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Research article

Survival analysis of newly extended Weibull data via adaptive progressive Type-II censoring and its modeling to Carbon fiber and electromigration

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Abstract: A new version of the three-parameter Weibull model, called the new extended Weibull (NEW), has been introduced in the literature to provide an increased or inverted bathtub failure rate. In a survival context, adaptive progressive Type-II censoring encourages the statistical inference's efficiency and minimizes the overall testing period during a lifetime experiment. By gathering a NEW sample from the proposed strategy, both likelihood and Bayesian inferential evaluations for the NEW model parameters of life were derived. For each unknown subject, asymptotic confidence intervals by normality and the log-transformed-normality approximations were constructed. Given independent uniform and gamma density priors against squared-error and general-entropy loss functions, Bayesian point and credible estimations were created utilizing several Monte-Carlo Markov-Chain techniques. To appreciate the usefulness of the acquired estimators, a comprehensive simulation analysis was performed by offering different experimental scenarios. To examine the superiority of the proposed model and to exhibit the viability of the proposed approaches in real practice, a pair of data examples, one based on the strength (in gigapascal) of carbon fibers and the other based on the failure times of conductors, were analyzed and the results were observed.

Keywords: new extended Weibull; survival analysis; adaptive progressive censoring; Markovian chain; Bayesian; data modeling
Mathematics Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05

1. Introduction

The Weibull distribution's versatility makes it a part of a collection of the most renowned lifespan models in reliability research. Although widely used in reliability analysis, the classical three-parameter Weibull distribution has inherent constraints when modeling complex failure behaviors. Specifically, it cannot adequately capture failure rates that exhibit both increasing and decreasing trends, particularly those that follow an upside-down bathtub shape. This limitation arises because the hazard rate function of the three-parameter Weibull model is strictly monotonic (either increasing, decreasing, or constant), making it unsuitable for systems where failure rates initially decrease due to early-life improvements and later increase due to aging effects. To overcome this limitation, several extensions of the Weibull distribution have been proposed in the literature, for example, the modified Weibull distribution by [1], the beta exponentiated Weibull distribution by [2], alpha power Weibull by [3] and [4], and recently the Q-Weibull distribution by [5].

Peng and Yan [6], introduced a new generalization of the Weibull distribution, which is referred to as the three-parameter new extended Weibull (NEW) distribution. Assume Y to be a random variable for the unit's lifetime that has NEW($\boldsymbol{\Delta}$) such that $\boldsymbol{\Delta} = (\mu, \theta, \sigma)^{\mathsf{T}}$ is the parameters vector. The additional term $e^{-\mu y^{-1}}$ makes the NEW distribution more flexible in modeling non-monotonic failure rates, including the upside-down bathtub shape. The respective cumulative distribution function (CDF) $F(\cdot)$, and probability density function (PDF) $f(\cdot)$ of the NEW distribution are given by

$$F(y;\boldsymbol{\Delta}) = 1 - e^{(-\sigma y^{\theta} e^{-\mu y^{-1}})},\ y > 0, \tag{1.1}$$

and

$$f(y;\boldsymbol{\Delta}) = \sigma(\mu + \theta y) y^{\theta-2} e^{(-\mu y^{-1} - \sigma y^{\theta} e^{-\mu y^{-1}})}, \tag{1.2}$$

where $\theta > 0$ and $\mu \geqslant 0$ control the shape of the NEW distribution while the parameter $\sigma > 0$ controls its scale. At a certain time $t > 0$, the reliability function (RF, say $R(\cdot)$) and the hazard rate function (HRF, say $h(\cdot)$), are

$$R(t;\boldsymbol{\Delta}) = \exp(-\sigma t^{\theta} e^{-\mu t^{-1}}),\ \ t > 0, \tag{1.3}$$

and

$$h(t;\boldsymbol{\Delta}) = \sigma(\mu + \theta t) t^{\theta-2} e^{-\mu t^{-1}}. \tag{1.4}$$

respectively. From (1.1) several sub-models can be derived as special members, such as: (i) Weibull (at $\mu = 0$), (ii) Rayleigh (at $\mu = 0$ and $\theta = 2$), and (iii) exponential (at $\mu = 0$ and $\theta = 1$) distributions. Using $\sigma = 1$ and some choices of μ and θ, Figure 1 displays several shapes of (1.2) and (1.4) functions. Figure 1 shows that the density of the NEW model can be symmetric with right (or left) skewness and that its HRF has an increasing or upside-down bathtub curve.

Many studies in the literature have considered this model in different scenarios; for example, Jia et al. [7] introduced the discrete NEW model; Azizi et al. [8] analyzed the bivariate NEW model from complete and censored data; and Jovanović et al. [9] discussed the reliability estimation problem of stress-strength of the NEW model. Peng and Yan [6] discussed various estimations of the NEW parameters. They also stated that the shape of NEW's hazard rate is quite simple and depends only on

a single parameter, whereas the shape of the hazard rate of the exponentiated Weibull and Marshall-Olkin extended Weibull distributions, suggested by Mudholkar and Srivastava [10] and Marshall and Olkin [11], respectively, has two parameters and produces a more complicated distribution.

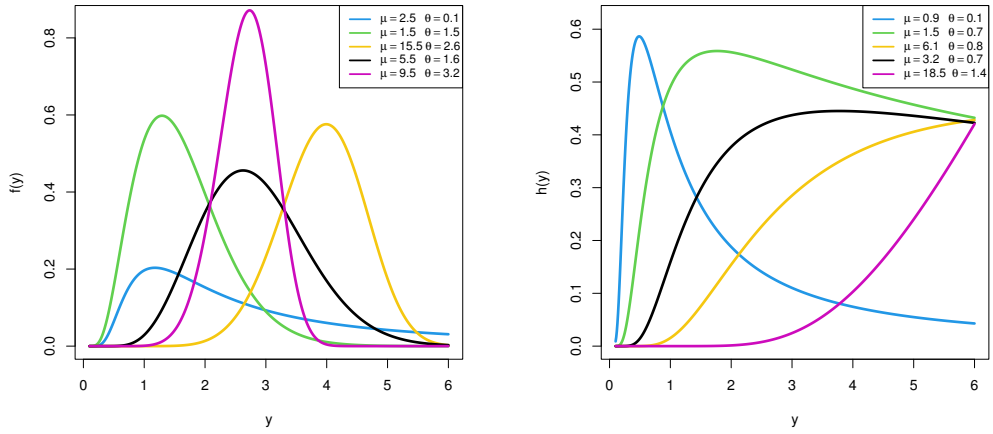


Figure 1. The PDF (left) and HRF (right) shapes of the NEW distribution.

Censoring mechanisms, from the point of view of cost, time, or both constraints, are beneficial for use in the statistical reliability analysis of a produced subject, such as a system or device. Type-I censoring has a predetermined time, while Type-II censoring has predetermined failure units. The progressive censoring scheme has been studied by many researchers, see [12–15], which allows the removal of units at different stages of an experiment. An extension version of the progressive censoring scheme, presented by Ng et al. [16], is called the adaptive progressively Type II censoring (A-PT2-C). This scheme ensures the ability to balance the total test time, test cost, and efficient sample size. However, as noted by Ng et al. [16], this censoring effectively estimates the parameters when the entire test time is unimportant. This mechanism ensures that the experimenter stops the experiment once the necessary units to fail k are observed and the total time of the test does not exceed the specified time (T). As a result, the A-PT2-C mechanism increases the accuracy of the acquired statistical inference, ensures that the experimenter controls the life test when he desires to obtain the required number of failures, and strikes a balance between the total test time, test cost, and effective sample size. This makes A-PT2-C particularly useful in reliability studies where long test durations may be impractical or costly. In addition, this mechanism dynamically adjusts the removal of surviving units based on observed failure times, thereby improving the precision of parameter estimation.

Suppose T is the threshold time point, k is the effective sample size, and $\mathbf{R} = (R_1, R_2, \ldots, R_k)$ is the progressive censoring. Let $Y_{i:k:n}$, $i = 1, \ldots, k$ be the time of the ith failed item. At the time $Y_{1:k:n}$ occurs, R_1 of $(n-1)$ live units are selected randomly and dropped out of the test. Next, when $Y_{2:k:n}$ occurs, $n - R_1 - 2$ survival units are randomly removed, and so on. However, if $Y_{k:k:n} < T$, the experiment stops at $Y_{k:k:n}$; otherwise, the experiment stops at T. The adaptation suggested by Ng et al. [16], when $Y_{d:k:n} < T < Y_{d+1:k:n} < Y_{k:k:n}$, instructs the researcher to stop removing any living units by setting $R_i = 0$ for $i = d+1, d+2, \ldots, k-1$, and thus the number of staying items is given by $R_k = n - k - \sum_{i=1}^{d} R_i$.

Figure 2 illustrates the censoring framework under the A-PT2-C mechanism.

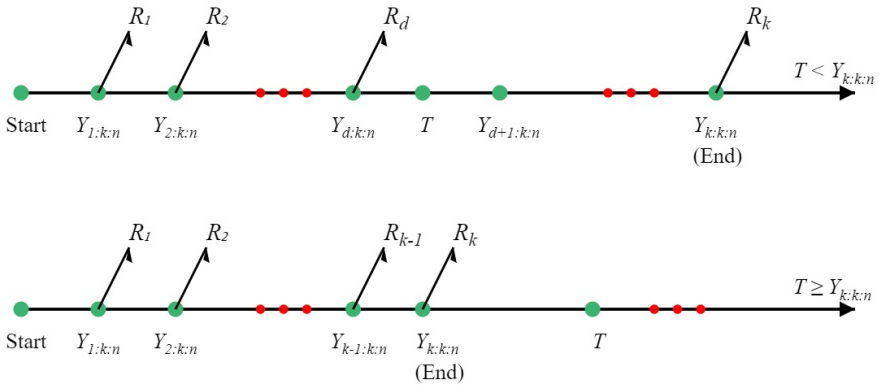


Figure 2. Illustration of A-PT2-C.

For various lifetime models, several authors have investigated the proposed strategy; see, for example, Lin et al. [17], Nassar and Abo-Kasem [18], Mohie El-Din et al. [19], Liu and Gui [20], Elshahhat and Nassar [21], Chen and Gui [22], Elshahhat et al. [23], Mohammed et al. [24], Alotaibi et al. [25], among others. Although the NEW distribution could be highly useful in reliability analysis as its hazard curves are increasing or upside-down bath-tubed, no work has yet studied the frequentist (or Bayes) estimation of μ, θ, σ, $R(t)$, or $h(t)$ in the presence of A-PT2-C data. As a result, the motivation for this work emerged from: (i) the NEW distribution's possibility to model a wide range of data with varying failure rates; (ii) the A-PT2-C's ability to increase the accuracy of the offered estimators; and (iii) several authors are studying the estimation of all unknown subjects. Before proceeding, and for distinction, Figure 3 shows a flowchart of all the estimation methodologies proposed in this work.

So, to close this gap, the main point of this study is fourfold:

- Derive the maximum likelihood (ML) and Bayes estimators of μ, θ, σ, $R(t)$, and $h(t)$.
- Implement Markov Chain Monte Carlo (MCMC) techniques with the square error loss (SEL) and the asymmetric general-entropy loss (GEL), to approximate the acquired Bayes estimators using independent gamma priors.
- Construct asymptotic confidence intervals (ACIs) of all unknown quantities via two normality approximations, namely: (i) the asymptotic distribution of ML estimators; and (ii) the asymptotic distribution of log-ML estimators. Create Bayes credible intervals (BCIs) and highest posterior density (HPD) intervals depending on the MCMC of μ, θ, σ, $h(t)$, and $R(t)$.
- Compare the proposed methodologies via extensive Monte Carlo simulations and report some general recommendations. Offering the usefulness of the NEW model and the applicability of the proposed methods by studying two different sets of real examples from the engineering field.

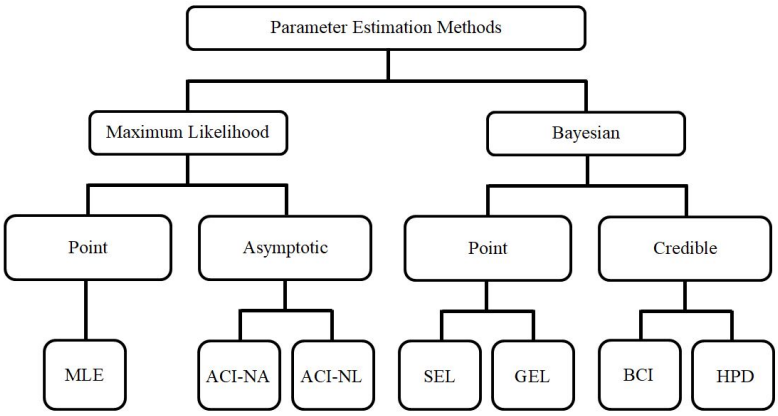


Figure 3. Flowchart of the proposed estimation approaches.

The remaining parts are presented as follows: The ML and Bayesian estimators are studied in Sections 2 and 3, respectively. In Section 4, simulation outputs are emphasized. Section 5 examines data applications. Conclusions are observed in Section 6.

2. Classical inference

One of the most common frequentist methods of estimation, named ML, is considered to obtain point and interval estimators of μ, θ, σ, $R(t)$, and $h(t)$.

2.1. ML estimator

Suppose $\{\mathbf{y}, \mathbf{R}\} = \{(y_{1:k:n}, R_1), \ldots, (y_{d:k:n}, R_d), T, (y_{d+1:k:n}, 0), \ldots, (y_{k-1:k:n}, 0), (y_{k:k:n}, R_k)\}$ is an A-PT2-C sample with a continuous distribution, hence, the joint likelihood function of this data type is

$$L(\boldsymbol{\Delta}|\mathbf{y}) = C \prod_{i=1}^{k} f(y_i, \boldsymbol{\Delta}) \prod_{i=1}^{d} [1 - F(y_i, \boldsymbol{\Delta})]^{R_i} [1 - F(y_k, \boldsymbol{\Delta})]^{R_k}, \tag{2.1}$$

where y_i is used for simplicity and $C = \prod_{i=1}^{k} \left[n - i + 1 - \sum_{j=1}^{\max\{i-1,d\}} R_j\right]$.

Assume the lifetimes follow the new distribution with CDF and PDF as written by Eqs (1.1) and (1.2) respectively. Then, the likelihood function can be expressed after ignoring any constant term as

$$L(\boldsymbol{\Delta}|\,\mathbf{y}) \propto \sigma^k \exp\left(-\left[\sigma\xi(\mathbf{y};\mu,\theta) + \mu \sum_{i=1}^{k} y_i^{-1}\right]\right) \prod_{i=1}^{k} (\mu + \theta y_i) y_i^{\theta}, \tag{2.2}$$

where $\xi(\mathbf{y};\mu,\theta) = \sum_{i=1}^{k} y_i^{\theta} e^{-\mu y_i^{-1}} + \sum_{i=1}^{d} R_i y_i^{\theta} e^{-\mu y_i^{-1}} + R_k y_k^{\theta} e^{-\mu y_k^{-1}}$.

Equivalently, the log-likelihood function ($l(\cdot) = \log L(\cdot)$) is expressed as

$$l(\boldsymbol{\Delta}|\,\mathbf{y}) \propto k \log(\sigma) - \sigma\xi(\mathbf{y};\theta,\mu) - \mu \sum_{i=1}^{k} y_i^{-1} + \theta \sum_{i=1}^{k} y_i + \sum_{i=1}^{k} \log(\mu + \theta y_i). \tag{2.3}$$

The ML estimators (MLEs) of μ, θ and σ which are referred to as $\hat{\mu}$, $\hat{\theta}$ and $\hat{\sigma}$, respectively, are derived by obtaining the following first (partial) derivatives:

$$\frac{\partial l}{\partial \mu} = -\sigma\xi_{\mu}^{*} - \sum_{i=1}^{k} y_i^{-1} + \sum_{i=1}^{k} (\mu + \theta y_i)^{-1}, \tag{2.4}$$

$$\frac{\partial l}{\partial \theta} = -\sigma \xi_{\theta}^{*} + \sum\nolimits_{i=1}^{k} y_i \left[1 + (\mu + \theta y_i)^{-1}\right], \tag{2.5}$$

and

$$\frac{\partial l}{\partial \sigma} = \sigma^{-1} k - \xi(\mathbf{y}; \mu, \theta),$$

respectively, where
$\xi_{\mu}^{*} = -\sum_{i=1}^{k} y_i^{\theta-1} e^{-\mu y_i^{-1}} - \sum_{i=1}^{d} R_i y_i^{\theta-1} e^{-\mu y_i^{-1}} - R_k y_k^{\theta-1} e^{-\mu y_k^{-1}}$
and
$\xi_{\theta}^{*} = \sum_{i=1}^{k} y_i^{\theta} \log(y_i) e^{-\mu y_i^{-1}} + \sum_{i=1}^{d} R_i y_i^{\theta} \log(y_i) e^{-\mu y_i^{-1}} + R_k y_k^{\theta} \log(y_k) e^{-\mu y_k^{-1}}$.

