresponding severity. However, given the mathematical difficulty that the latter scenario would entail, this can be replaced by transferring this hypothesis to the risk profiles, which retain characteristics of the former case. Even though when calculating bonus-malus premiums in the automobile insurance portfolio, independence between the number of claims and the amount of the claim is usually assumed, some recent works such as [39] have discussed the calculation of bonus-malus premiums for the compound loss by incorporating a certain degree of dependence between the frequency and severity. However, in practice, the correlation between both variables is usually low. For example, by considering the automobile insurance portfolio used in [40] and taken from [41] that corresponds to the years 2004-05 and contains data of 67,856 insureds, of whom 4,624 filed a claim, i.e., a positive claims amount, the correlation between the number of claims and the amount for the whole portfolio is 0.4818. Nevertheless, when it is restricted to the 4,624 policyholders that have declared at least one claim, the correlation drops to 0.0762, consistent with the values shown in Table 3.

6. Final comments and future works

We have presented an excess-of-loss reinsurance model derived from the compound Poisson distribution. The classical Pareto distribution was assumed for the severity of claims, and dependence between the number of claims and claims size was placed on the risk profiles. In this work, initially, we considered independence between the risk profiles whose parameters follow gamma and shifted Erlang distributions. Next, the hypothesis of independence was broken by assuming some degree of dependence between the random variables associated with these risk profiles via a prior bivariate distribution based on the Sarmanov-Lee probabilistic family. This model was used to study the effect of incorporating some degree of dependence of these variables on the collective and Bayesian net premiums. The results obtained lead us to conclude that even at moderate levels of correlation between the risk profiles, these aforementioned reinsurance premiums are more sensitive than those computed, assuming the independence of the risk profiles.

The methodology proposed in this paper is suitable for application in the collective model, in which the two relevant variables used in calculating premiums are used simultaneously: the number of claims and the amount associated with them. The difficulty of working simultaneously with these two variables is well-known, especially when obtaining the total claim amount distribution. Two scenarios that should be considered are, on the one hand, the study of the sensitivity of the obtained premiums to slight modifications in the choice of the prior distribution. To that end, we are discussing carrying out a Bayesian robustness analysis (local or global). On the other hand, choosing the Sarmanov-Lee family to construct the prior distribution that allows for a certain degree of dependence between the two parameters can be very rigid (it has the advantage of being almost conjugate) and could be replaced by a bivariate distribution based on another type of copula.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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References

1. H. Albrecher, J. Beirlant, J. L. Teugels, *Reinsurance: actuarial and statistical aspects*, Wiley, 2017.
2. A. J. Mata, Princing excess of loss reinsurance with reinstatements, *ASTIN Bull.*, **30** (2000), 349–368. https://doi.org/10.2143/AST.30.2.504640
3. M. Kaluszka, Truncated stop loss as optimal reinsurance agreement in one-period models, *ASTIN Bull.*, **35** (2005), 337–349. https://doi.org/10.2143/AST.35.2.2003456
4. J. Cai, Stop-loss premium, In: *Encyclopedia of actuarial science*, John Wiley & Sons, New York, 2006, 1–5. https://doi.org/10.1002/9780470012505.tas038
5. M. D. L. Centeno, Retention and reinsurance programmes, In: *Encyclopedia of actuarial science*, John Wiley & Sons, New York, 2006, 1–9. https://doi.org/10.1002/9780470012505.tar028
6. A. Zanotto, G. P. Clemente, An optimal reinsurance simulation model for non-life insurance in the Solvency II framework, *Eur. Actuar. J.*, **12** (2022), 89–123. https://doi.org/10.1007/s13385-021-00281-2
7. J. Cao, D. Li, V. R. Young, B. Zou, Reinsurance games with two reinsurers: tree versus chain, *Eur. J. Oper. Res.*, **310** (2023), 928–941. https://doi.org/10.1016/j.ejor.2023.04.005
8. M. Fackler, The global financial crisis-risk transfer, insurance layers, and (lack of?) reinsurance culture, *British Actuarial J.*, **28** (2023), 1–22. https://doi.org/10.1017/S1357321723000120
9. D. Yao, J. Zhu, Optimal reinsurance under a new design: two layers and multiple reinsurers, *Quant. Finance*, **24** (2024), 655–676. https://doi.org/10.1080/14697688.2024.2349019
10. O. Hesselager, A class of conjugate priors with applications to excess-of-loss reinsurance, *ASTIN Bull.*, **23** (1993), 77–93. https://doi.org/10.2143/AST.23.1.2005102
11. M. Rytgaard, Estimation in the Pareto distribution, *ASTIN Bull.*, **20** (1990), 201–216. https://doi.org/10.2143/AST.20.2.2005443
12. R. D. Reiss, M. Tomas, A new class of Bayesian estimators in Paretian excess-of-loss reinsurance, *ASTIN Bull.*, **29** (1999), 339–349. https://doi.org/10.2143/AST.29.2.504620
13. M. Denuit, J. Dhaene, M. Goovaerts, R. Kaas, *Actuarial theory for dependent risks measures, orders and models*, John Wiley & Sons, Ltd., 2005. https://doi.org/10.1002/0470016450

