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Research article

On perturbed-$\mathcal{S}_\tau$-contractions

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Abstract: This study aims to present novel fixed-point results within the structure of a newly introduced abstract structure known as perturbed metric spaces. As expected, these spaces naturally extend and generalize the classical metric spaces. Consequently, the key results of this study broaden, refine, and broaden the existing fixed-point results in the published outcomes.

Keywords: abstract metric space; fixed-point; metric spaces; perturbed metric spaces; self-mappings
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1. Introduction and Preliminaries

In mathematics, perturbation theory, particularly in applied mathematics, refers to techniques for estimating a problem's solution by beginning with the precise solution of a related and simpler problem. Roughly speaking, a critical aspect of the intermediate phase of the technique is separating the problem into "solvable" and "perturbative" parts.

This work examines the specific instance of this theory, namely perturbed metric spaces [1]. More precisely, the difficulty in effectively separating the gauge (perturbed) and solvable part is handled by the perturbed-metric technique. In fact, perturbed metric spaces have been widely used to understand and analyze several distinct phenomena of physics.

fixed-point theory (FPT) has a pivotal mission in resolving a number of distinct linear and nonlinear (differential/integral) equations. Roughly speaking, the idea of Picard iteration (successive approximation method) was adapted as a Banach fixed-point theorem (BFPTh), and it has been used to solve many real-world problems.

In this work, we examine the $\mathcal{S}_\tau$-contraction [2, 3], a natural and significant generalization of the Banach contraction mapping concept. This notion was improved in [4], to examine the rational contractions in the literature. Bilateral contractions have been unified in [5]. Generalized Suzuki type contraction was investigated via extended $\mathcal{S}_\tau$-contraction in [6]. Further, generalized interpolative contraction of various type was examined in [7]. In b-metric structure, non-unique fixed point theorems were reconsidered in [8]. In [9], the notion of $\mathcal{S}_\tau$-contraction has been applied to the best proximity theory. In this study, we engage the simulation maps to introduce perturbed $\mathcal{S}_\tau$-contraction in the construction on a perturbed metric space. Thereby, we observe advanced fixed-point theorems in this construction.

Fixed-point theory (FPT) has a pivotal mission in resolving a number of distinct linear and nonlinear (differential/integral) equations. Roughly speaking, the idea of Picard iteration (successive approximation method) was adapted as a Banach fixed-point theorem (BFPTh), and it has been used to solve many real-world problems [10,11].

All across the paper, we fix M as an arbitrary non-empty set for the sake of this study. In addition, the collection of all real numbers shall be represented as the letter $\mathbb{R}$. Likewise, the letters $\mathbb{N}$ (respectively, $\mathbb{R}^+$) are kept to indicate the set of positive integers (respectively, real) numbers. Furthermore, the symbol $\mathbb{N}_0$ (respectively, $\mathbb{R}_0^+$) is reserved for non-negative integer (respectively, real) numbers, more precisely, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{R}_0^+ = \{0\} \cup \mathbb{R}^+$. Supplementarily, the capital letters "C, M, F, S, T, P" are kept for "complete,metric, Fixed, Space, Theory, Perturbated." In particular, "CPMS" represents "Complete Perturbed Metric Space", and so on.

The following definition introduces the recently proposed concept of perturbed metric spaces in [1]:

Definition 1.1. *[1] Assume that Δ and P are mappings defined on $M \times M$ with values in $[0, \infty)$. We assert that Δ constitutes a perturbed metric on M in relation to P, if*

$$\Delta - P : M \times M \to \mathbb{R},$$
$$(m, r) \longrightarrow \Delta(m, s) - P(m, s)$$

defines a metric on M, meaning that for every $m, s, o \in M$,

(i) $(\Delta - P)(m, s) \geq 0$;
(ii) $(\Delta - P)(m, s) = 0 \Leftrightarrow m = s$;
(iii) $(\Delta - P)(m, s) = (\Delta - P)(s, m)$;
(iv) $(\Delta - P)(m, s) \leq (\Delta - P)(m, o) + (\Delta - P)(o, s)$.