It is obvious, from Eqs (2.4) and (2.5), that the system of nonlinear equations requires a numerical iterative technique to solve it and evaluate the desired estimators. Numerically, with R(4.2.2) software language, and the 'maxLik' command is employed. For more details, one may refer to Henningsen and Toomet [26] which implements the Newton-Raphson method in ml maximization.

Remark 1: Using (2.2), we've extended related works in the literature and they are obtained as special cases, such as:

- Peng and Yan [6]'s results, in the case of NEW distribution from PT2-C data, by setting $T \to \infty$.
- Ng et al. [16]'s results, in the case of exponential distribution from A-PT2-C data, by setting $(\mu, \theta) = (0, 1)$.
- Lin et al. [17]'s results, in the case of Weibull distribution from A-PT2-C data, by setting $\mu = 0$.
- Wu [27]'s results, in the case of Weibull distribution from PT2-C data, by setting $\mu = 0$ and $T \to \infty$.

Once $\hat{\mu}$, $\hat{\theta}$, and $\hat{\sigma}$ are evaluated, following the invariance property, the MLE for $R(t)$ along with $h(t)$ are obtained from (1.3) and (1.4) so that

$$\hat{R}(t) = \exp(-\hat{\sigma} t^{\hat{\theta}} e^{-\hat{\mu} t^{-1}}) \quad \text{and} \quad \hat{h}(t) = \hat{\sigma}(\hat{\mu} + \hat{\theta} t) t^{\hat{\theta}-2} e^{-\hat{\mu} t^{-1}},$$

respectively, for $t > 0$.

2.2. Approximate interval estimator

In this subsection, the asymptotic normality property of the MLE and the log-transformed-MLE for μ, θ, σ, $R(t)$, and $h(t)$ are employed to create the associated bounds of ACIs. First, we need to derive the estimate of the variance of each MLE. It is known, from the general large sample theory, that the MLEs $\hat{\mu}$, $\hat{\theta}$, and $\hat{\sigma}$ are normally approximate (NA) distributed, $\hat{\Delta} \sim N(\Delta, \mathbf{I}^{-1}(\Delta))$, with mean Δ and $\mathbf{I}^{-1}(\cdot)$ as the variance-covariance matrix. Taking second-partially derivatives of (2.3), locally at $\Delta = \hat{\Delta}$, the Fisher components l_{ij}, i and $j = 1, 2, 3$ are listed in Appendix 1. So, the estimated asymptotic variances and covariances of $\hat{\mu}$, $\hat{\theta}$, and $\hat{\sigma}$ are given by

$$\mathbf{I}^{-1}(\hat{\Delta}) \cong \begin{bmatrix} -l_{11} & -l_{12} & -l_{13} \\ & -l_{22} & -l_{23} \\ & & -l_{33} \end{bmatrix}^{-1}_{(\Delta=\hat{\Delta})} = \begin{bmatrix} \hat{v}_{11} & \hat{v}_{12} & \hat{v}_{13} \\ & \hat{v}_{22} & \hat{v}_{23} \\ & & \hat{v}_{33} \end{bmatrix}. \tag{2.6}$$

Thus, the $100(1-\epsilon)\%$ ACIs based on the normal approximation (say ACI-NA) of the MLEs $\hat{\mu}$, $\hat{\theta}$ and $\hat{\sigma}$ can be constructed, respectively, as

$$\left(\hat{\mu} \mp z_{\frac{\epsilon}{2}} \sqrt{\hat{v}_{11}}\right), \left(\hat{\theta} \mp z_{\frac{\epsilon}{2}} \sqrt{\hat{v}_{22}}\right), \quad \text{and} \quad \left(\hat{\sigma} \mp z_{\frac{\epsilon}{2}} \sqrt{\hat{v}_{33}}\right),$$

where $z_{\epsilon/2}$, with upper probability $\epsilon/2$, is the standard normal distribution's percentile.

On the other side, in terms of obtaining ACIs for $R(t)$ and $h(t)$, the delta method is used to estimate the variances of their MLEs $\hat{R}(t)$ and $\hat{h}(t)$. However, the approximate variances for $\widehat{var}(\hat{R}(t))$ and $\widehat{var}(\hat{h}(t))$ are expressed as

$$\widehat{var}(\hat{R}(t)) \approx [\Sigma_R \mathbf{I}^{-1}(\hat{\Delta})\Sigma_R^{\top}] \quad \text{and} \quad \widehat{var}(\hat{h}(t)) \approx [\Sigma_h \mathbf{I}^{-1}(\hat{\Delta})\Sigma_h^{\top}],$$

respectively. To establish this goal, we need to obtain Σ_R and Σ_h as

$$\Sigma_R = \left[\frac{\partial R}{\partial \mu}, \frac{\partial R}{\partial \theta}, \frac{\partial R}{\partial \sigma}\right]_{(\hat{\mu},\hat{\theta},\hat{\sigma})} \quad \text{and} \quad \Sigma_h = \left[\frac{\partial h}{\partial \mu}, \frac{\partial h}{\partial \theta}, \frac{\partial h}{\partial \sigma}\right]_{(\hat{\mu},\hat{\theta},\hat{\sigma})},$$

such as
$\frac{\partial R}{\partial \mu} = \sigma t^{\theta-1} e^{-\mu t^{-1}} R(t;\mathbf{\Delta})$, $\frac{\partial R}{\partial \theta} = -\sigma t^{\theta} \log(t) e^{-\mu t^{-1}} R(t;\mathbf{\Delta})$, $\frac{\partial R}{\partial \sigma} = t^{\theta} e^{-\mu t^{-1}} R(t;\mathbf{\Delta})$,
$\frac{\partial h}{\partial \mu} = \sigma t^{\theta-2}\left(1 - t^{-1}(\mu + \theta t)\right) e^{-\mu t^{-1}}$, $\frac{\partial h}{\partial \theta} = \sigma t^{\theta-1} e^{-\mu t^{-1}}\left(1 + (\mu + \theta t) t^{-1} \log(t)\right)$, and $\frac{\partial h}{\partial \sigma} = \sigma^{-1} h(t;\mathbf{\Delta})$,
where $R(t;\mathbf{\Delta})$ and $h(t;\mathbf{\Delta})$ are available in (1.3) and (1.4), respectively.

Hence, the $(1-\epsilon)100\%$ ACIs for $R(t)$ and $h(t)$ are created, respectively, as

$$\hat{R}(t) \mp z_{\frac{\epsilon}{2}} \sqrt{\widehat{var}(\hat{R}(t))}\ , \ \hat{h}(t) \mp z_{\frac{\epsilon}{2}} \sqrt{\widehat{var}(\hat{h}(t))}.$$

Practically, the main weakness of the conventional ACI-NA is that it sometimes produces a negative lower bound for a parameter that supports positive values. To handle this drawback, instead of replacing a negative value with zero, Meeker and Escobar [28] proposed the ACI based on the normally log-transformation-based (NL) of the MLE (say ACI-NL) for unknown parameter(s) that take positive values. Comparing the ACI-NA and ACI-NL methods, they also mentioned that the ACI has a higher coverage probability based on the former than the latter. On the other hand, the log transformation reduces the skewness often present in the sampling distribution of MLEs, making the normal approximation more accurate for finite samples. Hence, the $100(1-\epsilon)\%$ ACI-NL of μ, θ and σ are given by

$$\exp\left(\log(\hat{\mu}) \mp z_{\frac{\epsilon}{2}} \frac{\sqrt{\hat{v}_{11}}}{\hat{\mu}}\right), \exp\left(\log(\hat{\theta}) \mp z_{\frac{\epsilon}{2}} \frac{\sqrt{\hat{v}_{22}}}{\hat{\theta}}\right), \quad \text{and} \quad \exp\left(\log(\hat{\sigma}) \mp z_{\frac{\epsilon}{2}} \frac{\sqrt{\hat{v}_{33}}}{\hat{\sigma}}\right),$$

respectively. Similarly, the $(1-\epsilon)100\%$ two-sided intervals for $h(t)$ and $R(t)$ are created as

$$\exp\left(\log(\hat{R}(t)) \mp z_{\frac{\epsilon}{2}} \frac{\sqrt{\widehat{var}(\hat{R}(t))}}{\hat{R}(t)}\right) \quad \text{and} \quad \exp\left(\log(\hat{h}(t)) \mp z_{\frac{\epsilon}{2}} \frac{\sqrt{\widehat{var}(\hat{h}(t))}}{\hat{h}(t)}\right),$$

respectively.

3. Bayesian inference

The Bayesian estimations of μ, θ, σ, $h(t)$, $R(t)$ are provided in this section, concerning symmetric and asymmetric loss functions. First, we suppose μ, θ, and σ are stochastically independent to each other.

3.1. Prior and loss functions

Following Peng and Yan [6], the prior distributions of μ and θ are the uniform distributions, denoted by $\eta_1(\cdot)$ and $\eta_2(\cdot)$, respectively, as

$$\eta_1(\mu;\varepsilon_1) \propto \varepsilon_1^{-1}, \qquad \varepsilon_1 > \mu > 0, \tag{3.1}$$

$$\eta_2(\theta;\varepsilon_2) \propto \varepsilon_2^{-1}, \qquad \varepsilon_2 > \theta > 0, \tag{3.2}$$

and the natural PDF of σ is the gamma, say $\mathcal{G}(\sigma; a_1, a_2)$, density as

$$\mathcal{G}(\sigma; a_1, a_2) \propto \sigma^{a_1-1}e^{-a_2\sigma}, \qquad \sigma > 0, \tag{3.3}$$

where the hyperparameters ε_i and $a_i > 0$ for $i = 1, 2$ are selected to give prior information about μ, θ, and σ. It is important to remember that (i) the gamma density has been utilized to adapt support for σ because it is straightforward, concise, and flexible; (ii) there is no information about μ and θ, so less information will be available regarding these parameters. It has also been chosen due to its conjugacy properties, which simplify posterior computation and ensure analytical tractability. Since any suggested values of ε_1 and ε_2 do not affect the Bayesian estimates, to avoid computational dilemmas, Peng and Yan [6] used $\varepsilon_i = 1$ for $i = 1, 2$.

Consequently, the joint prior PDF ($\Pi(\cdot)$) for μ, θ, and σ is

$$\begin{aligned}\Pi(\boldsymbol{\Delta}) &\propto \eta_1(\mu;\varepsilon_1) \times \eta_2(\theta;\varepsilon_2) \times \mathcal{G}(\sigma; a_1, a_2)\\ &\propto \varepsilon_1^{-1}\varepsilon_2^{-1}\sigma^{a_1-1}e^{-a_2\sigma}, \quad \mu, \theta, \sigma > 0.\end{aligned} \tag{3.4}$$

A major issue in Bayesian estimation is determining the loss function since it may explore overestimation and underestimation of the study. One of the most commonly seen symmetric losses in Bayesian theory is the SEL function, which posits the overestimation and underestimation to be handled evenly because it is simple to implement and very simple for inferential concerns. However, the SEL (say $\mathcal{L}_S$) and its Bayes estimator (say $\tilde{\rho}_S(\cdot)$ of $\rho(\cdot)$), where $\rho(\cdot)$ denotes a parametric function of μ, θ and σ, are given by

$$\mathcal{L}_S(\rho(\Theta), \tilde{\rho}(\Theta)) = (\tilde{\rho}(\Theta) - \rho(\Theta))^2, \tag{3.5}$$

and

$$\tilde{\rho}_S(\Theta) = \int_\mu \int_\theta \int_\sigma \rho(\Theta)\Pi(\boldsymbol{\Delta})L(\boldsymbol{\Delta}|\,\mathbf{y})\,\mathrm{d}\sigma\mathrm{d}\theta\mathrm{d}\mu,$$

respectively, for more information, refer to Martz and Waller [29].

Additionally, GEL is one of the most generally asymmetric loss functions. The main feature, if one treats the GEL function, is that it has diverse importance for overestimation and underestimation. However, the GEL (say $\mathcal{L}_G$) and its Bayes estimator (say $\tilde{\rho}_G(\cdot)$ of $\rho(\cdot)$), respectively, are given by

$$\mathcal{L}_G(\rho(\Theta), \tilde{\rho}(\Theta)) \propto \left(\frac{\tilde{\rho}(\Theta)}{\rho(\Theta)}\right)^{\nu} - \nu \log\left(\frac{\tilde{\rho}(\Theta)}{\rho(\Theta)}\right) - 1, \ \nu \neq 0. \tag{3.6}$$

and

$$\tilde{\rho}_G(\Theta) = \left[\int_{\mu}\int_{\theta}\int_{\sigma} (\rho(\Theta))^{-\nu}\, \Pi(\boldsymbol{\Delta}) L(\boldsymbol{\Delta}|\,\mathbf{y})\, \mathrm{d}\sigma \mathrm{d}\theta \mathrm{d}\mu\right]^{-1/\nu},$$

for further details, refer to Dey et al. [30]. Without loss of generality, we've utilized two well-known functions, SEL and GEL. Additionally, one can easily incorporate other losses.

3.2. Posterior distribution

The joint posterior density, say $\Omega(\cdot)$, is given by

$$\Omega(\boldsymbol{\Delta}|\,\mathbf{y}) = \frac{\Pi(\boldsymbol{\Delta}) L(\boldsymbol{\Delta}|\,\mathbf{y})}{\int \Pi(\boldsymbol{\Delta}) L(\boldsymbol{\Delta}|\,\mathbf{y})\, d\Delta}. \tag{3.7}$$

Now, from (2.2) and (3.4), joint posterior PDF, ($\Omega(\cdot)$), for μ, θ and σ is

$$\Omega(\boldsymbol{\Delta}|\,\mathbf{y}) = \mathcal{D}^{-1}\sigma^{k+a_1-1} \exp\left(-\left[\sigma(a_2 + \xi(\mathbf{y};\mu,\theta)) + \mu\sum\nolimits_{i=1}^{k} y_i^{-1}\right]\right)\prod\nolimits_{i=1}^{k}(\mu+\theta y_i)y_i^{\theta}, \tag{3.8}$$

where

$$\mathcal{D} = \int_{\mu}\int_{\theta}\int_{\sigma} \sigma^{k+a_1-1} \exp\left(-\left[\sigma(a_2 + \xi(\mathbf{y};\mu,\theta)) + \mu\sum\nolimits_{i=1}^{k} y_i^{-1}\right]\right)\prod\nolimits_{i=1}^{k}(\mu+\theta y_i)y_i^{\theta}\mathrm{d}\sigma\mathrm{d}\theta\mathrm{d}\mu.$$

The analytical solution for the Bayes estimations using SEL and GEL functions for μ, θ, σ, $R(t)$, or $h(t)$ is not available. To solve this problem, we propose to consider the Gibbs sampler, which produces a Markov chain, whose constant density is the objective posterior density, and to approximate Bayes estimates, see Geman and Geman [31]. Thus, after a finite number of iterations, we're guaranteed that the sequential values for the Markov chain are converging to the desired posterior.

Now, to draw a sample from (3.8) via the Gibbs sampling procedure, the conditional PDF of each unknown parameter given the data must first be obtained.

Remark 2: The conditional PDF of σ, denoted by $\sigma|\mathbf{y},\mu,\theta$, has gamma density as

$$\sigma|\mathbf{y},\mu,\theta \sim \mathcal{G}(\sigma; k + a_1, a_2 + \xi(\mathbf{y};\mu,\theta)), \tag{3.9}$$

where $(k + a_1)$ is the shape parameter and $(a_2 + \xi(\mathbf{y};\mu,\theta))$ is the scale parameter.

Remark 3: The conditional PDF of μ given the data $\mathbf{y}$, say $\mathcal{H}_{\mu}^{*}(\cdot)$, is a log-concave function as

$$\mathcal{H}_{\mu}^{*}(\mu|\mathbf{y},\theta,\sigma) \propto \exp\left(-\left[\sigma\xi(\mathbf{y};\mu,\theta) + \mu\sum\nolimits_{i=1}^{k} y_i^{-1}\right]\right)\prod_{i=1}^{k}(\mu+\theta y_i). \tag{3.10}$$

Similarly, the conditional density of θ given the data $\mathbf{y}$, say $\mathcal{H}^*_\theta(\cdot)$, is also a log-concave function as

$$\mathcal{H}^*_\theta(\theta|\mathbf{y},\mu,\sigma) \propto \exp\left(-\sigma\xi(\mathbf{y};\mu,\theta)\right)\prod_{i=1}^{k}(\mu+\theta y_i)y_i^{\theta}. \tag{3.11}$$

Proof. See Appendix 2.

