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Research article

A Faber-Krahn inequality for nonhomogeneous wedge-like membranes

Abdelhalim Hasnaoui* **and Abdelhamid Zaghdani**

Mathematics Department, College of Sciences and Arts, Northern Border University, Rafha, Saudi Arabia

* **Correspondence:** Email: halim.hasnaoui1@gmail.com.

Abstract: In this paper, we adapt B. Schwarz and Banks-Krein techniques to obtain a new version of Faber-Krahn inequality for the first Dirichlet eigenvalue of the Laplacian in wedge like membranes with continuous mass density function.

Keywords: Dirichlet eigenvalues; wedge-like membranes; Faber-Krahn inequality; weighted symmetrization; Rayleigh quotient
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1. Introduction

The Faber-Krahn inequality, named also the Rayleigh-Faber-Krahn inequality, states that the ball minimizes the fundamental eigenvalue of the Dirichlet Laplacian among bounded domains with fixed volume. It was conjectured by Lord Rayleigh [1] and then proved independently by Faber [2] and Krahn [3]. One of the extensions of this inequality is the result established by B. Schwarz [4] for nonhomogeneous membranes, which is stated as follows: Let $\lambda_1(p)$ be principal frequency of a nonhomogeneous membrane D with positive density function p, then

$$\lambda_1(p^\star) \leq \lambda_1(p), \tag{1.1}$$

where $p^\star$ is the Schwartz symmetrization of p and $\lambda_1(p^\star)$ is the first eigenvalue of the symmetrized problem in the disk $D^\star$. Our aim in this paper is to give a version of the B. Schwarz inequality for the case of bounded domains completely contained in a wedge of angle $\frac{\pi}{\alpha}$, $\alpha \geq 1$. Such bounded domains are called wedge-like membranes. The method for proving our result requires a weighted version of decreasing rearrangement tailored to the case of wedge-like membranes. This technique was first introduced by Payne and Weinberger [5], and then studied and used to improve many classical inequalities by several authors, see for example [6–10]. An interesting feature of this method is that it leads to an improvement of classical inequalities for certain domains, as shown by Payne and

Weiberger [5] and Hasnaoui and Hermi [11]. Also, the results of this method have the interpretation of being the usual results in dimension $d = 2\alpha + 2$ for domains with axial or bi-axial symmetry, see [9, 11, 12]. We are also interested in a new version of the Banks-Krein inequality [13] where the numerical value of the lower bound of the first eigenvalue is given by the first positive root of an equation involving the Bessel function J_α.

2. Main results

Before stating our results, we need to introduce some notations and preliminary tools. Letting $\alpha \geq 1$, we will denote by $\mathcal{W}$ the wedge defined in polar coordinates (r, θ) in $\mathbb{R}^2$ by

$$\mathcal{W} = \Big\{(r, \theta) \,\Big|\, 0 < r,\ 0 < \theta < \frac{\pi}{\alpha}\Big\}. \tag{2.1}$$

For $\tau \geq 0$, we define the sector of radius τ by

$$S_\tau = \Big\{(r, \theta) \,\Big|\, 0 < r < \tau,\ 0 < \theta < \frac{\pi}{\alpha}\Big\}. \tag{2.2}$$

We also define

$$h(r, \theta) = r^\alpha \sin \alpha\theta, \tag{2.3}$$

which is a positive harmonic function in $\mathcal{W}$ vanishing on the boundary $\partial\mathcal{W}$. From that, we introduce the weighted measure μ defined by

$$\mu(D) = \int_D d\mu = \int_D h^2 dx, \tag{2.4}$$

for all bounded domains $D \subset \mathcal{W}$.

Throughout this paper, we denote by $\lambda_1(w)$ the first eigenvalue of the problem

$$\mathcal{P}_1 : \begin{cases} \Delta u + \lambda w u &= 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D, \end{cases}$$

where D is a bounded domain completely contained in $\mathcal{W}$ and w is a positive continuous function on D. It has been shown that this problem has a countably infinite discrete set of positive eigenvalues, and the first eigenvalue $\lambda_1(w)$ is simple and has an eigenfunction u of constant sign, see for example [14] . We will assume that $u > 0$ in D. Hence, the first eigenfunction can be represented as

$$u = v\, h, \tag{2.5}$$

where v is a positive smooth function vanishing on $\partial D \cap \mathcal{W}$.