The function P is a perturbed map, while $d = \Delta - P$ represents the standard associated metric. The structure (M, Δ, P) is thus called a perturbed metric space.

Remark 1.1. *A perturbed metric on M is not necessarily a metric on M. It is appropriate to regard $\Delta : M \times M \to [0, \infty)$ as $\Delta(m, s) = (m + s)^2 + \arctan|m - s|$ for any $m, s \in M$. It is evident that for the perturbed map (in short, p-map) $P : M \times M \to [0, \infty)$ described as $P(m, s) = (m + s)^2$, the function $d(m, s) = \arctan|m - s|$ serves as a standard metric, where $d = \Delta - P$ for all $m, s \in M$. However, Δ itself does not define a metric, as it fails to satisfy the identity property; specifically, $\Delta(2, 2) = 16 \neq 0$.*

We present the following basic properties of perturbed metric spaces, as established by Jleli and Samet [1]:

Proposition 1.1. *Let $\Delta, P, Q : M \times M \to [0, \infty)$ be three given mappings and $\alpha > 0$.*

(i) Let two triples (M, Δ, P) *and* (M, Δ, Q) *indicates two PMSs. Then, the triple* $\left(M, \Delta, \frac{P+Q}{2}\right)$ *forms a PMS, too.*
(ii) The triple $(M, \alpha\Delta, \alpha P)$ *forms a PMS, whenever the triple* (M, Δ, P) *forms a PMS.*

We now present several topological concepts in perturbed metric spaces as provided in [1].

Definition 1.2. *Let the triple* (M, Δ, P) *form a PMS. Consider a sequence* $\{m_n\}$ *in* M*, and a self-mapping* $\mathscr{F}$ *defined on* M.

(i) If $\{m_n\}$ *is a convergent sequence in a standard metric space* (M, d)*, where* $d = \Delta - P$*, then* $\{m_n\}$ *is a perturbed convergent (p-convergent) sequence in* (M, Δ, P).
(ii) If a sequence $\{m_n\}$ *is a Cauchy (p-Cauchy) in the context of standard metric space* (M, d)*, then* $\{m_n\}$ *is a perturbed Cauchy sequence in* (M, Δ, P).
(iii) A triple (M, Δ, P) *is named CPMS, if* (M, d) *is a CMS.*
(iv) A map $\mathscr{F}$ *is a perturbed continuous (in short, p-continuous) if* $\mathscr{F}$ *is continuous in the setting of the corresponding standard-MS* (M, d).

This is a generalization of Banach's fixed-point theorem, extending it from conventional metric spaces to perturbed metric spaces, as demonstrated in [1]:

Theorem 1.1. *Let* $\mathscr{F}$ *be a given self-mapping over a CMPS* (M, Δ, P)*. Presume the subsequent restrictions are true:*

(i) $\mathscr{F}$ *is a p-continuous map,*
(ii) Exists $\lambda \in (0, 1)$ *providing the inequality*

$$\Delta(\mathscr{F}m, \mathscr{F}n) \leq \lambda\Delta(m, n),$$

holds for any $m, n \in M$.

Then $\mathscr{F}$ *admits one and only one fixed-point.*

The idea of the simulation map work was presented in [2]:

Definition 1.3. *[2] If the mapping* $\tau : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *meets the following criteria, it is referred to as a simulation map:*

($\tau 1$) $\tau(0, 0) = 0$;
($\tau 2$) $\tau(m, r) < r - m$ *for all* $m, r > 0$;
($\tau 3$) *If* $\{m_n\}$ *and* $\{r_n\}$ *are sequences taking values in* $(0, \infty)$ *and satisfy*

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} r_n = L > 0,$$

yields

$$\limsup_{n\to\infty} \tau(m_n, r_n) < 0.$$

Let $\mathcal{T}$ be kept to present the collection of all simulation maps. We extrapolate from ($\tau 1$) that $\tau(b, b) < 0$ for any $b > 0$.