As a result, to develop the Bayes estimates of σ, we directly generate samples of σ from (3.9) via any gamma simulator. It is also noticeable, from (3.10) and (3.11), that the conditional PDFs for μ and θ can't be tended mathematically to which standard model. For this purpose, when the posterior PDF is complex and its evaluation is computationally demanding, Gilks et al. [32] proposed the adaptive rejection Metropolis sampling (ARMS) strategy. We thus employ the ARMS to adopt the Bayes estimates of μ and θ. Other sampling techniques to draw MCMC samples from the log-concave density, e.g., Devroye's algorithm (by Devroye [33]) Metropolis-Hastings algorithm (by Metropolis et al. [34]), or adaptive rejection algorithm (by Gilks and Wild [35]), can be easily incorporated; for additional details see Gelman et al. [36].

3.3. Adaptive rejection Metropolis sampling

To collect samples from the posterior PDFs of μ, θ and σ, we present the following procedure:

Step 1. Set starting values of (μ,θ,σ), say $(\mu^0,\theta^0,\sigma^0)$.

Step 2. Let $j=1$.

Step 3. Create σ^j from $\mathcal{G}(\sigma;k+a_1,a_2+\xi(\mathbf{y};\mu^{j-1},\theta^{j-1}))$.

Step 4. Generate μ^j by the ARMS algorithm from $\mathcal{H}^*_\mu\left(\mu|\mathbf{y},\theta^{j-1},\sigma^j\right)$.

Step 5. Generate θ^j by the ARMS algorithm from $\mathcal{H}^*_\theta\left(\theta|\mathbf{y},\mu^j,\sigma^j\right)$.

Step 6. Replace μ, θ and σ in (1.3) and (1.4) by their μ^j, θ^j and σ^j to calculate $R^j(t)$ and $h^j(t)$ for $t>0$.

Step 7. Assume $j=j+1$.

Step 8. Redo the steps from 3 to 7, $\mathcal{B}$ number of times, and get μ^j, θ^j, σ^j, $R^j(t)$, and $h^j(t)$ with $j=1,2,\ldots,\mathcal{B}$. After that, eliminate first $\mathcal{B}^\circ$ draws as a burn in.

Step 9. Find the Bayesian estimators for μ, θ, σ, $R(t)$, or $h(t)$, which is abbreviated in the following formula by δ, relative to the SEL and GEL functions, so that

$$\tilde{\delta}_S = \frac{1}{\mathcal{B}-\mathcal{B}^\circ}\sum_{j=\mathcal{B}^\circ+1}^{M}\delta^j,$$

$$\tilde{\delta}_G = \left[\frac{1}{\mathcal{B}-\mathcal{B}^\circ}\sum_{j=\mathcal{B}^\circ+1}^{\mathcal{B}}\left(\delta^j\right)^{-\nu}\right]^{-1/\nu},\ \nu\neq 0,$$

respectively.

Step 10. Create the BCI of δ by arranging the MCMC samples in an ascending way so that $\delta_{(\mathcal{B}^\circ+1)},\delta_{(\mathcal{B}^\circ+2)},\ldots,\delta_{(\mathcal{B})}$. Hence, the $100(1-\epsilon)\%$ BCI estimator for δ is

$$\left(\delta_{(\mathcal{B}-\mathcal{B}^\circ)(\epsilon/2)},\delta_{(\mathcal{B}-\mathcal{B}^\circ)(1-(\epsilon/2))}\right).$$

Step 11. Construct the HPD interval for δ by arranging the MCMC samples in an ascending way as $\delta_{(\mathcal{B}^\circ+1)}, \delta_{(\mathcal{B}^\circ+2)}, \ldots, \delta_{(\mathcal{B})}$. Next, the $(1-\epsilon)100\%$ interval estimation for δ can be given as

$$(\delta_{(j^*)}, \delta_{(j^*+(1-\epsilon)(\mathcal{B}-\mathcal{B}^\circ))}),$$

such that

$$\delta_{(j^*+[(1-\epsilon)(\mathcal{B}-\mathcal{B}^\circ)])} - \delta_{(j^*)} = \min_{1\leqslant j\leqslant \epsilon(\mathcal{B}-\mathcal{B}^\circ)} (\delta_{(j+[(1-\epsilon)(\mathcal{B}-\mathcal{B}^\circ)])} - \delta_j),$$

and $[q]$ stands for the greatest value equal or less than to q, for additional information, see Chen and Shao [37].

4. Monte Carlo comparisons

For evaluating the behavior of the acquired point and interval estimators (from both classical and Bayesian setups) of μ, θ, σ, $R(t)$, and $h(t)$, extensive Monte Carlo simulations are implemented. To achieve this goal, depends on different options of T, which represents the threshold time. Let n be the number of the experimental units, k be the size of censored units, and $\mathbf{R}$ be the progressive pattern, also suppose that large 1,000 A-PT2-C samples are collected from the NEW$(0.4, 0.8, 0.2)$ distribution. Taking $t = 0.1$, the plausible values of $R(t)$ and $h(t)$ are selected to be 0.87453 and 0.16087, respectively. All theoretical results are examined using various combinations of T, n, k, and $\mathbf{R}$ so that $n = (40,\ 80)$, $T = (2,\ 5)$, and k is specified as a percentage failure for every n as FP%$=\frac{n}{k} \times 100\%$, namely: FP%=50 and 75%. To highlight the performance of the removal pattern, different fashions of the progressive censoring $(R_1, R_2, \ldots, R_k)$ are also adopted, namely:

$$\begin{aligned}
&\text{Scheme-1} : (n-k, 0^{k-1});\\
&\text{Scheme-2} : (0^{\frac{k}{2}-1}, n-k, 0^{\frac{k}{2}});\\
&\text{and}\\
&\text{Scheme-3} : (0^{k-1}, n-k),
\end{aligned}$$

where 0^{k-1} (for example) means 0 repeated $k-1$ times.

To get a A-PT2-C sample from the NEW distribution, after assigning values of T, n, k, and $R_i,\ i = 1, 2, \ldots, k$, do the next steps:

Step 1: Simulate a usual PT2-C sample as:

i. Obtain k independent observations called $\psi_1, \psi_2, \ldots, \psi_k$.
ii. Set $q_i = \psi_i^{\left(i+\sum_{j=k-i+1}^{k} R_j\right)^{-1}},\ i = 1, 2, \ldots, k$.
iii. Suppose $u_i = 1 - (q_k q_{k-1} \cdots q_{k-i+1})$ with $i = 1, 2, \ldots, k$. Thus, $u_i,\ i = 1, 2, \ldots, k$ is the PT2-C sample with size k following $U(0, 1)$ distribution.
iv. Set $Y_i = F^{-1}(u_i; \mu, \theta, \sigma),\ i = 1, 2, \ldots, k$, as the PT2-C sample with NEW(μ, θ, σ).

Step 2: Obtain d at T, where $Y_d < T < Y_{d+1}$, and discard the staying sample $Y_i,\ i = d+2, \ldots, k$.

Step 3: From $[1 - F(y_{d+1})]^{-1} f(y)$, obtain $Y_{d+2}, \ldots, Y_k$.

To calculate the proposed Bayes objectives, with the Gibbs technique discussed in Section 3, the first 2,000 variates out of a total of 12,000 MCMC samples are abandoned as burn in. So, from the staying 10,000 MCMC samples, the Bayesian estimation for μ, θ, σ, $R(t)$ or $h(t)$ under the SEL and GEL (for $(\nu = -2, +2)$) along with their 95% BCI/HPD intervals are rated. To see the effects of the suggested priors on the Bayes' computations, besides $(a_1, a_2) = (1, 5)$, we have taken ε_i, $i = 1, 2$. According to Kundu [38]'s idea, the specified values of a_1 and a_2 are set so that the prior expectation equals the actual value of σ. Recall that the selected values ε_1 and ε_2 do not affect the Bayes results, and the hyperparameters a_i, $i = 1, 2$ are set so that the prior average reflects the actual value of σ. To assign the starting point values of μ, θ, and σ used in frequentist and Bayes MCMC calculations, it should be noted here that the suggested plausible values of μ, θ, and σ are taken as initial guesses to run ML optimizations while the fitted MLEs of μ, θ, and σ (in each iteration) are utilized as initial guesses to run the proposed MCMC sampler.

However, the average for the point estimators (Av.Es) for μ, θ, σ, $R(t)$ or $h(t)$, denoted by π, is written by

$$\text{Av.E}(\hat{\hat{\pi}}_\iota) = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \hat{\hat{\pi}}_\iota^{(i)},\ \iota = 1, \ldots, 5,$$

such that $\mathcal{K}$ is the replications number, $\hat{\hat{\pi}}^{(i)}$ is a point estimate of π at the ith simulated sample, $\pi_1 = \mu$, $\pi_2 = \theta$, $\pi_3 = \sigma$, $\pi_4 = R(t)$ and $\pi_5 = h(t)$.

The evaluation for point estimators of π is performed upon two standards, (i) root of mean squared error (RMSE), and (ii) mean relative absolute bias (MRAB), for $\iota = 1, \ldots, 5$, as

$$\text{RMSE}(\hat{\hat{\pi}}_\iota) = \sqrt{\frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \left(\hat{\hat{\pi}}_\iota^{(i)} - \pi_\iota\right)^2} \quad \text{and} \quad \text{MRAB}(\hat{\hat{\pi}}_\iota) = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \frac{1}{\pi_\iota} \left|\hat{\hat{\pi}}_\iota^{(i)} - \pi_\iota\right|,$$

respectively.

Comparing interval estimates of π is also performed depending on two criteria, namely: (i) average confidence length (ACL), and (ii) coverage percentage (CP), for $\iota = 1, \ldots, 5$, as

$$\text{ACL}_{(1-\epsilon)\%}(\pi_\iota) = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \left(\mathcal{U}_{\hat{\hat{\pi}}_\iota^{(i)}} - \mathcal{L}_{\hat{\hat{\pi}}_\iota^{(i)}}\right) \quad , \quad \text{CP}_{(1-\epsilon)\%}(\pi_\iota) = \frac{1}{\mathcal{K}} \sum_{i=1}^{\mathcal{K}} \boldsymbol{J}_{\left(\mathcal{L}_{\hat{\hat{\pi}}_\iota^{(i)}}; \mathcal{U}_{\hat{\hat{\pi}}_\iota^{(i)}}\right)}(\pi_\iota),$$

respectively, where $\boldsymbol{J}(\cdot)$ is the indicator operator and $(\mathcal{L}(\cdot), \mathcal{U}(\cdot))$ denotes the interval bounds of $100(1-\epsilon)\%$ an interval estimate of π_ι.

Using '`maxLik`' with '`armspp`' libraries suggested by Henningsen and Toomet [26] and Bertolacci [39], respectively, the point and interval estimates along with their RMSEs, MRABs, ACLs, and CPs for the parameters μ, θ, σ, and the functions $R(t)$ and $h(t)$ are obtained. These libraries were installed in the R programming software (version 4.2.2) and their investigations were performed on a laptop with a Core(TM) i5-5200M processor and 8.00 GB of RAM. Furthermore, we recommend performing a sensitivity analysis to measure the validity of the proposed censoring experimental tests.

To distinguish, in Tables 1–5, the Av.Es, RMSEs, and MRABs are listed in the first, second, and third columns, respectively. Further, in Tables 6–10, the ACLs and CPs are listed in the first and second, respectively.

From Tables 1–10, based on the smallest value for RMSE, MRAB, and ACL, and the biggest values of CP, one can adopt the below remarks:

- Overall, the acquired point and interval estimation for the NEW distribution parameters μ, θ and σ, or its reliability and hazard functions $R(t)$ and $h(t)$, perform well.
- As n (or FP%) increases, the accuracy of the estimation results becomes better. The same fact is also noted as $\sum_{i=1}^{k} R_i$ decreases.
- As T increases, it is noted that
  - The RMSEs and MRABs of μ, σ and $R(t)$ increase, while those of θ and $h(t)$ decrease.
  - The ACLs of μ and θ increase, while those of σ, $R(t)$ and $h(t)$ decrease. The opposite behavior of the same parameters, in the case of their simulated CP values, is reached.
- Comparing the suggested estimation methods, as the MCMC method has further prior information, the performance of the Bayes results for all parameters is strongly higher when comparing them to those produced by the likelihood methods.
- Comparing the proposed loss functions in Bayesian analysis, it is clear that the GEL emerges as the better loss compared to its competitive symmetric loss.
- Comparing the proposed interval inferential methods, due to the fact that the Bayesian intervals have additional prior information, both BCI and HPD interval estimates of μ, θ, σ, $R(t)$ and $h(t)$ implement more satisfactorily than those created from the ACI-NA (or ACI-NL) method. It is also noted that the calculated interval limits of $R(t)$ lie in its theoretical range (0,1).
- On the other hand, in most cases, the best interval bounds for μ, θ and σ are developed by the ACI-NA method, while those for $R(t)$ and $h(t)$ are developed with the ACI-NL technique.
- In particular, in terms of the smallest ACL and highest CP values, the HPD intervals of μ, θ, σ, $R(t)$ or $h(t)$ behave more satisfactorily than others.
- Comparing the proposed censoring schemes, it is noticed that the suggested estimates of μ, θ, σ, $R(t)$ and $h(t)$ developed under Scheme-1 'left censoring' perform superior to others. This holds for the fact that the overall duration testing by Scheme-1 is high compared to any other, and the collected data develop more knowledge about μ, θ, σ, $R(t)$ and $h(t)$ rather than others.
- To sum up, the Bayes' estimation methodology via the Gibbs sampling (to obtain MCMC variates of σ) and the ARMS (to adopt MCMC variates of μ and θ) is recommended for evaluating the NEW lifetime model from the suggested mechanism.

Table 1. The point evaluations of μ.

T	n[FP%]	Scheme	MLE			SEL			GEL					
$\nu \rightarrow$									-2			+2		
2	40[50%]	1	0.2984	0.2358	0.7351	0.2057	0.0914	0.3737	0.2066	0.0892	0.3641	0.2027	0.0893	0.3661
		2	0.3229	0.2659	0.8154	0.2835	0.1245	0.5054	0.2839	0.1239	0.5019	0.2822	0.1232	0.4985
		3	0.5615	0.4723	1.2170	0.2966	0.1353	0.5724	0.2970	0.1348	0.5702	0.2955	0.1342	0.5671
	40[75%]	1	0.2957	0.2261	0.7330	0.1919	0.0871	0.3618	0.1926	0.0855	0.3554	0.1897	0.0859	0.3585
		2	0.2964	0.2478	0.7665	0.2677	0.1160	0.4734	0.2684	0.1149	0.4688	0.2656	0.1138	0.4642
		3	0.5229	0.4555	1.1154	0.2956	0.1248	0.5224	0.2961	0.1241	0.5201	0.2943	0.1230	0.5143
	80[50%]	1	0.2634	0.1345	0.4682	0.1686	0.0704	0.2987	0.1694	0.0682	0.2904	0.1663	0.0696	0.2980
		2	0.2784	0.1668	0.5379	0.2804	0.1092	0.4400	0.2810	0.1082	0.4374	0.2788	0.1068	0.4294
		3	0.5634	0.4178	1.0270	0.2749	0.1218	0.5075	0.2756	0.1207	0.5027	0.2729	0.1197	0.4984
	80[75%]	1	0.2610	0.1322	0.4583	0.1953	0.0702	0.2804	0.1963	0.0671	0.2672	0.1921	0.0674	0.2710
		2	0.2582	0.1341	0.4657	0.2645	0.1008	0.3997	0.2653	0.0991	0.3941	0.2620	0.0973	0.3848
		3	0.5381	0.4101	1.0037	0.2783	0.1133	0.4646	0.2790	0.1119	0.4598	0.2759	0.1103	0.4517
5	40[50%]	1	0.3142	0.2303	0.7380	0.2561	0.1135	0.4611	0.2567	0.1126	0.4580	0.2546	0.1121	0.4562
		2	0.3043	0.2351	0.7533	0.2649	0.1158	0.4619	0.2654	0.1149	0.4563	0.2633	0.1142	0.4535
		3	0.4187	0.3486	1.2133	0.2713	0.1212	0.4986	0.2717	0.1205	0.4951	0.2699	0.1199	0.4923
	40[75%]	1	0.3042	0.2199	0.7148	0.2336	0.1008	0.4050	0.2344	0.0991	0.3973	0.2310	0.0985	0.3953
		2	0.3024	0.2279	0.7369	0.2571	0.1112	0.4512	0.2579	0.1098	0.4454	0.2547	0.1088	0.4411
		3	0.3534	0.3207	1.0396	0.2621	0.1150	0.4731	0.2628	0.1138	0.4683	0.2599	0.1128	0.4642
	80[50%]	1	0.2679	0.1347	0.4656	0.2453	0.0908	0.3531	0.2459	0.0894	0.3479	0.2435	0.0884	0.3440
		2	0.2698	0.1382	0.4860	0.2618	0.0978	0.3814	0.2625	0.0964	0.3763	0.2598	0.0949	0.3689
		3	0.3959	0.2802	0.9913	0.2629	0.1019	0.4044	0.2634	0.1009	0.4000	0.2612	0.0998	0.3948
	80[75%]	1	0.2652	0.1329	0.4641	0.2263	0.0818	0.3166	0.2272	0.0795	0.3059	0.2234	0.0786	0.3030
		2	0.2676	0.1362	0.4783	0.2544	0.0954	0.3751	0.2553	0.0934	0.3677	0.2516	0.0916	0.3591
		3	0.2999	0.1788	0.6242	0.2580	0.0993	0.3943	0.2588	0.0976	0.3880	0.2554	0.0960	0.3805

Table 2. The point evaluations of θ.