Now, we introduce the weighted rearrangement with respect to the measure μ, which is one of the principal tools in our work. Let f be a measurable function defined in $D \subset \mathcal{W}$, and let S_{r_0} be the sector of radius r_0 such that

$$\mu(S_{r_0}) = \mu(D).$$

Furthermore, we denote the sector with the same measure as a measurable subset A of D by $A^\star$. The distribution function of f with respect to the measure μ is defined by

$$m_f(t) = \mu(\{(r,\theta) \in D; \quad |f(r,\theta)| > t\}), \qquad \forall t \in [0, \operatorname{ess\,sup} |f|]. \tag{2.6}$$

The decreasing rearrangement of f with respect to μ is given by

$$f^*(0) = \operatorname{ess\,sup} |f|,$$

$$f^*(s) = \inf\{t \geq 0; \ \ m_f(t) < s\}, \qquad \forall s \in (0, \mu(D)].$$

The weighted rearrangement of f is the function $f^\star$ defined on the sector S_{r_0} by

$$f^\star(r,\theta) = f^*(\mu(S_r)). \tag{2.7}$$

An explicit computation gives that $\mu(S_r) = \frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}$. Substituting this in (2.7), we obtain

$$f^\star(r,\theta) = f^*(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}). \tag{2.8}$$

Since $f^\star$ is a radial and nonincreasing, it follows that its level sets are sectors centered at the origin and have weighted measure equal to $m_f(t)$. We will, by abuse of notation, write $f^\star(r)$ instead of $f^\star(r,\theta)$. Recall that w is the density of the membrane D. Let $w^\star$ denotes the weighted rearrangement of w, and $\lambda_1(w^\star)$ denotes the lowest eigenvalue of the following symmetrized problem

$$\mathcal{P}_2 : \begin{cases} \Delta z + \lambda w^\star z &= 0 \quad \text{in } S_{r_0} \\ z &= 0 \quad \text{on } \partial S_{r_0}. \end{cases}$$

The following result compares the first eigenvalue of the problem $\mathcal{P}_1$ with that of the symmetrized problem $\mathcal{P}_2$.

Theorem 2.1. *If w is a positive continuous function defined on $D \subset \mathcal{W}$, then*

$$\lambda_1(w) \geq \lambda_1(w^\star). \tag{2.9}$$

See the following section for a proof of the theorem. Note that Theorem 2.1 includes the Payne-Weinberger inequality [5] as the special case $w = 1$. The result above is also a new version of the B. Schwarz inequality [4] for wedge-like membranes.

To state the second result in this paper, we need to assume that there is a real number P such that $0 \leq w^\star \leq P$. From that, we introduce the function $\bar{w}$ defined in S_{r_0} by

$$\bar{w}(r,\theta) = \begin{cases} P, & \text{for } r \in [0,\rho], \\ 0, & \text{for } r \in (\rho, r_0], \end{cases}$$

where ρ is chosen such that $\int_{S_{r_0}} w^\star \, d\mu = \int_{S_{r_0}} \bar{w} \, d\mu$.

Theorem 2.2. *Assume that $0 \leq w^\star \leq P$. The first eigenvalue of the problem $\mathcal{P}_2$ satisfies the inequality*

$$\lambda_1(w^\star) \geq \lambda_1(\bar{w}), \tag{2.10}$$

where $\lambda_1(\bar{w})$ is the first eigenvalue of the problem

$$\mathcal{P}_3 : \begin{cases} \Delta\psi + \lambda \bar{w} \psi &= 0 \quad \textit{in } S_{r_0} \\ \psi &= 0 \quad \textit{on } \partial S_{r_0}. \end{cases}$$

The proof of Theorem 2.2 is detailed in the third section. In fact, this result together with the corollary below extend the Banks-Krein theorem [13] to the case of wedge like membranes. The classical version of our result was first proved for vibrating strings by Krein [15] and then extended to planar domains by Banks [13].