A. Roldán et al. [12] extended and refined Definition 1.3 to broaden the sets of simulation maps by replacing assumption ($\tau 3$) with the following restriction:

$(\tau 3)'$ contemplate two positive sequences $\{m_n\}$ and $\{r_n\}$ so that if

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} r_n > 0 \text{ and } m_n < r_n,$$

then

$$\limsup_{n\to\infty} \tau(m_n, r_n) < 0.$$

Within the taking after, we display a few illustrations of the recreation work given in [2]:

Example 1.1. *Presuppose that the maps r, e continuous self-maps on $[0,\infty)$ so that $e(t) = r(t) = 0 \iff t = 0$ and $e(t) < t \le r(t)$ for each $t > 0$. For $i = 1,2,3$, we define $\tau_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, as*

- (i) *For each $t, s \in [0,\infty)$, the map$\tau_1(t,s) = g(s) - r(t)$ belongs to the class $\mathcal{T}$.*
- (ii) *For each $t, s \in [0,\infty)$, describe$\tau_2(t,s) = s - \frac{H(t,s)}{F(t,s)} t$, where $H, F : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous in both variables and provide $H(t,s) > F(t,s)$ for each $t, s > 0$.*
- (iii) *For each $t, s \in [0,\infty)$, the function $\tau_3(t,s) = s - p(s) - t$ is in $\mathcal{T}$, where $p : [0,\infty) \to [0,\infty)$ is a continuous function with the property that $p(t) = 0$ if and only if $t = 0$.*

In what follows, the concept of an $\mathcal{S}_\tau$-contraction, introduced by Khojasteh et al. [2], shall be described:

Definition 1.4. *Let $\mathscr{F}$ be a self-maps on a standard-MS (M,d) with $\tau \in \mathcal{T}$. For any $m, r \in M$, the mapping $\mathscr{F}$ is called an $\mathcal{S}_\tau$-contraction if*

$$\tau(d(\mathscr{F}m, \mathscr{F}r), d(m,r)) \ge 0. \tag{1.1}$$

Remark 1.2. *[2]*

- (i) *The Banach contraction is an obvious illustration of S_τ-contraction. It is adequate to set $\lambda \in [0,1)$ and $\tau(r,m) = \lambda m - r$ for all $m, r \in [0,\infty)$ in (1.1).*
- (ii) *The inequality follows for any $b \ge a > 0$. Subsequently, for each distinct combination of $m, r \in M$, it infers*

$$d(\mathscr{F}m, \mathscr{F}r) < d(m,r).$$

In brief, each $\mathcal{S}_\tau$-contraction map is continuous.

Below, we outline the key results, facts, and findings as established in [2]:

Lemma 1.1. *[2] Let a self-map $\mathscr{F}$ over MS (M,d) be an $\mathcal{S}_\tau$-contraction. Then,*

1) *If $\mathscr{F}$ admits a fixed point, it has to be unique.*
2) *The operator $\mathscr{F}$ is asymptotically regular for any point $m \in M$.*
3) *For each $n \in \mathbb{N}$, if we describe $m_n = \mathscr{F}(m_{n-1})$ with an arbitrary initial value $m_0 \in M$ (i.e., setting $m_0 = m$ for some chosen m), then the sequence $\{m_n\}$ generated by the Picard operator $\mathscr{F}$ is bounded.*

Theorem 1.2. *Let a self-map $\mathscr{F}$ over MS (M,d) be an $\mathcal{S}_\tau$-contraction. Necessarily, it admits a unique fixed point $m^* \in M$. Moreover, the sequence $\{m_n\}$, described as $m_n = \mathscr{F}(m_{n-1})$ for all $n \in \mathbb{N}$ with arbitrarily chosen initial point $m_0 \in M$, converges to the fixed point m^* of $\mathscr{F}$.*

2. Main results

Let Σ denote the class of auxiliary functions $\varsigma : [0,\infty) \to [0,\infty)$ satisfying the subsequent constraints:

(ς_1) ς is non-decreasing;

(ς_2) $\sum_{n=1}^{+\infty} \varsigma^n(t) < \infty$ for each $t > 0$, in which ς^n denotes the n^{th} iterate of ς.