14. O. V. Sarmanov, Generalized normal correlation and two-dimensional Frechet classes, *Soviet Math. Doklady*, **25** (1966), 1207–1222.
15. M. L. T. Lee, Properties and applications of the Sarmanov family of bivariate distributions, *Commun. Stat.-Theory Methods*, **25** (1996), 1207–1222. https://doi.org/10.1080/03610929608831759
16. A. Hernández-Bastida, M. P. Fernández, E. Gómez-Déniz, The net Bayes premium with dependence between the risk profiles, *Insur.: Math. Econ.*, **45** (2009), 247–254. https://doi.org/10.1016/j.insmatheco.2009.07.002
17. Z. Bahraoui, C. Bolancé, E. Pelican, R. Vernic, On the bivariate Sarmanov distribution and copula. An application on insurance data using truncated marginal distributions, *SORT*, **39** (2015), 209–230.
18. C. Bolancé, R. Vernic, Multivariate count data generalized linear models: three approaches based on the Sarmanov distribution, *Insur.: Math. Econ.*, **85** (2019), 89–103. https://doi.org/10.1016/j.insmatheco.2019.01.001
19. E. C. K. Cheung, W. Ni, R. Oh, J. K. Woo, Bayesian credibility under a bivariate prior on the frequency and the severity of claims, *Insur.: Math. Econ.*, **100** (2021), 274–295. https://doi.org/10.1016/j.insmatheco.2021.06.003
20. H. H. Panjer, G. E. Willmot, Compound Poisson models in actuarial risk theory, *J. Econ.*, **23** (1983), 63–76. https://doi.org/10.1016/0304-4076(83)90075-1
21. H. H. Panjer, Recursive evaluation of a family of compound distributions, *ASTIN Bull.*, **12** (1981), 22–26. https://doi.org/10.1017/S0515036100006796
22. T. Rolski, H. Schmidli, V. Schmidt, J. Teugel, *Stochastic processes for insurance and finance*, John Wiley & Sons, Hoboken, NJ., 1999. https://doi.org/10.1002/9780470317044
23. C. M. Ramsay, The distribution of compound sums of Pareto distributed losses, *Scand. Actuar. J.*, **2009** (2009), 27–37. https://doi.org/10.1080/03461230802627835
24. B. Roehner, P. Winiwarter, Aggregation of independent Paretian random variables, *Adv. Appl. Probabil.*, **17** (1985), 465–469. https://doi.org/10.2307/1427153
25. W. R. Heilmann, Decision theoretic foundations of credibility theory, *Insur.: Math. Econ.*, **8** (1989), 77–95. https://doi.org/10.1016/0167-6687(89)90050-4
26. E. Gómez-Déniz, A generalization of the credibility theory obtained by using the weighted balanced loss function, *Insur.: Math. Econ.*, **42** (2008), 850–854. https://doi.org/10.1016/j.insmatheco.2007.09.002
27. D. C. M. Dickson, *Insurance risk and ruin*, Cambridge University Press, Cambridge, 2005. https://doi.org/10.1017/CBO9780511624155
28. M. Makarov, Applications of exact extreme value theorem, *J. Oper. Risk*, **2** (2007), 115–120.
29. P. V. Shevchenko, M. V. Wüthrich, The structural modelling of operational risk via Bayesian inference: combining loss data with expert opinions, *J. Oper. Risk*, **1** (2006), 3–26.
30. D. Drouet Mari, S. Kotz, *Correlation and dependence*, Singapore: Imperial Press, 2001. https://doi.org/10.1142/p226