Corollary 2.3. *Let* $0 \leq w \leq P$. *Then,*

$$\lambda_1(w) \geq \lambda_1(\bar{w}), \tag{2.11}$$

where $\lambda_1(\bar{w})$ *is the first positive solution of the equation*

$$J_\alpha(\sqrt{\lambda_1(\bar{w})P}\rho) + \sqrt{\lambda_1(\bar{w})P}\frac{\rho}{\alpha}J'_\alpha(\sqrt{\lambda_1(\bar{w})P}\rho)\frac{1-(\frac{\rho}{r_0})^{2\alpha}}{1+(\frac{\rho}{r_0})^{2\alpha}} = 0. \tag{2.12}$$

See the third section for a proof of the corollary.

At the end of this section, we give an appropriate variational characterization to the eigenvalue $\lambda_1(w)$ for the case of wedge-like domains. To begin, consider the functional space $W(D, d\mu)$ which is the set of measurable functions ϕ satisfying the following conditions:

(i) $\int_D |\nabla\phi|^2 d\mu + \int_D |\phi|^2 d\mu < +\infty$.

(ii) There exists a sequence of functions $\phi_n \in C^1(\overline{D})$ such that $\phi_n = 0$ on $\partial D \cap \mathcal{W}$ and

$$\lim_{n\to+\infty} \int_D |\nabla(\phi - \phi_n)|^2 d\mu + \int_D |\phi - \phi_n|^2 d\mu = 0.$$

For more details about this space, see [9] .

Lemma 2.4. *The first eigenvalue of the problem* $\mathcal{P}_1$ *can be defined via the weighted variational characterization*

$$\lambda_1(w) = \min_{\phi\in W(D,d\mu)} \frac{\int_D |\nabla\phi|^2 \, d\mu}{\int_D w\phi^2 \, d\mu}. \tag{2.13}$$

The proof of Lemma 2.4 is detailed in the following section.

3. Proof of Theorems

3.1. Proof of Theorem 2.1

In this part, we will prove Theorem 2.1 by showing that the weighted symmetrization decreases the numerator and increases the denominator of the Rayleigh quotient. For the numerator, we have the following weighted version of the Pólya-Szegő inequality.

Proposition 3.1. *Let* f *be a nonnegative function in* $W(D, d\mu)$. *Then,* $f^\star \in W(S_{r_0}, d\mu)$, *and*

$$\int_D |\nabla f|^2 d\mu \geq \int_{S_{r_0}} |\nabla f^\star|^2 d\mu. \tag{3.1}$$

The complete and detailed proof of Proposition 3.1 is given in [9] for more general cases $d\mu = h^k dx, \; k > 1$. For the denominator, we need the following lemma.

Lemma 3.2. *Let D be a bounded domain completely contained in $\mathcal{W}$ and f be a μ-integrable function defined in D. Let Ω be a measurable subset of D. Then,*

$$\int_{\Omega} f d\mu \leq \int_{S_{r_1}} f^{\star} d\mu, \tag{3.2}$$

where S_{r_1} is the sector satisfying $\mu(S_{r_1}) = \mu(\Omega)$.

Proof. If g denotes the restriction of f to Ω, we have

$$m_g(t) = \mu(\{(r,\theta) \in D; \quad |f(r,\theta)| > t\} \cap \Omega).$$

Thus, if $s \in [m_f(t), \mu(\Omega)]$, then $m_g(t) < s$. Hence,

$$\{t \geq 0; \;\; m_f(t) < s\} \subset \{t \geq 0; \;\; m_g(t) < s\} \tag{3.3}$$

and so

$$\inf\{t \geq 0; \;\; m_g(t) < s\} \leq \inf\{t \geq 0; \;\; m_f(t) < s\}, \tag{3.4}$$

which is exactly the inequality $g^*(s) \leq f^*(s)$.