If $\varsigma \in \Sigma$, then it is referred to as a (c)-comparison function. A notable property of a (c)-comparison function ς is that $\varsigma(t) < t$ for all $t > 0$. Further details regarding these functions can be found in [13].

We introduce the $\mathcal{S}_\tau$-contraction, a new contraction restriction in the structure of PMS, as follows:

Definition 2.1. *Let $\mathscr{F} : M \to M$ be a mapping, where (M, D, P) is a perturbed metric space and $\varsigma \in \Sigma$. Then $\mathscr{F}$ is said to be a perturbed $\mathcal{S}_\tau$-contraction if the subsequent restriction holds:*

$$\tau(D(\mathscr{F}m, \mathscr{F}l),\ \varsigma(D(m,l))) \geq 0, \tag{2.1}$$

for all $m, l \in M$.

Lemma 2.1. *Let $\mathscr{F} : M \to M$ be a perturbed $\mathcal{S}_\tau$-contraction on the PMS (M, D, P). Then, if a fixed point of $\mathscr{F}$ exists in M, it is unique.*

Proof. Suppose, contrariwise, the assertion does not hold. Then there exist two distinct fixed points $m, l \in M$ such that $\mathscr{F}(m) = m \neq l = \mathscr{F}(l)$. Consequently, we have $D(m,l) > 0$. Using the inequality (2.1), it follows that

$$0 \leq \tau(D(\mathscr{F}m, \mathscr{F}l), \varsigma(D(m,l)) < \varsigma(D(m,l)) - D(m,l).$$

In addition, the fact that $\varsigma(t) < t$ for any $t > 0$, proffers that

$$D(m,l) \leq \varsigma(D(m,l)) < D(m,l),$$

a contradiction. □

Theorem 2.1. *Let $\mathscr{F} : M \to M$ be a perturbed $\mathcal{S}_\tau$-contraction defined on a complete perturbed metric space (M, D, P). Then, $\mathscr{F}$ admits a unique fixed point $m^* \in M$. Moreover, for any initial point $m_0 \in M$, the constructed sequence $\{m_n\}$, described by $m_n = \mathscr{F}(m_{n-1})_{n\in\mathbb{N}}$, converges to m^*.*

Proof. It suffices to prove the existence of a fixed point for the mapping $\mathscr{F}$, as the uniqueness of the fixed point is ensured by Lemma 2.1. To illustrate the existence of a fixed point, we construct an iterative sequence beginning with an arbitrary point $m \in M$. We denote this initial point by $m_0 := m$ and define the sequence $\{m_n\}$ recursively by setting $m_n = \mathscr{F}m_{n-1}$ for all $n \in \mathbb{N}$.

It is assumed that no two consecutive terms in the sequence are equal. If, contrariwise, we assume that there exists some $k \in \mathbb{N}$ for which the terms k^{th} and $(k+1)^{th}$ coincide, then nothing remains to be proven. Indeed, we have $\mathscr{F}^k m = \mathscr{F}^{k-1} m$, which means $\mathscr{F}l = l$, where $l = \mathscr{F}^{k-1} m$, thus establishing the desired fixed point.