T	n[FP%]	Scheme	MLE			SEL			GEL					
$\nu \rightarrow$									-2			+2		
2	40[50%]	1	0.8106	0.2245	0.2218	0.8101	0.0956	0.0932	0.8102	0.0947	0.0923	0.8098	0.0946	0.0922
		2	0.8034	0.4313	0.5124	0.7233	0.3473	0.3258	0.7237	0.3465	0.3252	0.7224	0.3457	0.3240
		3	0.2556	0.6307	0.7199	0.5447	0.4658	0.4041	0.5448	0.4656	0.4034	0.5444	0.4656	0.4034
	40[75%]	1	0.7966	0.1724	0.1724	0.8048	0.0752	0.0761	0.8049	0.0741	0.0750	0.8045	0.0740	0.0750
		2	0.7514	0.1885	0.1867	0.7967	0.0887	0.0877	0.7968	0.0875	0.0863	0.7963	0.0875	0.0863
		3	0.3662	0.5756	0.6466	0.9901	0.2973	0.2713	0.9908	0.1202	0.2705	0.9893	0.2958	0.2695
	80[50%]	1	0.8045	0.1465	0.1443	0.8060	0.0669	0.0670	0.8061	0.0656	0.0658	0.8057	0.0655	0.0658
		2	0.8007	0.1492	0.1463	0.8028	0.0860	0.0824	0.8030	0.0845	0.0807	0.8023	0.0844	0.0807
		3	0.2874	0.5213	0.6408	0.8446	0.1210	0.1112	0.9904	0.2966	0.1103	0.8443	0.1199	0.1101
	80[75%]	1	0.7998	0.1209	0.1184	0.8046	0.0557	0.0550	0.8047	0.0541	0.0534	0.8042	0.0541	0.0534
		2	0.7727	0.1464	0.1339	0.7915	0.0616	0.0616	0.7917	0.0598	0.0597	0.7911	0.0599	0.0598
		3	0.3292	0.4936	0.5905	0.8077	0.0779	0.0766	0.8078	0.0767	0.0754	0.8073	0.0766	0.0753
5	40[50%]	1	0.7742	0.2101	0.2090	0.8032	0.1052	0.0993	0.8033	0.1043	0.0986	0.8029	0.1043	0.0986
		2	0.7900	0.2300	0.2270	0.8014	0.1319	0.1228	0.8015	0.1309	0.1218	0.8009	0.1308	0.1219
		3	0.6821	0.3572	0.3572	0.8432	0.1925	0.1816	0.8436	0.1908	0.1801	0.8419	0.1901	0.1796
	40[75%]	1	0.7750	0.1820	0.1769	0.7992	0.0866	0.0806	0.7993	0.0856	0.0793	0.7988	0.0855	0.0794
		2	0.7830	0.1947	0.1870	0.7916	0.0956	0.0913	0.7917	0.0945	0.0902	0.7912	0.0945	0.0903
		3	0.5860	0.3305	0.3435	0.9331	0.1903	0.1797	0.9333	0.1895	0.1788	0.9325	0.1889	0.1782
	80[50%]	1	0.7773	0.1440	0.1422	0.7977	0.0690	0.0674	0.7978	0.0677	0.0661	0.7974	0.0677	0.0661
		2	0.7877	0.1543	0.1521	0.7930	0.0846	0.0829	0.7932	0.0831	0.0815	0.7926	0.0831	0.0815
		3	0.6712	0.2566	0.2608	0.8093	0.1790	0.1644	0.8097	0.1782	0.1632	0.8081	0.1777	0.1626
	80[75%]	1	0.7900	0.1254	0.1240	0.8022	0.0607	0.0598	0.8023	0.0592	0.0584	0.8019	0.0592	0.0584
		2	0.7935	0.1312	0.1295	0.7966	0.0671	0.0662	0.7967	0.0655	0.0647	0.7962	0.0655	0.0647
		3	0.5692	0.3126	0.2089	0.9189	0.1381	0.1348	0.9191	0.1358	0.1326	0.9183	0.1354	0.1324

Table 3. The point evaluations of σ.

T	n[FP%]	Scheme	MLE			SEL			GEL					
$\nu \rightarrow$									-2			+2		
2	40[50%]	1	0.4428	0.1940	0.3260	0.3545	0.1559	0.2624	0.3573	0.1460	0.2457	0.3516	0.1450	0.2441
		2	0.4516	0.1845	0.3567	0.3476	0.1639	0.2871	0.3504	0.1534	0.2687	0.3448	0.1524	0.2670
		3	0.4687	0.2231	0.4186	0.3679	0.1885	0.3370	0.3708	0.1802	0.3222	0.3657	0.1791	0.3202
	40[75%]	1	0.4372	0.1543	0.2719	0.3432	0.1304	0.2188	0.3459	0.1220	0.2048	0.3404	0.1212	0.2035
		2	0.4558	0.1587	0.2828	0.3578	0.1341	0.2277	0.3607	0.1255	0.2131	0.3549	0.1247	0.2117
		3	0.4566	0.1872	0.3397	0.3584	0.1582	0.2735	0.3613	0.1512	0.2698	0.3563	0.1503	0.2615
	80[50%]	1	0.4276	0.1108	0.2091	0.3357	0.0937	0.1683	0.3383	0.0877	0.1575	0.3330	0.0871	0.1565
		2	0.4196	0.1145	0.2199	0.3294	0.0968	0.1770	0.3320	0.0906	0.1657	0.3267	0.0900	0.1646
		3	0.3109	0.1653	0.3348	0.2440	0.1397	0.2695	0.2460	0.1335	0.2577	0.2426	0.1327	0.2560
	80[75%]	1	0.4171	0.0924	0.1745	0.3275	0.0781	0.1405	0.3301	0.0731	0.1315	0.3248	0.0726	0.1307
		2	0.4301	0.0945	0.1778	0.3376	0.0799	0.1431	0.3403	0.0748	0.1339	0.3349	0.0743	0.1331
		3	0.2848	0.1382	0.3080	0.2235	0.1168	0.2480	0.2253	0.1117	0.2371	0.2222	0.1110	0.2356
5	40[50%]	1	0.4735	0.1994	0.3615	0.3717	0.1685	0.2910	0.3747	0.1611	0.2782	0.3695	0.1600	0.2765
		2	0.4791	0.2219	0.3959	0.3761	0.1875	0.3187	0.3791	0.1793	0.3047	0.3738	0.1782	0.3028
		3	0.6447	0.3320	0.6380	0.5061	0.2806	0.5136	0.5101	0.2626	0.4807	0.5020	0.2609	0.4776
	40[75%]	1	0.4561	0.1753	0.3116	0.3580	0.1481	0.2509	0.3609	0.1416	0.2398	0.3559	0.1407	0.2383
		2	0.4679	0.1845	0.3370	0.3673	0.1559	0.2713	0.3702	0.1491	0.2594	0.3651	0.1481	0.2577
		3	0.6530	0.2933	0.5347	0.5126	0.2478	0.4541	0.5150	0.2320	0.4182	0.5118	0.2305	0.4152
	80[50%]	1	0.4387	0.1248	0.2356	0.3444	0.1055	0.1896	0.3471	0.1008	0.1813	0.3423	0.1002	0.1801
		2	0.4359	0.1413	0.2754	0.3422	0.1194	0.2217	0.3449	0.1142	0.2119	0.3401	0.1135	0.2106
		3	0.5576	0.2844	0.5083	0.4377	0.2265	0.3918	0.4412	0.2197	0.3830	0.4342	0.2188	0.3805
	80[75%]	1	0.4273	0.1068	0.2039	0.3355	0.0903	0.1641	0.3381	0.0863	0.1569	0.3334	0.0857	0.1559
		2	0.4270	0.1170	0.2228	0.3352	0.0989	0.1794	0.3378	0.0945	0.1715	0.3332	0.0939	0.1704
		3	0.5125	0.2444	0.4428	0.4023	0.2065	0.3565	0.4055	0.1933	0.3337	0.3991	0.1920	0.3315

Table 4. The point evaluations of $R(t)$.

T	n[FP%]	Scheme	MLE			SEL			GEL					
$\nu \rightarrow$									-2			+2		
2	40[50%]	1	0.8688	0.0781	0.0767	0.8788	0.0470	0.0450	0.8814	0.0337	0.0329	0.8708	0.0309	0.0307
		2	0.8749	0.0791	0.0778	0.8607	0.0506	0.0482	0.8666	0.0358	0.0337	0.8448	0.0348	0.0319
		3	0.8567	0.0930	0.0879	0.8712	0.0564	0.0543	0.8778	0.0426	0.0413	0.8662	0.0419	0.0402
	40[75%]	1	0.8687	0.0756	0.0739	0.8751	0.0433	0.0407	0.8779	0.0272	0.0281	0.8666	0.0287	0.0269
		2	0.8663	0.0764	0.0746	0.8759	0.0468	0.0445	0.8787	0.0308	0.0313	0.8675	0.0295	0.0287
		3	0.8448	0.0826	0.0812	0.8636	0.0523	0.0489	0.8669	0.0332	0.0334	0.8534	0.0301	0.0324
	80[50%]	1	0.8671	0.0741	0.0731	0.8751	0.0335	0.0313	0.8814	0.0262	0.0247	0.8710	0.0247	0.0238
		2	0.8640	0.0744	0.0739	0.8770	0.0372	0.0350	0.8841	0.0298	0.0296	0.8740	0.0285	0.0273
		3	0.8562	0.0805	0.0794	0.8589	0.0416	0.0384	0.8700	0.0320	0.0317	0.8572	0.0290	0.0297
	80[75%]	1	0.8673	0.0733	0.0715	0.8788	0.0327	0.0307	0.8779	0.0192	0.0177	0.8668	0.0233	0.0226
		2	0.8650	0.0736	0.0718	0.8797	0.0311	0.0313	0.8816	0.0274	0.0266	0.8689	0.0240	0.0231
		3	0.8668	0.0791	0.0780	0.8482	0.0386	0.0373	0.8624	0.0296	0.0292	0.8399	0.0256	0.0254
5	40[50%]	1	0.8621	0.0770	0.0755	0.8739	0.0524	0.0498	0.8768	0.0313	0.0299	0.8653	0.0301	0.0299
		2	0.8625	0.0787	0.0770	0.8724	0.0612	0.0585	0.8778	0.0332	0.0325	0.8664	0.0324	0.0324
		3	0.9091	0.0794	0.0792	0.8746	0.0619	0.0634	0.8775	0.0362	0.0351	0.8660	0.0355	0.0346
	40[75%]	1	0.8644	0.0762	0.0748	0.8770	0.0473	0.0447	0.8750	0.0297	0.0286	0.8631	0.0290	0.0288
		2	0.8724	0.0774	0.0757	0.8720	0.0490	0.0466	0.8754	0.0306	0.0295	0.8636	0.0301	0.0301
		3	0.8647	0.0785	0.0769	0.8749	0.0510	0.0484	0.8798	0.0329	0.0312	0.8688	0.0325	0.0296
	80[50%]	1	0.8614	0.0754	0.0743	0.8725	0.0346	0.0324	0.8754	0.0272	0.0250	0.8654	0.0231	0.0222
		2	0.8637	0.0766	0.0744	0.8728	0.0411	0.0387	0.8757	0.0285	0.0260	0.8649	0.0250	0.0242
		3	0.9068	0.0776	0.0756	0.8707	0.0544	0.0582	0.8780	0.0298	0.0271	0.8618	0.0264	0.0263
	80[75%]	1	0.8644	0.0747	0.0736	0.8739	0.0339	0.0318	0.8738	0.0260	0.0245	0.8638	0.0233	0.0224
		2	0.8655	0.0748	0.0737	0.8735	0.0367	0.0345	0.8767	0.0267	0.0255	0.8641	0.0244	0.0236
		3	0.8696	0.0755	0.0742	0.8752	0.0382	0.0364	0.8764	0.0277	0.0266	0.8668	0.0252	0.0245

Table 5. The point evaluations of $h(t)$.

T	n[FP%]	Scheme	MLE			SEL			GEL					
$\nu \rightarrow$									-2			+2		
2	40[50%]	1	0.3754	0.2054	0.4801	0.3514	0.0932	0.2080	0.4056	0.0420	0.1045	0.4012	0.0419	0.1040
		2	0.3689	0.2070	0.4840	0.3164	0.1182	0.2437	0.3653	0.0599	0.1285	0.3613	0.0597	0.1497
		3	0.4012	0.2164	0.5062	0.3671	0.1333	0.2965	0.4239	0.0641	0.1675	0.4192	0.0638	0.1668
	40[75%]	1	0.3774	0.2051	0.4792	0.3544	0.0927	0.1945	0.4093	0.0417	0.1012	0.4048	0.0415	0.1008
		2	0.3706	0.2056	0.4817	0.3524	0.1034	0.2190	0.4069	0.0453	0.1145	0.4025	0.0452	0.1280
		3	0.3704	0.2133	0.4983	0.3644	0.1315	0.2767	0.4208	0.0611	0.1503	0.4162	0.0608	0.1449
	80[50%]	1	0.3774	0.2039	0.4762	0.3541	0.0773	0.1618	0.4089	0.0411	0.0978	0.4044	0.0409	0.0975
		2	0.3699	0.2061	0.4807	0.3503	0.0845	0.1798	0.4045	0.0439	0.1075	0.4000	0.0437	0.1070
		3	0.4708	0.2121	0.4965	0.3736	0.1263	0.2656	0.4314	0.0554	0.1455	0.4266	0.0552	0.1382
	80[75%]	1	0.3589	0.1997	0.4619	0.3567	0.0620	0.1293	0.4119	0.0388	0.0945	0.4074	0.0387	0.0941
		2	0.3779	0.2045	0.4775	0.3535	0.0692	0.1460	0.4082	0.0424	0.1063	0.4037	0.0422	0.1058
		3	0.4473	0.2113	0.4943	0.3667	0.1000	0.2356	0.4234	0.0544	0.1388	0.4187	0.0541	0.1140
5	40[50%]	1	0.3947	0.2070	0.4842	0.3541	0.1323	0.2744	0.4088	0.0455	0.1150	0.4068	0.0453	0.1148
		2	0.2463	0.2076	0.4857	0.3518	0.1420	0.3446	0.4062	0.0466	0.1185	0.4041	0.0464	0.1183
		3	0.3912	0.2116	0.4948	0.3547	0.1560	0.3293	0.4095	0.0552	0.1405	0.4075	0.0551	0.1402
	40[75%]	1	0.3831	0.2065	0.4829	0.3554	0.1101	0.2370	0.4105	0.0443	0.1118	0.4084	0.0442	0.1116
		2	0.3642	0.2070	0.4842	0.3539	0.1123	0.2407	0.4087	0.0453	0.1151	0.4067	0.0451	0.1149
		3	0.3830	0.2105	0.4918	0.3620	0.1233	0.2607	0.4180	0.0538	0.1322	0.4159	0.0536	0.1319
	80[50%]	1	0.3847	0.2058	0.4810	0.3569	0.0878	0.1849	0.4122	0.0431	0.1059	0.4101	0.0429	0.1057
		2	0.3818	0.2066	0.4829	0.3559	0.1041	0.2235	0.4109	0.0448	0.1106	0.4089	0.0446	0.1104
		3	0.2429	0.2081	0.4866	0.3588	0.1357	0.3442	0.4143	0.0478	0.1210	0.4122	0.0476	0.1207
	80[75%]	1	0.3765	0.2047	0.4780	0.3580	0.0777	0.1648	0.4134	0.0415	0.0995	0.4113	0.0414	0.0993
		2	0.3756	0.2058	0.4808	0.3569	0.0863	0.1833	0.4121	0.0439	0.1064	0.4100	0.0437	0.1062
		3	0.3708	0.2064	0.4819	0.3649	0.0966	0.2142	0.4214	0.0458	0.1098	0.4193	0.0456	0.1095