Thus,

$$\int_{\Omega} f d\mu = \int_{\Omega} g d\mu = \int_0^{\mu(\Omega)} g^*(s) ds \leq \int_0^{\mu(\Omega)} f^*(s) ds. \tag{3.5}$$

Now, by the change of variable $s = \frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}$, we have

$$\begin{aligned}
\int_{S_{r_1}} f^{\star} d\mu &= \int_0^{r_1} \int_0^{\frac{\pi}{\alpha}} f^*\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right) r^{2\alpha+1} \sin^2 \alpha\theta \, dr d\theta &(3.6)\\
&= \frac{\pi}{2\alpha} \int_0^{r_1} f^*\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right) r^{2\alpha+1} \, dr &(3.7)\\
&= \int_0^{\mu(S_{r_1})} f^*(s) \, ds &(3.8)\\
&= \int_0^{\mu(\Omega)} f^*(s) \, ds, &(3.9)
\end{aligned}$$

which proves the lemma. □

Now, we are finally in a position to complete the proof of Theorem 2.1. Recall the function v defined by (2.5). Let $0 \leq c_0 \leq w \leq c_1 \leq \infty$ and χ_{Ω} denotes the characteristic function of a subset Ω of the domain D. Then,

$$\begin{aligned}
\int_D w v^2 d\mu &= \int_D v^2 h^2 \int_0^{c_1} \chi_{\{w>t\}} \, dt \, dx &(3.10)\\
&= \int_0^{c_1} \int_D v^2 \chi_{\{w>t\}} h^2 \, dx \, dt &(3.11)
\end{aligned}$$

$$= \int_0^{c_0}\int_D v^2\chi_{\{w>t\}}h^2\,dx\,dt + \int_{c_0}^{c_1}\int_D v^2\chi_{\{w>t\}}h^2\,dx\,dt \tag{3.12}$$

$$= c_0\int_D v^2h^2\,dx + \int_{c_0}^{c_1}\int_{\{w>t\}} v^2h^2\,dx\,dt. \tag{3.13}$$

By Lemma 3.2 and the fact that $\int_D v^2h^2\,dx = \int_{S_{r_0}}(v^\star)^2h^2\,dx$, we obtain

$$\int_D wv^2d\mu \le c_0\int_{S_{r_0}}(v^\star)^2h^2\,dx + \int_{c_0}^{c_1}\int_{\{w>t\}^\star}(v^2)^\star h^2\,dx\,dt. \tag{3.14}$$

Now, using the equalities $(v^\star)^2 = (v^2)^\star$ and $\{w > t\}^\star = \{w^\star > t\}$ in the second term on the right-hand side of the last inequality, we deduce that

$$\int_D wv^2d\mu \le c_0\int_{S_{r_0}}(v^\star)^2h^2\,dx + \int_{c_0}^{c_1}\int_{\{w^\star>t\}}(v^\star)^2h^2\,dx\,dt \tag{3.15}$$

$$= \int_{S_{r_0}} w^\star(v^\star)^2d\mu. \tag{3.16}$$

The last equality was obtained by applying the same computation in (3.10) to $v^\star$ and $w^\star$. Finally, using Proposition 3.1 and inequality (3.15), we obtain that $v^\star \in W(S_{r_0}, d\mu)$ and

$$\lambda_1(w) = \frac{\int_D |\nabla v|^2\,d\mu}{\int_D wv^2d\mu} \ge \frac{\int_{S_{r_0}}|\nabla v^\star|^2\,d\mu}{\int_{S_{r_0}} w^\star(v^\star)^2d\mu} \ge \min_{\phi\in W(S_{r_0},d\mu)}\frac{\int_{S_{r_0}}\left|\nabla\phi\right|^2d\mu}{\int_{S_{r_0}} w^\star\phi^2\,d\mu} = \lambda_1(w^\star).$$

The proof of Theorem 2.1 is now complete. □

3.2. Proof of Theorem 2.2

The following lemmas are essential for the proof of our theorem.