31. S. Kotz, R. van Dorp, A versatile bivariate distribution on a bounded domain: another look at the product moment correlation, *J. Appl. Stat.*, **29** (2002), 1165–1179. https://doi.org/10.1080/0266476022000011247
32. J. M. Sarabia, E. Gómez-Déniz, Construction of multivariate distributions: a review of some recent results, *Stat. Oper. Res. Trans.*, **32** (2008), 3–36.
33. L. Cohen, Probability distributions with given multivariate marginals, *J. Math. Phys.*, **25** (1984), 2402–2403.
34. D. Morgenstern, Einfache beispiele zweidimensionaler verteilungen, *Mitteilungsbl. Math. Stat.*, **8** (1956), 234–235.
35. D. J. G. Farlie, The performance of some correlation coeficients for a general bivariate distribution, *Biometrika*, **47** (1960), 307–323. https://doi.org/10.2307/2333302
36. E. J. Gumbel, Bivariate exponential distributions, *J. Am. Stat. Assoc.*, **55** (1960), 698–707.
37. B. C. Arnold, S. J. Press, Bayesian estimation and prediction for pareto data, *J. Am. Stat. Assoc.*, **84** (1989), 1079–1084.
38. R. V. Hogg, S. A. Klugman, *Loss distributions*, John Wiley & Sons: Hoboken, NJ, USA., 1984. https://doi.org/10.1002/9780470316634
39. H. Jeong, E. A. Valdez, Predictive compound risk models with dependence, *Insur.: Math. Econ.*, **94** (2020), 182–195. https://doi.org/10.1016/j.insmatheco.2020.07.011
40. E. Gómez-Déniz, Bivariate credibility bonus-malus premiums distinguishing between two types of claims, *Insur.: Math. Econ.*, **70** (2016), 117–124. https://doi.org/10.1016/j.insmatheco.2016.06.009
41. P. de Jong, G. Z. Heller, *Generalized linear models for insurance data*, Cambridge University Press, 2008.

Appendix

Proof of Lemma 2

First, we have that

$$\begin{aligned}\mathbb{E}[\Lambda \exp(-\Lambda)] &= \frac{\tau^{\nu}}{\Gamma(\nu)}\int_0^{\infty} \lambda \exp(-\lambda)\lambda^{\nu-1}\exp(-\tau\lambda)\,d\lambda \\ &= \frac{\tau^{\nu}}{\Gamma(\nu)}\frac{\Gamma(\nu+1)}{(\tau+1)^{\nu+1}} = \frac{\nu}{\tau}(1+\tau^{-1})^{-\nu-1} = \frac{\nu}{\tau}\delta_1(\nu+1,\tau).\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}\mathbb{E}[\Psi \exp(-\Psi)] &= \frac{\xi^{\gamma}}{\Gamma(\gamma)}\int_k^{\infty} \psi \exp(-\psi)(\psi-k)^{\gamma-1}\exp[-\xi(\psi-k)]\,d\psi \\ &= \frac{\xi^{\gamma}}{\Gamma(\gamma)}\exp(-k)\int_k^{\infty} \psi(\psi-k)^{\gamma-1}\exp[-(\xi+1)(\psi-k)]\,d\psi \\ &= \left(\frac{\xi}{\xi+1}\right)^{\gamma}\left(k+\frac{\gamma}{\xi+1}\right)\exp(-k)\end{aligned}$$

$$= \left(k + \frac{\gamma}{\xi+1}\right)\delta_2(\xi,\gamma)\exp(-k).$$

Proof of Proposition 7

By taking logarithm in (4.8), we get that

$$\begin{aligned}\log\rho(\Lambda,\Psi) &= \log\omega + \frac{1}{2}\log(\nu\gamma) - (\nu+1)\log\left(\frac{\tau+1}{\tau}\right) + \gamma\log\left(\frac{\xi}{1+\xi}\right)\\ &\quad - \log\tau - \log(1+\xi) - 1,\end{aligned}$$

from which we get, after some simple algebra, that

$$\begin{aligned}\frac{\partial\rho(\Lambda,\Psi)}{\partial\tau} &= \frac{\rho(\Lambda,\Psi)}{\tau}\frac{\nu-\tau}{1+\tau},\\ \frac{\partial\rho(\Lambda,\Psi)}{\partial\nu} &= \rho(\Lambda,\Psi)\left[\frac{1}{2\nu} - \log\left(1+\frac{1}{\tau}\right)\right],\\ \frac{\partial\rho(\Lambda,\Psi)}{\partial\gamma} &= \rho(\Lambda,\Psi)\left[\frac{1}{2\gamma} + \log\left(\frac{\xi}{1+\xi}\right)\right],\\ \frac{\partial\rho(\Lambda,\Psi)}{\partial\xi} &= \rho(\Lambda,\Psi)\frac{\gamma-\xi}{\xi(1-\xi)}.\end{aligned}$$

Now, the result of Proposition 7 is almost direct.