Table 6. The 95% interval evaluations of μ.

n[FP%]	Scheme	ACI-NA	ACI-NL	BCI	HPD	ACI-NA	ACI-NL	BCI	HPD
$T \rightarrow$		2				5			
40[50%]	1	0.859	0.938	0.395	0.963	0.885	0.929	0.385	0.953
		0.962	0.931	0.371	0.965	0.994	0.924	0.375	0.955
	2	0.879	0.936	0.406	0.961	0.872	0.927	0.388	0.951
		0.987	0.930	0.391	0.963	1.032	0.922	0.381	0.951
	3	0.986	0.929	0.480	0.956	1.073	0.918	0.405	0.947
		1.025	0.926	0.436	0.959	1.113	0.916	0.395	0.949
40[75%]	1	0.800	0.946	0.357	0.968	0.804	0.934	0.342	0.959
		0.891	0.938	0.332	0.970	0.912	0.929	0.331	0.960
	2	0.810	0.945	0.372	0.967	0.814	0.931	0.353	0.958
		0.915	0.936	0.342	0.969	0.919	0.927	0.342	0.959
	3	0.834	0.943	0.391	0.965	0.907	0.925	0.360	0.955
		0.949	0.933	0.369	0.967	0.977	0.922	0.350	0.957
80[50%]	1	0.551	0.957	0.290	0.973	0.554	0.947	0.311	0.963
		0.711	0.951	0.282	0.974	0.799	0.941	0.303	0.964
	2	0.592	0.955	0.320	0.970	0.595	0.943	0.319	0.961
		0.769	0.948	0.312	0.971	0.913	0.937	0.312	0.961
	3	0.613	0.954	0.370	0.968	0.674	0.938	0.348	0.958
		0.905	0.937	0.355	0.969	0.960	0.931	0.340	0.960
80[75%]	1	0.533	0.961	0.257	0.976	0.530	0.951	0.265	0.966
		0.604	0.955	0.236	0.977	0.658	0.945	0.257	0.968
	2	0.550	0.958	0.301	0.974	0.552	0.948	0.279	0.964
		0.650	0.952	0.287	0.975	0.705	0.941	0.271	0.965
	3	0.575	0.957	0.344	0.971	0.611	0.944	0.284	0.961
		0.727	0.948	0.318	0.973	0.814	0.939	0.275	0.963

Table 7. The 95% interval evaluations of θ.

n[FP%]	Scheme	ACI-NA ACI-NL		BCI HPD		ACI-NA ACI-NL		BCI HPD	
$T \rightarrow$		2				5			
40[50%]	1	0.797	0.942	0.284	0.967	0.811	0.936	0.291	0.961
		0.839	0.937	0.275	0.969	0.852	0.931	0.286	0.963
	2	0.818	0.940	0.313	0.964	0.831	0.934	0.307	0.958
		0.860	0.935	0.309	0.965	0.872	0.928	0.303	0.959
	3	1.023	0.932	0.465	0.953	1.262	0.924	0.514	0.947
		1.137	0.930	0.458	0.954	1.498	0.919	0.507	0.949
40[75%]	1	0.538	0.951	0.257	0.970	0.657	0.945	0.256	0.964
		0.549	0.950	0.254	0.971	0.680	0.943	0.253	0.965
	2	0.652	0.947	0.284	0.967	0.664	0.941	0.304	0.961
		0.685	0.945	0.280	0.967	0.686	0.939	0.301	0.961
	3	0.670	0.946	0.440	0.956	0.964	0.940	0.499	0.950
		0.691	0.944	0.434	0.958	0.978	0.938	0.493	0.951
80[50%]	1	0.490	0.956	0.237	0.974	0.549	0.950	0.246	0.968
		0.506	0.955	0.234	0.975	0.560	0.948	0.243	0.969
	2	0.566	0.952	0.275	0.969	0.558	0.946	0.281	0.963
		0.574	0.950	0.272	0.970	0.570	0.944	0.278	0.965
	3	0.569	0.952	0.311	0.963	0.788	0.946	0.363	0.957
		0.578	0.951	0.302	0.965	0.796	0.945	0.358	0.960
80[75%]	1	0.320	0.967	0.211	0.977	0.456	0.961	0.226	0.971
		0.344	0.965	0.208	0.978	0.463	0.960	0.213	0.973
	2	0.453	0.960	0.259	0.972	0.461	0.954	0.279	0.966
		0.460	0.959	0.256	0.973	0.467	0.954	0.276	0.967
	3	0.463	0.959	0.285	0.968	0.534	0.953	0.358	0.962
		0.469	0.957	0.281	0.970	0.566	0.951	0.353	0.964

Table 8. The 95% interval evaluations of σ.

n[FP%]	Scheme	ACI-NA ACI-NL		BCI HPD		ACI-NA ACI-NL		BCI HPD	
$T \rightarrow$		2				5			
40[50%]	1	0.712	0.946	0.620	0.965	0.626	0.949	0.535	0.974
		0.797	0.939	0.616	0.966	0.680	0.947	0.529	0.976
	2	0.795	0.940	0.648	0.959	0.742	0.945	0.5854	0.968
		0.915	0.932	0.641	0.960	0.830	0.938	0.5796	0.970
	3	0.841	0.936	0.673	0.955	0.827	0.942	0.607	0.965
		0.949	0.928	0.662	0.956	0.926	0.930	0.598	0.966
40[75%]	1	0.630	0.954	0.593	0.973	0.553	0.959	0.467	0.983
		0.685	0.950	0.588	0.974	0.673	0.951	0.457	0.985
	2	0.672	0.9490	0.615	0.968	0.649	0.952	0.518	0.978
		0.739	0.9430	0.608	0.970	0.714	0.946	0.504	0.980
	3	0.800	0.941	0.624	0.963	0.701	0.947	0.562	0.973
		0.905	0.933	0.618	0.964	0.776	0.942	0.558	0.975
80[50%]	1	0.453	0.960	0.411	0.978	0.404	0.965	0.341	0.988
		0.475	0.958	0.392	0.979	0.419	0.964	0.339	0.989
	2	0.523	0.957	0.463	0.975	0.463	0.962	0.391	0.985
		0.559	0.954	0.453	0.976	0.485	0.960	0.384	0.986
	3	0.592	0.952	0.520	0.971	0.539	0.957	0.455	0.980
		0.613	0.950	0.512	0.972	0.575	0.954	0.448	0.982
80[75%]	1	0.408	0.964	0.353	0.983	0.328	0.971	0.277	0.991
		0.425	0.962	0.347	0.984	0.349	0.969	0.272	0.991
	2	0.439	0.9590	0.377	0.978	0.431	0.961	0.364	0.988
		0.459	0.9560	0.370	0.979	0.451	0.958	0.361	0.989
	3	0.548	0.954	0.474	0.971	0.471	0.959	0.332	0.981
		0.573	0.951	0.466	0.973	0.493	0.957	0.326	0.982

Table 9. The 95% interval evaluations of $R(t)$.

n[FP%]	Scheme	ACI-NA		BCI		ACI-NA		BCI	
		ACI-NL		HPD		ACI-NL		HPD	
$T \rightarrow$		2				5			
40[50%]	1	0.242	0.943	0.167	0.965	0.232	0.948	0.159	0.969
		0.240	0.945	0.165	0.967	0.231	0.948	0.157	0.968
	2	0.248	0.941	0.179	0.9610	0.235	0.945	0.175	0.965
		0.246	0.943	0.177	0.9630	0.233	0.946	0.172	0.966
	3	0.261	0.937	0.207	0.957	0.238	0.943	0.210	0.961
		0.259	0.938	0.206	0.958	0.235	0.944	0.200	0.963
40[75%]	1	0.226	0.952	0.164	0.968	0.228	0.950	0.125	0.972
		0.225	0.953	0.162	0.970	0.227	0.951	0.123	0.971
	2	0.228	0.951	0.173	0.964	0.230	0.949	0.166	0.971
		0.226	0.953	0.172	0.966	0.229	0.949	0.164	0.970
	3	0.256	0.945	0.183	0.961	0.233	0.945	0.182	0.968
		0.254	0.946	0.181	0.963	0.232	0.946	0.180	0.967
80[50%]	1	0.221	0.953	0.124	0.976	0.226	0.952	0.115	0.975
		0.219	0.955	0.122	0.978	0.224	0.951	0.113	0.974
	2	0.225	0.952	0.133	0.972	0.227	0.951	0.124	0.973
		0.223	0.954	0.131	0.975	0.226	0.950	0.122	0.974
	3	0.237	0.948	0.152	0.968	0.230	0.948	0.153	0.972
		0.233	0.950	0.151	0.970	0.228	0.947	0.151	0.973
80[75%]	1	0.216	0.957	0.119	0.983	0.221	0.955	0.092	0.987
		0.215	0.958	0.118	0.984	0.220	0.954	0.091	0.988
	2	0.221	0.954	0.125	0.980	0.223	0.953	0.118	0.985
		0.220	0.955	0.123	0.982	0.222	0.954	0.116	0.986
	3	0.228	0.951	0.129	0.978	0.225	0.952	0.130	0.981
		0.226	0.953	0.127	0.980	0.223	0.953	0.128	0.980

Table 10. The 95% interval evaluations of $h(t)$.

n[FP%]	Scheme	ACI-NA		BCI		ACI-NA		BCI	
		ACI-NL		HPD		ACI-NL		HPD	
$T \rightarrow$		2				5			
40[50%]	1	0.673	0.950	0.445	0.967	0.671	0.952	0.428	0.974
		0.671	0.952	0.419	0.970	0.667	0.953	0.406	0.976
	2	0.679	0.948	0.512	0.959	0.677	0.949	0.496	0.962
		0.676	0.949	0.482	0.963	0.673	0.951	0.466	0.965
	3	0.709	0.944	0.604	0.956	0.693	0.946	0.591	0.958
		0.704	0.945	0.570	0.959	0.688	0.949	0.550	0.961
40[75%]	1	0.671	0.953	0.439	0.970	0.669	0.955	0.317	0.976
		0.668	0.954	0.417	0.973	0.664	0.957	0.297	0.980
	2	0.679	0.950	0.480	0.965	0.673	0.951	0.441	0.968
		0.673	0.952	0.451	0.967	0.668	0.953	0.419	0.971
	3	0.703	0.947	0.504	0.962	0.687	0.948	0.490	0.963
		0.701	0.948	0.480	0.964	0.683	0.950	0.461	0.967
80[50%]	1	0.665	0.958	0.328	0.977	0.659	0.960	0.300	0.980
		0.660	0.960	0.319	0.978	0.655	0.961	0.292	0.981
	2	0.670	0.952	0.340	0.975	0.671	0.953	0.335	0.977
		0.667	0.953	0.330	0.976	0.665	0.955	0.324	0.978
	3	0.691	0.949	0.417	0.973	0.680	0.950	0.410	0.974
		0.682	0.951	0.399	0.971	0.677	0.952	0.391	0.972
80[75%]	1	0.658	0.960	0.299	0.980	0.601	0.964	0.216	0.986
		0.656	0.961	0.291	0.981	0.596	0.965	0.209	0.987
	2	0.667	0.957	0.328	0.979	0.661	0.958	0.299	0.983
		0.662	0.958	0.318	0.980	0.658	0.959	0.291	0.984
	3	0.673	0.953	0.358	0.977	0.667	0.957	0.332	0.979
		0.671	0.954	0.350	0.978	0.662	0.956	0.322	0.981

5. Real-world applications

To examine the efficacy and fitness of the suggested estimation methods, two real-world scenarios illustrate the potentiality of this model, which is to perform analyses for those real data sets that are gathered from the engineering experiment.

5.1. Carbon fibers data

Carbon fiber was widely used in the industry due to its great tensile strength, low density, strong thermal properties, electrical conductivities, chemical stability, and high thermal. So, fibers are used today to produce several components that are needed with light weight and high strength. This application examines the strength of single carbon fibers measured in gigapascal pascal (GPa) at gauge lengths of 20 mm under stress, and impregnated thousand-carbon fibers; refer to Table 11. The dataset was provided by Badar and Priest [40] and discussed by Kundu and Raqab [41].

Table 11. Strength data in GPa for 74 carbon fibers.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.809	2.818	2.818	2.821	2.821	2.848
2.848	2.880	2.880	2.954	3.012	3.067	3.084	3.090	3.096	3.128
3.233	3.433	3.585	3.585						

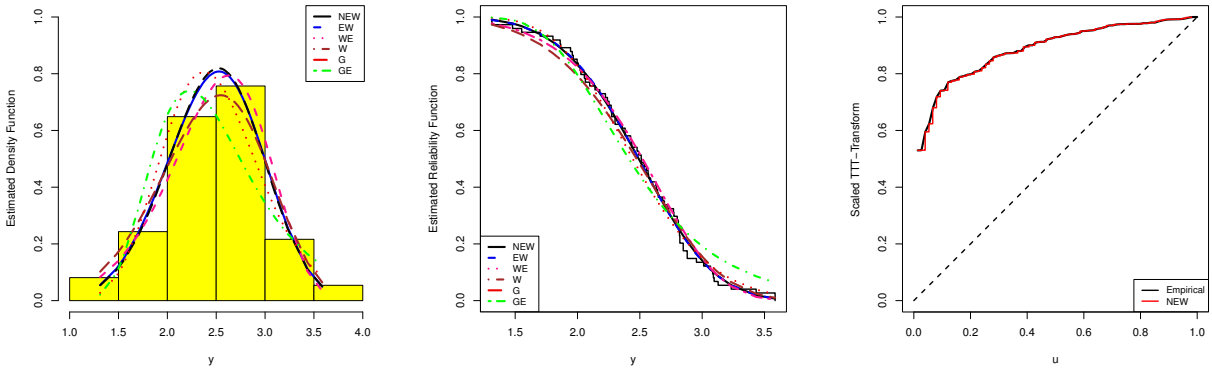
Now, based on six different criteria, which are: the negative log likelihood (NL), consistent Akaike (CAI), Akaike (A), Bayesian (B), Hannan-Quinn (HQ), and the Kolmogorov-Smirnov (KS) test with the corresponding P-value methods, the NEW lifetime distribution is compared to other common distributions in the literature as competitors, namely: Harris extended-exponential (HEE(μ, θ, σ)), Marshall-Olkin extended-exponential (MOEE(μ, θ, σ)), Marshall-Olkin Gompertz (MOG(μ, θ, σ)), exponentiated Weibull (EW(μ, θ, σ)), Weibull-exponential (WE(μ, θ, σ)), inverse Weibull (IW(θ, σ)), Weibull (W(θ, σ)), gamma (G(θ, σ)), and generalized exponential (GE(θ, σ)) distributions. The CDFs, for $y > 0$ and $\mu, \theta, \sigma > 0$, for the competing models of the NEW distribution, are reported in Table 12. For each model, through the method of ML, each parameter with its standard error (St.Er) in addition to the goodness-of-fit-criteria are evaluated, see Table 13. The best model of fit must give the lowest value of all the given criteria, and give the highest value of P. It is evident, from Table 13, that the NEW model fits the carbon fibers data superior to the other competing models. Also, in Figure 4, different plots are treated to reinforce the numerical findings indexed in Table 13, namely: (i) estimated densities with data histograms, (ii) estimated reliability functions, and (iii) total time on a test (TTT) plots of all competitive life models. It indicates, however, that the NEW model fits the carbon fibers data set satisfactorily and validates the numerical findings. It also reveals that the failure rate shape for the carbon fibers data increases, since the NEW distribution can present an increasing failure rate, which means that it can be adequately used to fit the carbon fibers data. For more information on the data representations and visual aids proposed here, we recommend following the workshop presented by Elshahhat [42].

Table 12. Competing lifetime distributions for the NEW distribution.

Model	CDF	Author(s)
HEE	$1-(\sigma\exp(-\mu\theta y)(1-(1-\sigma)\exp(-\mu\theta y)))^{-\frac{1}{\mu}}$	Pinho et al. [43]
MOEE	$(1-\exp(1-(1+\mu y)^{\theta}))(1-(1-\sigma)\exp(1-(1+\mu y)^{\theta}))^{-1}$	Lemonte et al. [44]
MOG	$(1-\exp(-\frac{\mu}{\theta}e^{\mu y}-1)))(1-(1-\sigma)\exp(\frac{\mu}{\theta}(1-e^{\mu y}))^{-1}$	Eghwerido et al. [45]
EW	$(1-\exp(-(\sigma y)^{\mu}))^{\theta}$	Mudholkar and Srivastava [10]
WE	$1-\exp(-\sigma(\exp(\mu y)-1)^{\theta})$	Oguntunde et al. [46]
IW	$e^{-\sigma y^{-\theta}}$	Ramos et al. [47]
W	$1-\exp(-\sigma y^{\theta})$	Weibull [48]
G	$\frac{1}{\Gamma(\theta)}\gamma(\theta,\mu y)$	Johnson et al. [49]
GE	$(1-\exp(-\sigma y))^{\theta}$	Gupta and Kundu [50]

Table 13. Fit results for the NEW and other models for carbon fibers data.