Lemma 3.3. *Let f_1, f_2, and Φ be μ-integrable functions over D, let $\Omega_1 = \{(r,\theta) \in D \mid f_1(r,\theta) \le f_2(r,\theta)\}$ and $\Omega_2 = \{(r,\theta) \in D \mid f_1(r,\theta) > f_2(r,\theta)\}$, and suppose*

$$\int_D f_1d\mu \ge \int_D f_2d\mu. \tag{3.17}$$

If $0 \le \Phi(r_1,\theta_1) \le \Phi(r_2,\theta_2)$ for all $(r_1,\theta_1) \in \Omega_1$, $(r_2,\theta_2) \in \Omega_2$, then

$$\int_D f_1\Phi d\mu \ge \int_D f_2\Phi d\mu. \tag{3.18}$$

The proof of this lemma is similar to the proof of Lemma 2.7 in [16].

Lemma 3.4. *The eigenfunction z_1 corresponding to the first eigenvalue $\lambda_1(w^\star)$ of the problem $\mathcal{P}_2$ can be written as $z_1 = \xi h$, where ξ is a radial function, which is radially decreasing.*

Proof. By Proposition 3.1 and inequality (3.15), it follows that

$$\lambda_1(w^\star) = \frac{\int_{S_{r_0}} \left|\nabla\xi\right|^2 d\mu}{\int_{S_{r_0}} w^\star \xi^2 \, d\mu} \geq \frac{\int_{S_{r_0}} \left|\nabla\xi^\star\right|^2 d\mu}{\int_{S_{r_0}} w^\star (\xi^\star)^2 \, d\mu}. \tag{3.19}$$

Since $\xi \in W(S_{r_0}, d\mu)$, then $\xi^\star \in W(S_{r_0}, d\mu)$ and is an admissible function for the weighted variational formula (2.13). Using this and inequality (3.19), we see

$$\lambda_1(w^\star) = \frac{\int_{S_{r_0}} \left|\nabla\xi^\star\right|^2 d\mu}{\int_{S_{r_0}} w^\star (\xi^\star)^2 \, d\mu}, \tag{3.20}$$

which means that $\xi^\star h$ is an eigenfunction as well. Finally, the simplicity of the first eigenvalue $\lambda_1(w^\star)$ implies that $z_1 = \xi h = \xi^\star h$, and so $\xi = \xi^\star$. Thus, ξ is radial and radially decreasing. This completes the proof of the lemma. □

Now, setting $\Omega_1 = \{(r,\theta) \in S_{r_0} \mid \rho < r < r_0,\ 0 < \theta < \frac{\pi}{\alpha}\}$ and $\Omega_2 = \{(r,\theta) \in S_{r_0} \mid 0 < r < \rho,\ 0 < \theta < \frac{\pi}{\alpha}\}$, it is not difficult to check that the functions $\bar{w}$ and $w^\star$ satisfy the same relationship as f_1 and f_2 of Lemma 3.3. Also, using Lemma 3.4, we obtain that ξ satisfies the same assumption as Φ, and then

$$\int_{S_{r_0}} w^\star \xi^2 \, d\mu \leq \int_{S_{r_0}} \bar{w} \xi^2 \, d\mu. \tag{3.21}$$

Using the above inequality, we obtain

$$\lambda_1(w^\star) = \frac{\int_{S_{r_0}} \left|\nabla\xi\right|^2 d\mu}{\int_{S_{r_0}} w^\star \xi^2 \, d\mu} \geq \frac{\int_{S_{r_0}} \left|\nabla\xi\right|^2 d\mu}{\int_{S_{r_0}} \bar{w} \xi^2 \, d\mu} \geq \min_{\phi \in W(S_{r_0}, d\mu)} \frac{\int_{S_{r_0}} \left|\nabla\phi\right|^2 d\mu}{\int_{S_{r_0}} \bar{w} \phi^2 \, d\mu} = \lambda_1(\bar{w}).$$

This completes the proof of the theorem. □

3.3. *Proof of Corollary 2.3*

Inequality (2.11) follows immediately from Theorems 2.1 and 2.2. To prove the equality (2.12), we first proceed as in [8] to obtain that the eigenfunction corresponding to $\lambda_1(\bar{w})$ is explicitly given by $\psi_1(r,\theta) = R(r) \sin \alpha\theta$, where the function R is defined on $[0, r_0]$ by

$$R(r) = \begin{cases} cJ_\alpha(\sqrt{\lambda_1(\bar{w})P} r), & \text{for } r \in [0, \rho], \\ \tilde{c}(r^{-\alpha} - r_0^{-2\alpha} r^\alpha), & \text{for } r \in (\rho, r_0]. \end{cases}$$