Proof of Proposition 9

Expression (4.10) is obtained after using (4.5) and the following computations

$$\begin{aligned}&\int_1^\infty\int_0^\infty e^{-\lambda-\psi}P(\lambda,\psi)\pi_1(\lambda|\nu,\tau)\pi_2(\psi|\gamma,\xi)\,d\lambda\,d\psi\\ &\qquad = \frac{\tau^\nu\xi^\gamma}{\Gamma(\nu)\Gamma(\gamma)e}\int_1^\infty\int_0^\infty P(\lambda,\psi)\lambda^{\nu-1}e^{-(\tau+1)\lambda}(\psi-1)^{\gamma-1}e^{-(\xi+1)(\psi-1)}\,d\lambda\,d\psi\\ &\qquad = \frac{1}{e}\left(\frac{\tau}{\tau+1}\right)^\nu\left(\frac{\xi}{\xi+1}\right)^\gamma P(\nu,\tau+1,\gamma,\xi+1)\\ &\qquad = \delta_1(\nu,\tau)\delta_2(\gamma,\xi)P(\nu,\tau+1,\gamma,\xi+1).\end{aligned}$$

$$\begin{aligned}&\int_1^\infty\int_0^\infty e^{-\lambda}P(\lambda,\psi)\pi_1(\lambda|\nu,\tau)\pi_2(\psi|\gamma,\xi)\,d\lambda\,d\psi\\ &\qquad = \frac{\tau^\nu}{\Gamma(\nu)}\frac{\xi^\gamma}{\Gamma(\gamma)}\int_1^\infty\int_0^\infty \P(\lambda,\psi)\lambda^{\nu-1}e^{-(\tau+1)\lambda}(\psi-1)^{\gamma-1}e^{-\xi(\psi-1)}\,d\lambda\,d\psi\\ &\qquad = \frac{\tau^\nu}{\Gamma(\nu)}\frac{\Gamma(\nu)}{(\tau+1)^\nu}\P(\nu,\tau+1,\gamma,\xi)\\ &\qquad = \delta_1(\nu,\tau)P(\nu,\tau+1,\gamma,\xi).\end{aligned}$$

$$\int_1^\infty \int_0^\infty e^{-\psi} P(\lambda,\psi)\pi_1(\lambda|\nu,\tau)\pi_2(\psi|\gamma,\xi)\, d\lambda\, d\psi$$
$$= \frac{\tau^\nu}{e\Gamma(\nu)} \frac{\xi^\gamma}{\Gamma(\gamma)} \int_1^\infty \int_0^\infty P(\lambda,\psi)\lambda^{\nu-1} e^{-\tau\lambda}(\psi-1)^{\gamma-1} e^{-(\xi+1)(\psi-1)}\, d\lambda\, d\psi$$
$$= \frac{\xi^\gamma}{e\Gamma(\gamma)} \frac{\Gamma(\nu)}{(\xi+1)^\gamma} P(\nu,\tau,\gamma,\xi+1)$$
$$= \delta_1(\nu,\tau)P(\nu,\tau+1,\gamma,\xi).$$

The Bayes premium is obtained by replacing in (4.9) the posterior distributions $\pi_1^*\pi_2^*$ by the corresponding Bayes premium obtained from the proposed model used.

Proof of Corollary 1

By using expression (3.7), we have

$$\begin{aligned} P(\nu,\tau+1,\gamma,\xi) &= \frac{\tau}{\tau+1} P(\nu,\tau,\gamma,\xi) \quad \text{and} \\ P(\nu,\tau+1,\gamma,\xi+1) &= \frac{\tau}{\tau+1} P(\nu,\tau,\gamma,\xi+1). \end{aligned}$$

Now, by expanding (4.10), we have

$$\begin{aligned} \mathbb{E}_\pi[P(\Lambda,\Psi)] = {} & P(\nu,\tau,\gamma,\xi) + \omega\delta_1(\mu,\tau)\delta_2(\xi,\gamma)\left[P(\nu,\tau,\gamma,\xi) + P(\nu,\tau+1,\gamma,\xi+1)\right. \\ & \left. -P(\nu,\tau,\gamma,\xi+1) - P(\nu,\tau+1,\gamma,\xi)\right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} \mathbb{E}_\pi[P(\Lambda,\Psi)] = {} & P(\nu,\tau,\gamma,\xi) + \omega\delta_1(\mu,\tau)\delta_2(\xi,\gamma)\left[P(\nu,\tau,\gamma,\xi) + \frac{\tau}{\tau+1}P(\nu,\tau,\gamma,\xi+1)\right. \\ & \left. -P(\nu,\tau,\gamma,\xi+1) - \frac{\tau}{\tau+1}P(\nu,\tau,\gamma,\xi)\right] \end{aligned}$$

and then the result easily follows.

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