Ergo, from now on, presume that for every $n \in \mathbb{N}$, $\mathscr{F}^n m \neq \mathscr{F}^{n-1}m$. Keeping condition (2.1) in mind, we derive that

$$\begin{aligned} 0 &\leq \tau(D(\mathscr{F}^{n+1}m, \mathscr{F}^n m), \varsigma(D(\mathscr{F}^n m, \mathscr{F}^{n-1}m))) \\ &= \tau(D(\mathscr{F}\mathscr{F}^n m, \mathscr{F}\mathscr{F}^{n-1}m), \varsigma(D(\mathscr{F}^n m, \mathscr{F}^{n-1}m))) \\ &< \varsigma(D(\mathscr{F}^n m, \mathscr{F}^{n-1}m)) - D(\mathscr{F}^{n+1}m, \mathscr{F}^n m). \end{aligned}$$

Since the consecutive terms are distinct, the inequality above implies

$$D(\mathscr{F}^{n+1}m, \mathscr{F}^n m) \leq \varsigma(D(\mathscr{F}^n m, \mathscr{F}^{n-1}m)) < D(\mathscr{F}^n m, \mathscr{F}^{n-1}m)). \tag{2.2}$$

Recursively, one can get from the inequality (2.2) that

$$D(\mathscr{F}^{n+1}m, \mathscr{F}^n m) \leq \varsigma^{n-1}(D(\mathscr{F}^2 m, \mathscr{F}m)). \tag{2.3}$$

Due to (2.2), we deduce that $\{D(\mathscr{F}^n m, \mathscr{F}^{n-1}m)\}$ is monotonically decreasing. It has to be convergent since the sequence consists of nonnegative real numbers. Set

$$\lim_{n\to\infty} D(\mathscr{F}^n m, \mathscr{F}^{n+1}m) = \kappa \geq 0.$$

We claim that $\kappa = 0$. Suppose, contrariwise, that $\kappa > 0$. Since $\mathscr{F}$ is a perturbed $\mathcal{S}_\tau$-contraction, by applying restriction ($\tau 3$), we obtain

$$0 \leq \lim_{n\to\infty} \sup \tau\big(D(\mathscr{F}^{n+1}m, \mathscr{F}^n m),\ D(\mathscr{F}^n m, \mathscr{F}^{n-1}m)\big) < 0,$$

a contradiction. Inevitably, $\kappa = 0$, that is, $\lim_{n\to\infty} D(\mathscr{F}^n m, \mathscr{F}^{n+1}m) = 0$. Thus, the map $\mathscr{F}$ is asymptotically regular. We assert that the obtained sequence is Cauchy. Employing the triangle inequality, we shall find

$$\begin{aligned} d(\mathscr{F}^{n+k}m, \mathscr{F}^n m) &\leq d(\mathscr{F}^{n+k}m, \mathscr{F}^{n+k-1}m) + d(\mathscr{F}^{n+k-1}m, \mathscr{F}^n m) \\ &\leq d(\mathscr{F}^{n+k}m, \mathscr{F}^{n+k-1}m) + d(\mathscr{F}^{n+k-1}m, \mathscr{F}^{n+k-2}m) + d(\mathscr{F}^{n+k-2}m, \mathscr{F}^n m) \\ &\leq d(\mathscr{F}^{n+k}m, \mathscr{F}^{n+k-1}m) + d(\mathscr{F}^{n+k-1}m, \mathscr{F}^{n+k-2}m) + \cdots + d(\mathscr{F}^{n-1}m, \mathscr{F}^n m) \\ &\leq \varsigma^{n+k-2}(d(\mathscr{F}^2 m, \mathscr{F}m)) + \varsigma^{n+k-3}(d(\mathscr{F}^2 m, \mathscr{F}m)) + \cdots + \varsigma^{n-1}(d(\mathscr{F}^2 m, \mathscr{F}m)) \\ &\leq \sum_{i=1}^{n+k-2} \varsigma^i(d(\mathscr{F}^2 m, \mathscr{F}m)) \leq \sum_{i=1}^{\infty} \varsigma^i(d(\mathscr{F}^3 m, \mathscr{F}^2 m)). \end{aligned}$$

Hence, using the restriction (ς_2) and letting $n, k \to \infty$, it follows that the sequence $\{\mathscr{F}^n m\}$ is Cauchy. Keeping in mind that the perturbed metric space is complete, we deduce that it converges to m^*.