Model	MLE(St.Er)			NL	A	CA	B	HQ	KS	
	μ	θ	σ						Statistic	*P*-value
NEW	5.9388(7.0814)	3.4070(2.7981)	0.3355(1.8194)	51.137	108.27	108.62	113.22	110.45	0.0598	0.9537
HEE	0.5800(0.2313)	3.2119(1.0139)	155.28(46.555)	63.738	133.48	133.82	140.39	136.23	0.1963	0.0665
MOEE	24.849(28.315)	67.816(26.836)	0.0275(0.0329)	53.091	112.18	112.52	119.09	114.94	0.0637	0.9419
MOG	0.2511(0.2791)	15.491(18.903)	0.9886(0.3980)	54.782	115.56	115.91	122.48	118.32	0.0738	0.8152
EW	0.4068(0.0446)	1.6369(0.9195)	4.4322(1.2970)	51.152	108.30	108.65	115.22	111.06	0.0600	0.9525
WE	0.2664(0.0709)	4.1130(0.4931)	0.8277(1.2611)	52.236	110.47	110.81	117.38	113.23	0.0766	0.7785
IW	-	4.1092(0.3231)	24.111(5.7406)	69.099	142.02	142.19	146.63	143.86	0.1453	0.0878
W	-	5.1255(0.3318)	0.0067(0.0023)	52.308	108.62	108.78	115.19	111.03	0.0743	0.8088
G	-	24.227(3.9558)	9.7796(1.6134)	53.165	110.33	110.50	114.94	112.17	0.0681	0.8821
GE	-	85.764(30.812)	1.9993(0.1695)	58.810	121.62	121.79	126.23	123.46	0.0982	0.4740

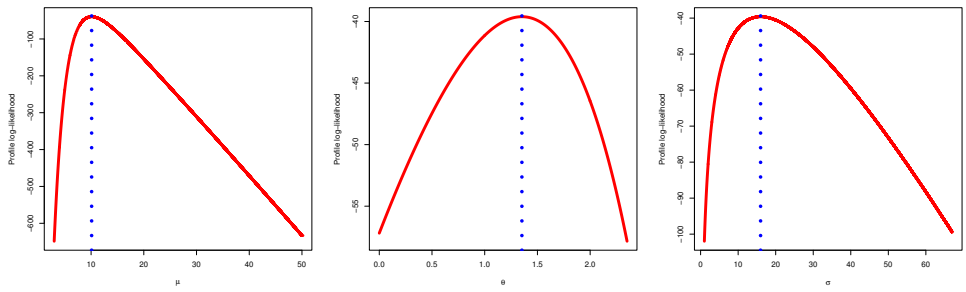


(a) (b) (c)

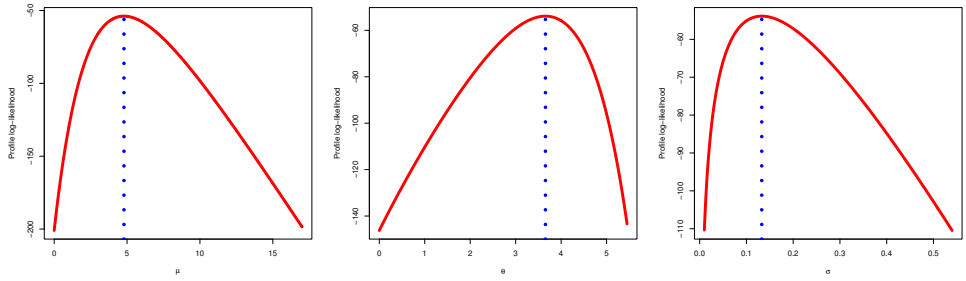
Figure 4. The fitted PDFs (a), fitted RFs (b), and TTT (c) plots of the NEW and its competing models from carbon fibers data.

From the carbon fibers dataset, and using $k = 34$ with various options of T and $\mathbf{R}$, the artificial A-PT2-C sampling schemes are created and recorded in Table 14. Using the Gibbs steps investigated in Section 3, we ignored the first 10,000 samples out of 50,000 MCMC samples. The calculated MLEs of μ, θ, and σ are chosen as initial guesses for employing the MCMC algorithms. Since prior information on μ and θ is not possible, the Bayesian estimators for μ, θ, σ, $R(t)$, and $h(t)$ (at $t = 1.5$), from a

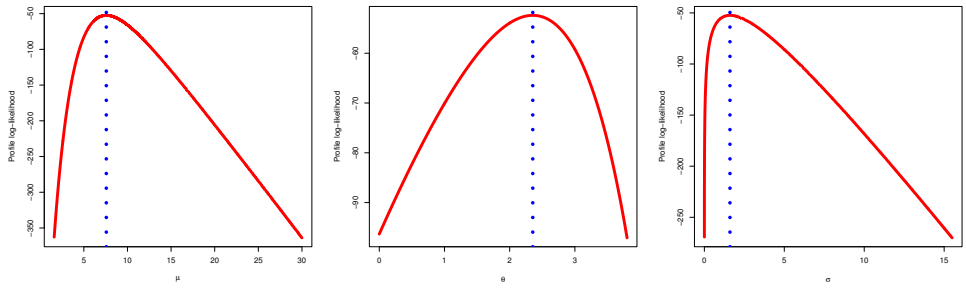
non-informative prior, are acquired against both SEL and GEL (for $\nu = (-5, -0.05, +5)$) functions. For the calculation logic, the hyperparameter values are specified to be 0.001. Table 15 displays the point estimation (with their St.Ers) of μ, θ, σ, $R(t)$, and $h(t)$. In Table 15, the point estimate (along with its St.Er) is displayed in the first and second columns, respectively. Additionally, the two interval bounds (with their interval lengths (ILs)), developed by 95% asymptotic (NA/NL) and Bayesian (BCI/HPD) intervals, of the same unknown quantities, are computed and presented in Table 16. To perform the MCMC method, the classical estimators' for μ, θ, and σ are taken as the starting guesses. Tables 15 and 16 stated, in charge of the lowest St.Er and IL values, that the given interval and point estimates calculated from the proposed Bayes MCMC technique act better than those calculated from the ML method.



(a) Sample 1



(b) Sample 2



(c) Sample 3

Figure 5. Profile log-likelihood functions for μ, θ, and σ from carbon fibers data.

To show that the attained MLEs exist and are unique, for all generated samples in Table 14, the profile log-likelihood of μ, θ and σ are shown in Figure 5. As we anticipated, Figure 5 supported the

numerical values reported in Table 15. One of the biggest issues when adopting the Gibbs process is proving Markovian convergence. For this objective, from each created sample, Figure 6 depicts both the trace plot and the density plot of all unknown parameters. In each subplot in Figure 6, the sample mean as well as the two bounds of 95% Bayes credible intervals are represented as blue solid (—) and dashed (- - -) horizontal lines, respectively. Figure 6 shows that the suggested Gibbs sampling is convergent adequately and indicates that the densities of μ, θ and σ are almost symmetric, while those of $R(t)$ and $h(t)$ are negatively and positively-skewed, respectively.

Again, based on 40,000 MCMC variates collected from all created samples, useful statistics, namely, mean, mode, three quartiles ($Q_i,\ i = 1, 2, 3$), standard deviation (St.D.), and skewness (Sk.) of μ, θ, σ, $R(t)$, and $h(t)$ are obtained; see Table 17. It supports the same numerical findings shown in Table 15 and confirms the same graphical facts shown in Figure 6.

Table 14. Three A-PT2-C samples from carbon fibers.

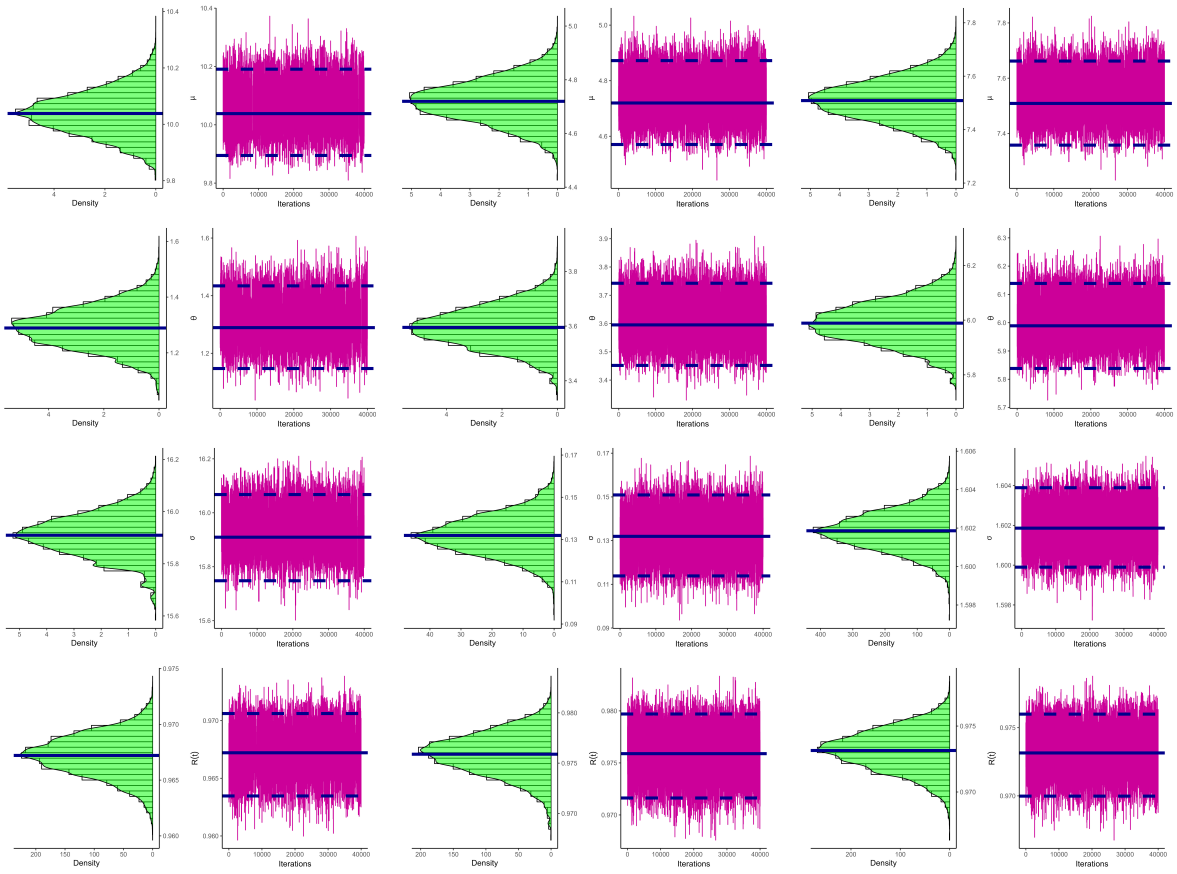
Sample	Scheme	$T(d)$	R_k	Censored Sample
1	$(10^4, 0^*30)$	1.4(2)	20	1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.490, 2.511
2	$(0^*15, 10^4, 0^*15)$	2.2(18)	10	1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.179, 2.224, 2.272, 2.274, 2.301, 2.382, 2.426, 2.434, 2.435, 2.490, 2.535, 2.554, 2.633, 2.642, 2.648, 2.684, 2.697
3	$(0^*30, 10^4)$	2.43(32)	20	1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.511

Table 15. Point estimates of μ, θ, σ, $R(t)$ and $h(t)$ from carbon fibers data.

Sample	Par.	MLE		SEL		GEL					
$\nu \rightarrow$						-5		-0.05		+5	
1	μ	10.105	1.0987	10.038	0.1009	10.040	0.0657	10.038	0.0671	10.037	0.0685
	θ	1.3527	0.5751	1.2891	0.0976	1.2976	0.0551	1.2871	0.0656	1.2763	0.0764
	σ	15.974	7.9715	15.909	0.1038	15.910	0.0641	15.909	0.0651	15.908	0.0661
	$R(1.5)$	0.9677	0.0154	0.9672	0.0019	0.9672	0.0005	0.9672	0.0005	0.9672	0.0005
	$h(1.5)$	0.1769	0.0636	0.1774	0.0101	0.1785	0.0016	0.1771	0.0002	0.1757	0.0012
2	μ	4.7867	14.5981	4.7198	0.1026	4.7223	0.0644	4.7192	0.0676	4.7159	0.0708
	θ	3.6552	6.6844	3.5952	0.0955	3.5983	0.0569	3.5945	0.0607	3.5906	0.0646
	σ	0.1327	1.5899	0.1320	0.0093	0.1332	0.0005	0.1316	0.0011	0.1300	0.0028
	$R(1.5)$	0.9763	0.0154	0.9759	0.0021	0.9759	0.0004	0.9759	0.0004	0.9759	0.0004
	$h(1.5)$	0.1097	0.0381	0.1097	0.0094	0.1113	0.0016	0.1093	0.0004	0.1073	0.0024
3	μ	7.5759	5.2159	7.5087	0.1031	7.5104	0.0655	7.5083	0.0675	7.5063	0.0696
	θ	2.3540	2.4615	2.2917	0.0969	2.2965	0.0575	2.2906	0.0634	2.2845	0.0695
	σ	1.6019	6.8020	1.6019	0.0010	1.6019	0.0000	1.6019	0.0000	1.6019	0.0000
	$R(1.5)$	0.9737	0.0137	0.9731	0.0016	0.9732	0.0005	0.9731	0.0006	0.9731	0.0006
	$h(1.5)$	0.1316	0.0479	0.1324	0.0077	0.1333	0.0018	0.1322	0.0006	0.1311	0.0005

Table 16. Interval estimation for μ, θ, σ, $R(t)$ and $h(t)$ with carbon fibers data.

Sample	Par.	ACI-NA			ACI-NL			BCI			HPD		
		Lower	Upper	IL	Lower	Upper	IL	Lower	Upper	IL	Lower	Upper	IL
1	μ	7.9517	12.259	4.3069	8.1657	12.505	4.3396	9.8948	10.191	0.2959	9.8932	10.187	0.2936
	θ	0.2256	2.4798	2.2542	0.5879	3.1122	2.5243	1.1472	1.4341	0.2869	1.1371	1.4216	0.2845
	σ	0.3505	31.598	31.248	6.0070	42.480	36.473	15.748	16.068	0.3195	15.771	16.085	0.3148
	$R(1.5)$	0.9376	0.9979	0.0602	0.9381	0.9983	0.0603	0.9635	0.9706	0.0071	0.9635	0.9706	0.0071
	$h(1.5)$	0.0523	0.3015	0.2492	0.0875	0.3578	0.2703	0.1591	0.1983	0.0391	0.1587	0.1977	0.0390
2	μ	0.0000	33.398	33.398	0.0121	38.877	38.865	4.5703	4.8723	0.3020	4.5686	4.8694	0.3008
	θ	0.0000	16.756	16.756	0.1015	31.690	31.589	3.4519	3.7425	0.2906	3.4395	3.7285	0.2890
	σ	0.0000	3.2488	3.2488	0.0014	6.5148	6.1534	0.1138	0.1509	0.0371	0.1135	0.1503	0.0368
	$R(1.5)$	0.9461	1.0064	0.0603	0.9466	1.0069	0.0603	0.9716	0.9797	0.0081	0.9718	0.9799	0.0080
	$h(1.5)$	0.0350	0.1844	0.1494	0.0555	0.2167	0.1612	0.0925	0.1293	0.0368	0.0922	0.1287	0.0365
3	μ	0.0000	17.799	17.799	1.9651	29.207	27.241	7.3578	7.6621	0.3042	7.3588	7.6624	0.3036
	θ	0.0000	7.1784	7.1784	0.3032	18.276	17.973	2.1468	2.4372	0.2903	2.1384	2.4253	0.2869
	σ	0.0000	14.934	14.934	0.0004	2.1394	2.1390	1.5999	1.6039	0.0040	1.6000	1.6039	0.0039
	$R(1.5)$	0.9469	1.0005	0.0537	0.9472	1.0009	0.0537	0.9700	0.9760	0.0060	0.9700	0.9759	0.0060
	$h(1.5)$	0.0378	0.2254	0.1876	0.0645	0.2684	0.2039	0.1182	0.1481	0.0299	0.1177	0.1475	0.0298



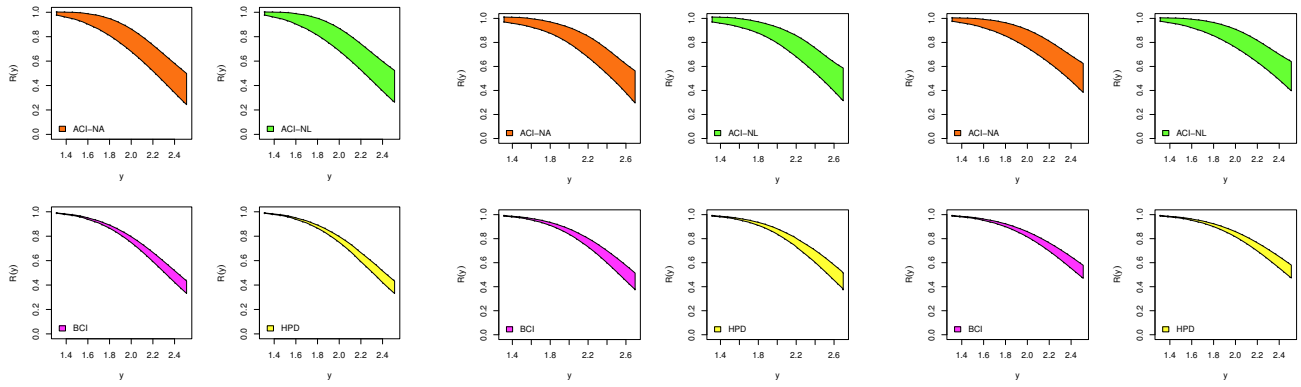
(a) Sample 1 (b) Sample 2 (c) Sample 3

Figure 6. Density and Trace (left and right) plots for μ, θ, σ, $R(t)$ and $h(t)$ from carbon fibers data.