Here, the constants c and $\tilde{c}$ satisfy the continuity of the function R and of its derivative. Now, since R is continuous at $r = \rho$, we see that

$$cJ_\alpha(\sqrt{\lambda_1(\bar{w})P} \rho) = \tilde{c}(\rho^{-\alpha} - r_0^{-2\alpha} \rho^\alpha). \tag{3.22}$$

The continuity of the derivative of R at $r = \rho$ gives

$$c\sqrt{\lambda_1(\bar{w})P}J'_\alpha(\sqrt{\lambda_1(\bar{w})P}\rho) = -\tilde{c}\frac{\alpha}{\rho}(\rho^{-\alpha} + r_0^{-2\alpha}\rho^{\alpha}). \tag{3.23}$$

Thus,

$$\tilde{c} = -\frac{\rho}{\alpha}\sqrt{\lambda_1(\bar{w})P}J'_\alpha(\sqrt{\lambda_1(\bar{w})P}\rho)\frac{1}{\rho^{-\alpha} + r_0^{-2\alpha}\rho^{\alpha}}. \tag{3.24}$$

Finally, plugging Eq (3.24) into (3.22), we obtain the desired result. The proof of Corollary 2.3 is now complete. □

3.4. Proof of Lemma 2.4

The first eigenvalue of problem $\mathcal{P}_1$ can be characterized by the Rayleigh principle

$$\lambda_1(w) = \min_{\varphi\in H_0^1(D)} \frac{\int_D |\nabla\varphi|^2\, dx}{\int_D w\varphi^2\, dx}. \tag{3.25}$$

Let $\phi \in W(D, d\mu)$. Using the fact that $\Delta h = 0$ and the divergence theorem, we obtain

$$\int_D |\nabla(\phi h)|^2\, dx = \int_D |\nabla\phi|^2 h^2 + |\nabla h|^2\phi^2 + 2\phi h\nabla h\cdot\nabla\phi\, dx = \int_D |\nabla\phi|^2 h^2\, dx. \tag{3.26}$$

Since the function ϕh belongs to the Sobolev space $H_0^1(D)$, then we can use it as a test function in the Rayleigh quotient (3.25). Applying (3.26), we get

$$\lambda_1(w) \le \frac{\int_D |\nabla(\phi h)|^2\, dx}{\int_D w\phi^2 h^2\, dx} = \frac{\int_D |\nabla\phi|^2\, d\mu}{\int_D w\phi^2\, d\mu}. \tag{3.27}$$

Now, if we write the first eigenfunction as in (2.5) and substitute it into (3.25), we obtain

$$\lambda_1(w) = \frac{\int_D |\nabla u|^2\, dx}{\int_D wu^2\, dx} = \frac{\int_D |\nabla(vh)|^2\, dx}{\int_D wv^2h^2\, dx} = \frac{\int_D |\nabla v|^2 h^2\, dx}{\int_D wv^2h^2\, dx} = \frac{\int_D |\nabla v|^2\, d\mu}{\int_D wv^2 d\mu},$$

which proves the lemma. □

4. Conclusions

The Dirichlet eigenvalues are known only for a limited number of regions, such as disks, sectors and rectangles. This lack of information has prompted many researchers to explore methods and techniques for estimating eigenvalues. In this paper, we have proved a new lower bound for the first Dirichlet eigenvalue of an arbitrarily shaped region with continuous mass density function and completely contained in a wedge. This lower bound has been given as the lowest positive root of the Eq (2.12). In our next projects, we aim to use the method of wedge like-membranes to improve the Z. Nehari inequality [17] . Additionally, We will adapt the increasing rearrangement techniques to offer a complementary results to those of this paper. Furthermore, the generalization of all these results to higher dimensions will be considered in future works.

Author contributions

The authors contributed equally and they both read and approved the final manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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