Consider the following inequality to demonstrate that m^* is the wanted fixed-point of $\{\mathscr{F}\}$ by presuming the contrary, (that is $\mathscr{F}m^* \neq m^*$ which is equivalent to $D(\mathscr{F}m^*, m^*) > 0$) such that

$$0 \leq \tau(D(\mathscr{F}m^*, \mathscr{F}^n m), \varsigma(D(m^*, \mathscr{F}^{n-1}m))) < \varsigma(D(m^*, \mathscr{F}^{n-1}m)) - D(\mathscr{F}m^*, \mathscr{F}^n m)$$

which turns into

$$D(\mathscr{F}m^*, \mathscr{F}^n m) \leq \varsigma(D(m^*, \mathscr{F}^{n-1}m)) < D(m^*, \mathscr{F}^{n-1}m).$$

As limit $n \to \infty$ in the subsequent inequality, we extrapolate

$$D(\mathscr{F}m^*, m^*) \leq D(m^*, m^*) = 0,$$

a contradiction. Ergo, $\mathscr{F}m^* = m^*$. □

Example 2.1. *Let $M = [1, \infty)$. We construct the function $P : M \times M \to [0, \infty)$ by setting*

$$P(m, n) = \frac{|m - n|^2}{1 + |m - n|}.$$

Let us suppose $\tau(b, a) = \frac{a}{2} - b$ and $\varsigma \in \Sigma$ be described as $\varsigma(t) = \frac{3}{5}t$ for all $t \in [0, \infty)$. We consider the mapping $\mathscr{F} : M \to M$ described as $\mathscr{F}(m) = \frac{m}{2} + \frac{1}{2}$. Also, we consider the mapping $D : M \times M \to [0, \infty)$ described as

$$D(m, n) = |m - n| + P(m, n),$$

where $P(m, n) = \frac{|m - n|^2}{1 + |m - n|}$. Observe that (M, D, P) is a perturbed metric space. In this case, the exact metric is the usual metric $d : M \times M \to [0, \infty)$ described as $d(m, n) = |m - n|$. Clearly, we have

$$\tau(D(\mathscr{F}m, \mathscr{F}n), \varsigma D(m, n)) \geq 0.$$

Remark also that D is not a metric on M. We have

$$D(1, 3) = \frac{10}{3}, D(3, 5) = \frac{10}{3} \text{ and } D(1, 5) = \frac{36}{5},$$

which shows that $D(1, 5) > D(1, 3) + D(3, 5)$.
Clearly, the mapping $\mathscr{F}$ meets all the requirements stated in Theorem 2.1. Its unique fixed point is $m = 1$.

Example 2.2. *Let $D : C([0, 1]) \times C([0, 1]) \to [0, \infty)$ be the mapping described as*

$$D(f, e) = \int_0^1 |f(t) - e(t)|dt + (f(0) - g(0))^2, \quad f, e \in C([0, 1]),$$

where $C([0, 1]) = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous on } [0, 1]\}$. In this setting, D constitutes a perturbed metric on $C([0, 1])$ corresponding to the perturbation map

$$P : C([0, 1]) \times C([0, 1]) \to [0, \infty)$$

given by

$$P(f, e) = (f(0) - g(0))^2, \quad f, e \in C([0, 1]).$$

Define mappings $\mathscr{F} : C([0, 1]) \to C([0, 1])$ as $\mathscr{F}f = \frac{f}{5}$ for all $f \in C([0, 1])$ and $\varsigma \in \Sigma$ be described as $\varsigma(t) = \frac{4t}{5}$ for all $t \in [0, \infty)$. It is an $\mathcal{S}_\tau$-contraction, where

$$\tau(b, a) = a - \sqrt{a} - b,$$

Indeed, if $f, e \in C([0, 1])$, then

$$\tau(D(\mathscr{F}f, \mathscr{F}g), \varsigma D(f, e)) \quad = \quad \tau(D\left(\frac{f}{2}, \frac{g}{2}\right), \frac{1}{2}D(f, e))$$

$$= \frac{\frac{1}{2}D(f,e) - D\left(\frac{f}{2}, \frac{g}{2}\right)}{2 + \frac{1}{2}D(f,e) + D\left(\frac{f}{2}, \frac{g}{2}\right)} \geq 0.$$

Ergo, $\mathscr{F}$ admits a unique fixed point $f \equiv 0 \in C([0,1])$, as all the restrictions of Theorem 2.1 are satisfied.