Table 17. Statistics for 40,000 Markovian variates of μ, θ, σ, $R(t)$ and $h(t)$ from carbon fibers data.

Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	St.D.	Sk.
1	μ	10.038	9.9055	9.9872	10.038	10.088	0.0756	0.1108
	θ	1.2891	1.2359	1.2375	1.2887	1.3392	0.0740	0.0705
	σ	15.909	15.716	15.856	15.909	15.964	0.0810	-0.0643
	$R(1.5)$	0.9672	0.9655	0.9660	0.9673	0.9685	0.0018	-0.1809
	$h(1.5)$	0.1774	0.1838	0.1703	0.1770	0.1839	0.0101	0.2297
2	μ	4.7198	4.5846	4.6677	4.7201	4.7717	0.0777	0.0183
	θ	3.5952	3.4400	3.5450	3.5943	3.6453	0.0744	0.0597
	σ	0.1320	0.1338	0.1257	0.1318	0.1382	0.0093	0.0492
	$R(1.5)$	0.9759	0.9749	0.9746	0.9760	0.9773	0.0021	-0.2492
	$h(1.5)$	0.1097	0.1100	0.1032	0.1093	0.1159	0.0094	0.2190
3	μ	7.5087	7.3737	7.4557	7.5095	7.5614	0.0783	0.0157
	θ	2.2917	2.1389	2.2415	2.2905	2.3417	0.0742	0.0380
	σ	1.6019	1.6020	1.6012	1.6018	1.6025	0.0010	0.0087
	$R(1.5)$	0.9731	0.9724	0.9721	0.9732	0.9742	0.0015	-0.1839
	$h(1.5)$	0.1324	0.1315	0.1271	0.1320	0.1375	0.0077	0.2054

One of the basic affairs in reliability analysis is how to distinguish the real action of the reliability parameter $R(t)$ at every data point. Thus, to highlight the effectiveness of the reliability parameter, Figure 7 shows the estimates provided by the asymptotic methods (ACI-NA-ACI-NL techniques) and Bayes' approaches (BCI-HPD interval techniques) using the three proposed samples. It is evidence that interval estimation of $R(t)$ created by the ACI-NA behave well compared to the ACI-NL, while the results created from the BCI and HPD interval techniques almost approach each other.



(a) Sample 1 (b) Sample 2 (c) Sample 3

Figure 7. Plots for reliability interval estimates from carbon fibers data.

5.2. Electromigration data

In actuality, failures in microcircuits can occur due to the movement of atoms in the circuit's conductors, a phenomenon known as electromigration. This application displays the lifetime analysis of 59 conductors recorded by an accelerated life test; see Table 18. This dataset does not include censored observations and was taken from Lawless [51].

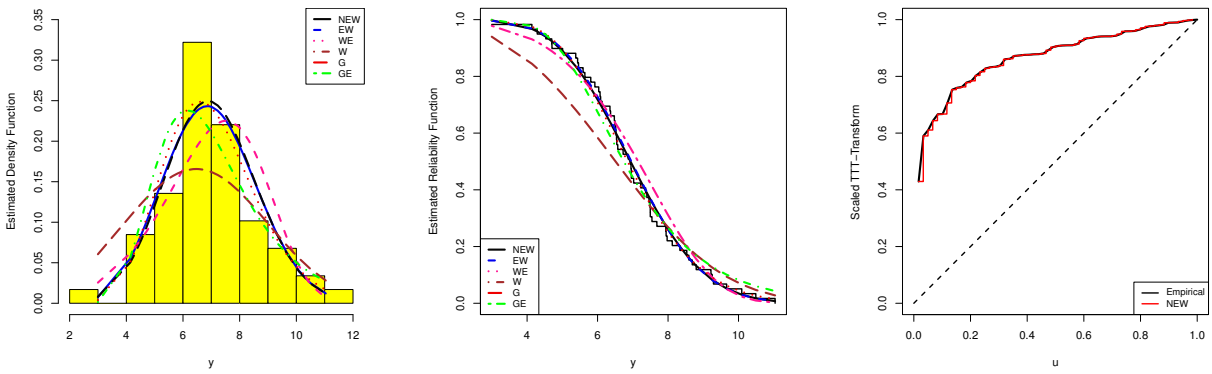
Table 18. Failure times from an accelerated life test of 59 conductors.

2.997	4.137	4.288	4.531	4.700	4.706	5.009	5.381	5.434	5.459
5.589	5.640	5.807	5.923	6.033	6.071	6.087	6.129	6.352	6.369
6.476	6.492	6.515	6.522	6.538	6.545	6.573	6.725	6.869	6.923
6.948	6.956	6.958	7.024	7.224	7.365	7.398	7.459	7.489	7.495
7.496	7.543	7.683	7.937	7.945	7.974	8.120	8.336	8.532	8.591
8.687	8.799	9.218	9.254	9.289	9.663	10.092	10.491	11.038	

Just like our calculations in Subsection 5.1, we shall compare the NEW distribution against the same competitors shown in Table 12. The MLEs with their St.Ers are listed in 19, in addition, the fitness measures: NL, A, CA, B, HQ, and KS distance with the *P*-value are provided in Table 19. The results in 19 indicate that the NEW distribution has a better fit to the conductors' data than other distributions. Figure 8 shows graphically the relative histograms of the conductors data set and the fitted PDFs, fitted/empirical RFs, and TTT plots. The same numerical findings are also supported by Figure 8. Figure 8(c) indicates that the TTT plot for the conductors data displays an increasing failure rate, so the proposed model can be adapted to fit this data type because it can present an increasing failure rate.

Table 19. Fit results of the NEW with competing models from conductors' data.

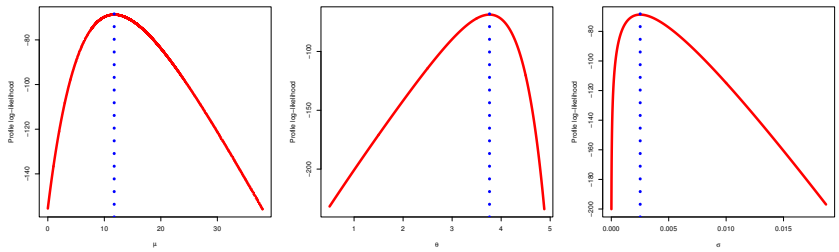
Model	MLE(St.Er)			NL	A	CA	B	HQ	KS	
	μ	θ	σ						Statistic	*P*-value
NEW	22.832(10.988)	1.5700(1.4875)	0.8922(3.9683)	111.31	227.66	227.88	231.82	229.28	0.0682	0.9291
HEE	0.7360(0.2791)	0.9650(0.2776)	174.79(67.749)	116.33	238.67	239.11	244.90	241.10	0.1474	0.1391
MOEE	3.9335(2.6170)	57.257(34.093)	0.0733(0.0647)	113.657	233.31	233.75	239.55	235.75	0.0766	0.8533
MOG	0.1059(0.0905)	9.3193(8.4746)	0.2855(0.1058)	116.797	239.59	240.03	245.83	242.03	0.1167	0.3694
EW	0.1610(0.0350)	2.4940(2.0190)	3.0657(1.1400)	111.34	228.62	229.06	234.85	231.05	0.0704	0.9122
WE	0.1335(0.0211)	2.9312(0.3529)	0.1754(0.1117)	114.10	234.20	234.64	240.44	236.64	0.1245	0.2948
IW	-	2.9749(0.2178)	206.25(76.331)	126.09	256.19	256.40	260.34	256.81	0.1706	0.0573
W	-	3.1119(0.0728)	0.0020(0.0002)	119.70	243.40	243.61	247.55	245.02	0.1996	0.0155
G	-	18.104(3.3030)	2.5933(0.4797)	111.83	228.87	229.31	235.11	231.31	0.0707	0.9091
GE	-	52.026(19.122)	0.6412(0.0622)	114.95	233.89	234.11	238.05	235.52	0.1042	0.5103



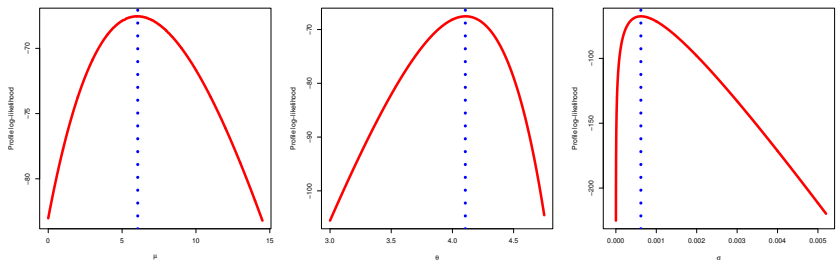
(a) (b) (c)

Figure 8. The fitted PDFs (a), fitted RFs (b), and TTT (c) curves for the NEW and its alternative models from conductors example.

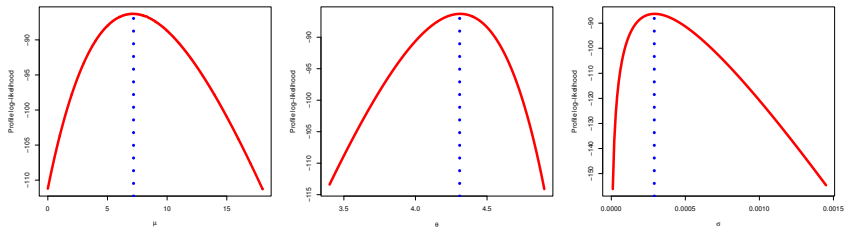
Now, by taking $k = 29$ and with different selections for T and $\mathbf{R}$, we create three A-PT2-C sampling schemes considering the complete conductors' data and reporting them in Table 20. For each generated A-PT2-C data, the maximum likelihood with 95% ACI-NA/ACI-NL intervals estimates as well as the Bayes based on 95% BCI/HPD intervals estimates of μ, θ, σ, $R(t)$, and $h(t)$ (at $t = 5$) are calculated. Following the Gibbs algorithm proposed in Section 3, to evaluate the Bayes' estimations, 50,000 variates, with the first 10,000 abandoned, are created. Then, the Bayes estimates against the SEL and GEL (for $\nu = (-5, -0.05, +5)$) are derived. Tables 21 and 22 report the point and interval results with the corresponding St.Ers and ILs for μ, θ, σ, $R(t)$, and $h(t)$. To collect the desired MCMC variates of μ, θ, and σ, the MLEs of μ, θ, and σ are selected as starting points. The profile log-likelihood curves are plotted and shown in Figure 9. It is noted, from Table 21, that the Bayes' estimates of μ, θ, σ, $R(t)$, and $h(t)$ perform better compared to the frequentist estimates. The interval limits of μ, θ, σ, $R(t)$, and $h(t)$ developed by the asymptotic procedure (ACI-NA/ACI-NL) and Bayes procedure (BCI/HPD) are very similar.



(a) Sample 1



(b) Sample 2



(c) Sample 3

Figure 9. The profile log-likelihood functions of μ, θ and σ from conductors data.

To assess whether the simulated MCMC samples achieve proper convergence, Figure 10 illustrates the density and trace plots for μ, θ, σ, $R(t)$, and $h(t)$, based on the generated samples in Table 20. The

figure confirms that the proposed Gibbs sampling method converges effectively and that the chosen burn-in size is sufficient to eliminate the influence of initial values. Additionally, these plots indicate that the estimates for μ, θ, and σ are approximately symmetrical, whereas the estimates for $R(t)$ and $h(t)$ exhibit negative and positive skewness, respectively.

Table 20. Three A-PT2-C samples from conductors data.

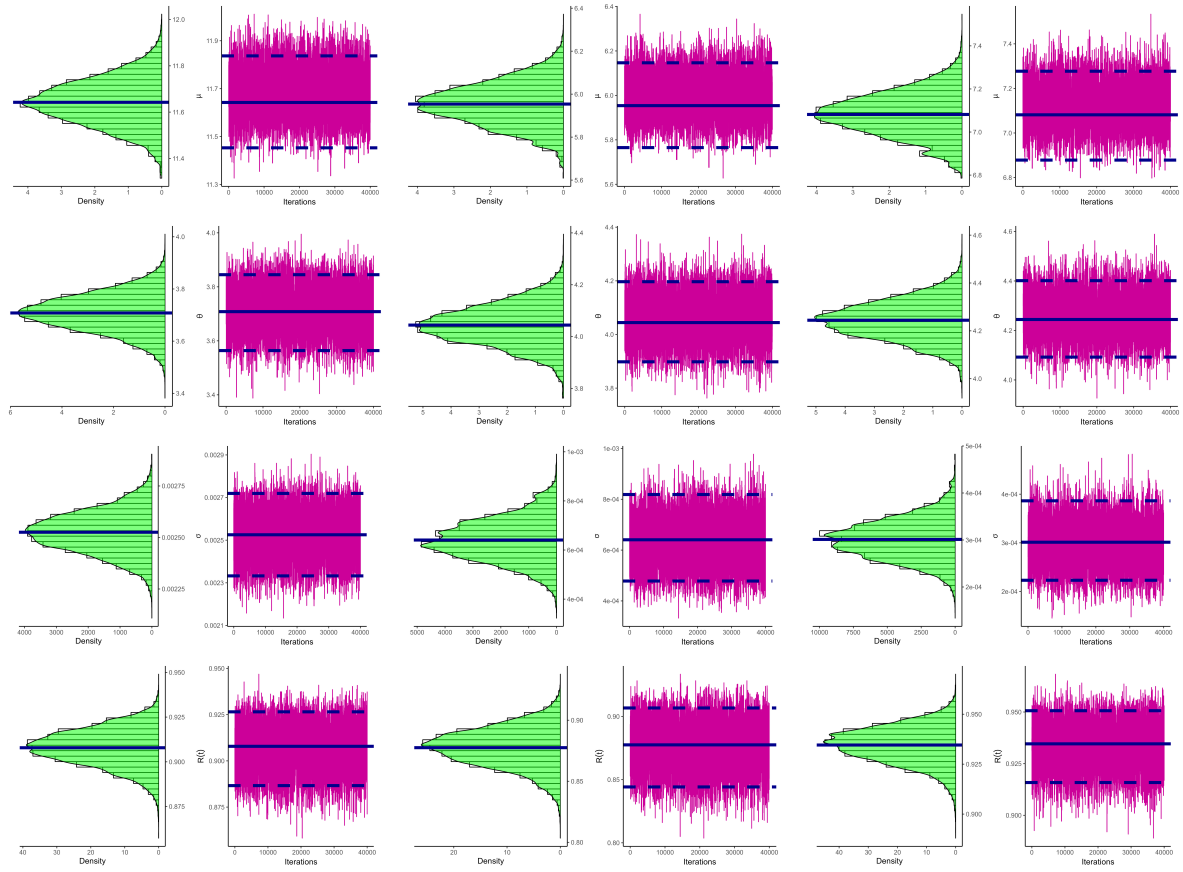
Sample	Scheme	$T(d)$	R_k	Censored sample
1	$(5^6, 0^*23)$	4.8(4)	10	2.997, 4.137, 4.288, 4.706, 5.009, 5.381, 5.434, 5.459, 5.589, 5.640, 5.807, 5.923, 6.087, 6.129, 6.352, 6.369, 6.476, 6.515, 6.522, 6.538, 6.545, 6.573, 6.725, 6.869, 6.923, 6.958, 7.024, 7.224, 7.459
2	$(0^*12, 5^6, 0^*11)$	6.2(15)	15	2.997, 4.137, 4.288, 4.531, 4.700, 4.706, 5.009, 5.381, 5.434, 5.459, 5.589, 5.640, 5.807, 5.923, 6.033, 6.492, 6.538, 6.545, 6.573, 6.725, 6.923, 6.948, 6.956, 6.958, 7.024, 7.224, 7.365, 7.398, 7.459
3	$(0^*23, 5^6)$	6.6(25)	20	2.997, 4.137, 4.288, 4.531, 4.700, 4.706, 5.009, 5.381, 5.434, 5.459, 5.589, 5.640, 5.807, 5.923, 6.033, 6.071, 6.087, 6.129, 6.352, 6.369, 6.476, 6.492, 6.515, 6.522, 6.538, 6.725, 6.923, 6.948, 6.956

Table 21. Point estimates of μ, θ, σ, $R(t)$ and $h(t)$ from conductors data.

Sample	Par.	MLE		SEL		GEL					
$\nu \rightarrow$						-5		-0.05		+5	
1	μ	11.745	9.6431	11.6424	0.1421	11.644	0.1006	11.642	0.1026	11.640	0.1047
	θ	3.7628	2.0566	3.7080	0.0897	3.7108	0.0520	3.7074	0.0554	3.7039	0.0589
	σ	0.0025	0.0133	0.0025	0.0001	0.0025	0.0000	0.0025	0.0000	0.0025	0.0000
	$R(5)$	0.9025	0.0345	0.9079	0.0116	0.9081	0.0056	0.9078	0.0053	0.9075	0.0050
	$h(5)$	0.1254	0.0331	0.1169	0.0171	0.1207	0.0047	0.1160	0.0093	0.1112	0.0141
2	μ	6.0586	6.8345	5.9539	0.1438	5.9572	0.1014	5.9531	0.1054	5.9490	0.1095
	θ	4.1053	1.3812	4.0444	0.0985	4.0473	0.0580	4.0437	0.0617	4.0399	0.0654
	σ	0.0006	0.0022	0.0006	0.0001	0.0007	0.0000	0.0006	0.0000	0.0006	0.0000
	$R(5)$	0.8725	0.0410	0.8775	0.0166	0.8781	0.0056	0.8774	0.0049	0.8767	0.0041
	$h(5)$	0.1450	0.0335	0.1371	0.0215	0.1429	0.0021	0.1357	0.0093	0.1283	0.0167
3	μ	7.1852	11.749	7.0816	0.1441	7.0844	0.1008	7.0809	0.1043	7.0773	0.1078
	θ	4.3088	2.1736	4.2443	0.1025	4.2473	0.0615	4.2436	0.0652	4.2398	0.0690
	σ	0.0003	0.0017	0.0003	0.0000	0.0003	0.0000	0.0003	0.0000	0.0003	0.0000
	$R(5)$	0.9316	0.0229	0.9346	0.0094	0.9348	0.0032	0.9346	0.0030	0.9343	0.0028
	$h(5)$	0.0815	0.0186	0.0767	0.0123	0.0801	0.0014	0.0759	0.0056	0.0717	0.0098

Table 22. Interval estimation for μ, θ, σ, $R(t)$ and $h(t)$ of conductors data.

Sample	Par.	ACI-NA			ACI-NL			BCI			HPD		
		Lower	Upper	IL	Lower	Upper	IL	Lower	Upper	IL	Lower	Upper	IL
1	μ	0.0000	30.645	30.645	2.3494	58.712	56.363	11.453	11.837	0.3838	11.460	11.842	0.3814
	θ	0.0000	7.7936	7.7936	1.2891	10.983	9.6943	3.5638	3.8449	0.2811	3.5669	3.8465	0.2797
	σ	0.0000	0.0286	0.0286	0.0000	79.552	79.552	0.0023	0.0027	0.0004	0.0023	0.0027	0.0004
	$R(5)$	0.8349	0.9701	0.1352	0.8374	0.9727	0.1353	0.8866	0.9265	0.0399	0.8884	0.9283	0.0398
	$h(5)$	0.0604	0.1903	0.1299	0.0747	0.2104	0.1357	0.0901	0.1483	0.0582	0.0882	0.1460	0.0579
2	μ	0.0000	19.454	19.454	0.6640	55.283	54.619	5.7651	6.1467	0.3816	5.7542	6.1328	0.3786
	θ	1.3983	6.8124	5.4141	2.1231	7.9382	5.8150	3.8981	4.1968	0.2987	3.8981	4.1965	0.2984
	σ	0.0000	0.0049	0.0049	0.0000	0.6235	0.6235	0.0005	0.0008	0.0003	0.0005	0.0008	0.0003
	$R(5)$	0.7921	0.9530	0.1608	0.7957	0.9568	0.1611	0.8442	0.9067	0.0625	0.8453	0.9075	0.0622
	$h(5)$	0.0793	0.2107	0.1314	0.0922	0.2281	0.1360	0.1012	0.1796	0.0784	0.1017	0.1800	0.0783
3	μ	0.0000	30.213	30.213	0.2915	77.126	76.835	6.8784	7.2771	0.3987	6.8777	7.2667	0.3890
	θ	0.0487	8.5689	8.5202	1.6032	11.581	9.9777	4.0923	4.4012	0.3089	4.0912	4.3993	0.3081
	σ	0.0000	0.0036	0.0036	0.0000	23.643	23.643	0.0002	0.0004	0.0002	0.0002	0.0004	0.0002
	$R(5)$	0.8867	0.9764	0.0897	0.8878	0.9775	0.0898	0.9159	0.9506	0.0348	0.9175	0.9521	0.0345
	$h(5)$	0.0449	0.1180	0.0731	0.0520	0.1276	0.0756	0.0566	0.1011	0.0446	0.0556	0.0995	0.0440

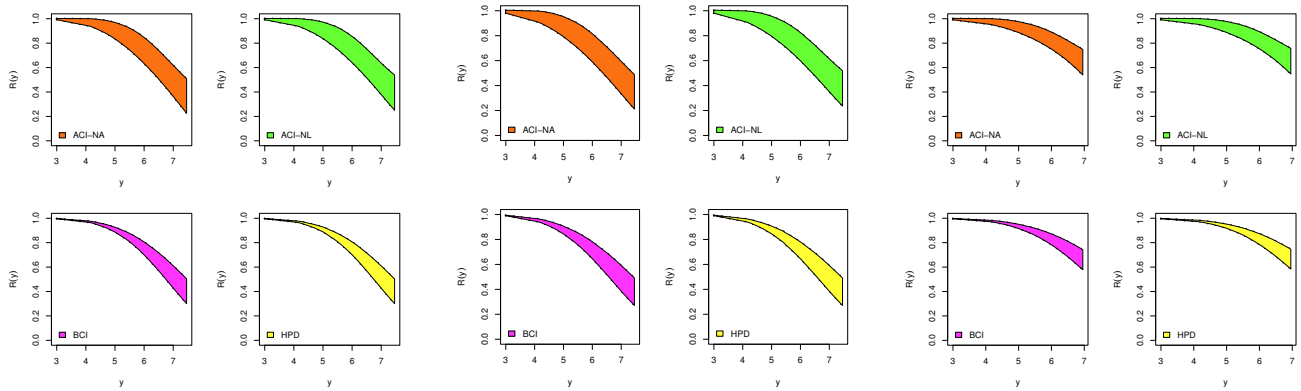


(a) Sample 1 (b) Sample 2 (c) Sample 3

Figure 10. Density and Trace (left and right) plots of μ, θ, σ, $R(t)$ and $h(t)$ for conductors data.

Table 23. Statistics for 40,000 Markovian variates of μ, θ, σ, $R(t)$ and $h(t)$ from conductors data.

Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	St.D.	Sk.
1	μ	11.642	11.413	11.576	11.642	11.709	0.0987	0.0290
	θ	3.7080	3.6597	3.6614	3.7090	3.7566	0.0711	-0.0943
	σ	0.0025	0.0026	0.0025	0.0025	0.0026	0.0001	-0.0005
	$R(5)$	0.9079	0.9093	0.9012	0.9082	0.9151	0.0103	-0.2244
	$h(5)$	0.1169	0.1130	0.1063	0.1163	0.1266	0.0149	0.2823
2	μ	5.9539	5.7112	5.8872	5.9543	6.0206	0.0986	0.0493
	θ	4.0444	3.9201	3.9916	4.0433	4.0952	0.0773	0.0486
	σ	0.0006	0.0008	0.0006	0.0006	0.0007	0.0001	0.1136
	$R(5)$	0.8775	0.8649	0.8671	0.8783	0.8884	0.0159	-0.2279
	$h(5)$	0.1371	0.1469	0.1231	0.1359	0.1497	0.0200	0.3129
3	μ	7.0816	6.8777	7.0149	7.0822	7.1484	0.1001	0.0170
	θ	4.2443	4.1072	4.1892	4.2449	4.2989	0.0797	0.0396
	σ	0.0003	0.0003	0.0003	0.0003	0.0003	0.0000	0.1651
	$R(5)$	0.9346	0.9396	0.9287	0.9351	0.9408	0.0089	-0.3104
	$h(5)$	0.0767	0.0683	0.0685	0.0761	0.0841	0.0114	0.3698



(a) Sample 1 (b) Sample 2 (c) Sample 3

Figure 11. Reliability interval estimates plots from conductors data.

In actual situations, it is preferable to know the interval performance of the reliability parameter. Thus, using all time points in Table 20, Figure 11 displays the interval estimation plots of $R(t)$ calculated via the ACI-NA, ACI-NL, BCI, and HPD procedures. It demonstrates that the length for the interval for the reliability function created from ACI-NL (or ACI-NA) approach is very similar, and the same results are consequently obtained under the case of HPD (or BCI) intervals.

To sum up, the analysis of the given real data sets reveals that the NEW lifetime model is an accurate model for describing engineering-application phenomena. Given that the empirical functions of both datasets exhibit an increasing trend, and since many competing models, such as the W and EW distributions, are capable of capturing such behavior, the results demonstrate that the proposed distribution achieves a superior fit for the two engineering applications. If the partitioner's major concern is the size of failed items, the NEW lifetime model under the suggested censoring is applicable and extremely beneficial in stopping the experimental process. Using the suggested approach, one may obtain adequate estimates of the reliability time parameters rather than utilizing the full sample,

which incurs high costs. Lastly, the proposed point and interval estimates, as shown by data sets from carbon fibers and conductors when performing an adaptive Type-II progressively strategy, revealed the applicability of the proposed estimation approaches to real-world phenomena.

6. Concluding remarks

This study utilizes both frequentist and Bayesian approaches to estimate the unknown parameters of the NEW distribution within the framework of adaptive Type-II progressive censoring. Approximate confidence intervals for all parameters were constructed using the asymptotic normality of the likelihood estimators and their log-transformed counterparts. Bayesian estimators were derived under gamma priors, employing both squared-error and general-entropy losses. To approximate the Bayes estimators and construct the corresponding credible intervals, we utilized Gibbs sampling via adaptive rejection Metropolis sampling. Through extensive numerical comparisons, we assessed the performance of the various estimators obtained. The simulation results demonstrated that the Bayesian approach consistently provides more accurate estimates compared to the frequentist method. To illustrate the practical applications of the proposed parameter estimation methods, we analyzed two real-world datasets involving the lifetimes of carbon fibers and conductors. In these applications, the NEW lifetime distribution was shown to offer a better fit than other alternatives including the conventional Weibull distribution, where different goodness of fit measures were used to indicate the results. Consequently, we recommend the use of the Bayesian estimation approach via Gibbs sampling for parameter estimation of the NEW distribution where the sample is adaptive Type-II progressively censored. We believe that the methodologies explored in this study will be valuable for data analysts and reliability engineers. Although this study focuses on analyzing the NEW parameters from such censoring, the inferential techniques presented here can be reimplemented for other lifetime models or in-depth model evaluation in future research.

Author contributions

Hanan Haj Ahmad: Funding acquisition (equal); Methodology (equal); Writing–original draft (equal); Writing–review & editing (equal); Osama E. Abo-Kasem: Methodology (equal); Project administration; Writing–review & editing (equal). Ahmed Rabaiah: Methodology (equal); Data curation (equal); Ahmed Elshahhat: Methodology (equal); Software (equal); Writing–original draft (equal). All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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Appendix 1

From (2.3), the Fisher elements $l_{ij},\ i, j = 1, 2, 3$ with respect to μ, θ and σ are

$$l_{11} = -\sigma\xi_{\mu}^{**} - \sum_{i=1}^{k}(\mu + \theta y_i)^{-2},$$

$$l_{22} = -\sigma\xi_{\theta}^{**} - \sum_{i=1}^{k} y_i(\mu + \theta y_i)^{-2},$$

$$l_{33} = -\frac{k}{\sigma^2},$$

$$l_{12} = -\sigma\xi_{\mu\theta}^{**} - \sum_{i=1}^{k} y_i(\mu + \theta y_i)^{-2},$$

$$l_{13} = -\xi_{\mu}^{*},$$

$$l_{23} = -\xi_{\theta}^{*},$$

with

$$\xi_{\mu}^{**} = \sum\nolimits_{i=1}^{k} y_i^{\theta-2} e^{-\mu y_i^{-1}} + \sum\nolimits_{i=1}^{d} R_i y_i^{\theta-2} e^{-\mu y_i^{-1}} + R_k y_k^{\theta-2} e^{-\mu y_k^{-1}}$$

$$\xi_{\theta}^{**} = \sum\nolimits_{i=1}^{k} y_i^{\theta} \log^2(y_i) e^{-\mu y_i^{-1}} + \sum\nolimits_{i=1}^{d} R_i y_i^{\theta} \log^2(y_i) e^{-\mu y_i^{-1}} + R_k y_k^{\theta} \log^2(y_k) e^{-\mu y_k^{-1}}$$

$$\xi_{\mu\theta}^{**} = -\sum\nolimits_{i=1}^{k} y_i^{\theta-1} \log(y_i) e^{-\mu y_i^{-1}} - \sum\nolimits_{i=1}^{d} R_i y_i^{\theta-1} \log(y_i) e^{-\mu y_i^{-1}} - R_k y_k^{\theta-1} \log(y_k) e^{-\mu y_k^{-1}}.$$

Appendix 2

To show that the conditional PDFs $\mathcal{H}^*_\mu(\mu|\mathbf{y},\theta,\sigma)$ and $\mathcal{H}^*_\theta(\theta|\mathbf{y},\mu,\sigma)$ of μ and θ, respectively, are both log-concave, following Jiang et al. [52], the second derivatives of $\log(\mathcal{H}^*_\mu(\mu|\mathbf{y},\theta,\sigma))$ and of $\log(\mathcal{H}^*_\theta(\theta|\mathbf{y},\mu,\sigma))$ with regard to μ and θ, respectively, are nonpositive. Consequently, from (3.10), we have

$$\frac{\partial \log(\mathcal{H}^*_\mu(\mu|\mathbf{y},\theta,\sigma))}{\partial\mu} = -\sigma\xi^*_\mu - \sum\nolimits_{i=1}^{k} y_i^{-1} + \sum\nolimits_{i=1}^{k}(\mu+\theta y_i)^{-1}, \tag{.1}$$

$$\frac{\partial^2 \log(\mathcal{H}^*_\mu(\mu|\mathbf{y},\theta,\sigma))}{\partial\mu^2} = -\sigma\xi^{**}_\mu - \sum\nolimits_{i=1}^{k}(\mu+\theta y_i)^{-2} < 0, \tag{.2}$$

where ξ^*_μ and ξ^{**}_μ are available in Appendix 1.

Similarly, from (3.11), we have

$$\frac{\partial \log(\mathcal{H}^*_\theta(\theta|\mathbf{y},\mu,\sigma))}{\partial\theta} = -\sigma\xi^*_\theta + \sum\nolimits_{i=1}^{k}\log(y_i) + \sum\nolimits_{i=1}^{k} y_i(\mu+\theta y_i)^{-1}, \tag{.3}$$

$$\frac{\partial^2 \log(\mathcal{H}^*_\theta(\theta|\mathbf{y},\mu,\sigma))}{\partial\theta^2} = -\sigma\xi^{**}_\theta - \sum\nolimits_{i=1}^{k} y_i^2(\mu+\theta y_i)^{-2} < 0, \tag{.4}$$

where ξ^*_θ and ξ^{**}_θ are available in Appendix 1.

It is clear, from (.1) and (.3), that the conditional PDFs of μ and θ, respectively, hold the log-concavity property. Thus, to create random variates from $\mathcal{H}^*_\mu(\mu|\mathbf{y},\theta,\sigma)$ and $\mathcal{H}^*_\theta(\theta|\mathbf{y},\mu,\sigma)$ of μ, and θ, we can use the highly efficient ARMS method.

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