Example 2.3. *Let $D : C([0,1]) \times C([0,1]) \to [0,\infty)$ be the mapping described as*

$$D(f,e) = \int_0^1 |f(t) - e(t)|dt + (f(0) - g(0))^2, \quad f,e \in C([0,1]),$$

where $C([0,1]) = \{f : [0,1] \to \mathbb{R} : f$ is continuous on $[0,1]\}$. Then D is a perturbed metric on $C([0,1])$ with respect to the perturbed map

$$P : C([0,1]) \times C([0,1]) \to [0,\infty)$$

given by

$$P(f,e) = (f(0) - g(0))^2, \quad f,e \in C([0,1]).$$

Let us define the mappings $\mathscr{F} : C([0,1]) \to C([0,1])$, $\varsigma \in \Sigma$ and $\tau : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\mathscr{F}(f)(t) = \begin{cases} \frac{1}{4} f(t) & \text{if } t > 0, \\ \frac{1}{2} f(0) & \text{if } t = 0. \end{cases}$$

$\varsigma(t) = \frac{t}{2}$ *and* $\tau(a,b) = \frac{b}{2} - a$.

Clearly, the mapping $\mathscr{F}$ is a perturbed $\mathcal{S}_\tau$-contraction. That is,

$$\tau(D(\mathscr{F}f, \mathscr{F}g), \frac{1}{2}D(f,e)) \geq 0,$$

which shows that $\mathscr{F}$ is a perturbed $\mathcal{S}_\tau$-contraction. Clearly, all the restrictions of our Theorem 2.1 are satisfied. The fixed-point of the mapping $\mathscr{F}$ is function $f^* = 0$.

3. Conclusions

This work presents novel fixed-point results formulated in the structure of PMS, a recent and natural generalization of classical metric spaces. Our results not only generalize some well-known fixed-point theorems but also offer improvements and refinements over existing work. This shows the usefulness of perturbed metric spaces in broadening the scope of fixed-point theory and provides a strong foundation for future studies in this direction.

Perturbed metric spaces open up several promising avenues for future research. One direction involves extending the current results to multivalued, cyclic, and generalized contractions, which play a crucial role in nonlinear analysis. Another interesting line of inquiry is exploring fixed-point theorems under additional structures, such as ordered or partial metric spaces, within the perturbed setting. Moreover, the potential applications of perturbed metric spaces are quite rich. In applied mathematics, they could be useful in the analysis of iterative algorithms, especially those involving perturbations or noise, such as in numerical approximation methods.

In computer science, fixed-point results in perturbed metric spaces may find relevance in program semantics, where the convergence of recursive procedures under non-exact conditions is studied. Furthermore, optimization problems, particularly those involving tolerance levels or uncertain constraints, could benefit from this flexible structure. The study of equilibrium problems, game theory, and dynamic systems with small perturbations also offers fertile ground for applying this theory. Overall, the perturbed metric space structure provides a versatile and powerful tool that not only deepens our understanding of fixed-point theory but also connects it meaningfully to real-world problems in various disciplines.

Author contributions

All the authors contributed significantly to the writing of this article. The authors read and approved the final form of the manuscript. Conceptualization, GA,EK,PS; methodology, GA,EK,PS; writing—review and editing, GA,EK,PS; supervision, EK. All the authors contributed significantly to the writing of this article. The authors read and approved the final form of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used any Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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